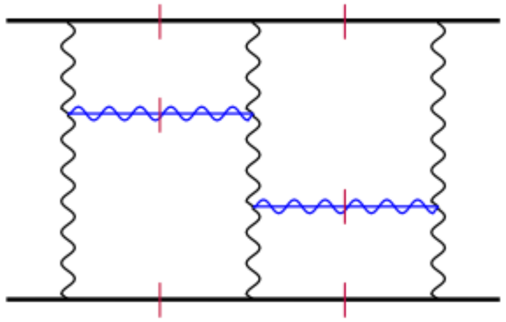
Gravity amplitudes in the high-energy limit

Vittorio Del Duca
INFN LNF



in collaboration with Francesco Alessio & Emanuele Rosi (INFN LNF)
Riccardo Gonzo (Queen Mary U.)
Ira Rothstein (Carnegie Mellon U.)
Michael Saavedra (UCLA)

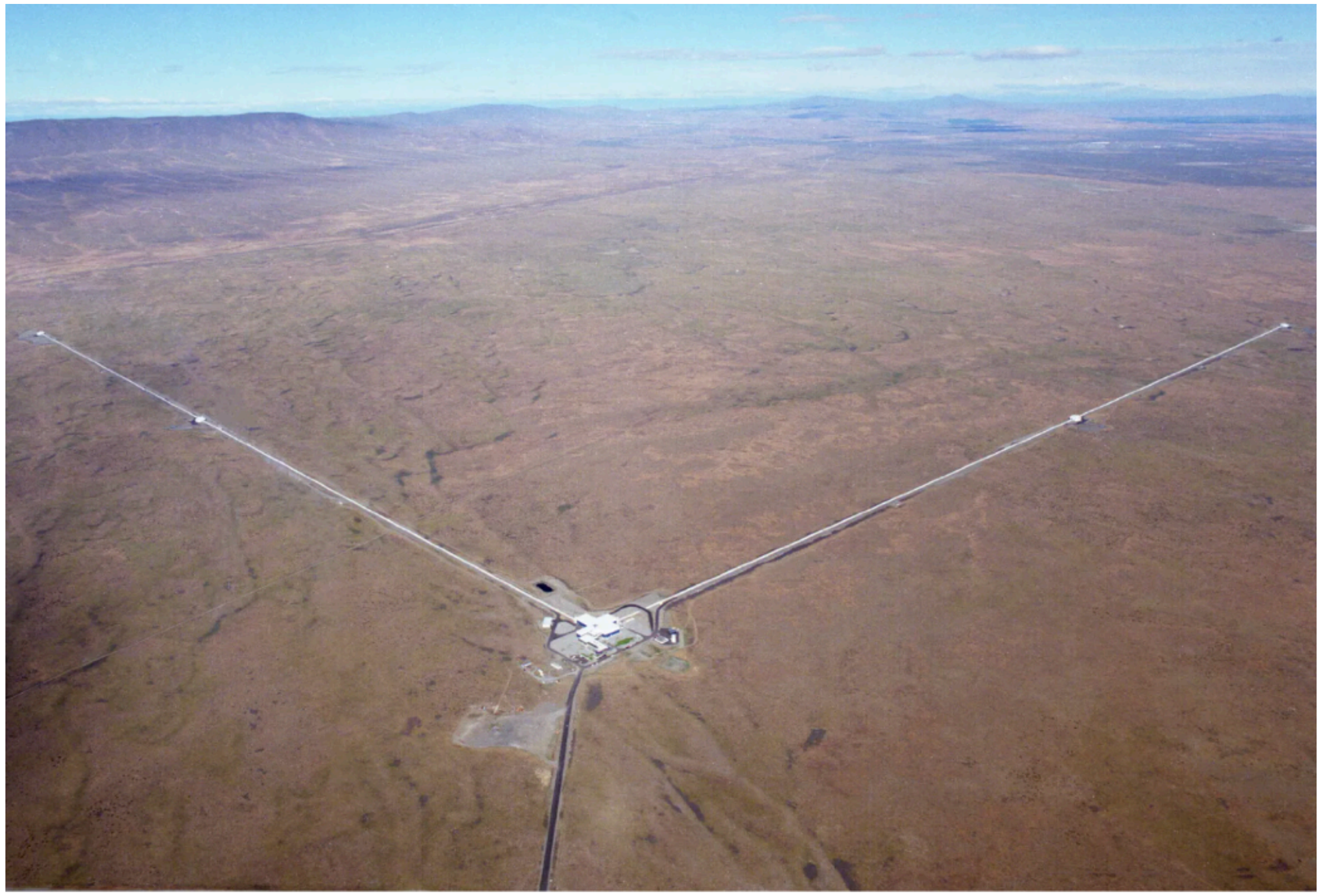
Rothstein Saavedra 2412.04428
Alessio VDD Gonzo Rosi Rothstein Saavedra 2511.11457
Alessio VDD Gonzo Rosi 2601.21687

Factorization in QCD and Beyond

Edinburgh 8 May 2026

Gravitational waves

- 1915: from GR, Einstein predicts GWs
- 2015: first GW signal, GW150914:
  two black holes, each about 30 $M_\odot$, $1.5 \cdot 10^9$ ly away



LIGO

KAGRA

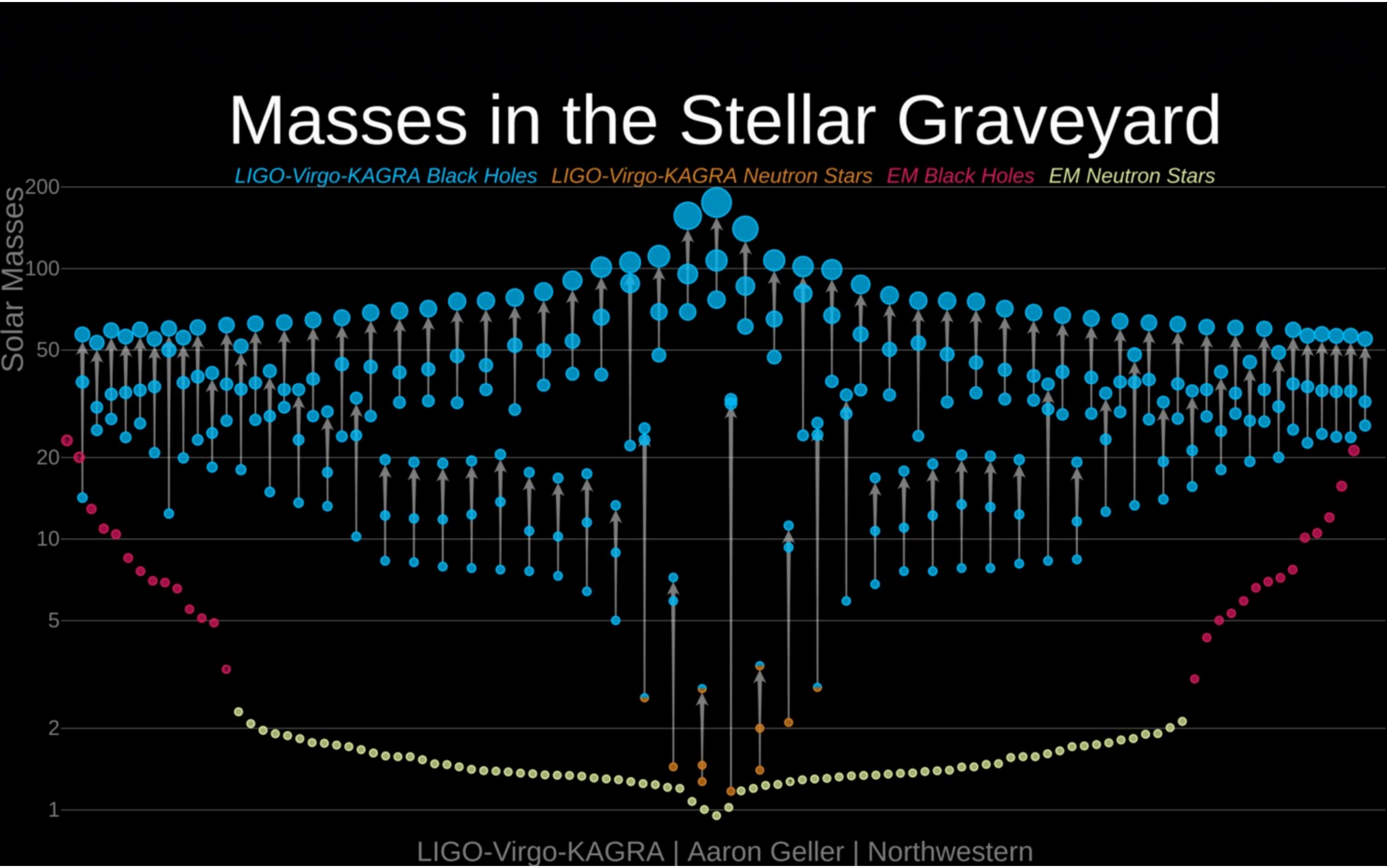
$f = 30$ Hz → $\lambda = 10^7$ m

$f = 10^4$ Hz → $\lambda = 3 \cdot 10^4$ m

mass $\lesssim 200\, M_\odot$

distance $z = 0.25 \sim 3.3 \cdot 10^9$ ly

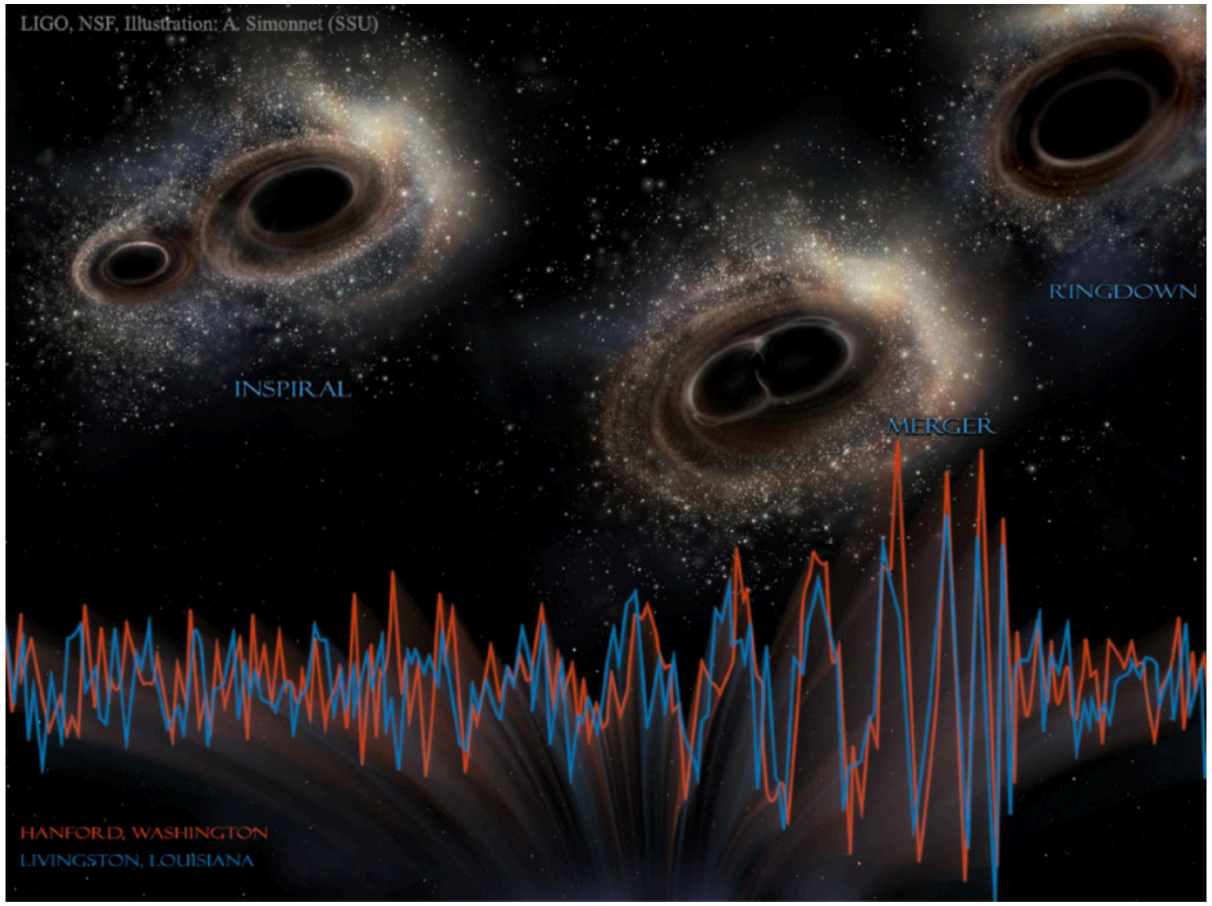
Virgo



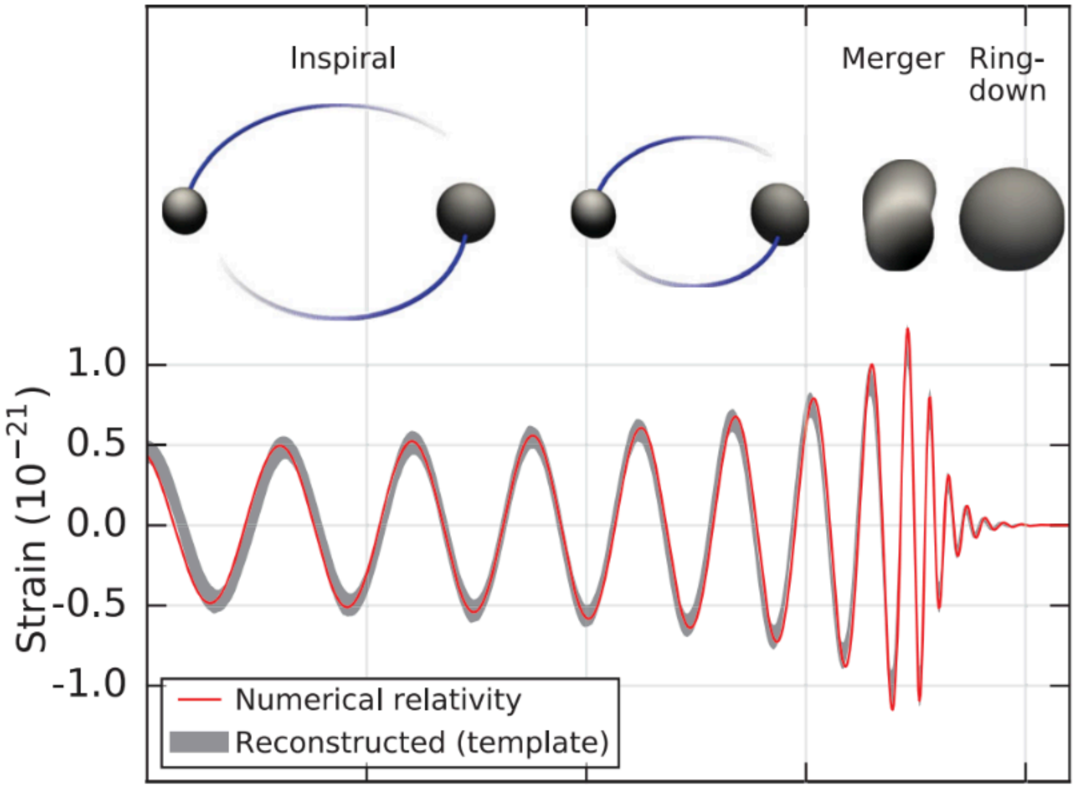
observed for < 200 $M_\odot$ so far

GW150914

LIGO/National Science Foundation/Aurore Simmonet/Somona State University

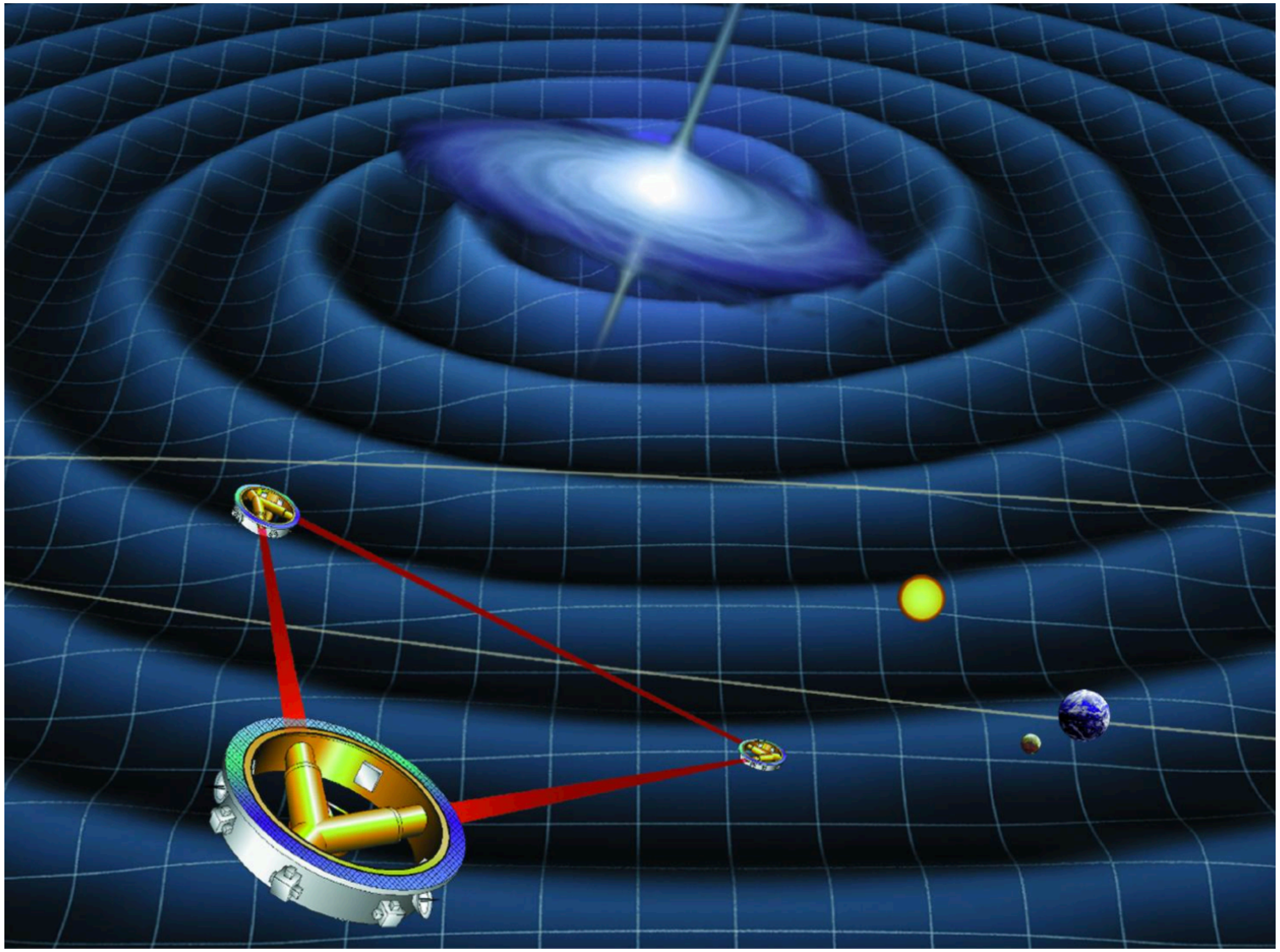


inspiral	merger	ringdown
perturbative expansions	NR	BHPT



LIGO & Virgo, PRL 116 (2016) 061102

Next-generation interferometers (ET, LISA, …)



LISA

launch ~ 2035

f = 0.1 Hz → $\lambda = 3 \cdot 10^9$ m

f = 10^{-4} Hz → $\lambda = 3 \cdot 10^{12}$ m

mass ~ 10^4 - 10^7 $M_\odot$ (MBH) distance z = 10 ~ $13.4 \cdot 10^9$ ly

M_1 ~ 10 $M_\odot$ M_2 ~ 10^5 $M_\odot$ (EMRI) distance z = 4 ~ $12 \cdot 10^9$ ly
inspiral with up to ~ 10^4 cycles

Parameter space

- time duration *T* of an inspiral waveform starting at initial GW frequency f_{in} scales like $T \propto \nu^{-1} f_{in}^{-8/3}$

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

- the number of cycles in inspiral phase is $O(m_1/m_2)$
- systems with $m_1/m_2 >> 1$ have a very long inspiral phase
- LVK detects systems with $m_1/m_2 < 30$
- ET will detect systems with $m_1/m_2 = O(1000)$
- LISA will detect systems (EMRI) with $m_1/m_2 = O(10^4)$

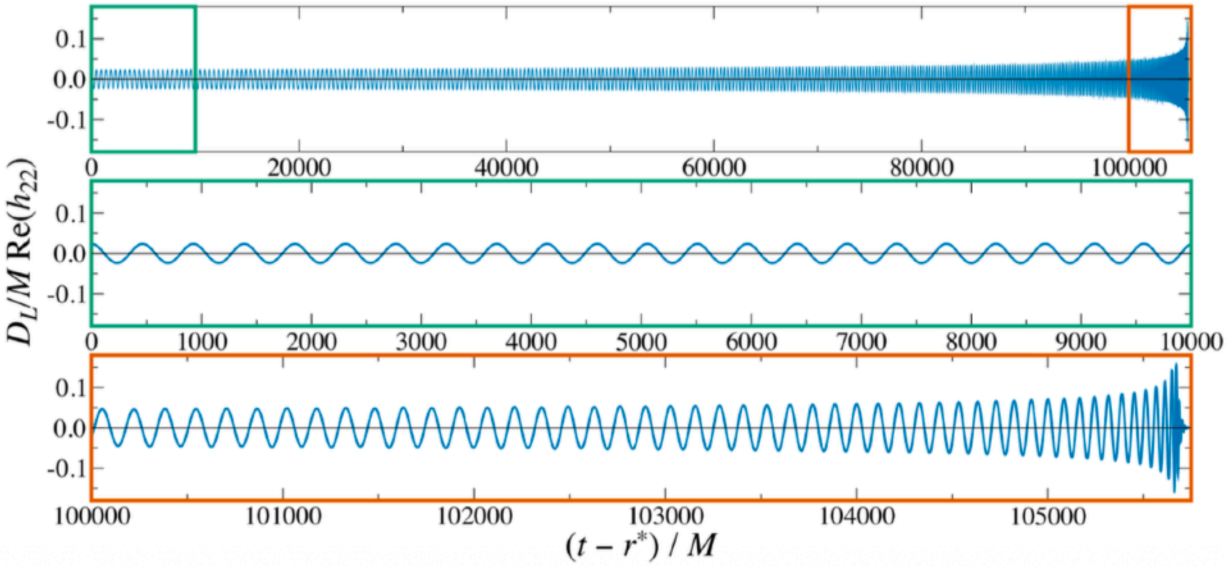


Figure. Example of NR simulation covering an 8 months signal with $m_1/m_2 = 1/7$, ~ 10^6 CPU hours. It covers one point in parameter space. [Szilagyi *et al.* 2015]

Black-hole binaries in classical gravity

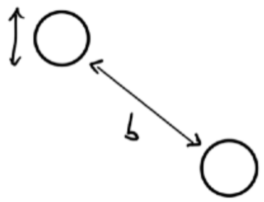
- elastic 2-body scattering $s + t + u = 2m_1^2 + 2m_2^2$

  $s = m_1^2 + m_2^2 + 2p_1 \cdot p_2$

- in classical gravity, impact parameter is much larger than De Broglie wavelength

  $|b| \gg \lambda = \dfrac{1}{|p|}$

  then angular momentum is large $J \sim |p \times b| \gg 1$



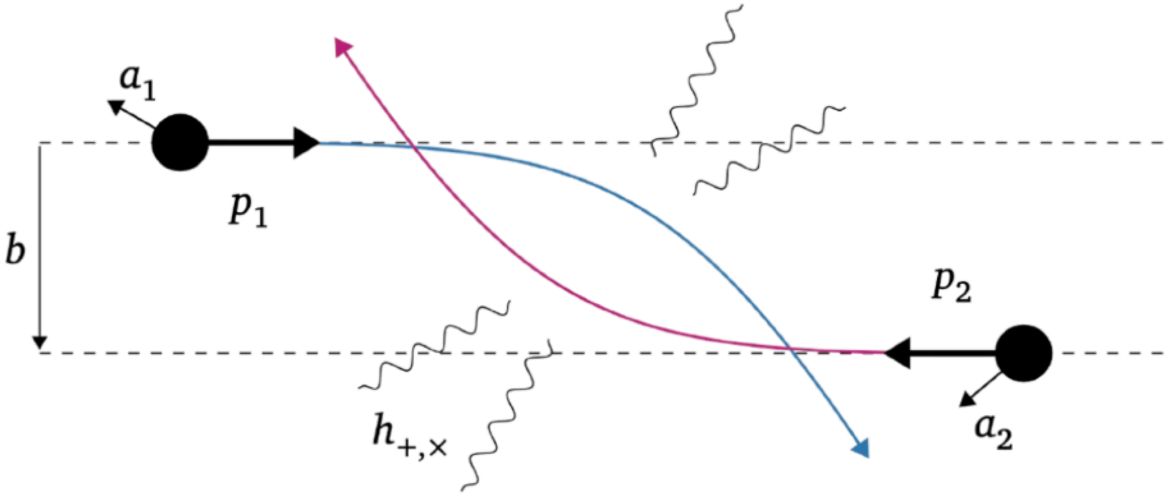
- since impact parameter is inverse of momentum transfer $|b| \sim \dfrac{1}{|q|}$

  $|p|, m_1, m_2 \sim J|q| \gg |q|$

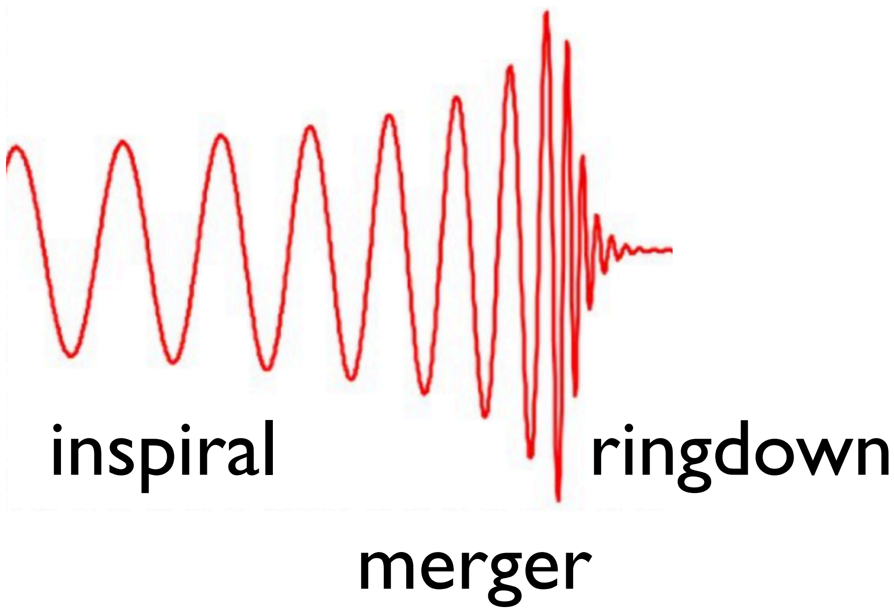
Bern Cheung Roiban Shen Solon Zeng 2019

- therefore the forward limit $s \sim m_1^2, m_2^2 \gg |t|$

  is naturally realised in 2-body scattering in classical gravity



Perturbative expansions



- interested in modelling inspiral phase with perturbative expansions

- Post Newtonian (PN) expansion: $v^2/c^2 << 1$

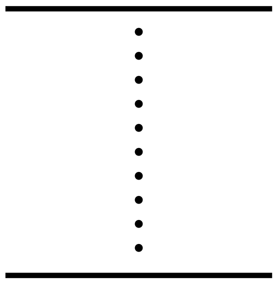
  (deals with 3-dim integrals in config. space)

  1 PN: Lorentz Droste 1917; Einstein Infeld Hoffmann 1938

  …

  6 PN: Bini Damour Geralico 2020-2021
  Blümlein Maier Marquard Schäfer 2020-2021

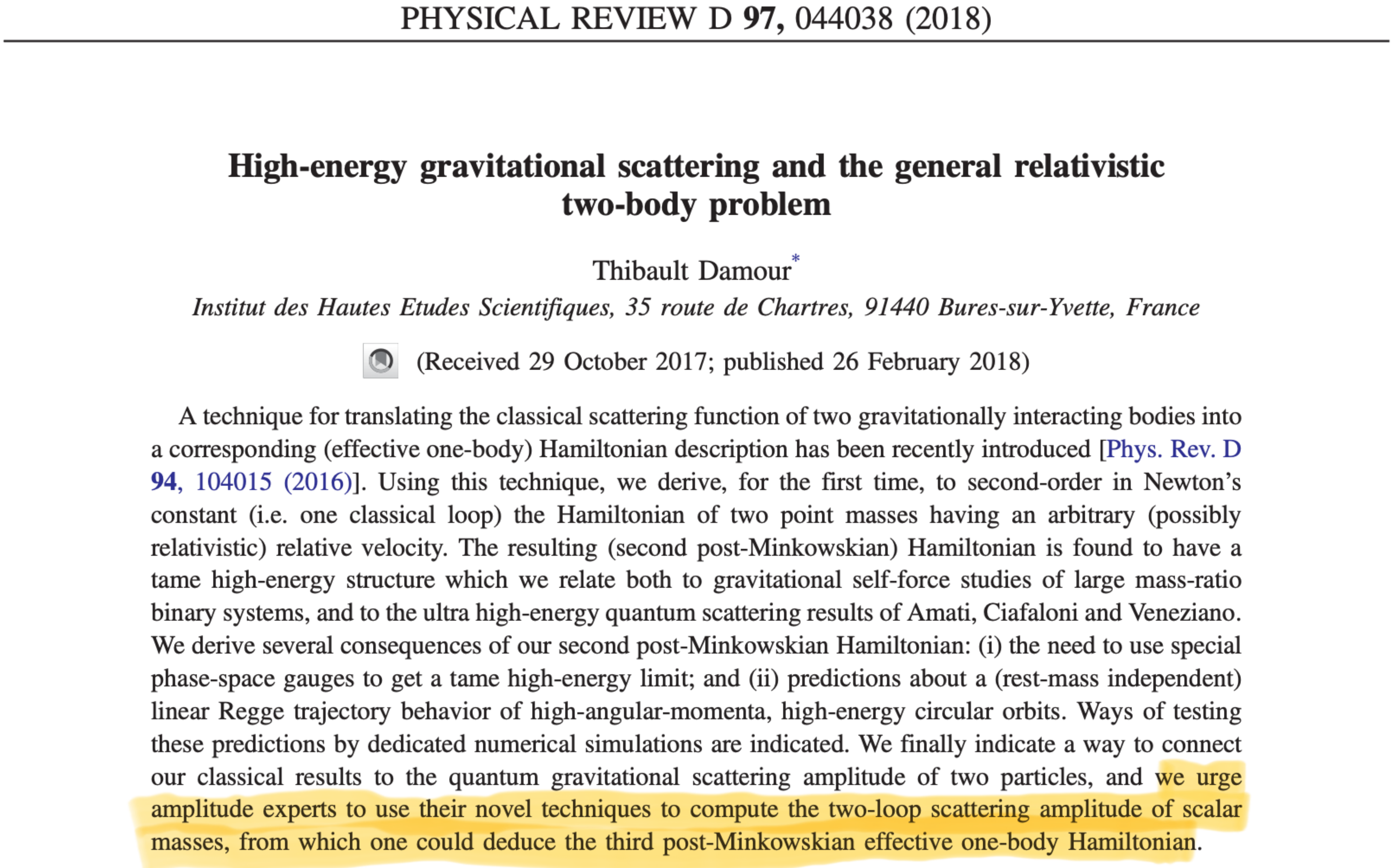
- Post Minkowskian (PM) expansion: $mG/r << 1$

  matches loop expansion of amplitude:
  1 PM = O(G) = tree level
  2 PM = O(G^2) = one loop, etc.

  2 PM: Westpfahl Goller 1979 … Cheung Rothstein Solon 1808.02489

- Virial theorem $mG/r \sim v^2$ connects PM to PN



PHYSICAL REVIEW D **97,** 044038 (2018)

High-energy gravitational scattering and the general relativistic two-body problem

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(Received 29 October 2017; published 26 February 2018)

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

Power scaling of PM expansion

4-pt amplitude for the scattering of two massive scalars minimally coupled to gravity

1PM: $A^{(0)}(s,t) \simeq G\, c^{(0)}_{0\mathrm{SF}}\, m_1^2 m_2^2$

$c^{(\ell)}_{n\mathrm{SF}} \equiv c^{(\ell)}_{n\mathrm{SF}}(s,t,m_1,m_2)$

2PM: $A^{(1)}(s,t) \simeq G^2\, c^{(1)}_{0\mathrm{SF}}\, m_1^2 m_2^2\,(m_1+m_2)$

3PM: $A^{(2)}(s,t) \simeq G^3 \left[c^{(2)}_{0\mathrm{SF}}\, m_1^2 m_2^2 (m_1^2+m_2^2) + c^{(2)}_{1\mathrm{SF}}\, m_1^3 m_2^3\right]$

4PM: $A^{(3)}(s,t) \simeq G^4 \left[c^{(3)}_{0\mathrm{SF}}\, m_1^2 m_2^2 (m_1^3+m_2^3) + c^{(3)}_{1\mathrm{SF}}\, m_1^3 m_2^3 (m_1+m_2)\right]$

5PM: $A^{(4)}(s,t) \simeq G^5 \left[c^{(4)}_{0\mathrm{SF}}\, m_1^2 m_2^2 (m_1^4+m_2^4) + c^{(4)}_{1\mathrm{SF}}\, m_1^3 m_2^3 (m_1^2+m_2^2) + c^{(4)}_{1\mathrm{SF}}\, m_1^4 m_2^4\right]$

Power scaling of PM expansion

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3PM: $A^{(2)}(s,t) \simeq G^3 \left[c_{0\mathrm{SF}}^{(2)}\, m_1^2 m_2^2 (m_1^2+m_2^2) + c_{1\mathrm{SF}}^{(2)}\, m_1^3 m_2^3\right]$

4PM: $A^{(3)}(s,t) \simeq G^4 \left[c_{0\mathrm{SF}}^{(3)}\, m_1^2 m_2^2 (m_1^3+m_2^3) + c_{1\mathrm{SF}}^{(3)}\, m_1^3 m_2^3 (m_1+m_2)\right]$

5PM: $A^{(4)}(s,t) \simeq G^5 \left[c_{0\mathrm{SF}}^{(4)}\, m_1^2 m_2^2 (m_1^4+m_2^4) + c_{1\mathrm{SF}}^{(4)}\, m_1^3 m_2^3 (m_1^2+m_2^2) + c_{1\mathrm{SF}}^{(4)}\, m_1^4 m_2^4\right]$

0SF

Power scaling of PM expansion

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0SF

1SF

Power scaling of PM expansion

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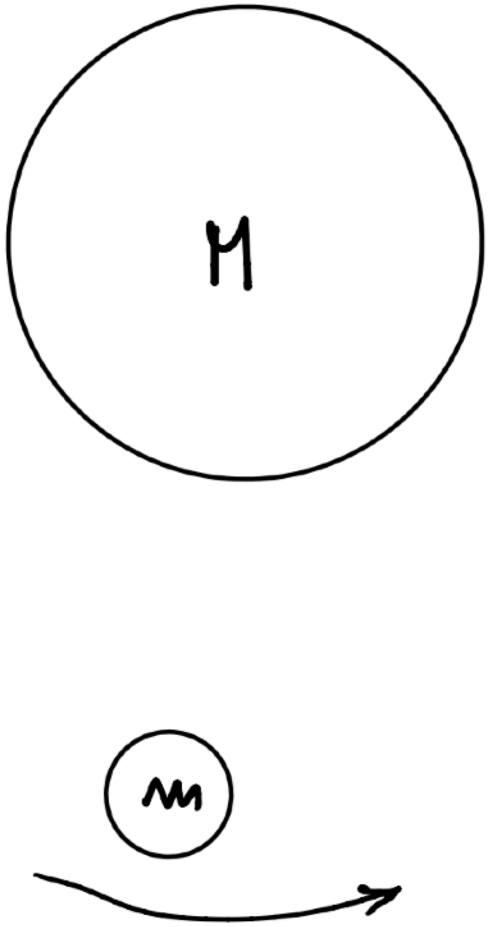
0SF

1SF

2SF

Self Force



- in Self Force (SF) expansion,
  one expands Einstein's equation in powers of m/M

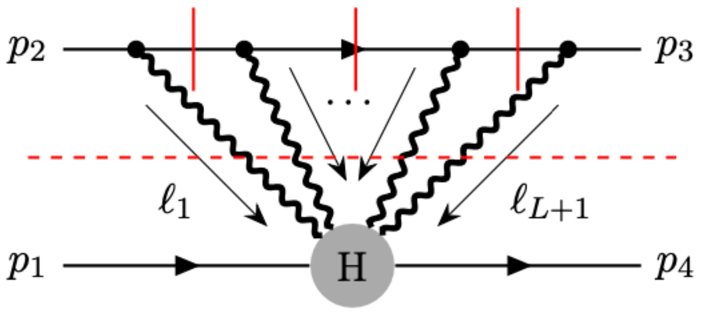
  0 SF = Probe limit = Schwarzschild
  1 SF: m/M
  2 SF: $(m/M)^2$

- Probe limit corresponds to Schwarzschild geometry:
  static (infinitely heavy) black hole

Probe limit

(*L*+1)-PM, 0 SF: $A^{(L)}_{0\mathrm{SF}}(s,t) \simeq G^{L+1} m_1^2 m_2^{L+2}\, y^2 (1 + O(1/y^2))$

$$y = \frac{s - m_1 - m_2}{2 m_1 m_2}$$



Brandhuber Chen Travaglini Wen 2108.04216 conjectured polynomial form, with unknown coefficients

Sasank Chava (2023, MSc thesis unpublished)

obtained polynomial coefficients leveraging geometric info
from geodesic eq. for test particle in a Schwarzschild background

Cheung Shah Solon 2010.08568
Damour 1710.10599

$$\mathcal{M}^0_{\text{probe}} = \frac{16\pi G_N m_1^2 m_2^2 (2y^2 - 1)}{(-q^2)},$$

$$\mathcal{M}^1_{\text{probe}} = 6\pi^2 G_N^2 m_1^2 m_2^3 \left(5y^2 - 1\right) (-q^2)^{-\frac{1}{2} - 2\epsilon},$$

$$\mathcal{M}^2_{\text{probe}} = \frac{2\pi G_N^3 m_1^2 m_2^4 (-q^2)^{-2\epsilon} \left(64y^6 - 120y^4 + 60y^2 - 5\right)}{3\epsilon \left(y^2 - 1\right)^2},$$

$$\mathcal{M}^3_{\text{probe}} = -\frac{35\pi^2 G_N^4 m_1^2 m_2^5 (-q^2)^{\frac{1}{2} - 3\epsilon} \left(33y^4 - 18y^2 + 1\right)}{8\left(y^2 - 1\right)},$$

$$\mathcal{M}^4_{\text{probe}} = -\frac{\pi G_N^5 m_1^2 m_2^6 (-q^2)^{1-4\epsilon} \left(1792y^{10} - 5760y^8 + 6720y^6 - 3360y^4 + 630y^2 - 21\right)}{40\epsilon \left(y^2 - 1\right)^4}$$

$$\mathcal{M}^5_{\text{probe}} = \frac{77\pi^2 G_N^6 m_1^2 m_2^7 (-q^2)^{3/2 - 5\epsilon} \left(221y^6 - 195y^4 + 39y^2 - 1\right)}{96\left(y^2 - 1\right)^2},$$

Bern Parra-Martinez Roiban Ruf Shen Solon Zeng 2021
Bjerrum-Bohr Planté Vanhove 2021

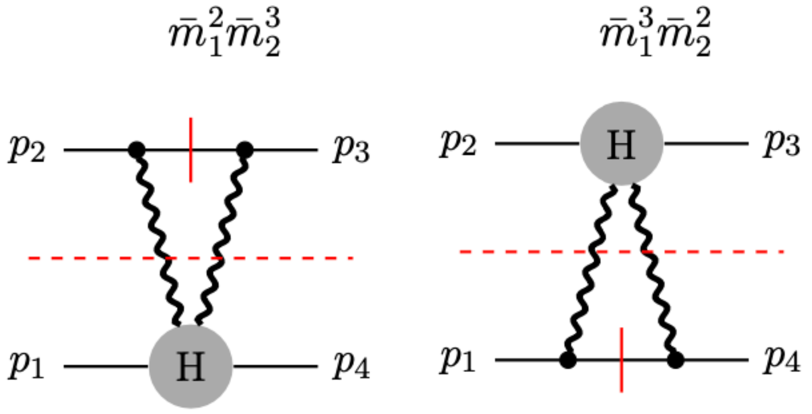
- an analytic resummation of next-to-probe (1 SF) terms is not yet known

Heavy mass effective field theory

Brandhuber Chen Travaglini Wen 2108.04216

2PM: $A^{(1)}(s,t) \simeq G^2 m_1^2 m_2^2\,(m_1+m_2)\,y^2(1+O(1/y^2))$

$$y = \frac{s-m_1-m_2}{2m_1m_2}$$

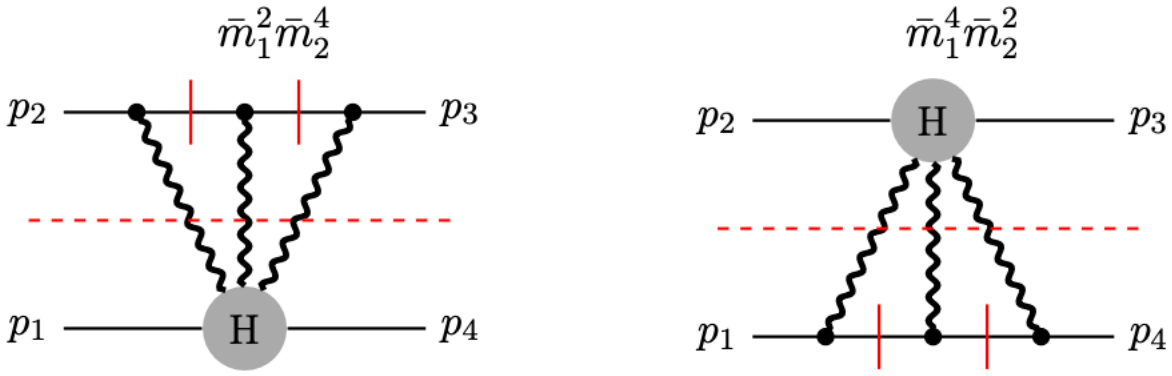


(probe limit)

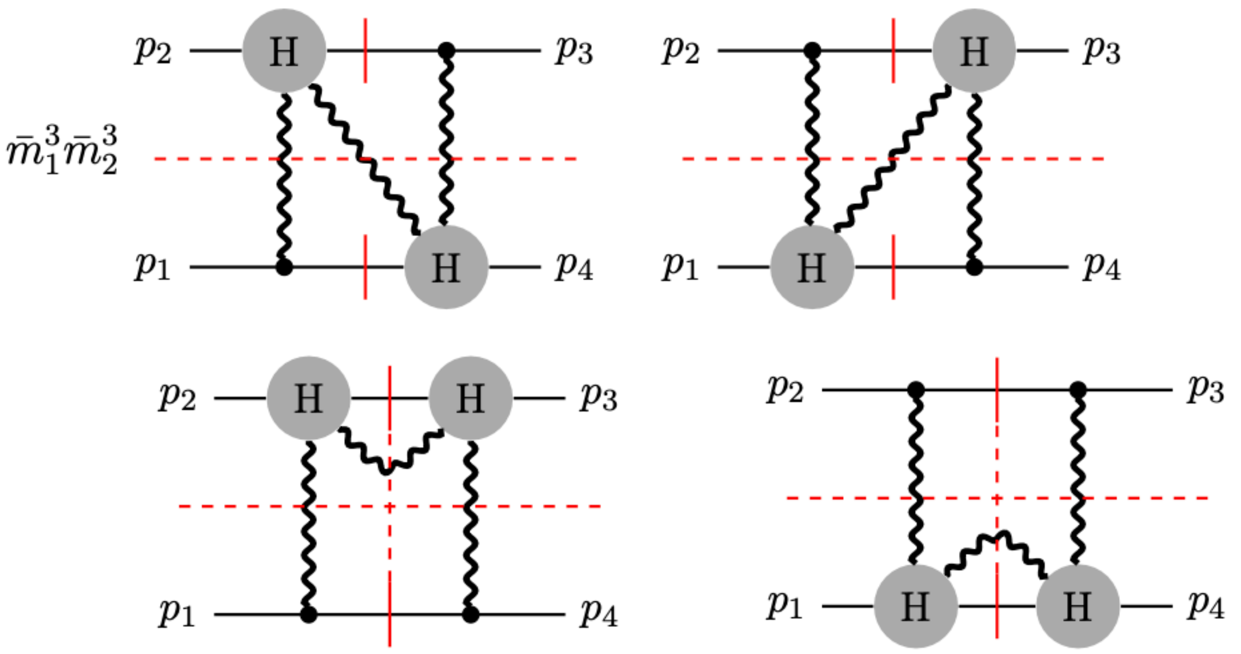
3PM, 0SF:

$$A^{(2)}_{0\mathrm{SF}}(s,t) \simeq G^3 m_1^2 m_2^2\,(m_1^2+m_2^2)\,y^2(1+O(1/y^2))$$

Bern Cheung Roiban Shen Solon Zeng 1901.04424 & 1908.01493



3PM, 1SF: (beyond the probe)



$$\mathrm{Re}A^{(2)}_{1\mathrm{SF}}(s,t) \simeq G^3 m_1^3 m_2^3\,y^3(1+O(1/y))$$

$$\mathrm{Im}A^{(2)}_{1\mathrm{SF}}(s,t) \simeq G^3 m_1^3 m_2^3\,y^3(\log(2y)+O(1/y))$$

radiation reaction needed
to cancel additional power of log(s/|t|)

(radiation reaction)

4 PM, 0 SF: $A^{(3)}_{0\mathrm{SF}}(s,t) \simeq G^4 m_1^2 m_2^2\,(m_1^3+m_2^3)\,y^2(1+O(1/y^2))$

4 PM, 1 SF: $A^{(3)}_{1\mathrm{SF}}(s,t) \simeq G^4 m_1^3 m_2^3\,(m_1+m_2)\,y^3\,f^{(3)}_{1\mathrm{SF}}(m_1,m_2,y)$

Bern Parra-Martinez Roiban Ruf Shen Solon Zeng 2101.07254 & 2112.10750
Dlapa Kälin Liu Neef Porto 2106.08276, 2112.11296 & 2210.05541

5 PM, 0 SF: $A^{(4)}_{0\mathrm{SF}}(s,t) \simeq G^5 m_1^2 m_2^2\,(m_1^4+m_2^4)\,y^2(1+O(1/y^2))$

5 PM, 1 SF: $A^{(4)}_{1\mathrm{SF}}(s,t) \simeq G^5 m_1^3 m_2^3\,(m_1^2+m_2^2)\,y^3\,f^{(4)}_{1\mathrm{SF}}(m_1,m_2,y)$

Driesse Jakobsen Klemm Mogull Nega Plefka Sauer Usovitsch 2403.07781 & 2411.11846
Dlapa Kälin Liu Porto 2506.20665

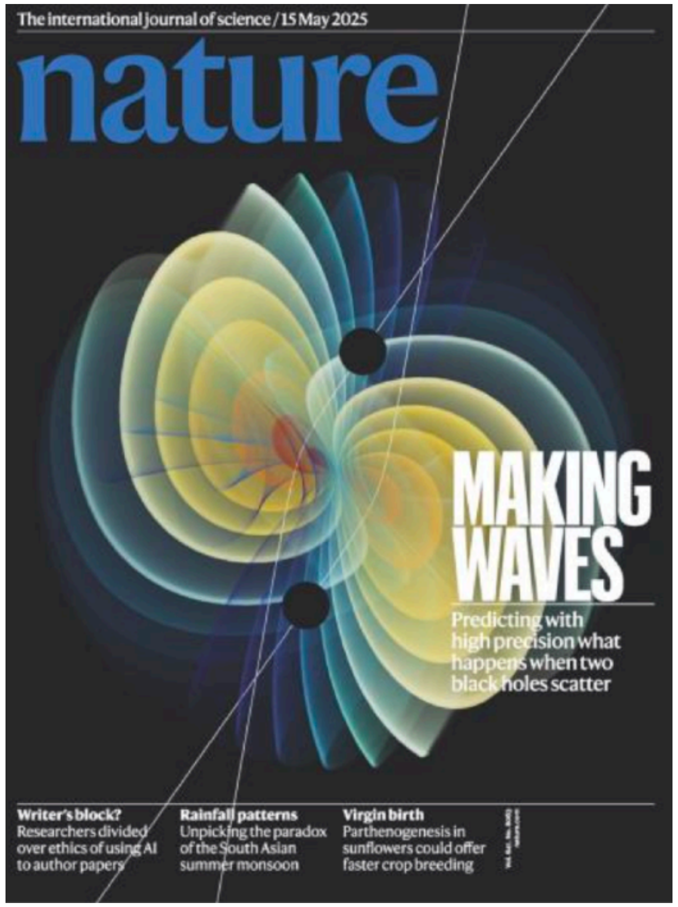
through world-line formalism

$$S = -\sum_{i=1}^{2}\frac{m_i}{2}\int d\tau_i g_{\mu\nu}(x_i(\tau_i))\,v_i^\mu(\tau_i)v_i^\nu(\tau_i)$$

5 PM, 2 SF: $A^{(4)}_{2\mathrm{SF}}(s,t) \simeq G^5 m_1^4 m_2^4\,y^4(\log^2(2y)+O(1/y))$

Alessio VDD Gonzo Rosi Rothstein Saavedra 2511.11457
(Regge limit, at leading log accuracy)

Driesse Jakobsen Mogull Nega Plefka Sauer Usovitsch 2601.16256
(only conservative)



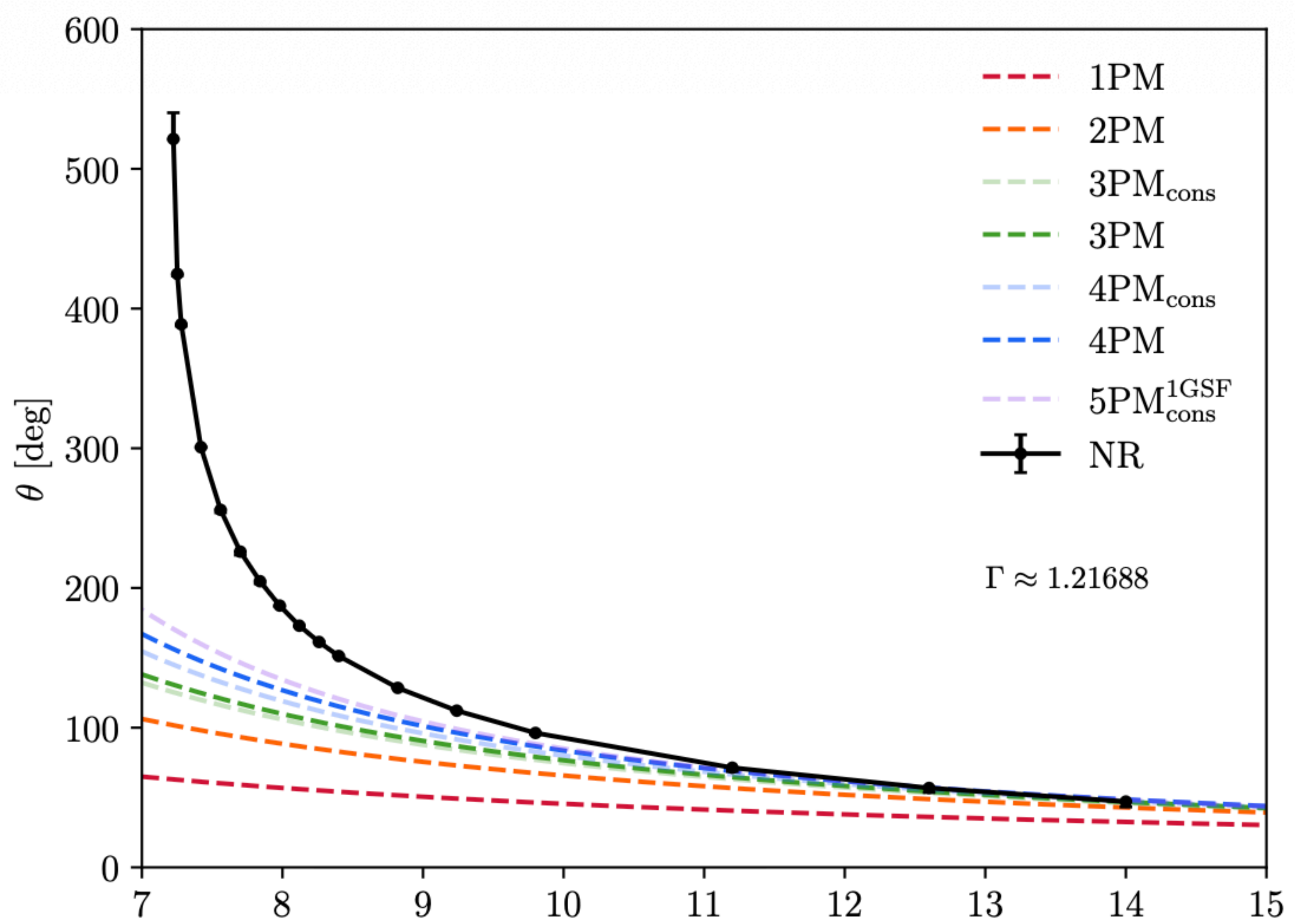
Writer's block? Researchers divided over ethics of using AI to author papers

Rainfall patterns Unpicking the paradox of the South Asian summer monsoon

Virgin birth Parthenogenesis in sunflowers could offer faster crop breeding

Comparison of PM to Numerical Relativity

deflection angle from NR data vs. PM



angular momentum

Pratten Schmidt Swain 2411.09652

Γ: energy in the NR simulation

“It has been shown that the non-resummed PM-expanded scattering angles demonstrate poor convergence towards NR. This motivates the exploration of *resummation* strategies… … in particular we focus on the approach to the high-energy limit for equal-mass non-spinning binaries, which proves to be challenging for all resummation schemes considered …”

- comparison worsens as we approach strong-field regime

Forward limits

- 2-body scattering in classical gravity naturally realises forward limit

  $s \sim m_1^2, m_2^2 \gg |t|$

- take high-energy (HE) limits of classical gravity
  - $s \gg m_1^2, m_2^2 \gg |t|$
  - $s \gg |t|, m_1^2, m_2^2$ Regge limit

    relative size of |t| and m^2 is immaterial, at leading log accuracy
  - $s \gg |t| \gg m_1^2, m_2^2$ massless limit

- In massless limit, we assume $Gs \gg 1 \gg G|t|$

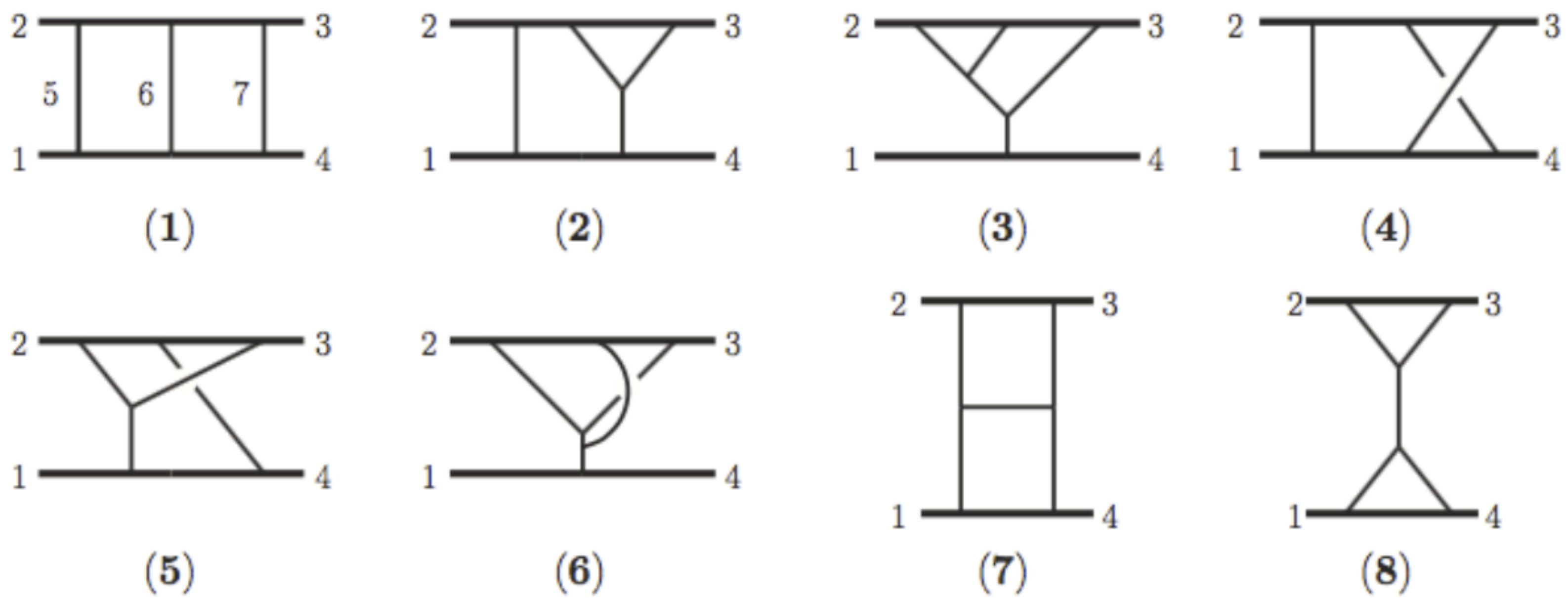
- Warning: HE limit and massless limit might not commute
  so smoothness of massless limit of HE limit must be checked

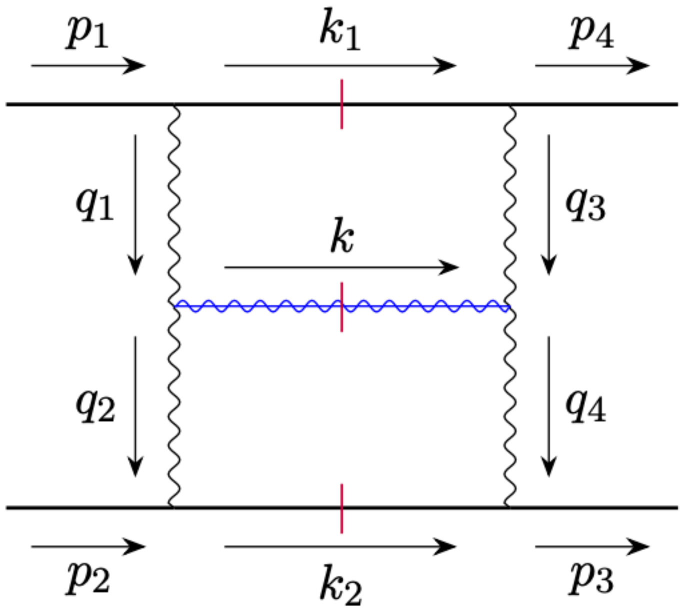
HE limits

- HE limit: large-y limit $y \to \dfrac{s}{2m_1m_2}$
- (n+1)PM 0SF (probe limit) amplitudes go like $G^{n+1}\, s^2\, (m_1^n + m_2^n)$
- 3PM 1SF amplitudes go like $G^3\, s^3\, \log(s)$ Amati Ciafaloni Veneziano 1990
  Di Vecchia Heissenberg Russo Veneziano 2008.12743
- 4PM 1SF amplitudes go like ?
- 5PM 1SF amplitudes go like ?
- 5PM 2SF amplitudes go like $G^5\, s^4\, \log^2(s)$

Alessio VDD Gonzo Rosi Rothstein Saavedra 2511.11457
Alessio VDD Gonzo Rosi 2601.21687

3 PM: Bern Cheung Roiban Shen Solon Zeng 1901.04424 & 1908.01493





H diagram

Amati Ciafaloni Veneziano 1990

Regge limit *s* >> |*t*| massless

$$\mathrm{Im} A^{(2)}(s,t) \simeq G^3 s^3 \log(s/t) \cdot (\text{poles in } \varepsilon)$$

$$\mathrm{Re} A^{(2)}(s,t) \simeq \frac{1}{\log(s/t)} \mathrm{Im} A^{(2)}(s,t)$$

Di Vecchia Heissenberg Russo Veneziano 2008.12743

massive

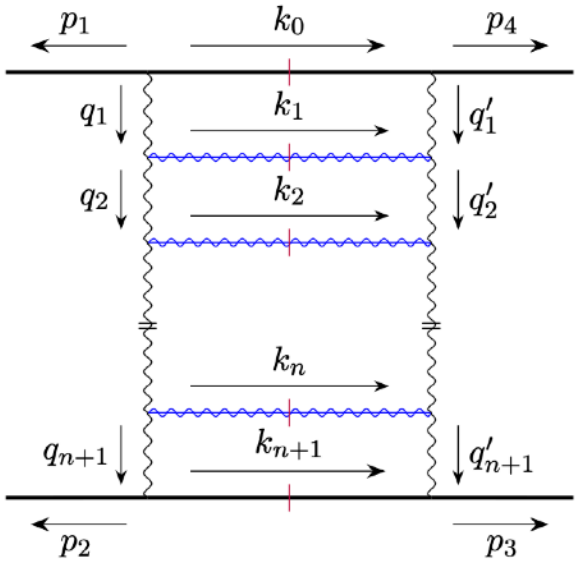
Regge theory of QCD and Gravity

- In the Regge limit, radiative corrections to 2→2 massless amplitudes display iterative patterns of the evolution in rapidity $y \simeq \log(s/|t|)$, either if the evolution occurs in the t channel or in the s channel

- In the Regge limit of QCD, the leading radiative corrections are associated to a gluon ladder exchanged in the t channel, the Reggeised or Glauber gluon. BFKL 1976-77

  t channel two-gluon ladder and s channel terms are logarithmically suppressed

$$\mathcal{A} \simeq \exp\left(i\pi \mathbf{T}_s^2 + \mathbf{T}_t^2 \log\left(\frac{s}{-t}\right)\right)$$



- In the Regge limit of gravity, the leading radiative corrections are due to the eikonal phase terms (Weinberg's soft gravitons).

  t channel one-graviton ladder is power suppressed in t/s

Bartels Lipatov Sabio-Vera 1208.3423
Melville Naculich Schnitzer White1306.6019

colour-kinematics duality

$$\mathbf{T}_s^2 \to s \qquad \mathbf{T}_t^2 \to t$$

- s channel ladders win over t channel ladders

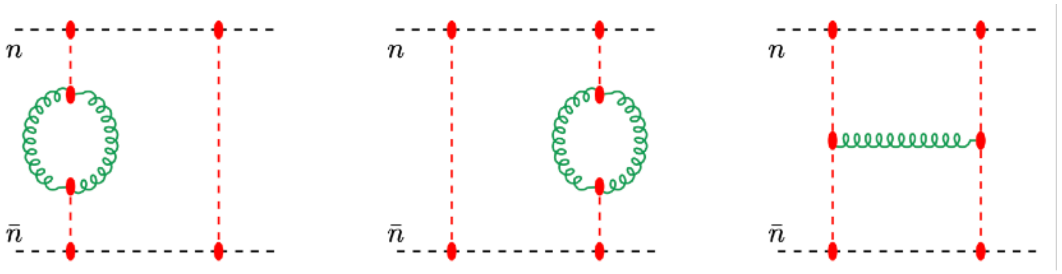
Gravity massless scattering: Glauber EFT

- Mimicking Glauber EFT of QCD   Rothstein Stewart 1601.04695

$$\mathcal{M}_{2\to2} = i\sum_{M} J_{(M)} \otimes S_{(M)} \otimes \bar{J}_{(M)}$$

Rothstein Saavedra 2412.04428 showed that the exchange of a *t*-channel two-graviton ladder is ruled by a rapidity RGE, whose anomalous dimension is Lipatov gravity (BFKL-like) kernel with graviton trajectory and graviton central-emission vertex (CEV)   Lipatov 1982

$$\nu\frac{d}{d\nu}S_{(N)} = -\gamma^{\nu}_{(N)} \otimes S_{(N)} - S_{(N)} \otimes \gamma^{\nu}_{(N)}$$



$$\gamma^{\nu}_{(M)} \sim \sum_{j} \omega_G(q_i) I_{\perp(M-1)} + \sum_{\text{Pairs } i,j} \mathcal{K}^{\text{GR}}(q_i, q_j; q)\, I_{\perp(M-2)}$$

- through $\hbar$ counting, easy to see that trajectory is quantum (powers of $\alpha_Q = G_N\, t$)

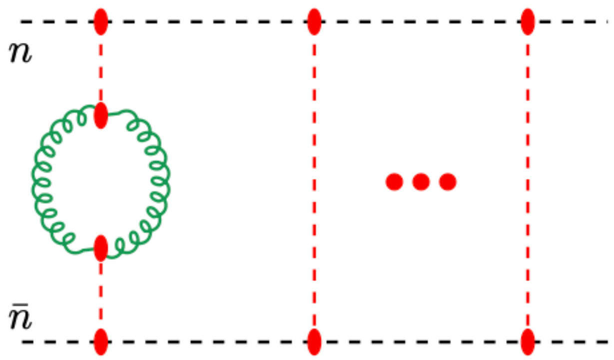
  in fact the whole two-graviton ladder is quantum,

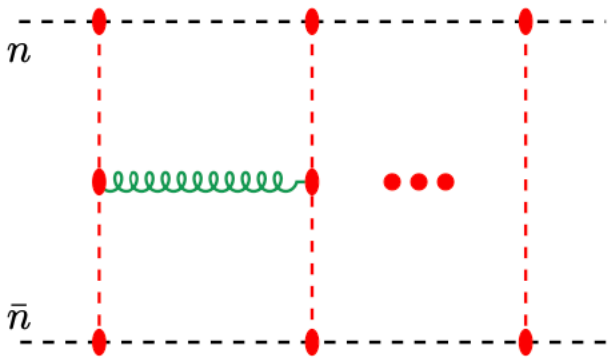
  except for one graviton CEV emission: the *H* diagram

$$\text{Im}\,\mathcal{M}^{(n+1)}_{2\to2}(s,q^2) \sim s^3 \log\left(\frac{s}{-t}\right)^n \frac{G}{q_\perp^2}\, G\, q_\perp^n\, (G\, q_\perp)^n$$

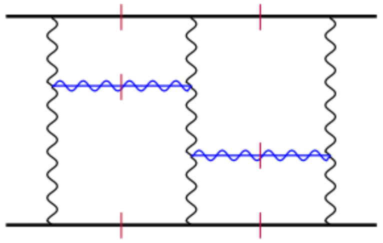
Gravity massless scattering: Glauber EFT

- Rothstein Saavedra 2412.04428 showed that the exchange of a *t*-channel multi-graviton ladder is ruled by a rapidity RGE, whose anomalous dimension is a convolution of gravity BFKL-like kernels





- through $\hbar$ counting, easy to see that the whole three-graviton ladder is quantum, except for the convolution of two gravity BFKL-like kernels



In fact, a *t*-channel ladder with *(n*+2) gravitons in the *s* channel features a classical term of (2n+3)-PM order, and provides a correction of $O((G^2\, s \log(s/|t|))^n)$ to the *H* diagram

Gravity massless scattering: shock waves

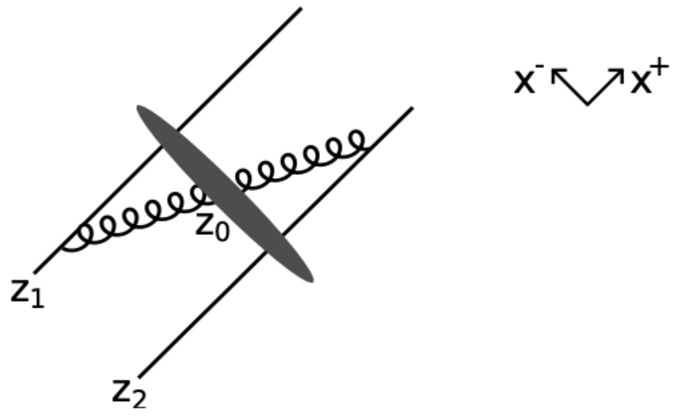
Alessio VDD Gonzo Rosi 2601.21687

- early work by Dray 't Hooft 1985
  Lodone Rychkov 0909.3519

  more recent by Raj Venugopalan 2311.03463, 2312.03507, 2406.10483

- colliding particles, projectile (target) moving in the + (-) light-cone dir
  with a transverse position z offset, $|\tilde{\Psi}_i\rangle \sim \tilde{\Phi}_i(z)\,|0\rangle$

  are modelled by infinite light-like Wilson lines



see Falcioni's & Gambuti's talks

$$\tilde{\Phi}_i(z) = \exp\left(i\frac{\kappa}{\hbar}\int_{-\infty}^{\infty} ds\, p_i^\mu p_i^\nu h_{\mu\nu}\left(sp_i + z\right)\right)$$

- because of motion in +/- directions, infinite Wilson lines yield rapidity divergences, regulated by rapidity cutoff η.
  Then states Φ renormalised by evolution equation, driven by boost Hamiltonian

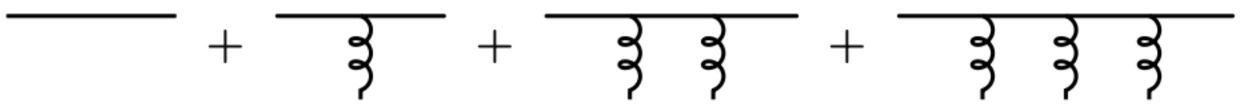
$$-\frac{d}{d\eta}\tilde{\Phi}_i^\eta = \hat{H}\,\tilde{\Phi}_i^\eta$$

gravity analog of Balitsky-JIMWLK

- parametrise Wilson line through Reggeised graviton expansion

$$\tilde{\Phi}_i(z) = \exp\left[i\frac{\kappa}{\hbar}\sqrt{\frac{s}{2}}\,\tilde{W}(z)\right] \qquad \tilde{W}(z) = \int_{-\infty}^{\infty} dx^+\, h_{++}(x^+, 0, z)\,.$$

$\Phi_i(q)|0\rangle =$



$+\cdots$

Caron-Huot 1309.6521
Caron-Huot Gardi Vernazza 1701.05241

Gravity massless scattering: shock waves

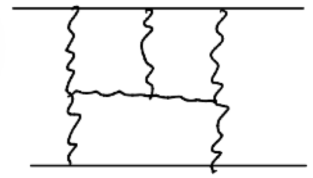
- amplitude driven by rapidity evolution between projectile & target

$$i\mathcal{M}^{\mathrm{MRK}}_{2\to 2}(s,q^2)\,\hat\delta^{(d)}\left(q-q'\right) = 2s\,\langle\Psi_j(q')|\,e^{-\hat H L}|\Psi_i(q)\rangle$$

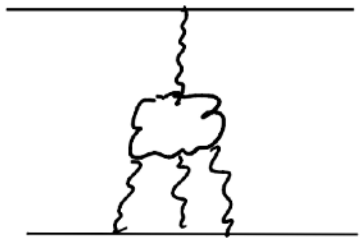
with boost Hamiltonian

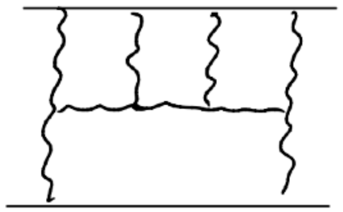
$$\hat H \begin{pmatrix} W \\ WW \\ WWW \\ \vdots \end{pmatrix} = \begin{pmatrix} \hat H_{1\to 1} & 0 & 0 & \cdots \\ 0 & \hat H_{2\to 2} & 0 & \cdots \\ 0 & 0 & \hat H_{3\to 3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} W \\ WW \\ WWW \\ \vdots \end{pmatrix} + \mathcal{O}(\kappa^4)$$

- amplitude is $s \leftrightarrow u$ crossing symmetric, thus off-diagonal *H* terms mixing even and odd number of Reggeised gravitons are forbidden $\hat H_{k,k\pm 2n+1} = 0$



- off-diagonal terms $\hat H_{k,k\pm 2n} = 0$ are also forbidden because either quantum or sub-leading





Gravity massless scattering: shock waves

- amplitude driven by rapidity evolution between projectile & target

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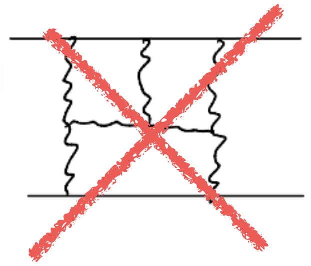

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Gravity massless scattering: shock waves

- amplitude driven by rapidity evolution between projectile & target

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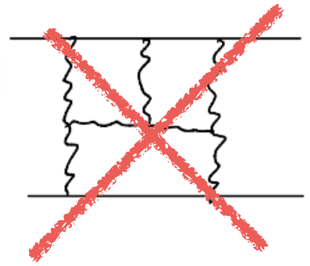

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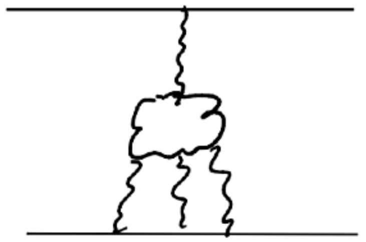

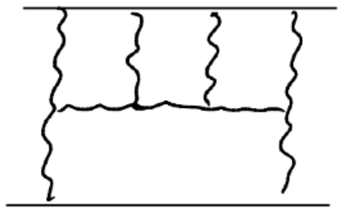

Gravity massless scattering: shock waves

- amplitude driven by rapidity evolution between projectile & target

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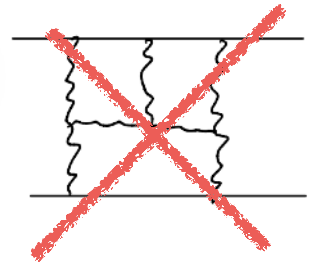

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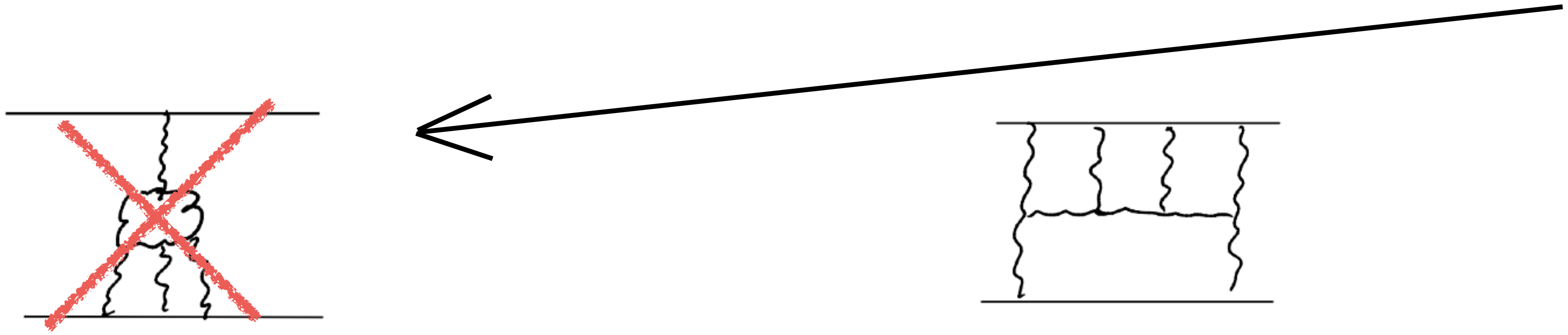

Gravity massless scattering: shock waves

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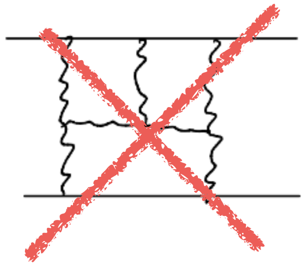

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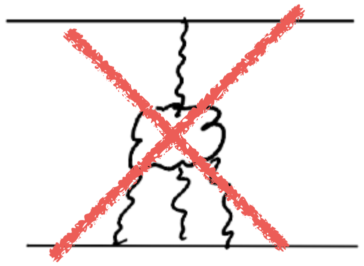

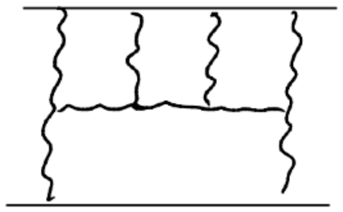

Gravity massless scattering: shock waves

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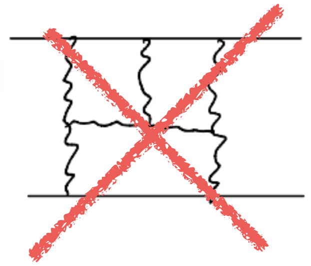

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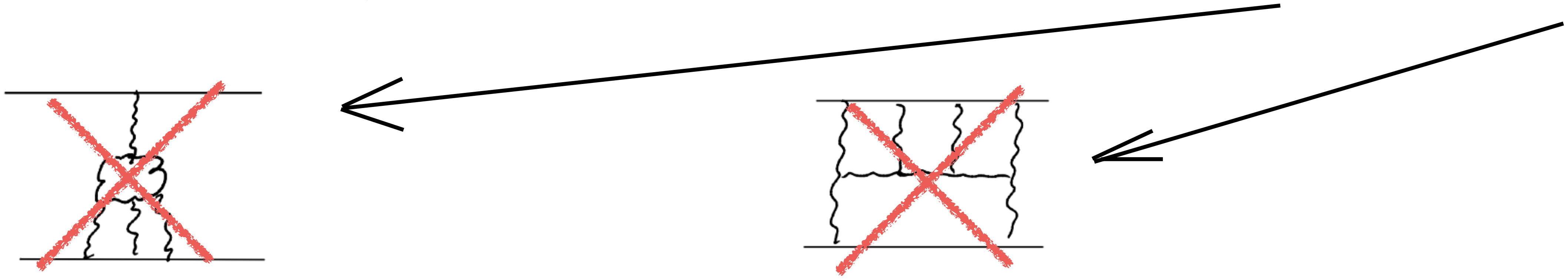

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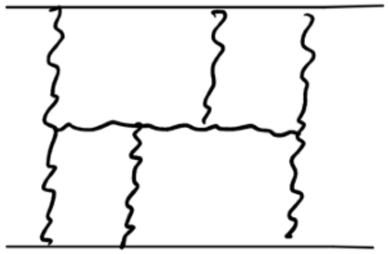
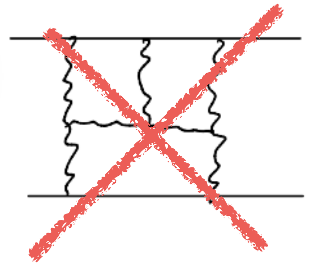


sub-leading

- amplitude is $s \leftrightarrow u$ crossing symmetric, thus off-diagonal *H* terms mixing even and odd number of Reggeised gravitons are forbidden $\hat{H}_{k,k\pm 2n+1} = 0$



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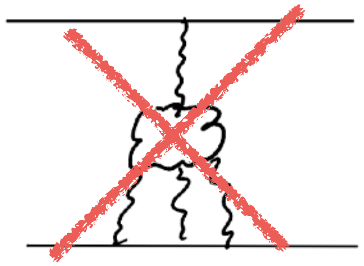
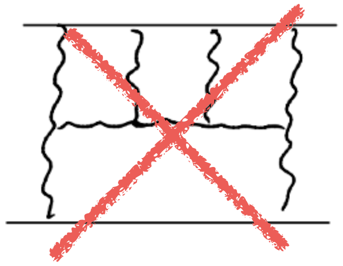


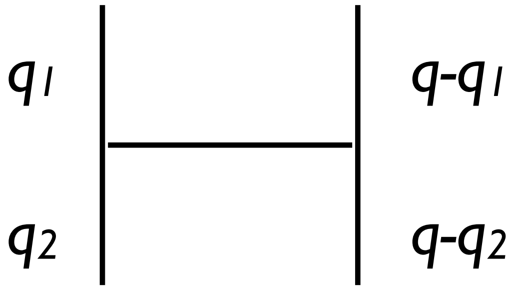


- $\hat{H}_{k,k} = \hat{\mathcal{R}}_1 + \hat{\mathcal{R}}_2$

$$\hat{\mathcal{R}}_1 = \int_{q_\perp} \alpha^{(1)}(q)\, W(q)\, \frac{\delta}{\delta W(q)}$$

dresses Reggeised graviton with Regge trajectory: quantum

$$\hat{\mathcal{R}}_2 = -\frac{\kappa^2}{8\pi\hbar} \int_{\ell_\perp, q_{1\perp}, q_{2\perp}} H_{22}(\ell; q_1, q_2) W(q_1+\ell) W(q_2-\ell) \frac{\delta}{\delta W(q_1)} \frac{\delta}{\delta W(q_2)}$$



exchanges soft graviton between 2 Reggeised gravitons

$$H_{22}(\ell;\, q_1, q_2) = \frac{\mathcal{H}_{\rm GR}(q_1, q_1+\ell; q_1+q_2)}{\vec{q}_1^{\,2}\, \vec{q}_2^{\,2}}$$

H_{GR}: Gravity (BFKL-like) kernel Lipatov 1982

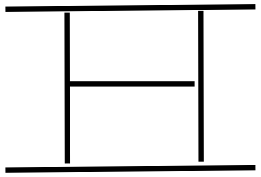
$$\mathcal{H}_{\rm GR}(q_1, q_2; q) = \Big[\mathcal{H}_{\rm YM}(q_1, q_2; q)\Big]^2 + \frac{4}{(q_{1\perp} - q_{2\perp})^4} \Big[q_{1\perp}^2 q_{2\perp}^2 (q_\perp - q_{1\perp})^2 (q_\perp - q_{2\perp})^2$$
$$- (q_\perp - q_{1\perp})^2 (q_\perp - q_{2\perp})^2 (q_{1\perp} \cdot q_{2\perp})^2 - q_{1\perp}^2 q_{2\perp}^2 \big[(q_\perp - q_{1\perp}) \cdot (q_\perp - q_{2\perp})\big]^2 \Big]$$

where graviton CEV is double copy of gluon CEV

$$\mathcal{H}_{\rm YM}(q_1, q_2; q) = -q_\perp^2 + \frac{q_{2\perp}^2 (q_\perp - q_{1\perp})^2 + q_{1\perp}^2 (q_\perp - q_{2\perp})^2}{(q_{1\perp} - q_{2\perp})^2}$$

BFKL kernel

H diagram



quantum

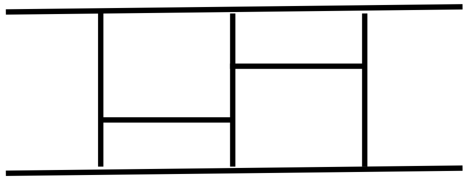
- 2 loops

$$\frac{i}{2s}\mathcal{M}^{(2)}_{2\to 2}(s,q^2)=\frac{L^2}{2}\langle\psi_{j,1}|\hat{\mathcal{R}}_1^2|\psi_{i,1}\rangle - L\,\langle\psi_{j,2}|(\hat{\mathcal{R}}_1+\hat{\mathcal{R}}_2)|\psi_{i,2}\rangle+\langle\psi_{j,3}|\psi_{i,3}\rangle$$

superclassical

- 4 loops

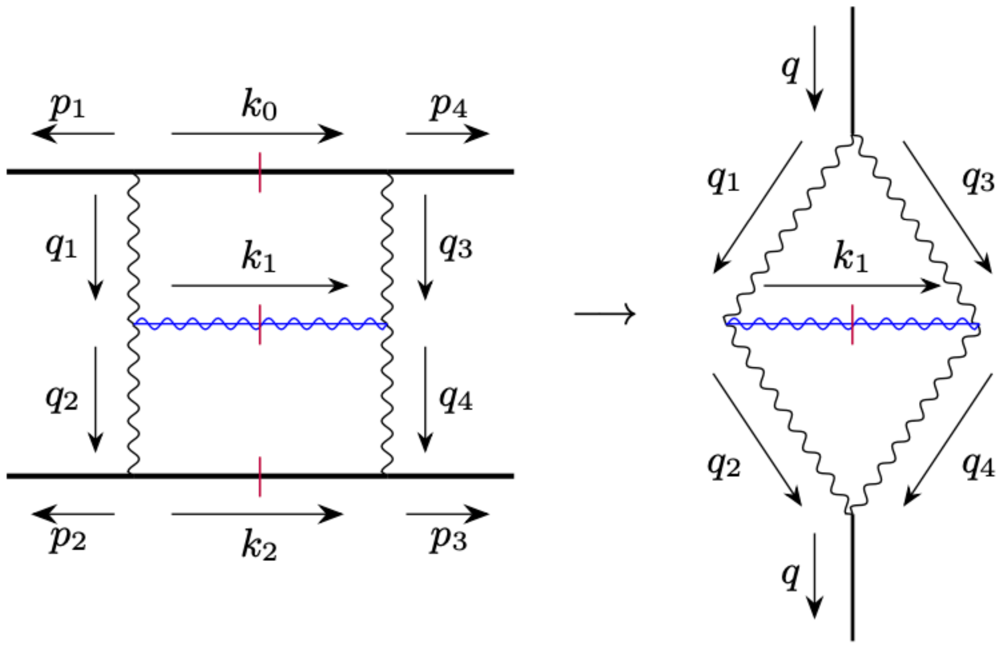
$$\begin{aligned}\frac{i}{2s}\mathcal{M}^{(4)}_{2\to 2}(s,q^2)=&\frac{L^4}{24}\langle\psi_{j,1}|\hat{\mathcal{R}}_1^4|\psi_{i,1}\rangle-\frac{L^3}{6}\langle\psi_{j,2}|\left(\hat{\mathcal{R}}_1+\hat{\mathcal{R}}_2\right)^3|\psi_{i,2}\rangle+\frac{L^2}{2}\langle\psi_{j,3}|\left(\hat{\mathcal{R}}_1+\hat{\mathcal{R}}_2\right)^2|\psi_{i,3}\rangle+\\&-L\,\langle\psi_{j,4}|(\hat{\mathcal{R}}_1+\hat{\mathcal{R}}_2)|\psi_{i,4}\rangle+\langle\psi_{j,5}|\psi_{i,5}\rangle\end{aligned}$$



Alessio VDD Gonzo Rosi 2601.21687

Gravity scattering: *s*-channel unitarity cuts

Revisiting the *H* diagram Amati Ciafaloni Veneziano 1990



- in MRK, the light-cone dof's decouple from the transverse dof's

compute discontinuity through 3-particle cut

$$2\mathrm{Im}\,\mathcal{M}^{(2)}_{2\to 2}(s,q^2) = \int \mathrm{d}\mathcal{P}_3\,|\mathcal{M}^{(0)}_{2\to 3}|^2$$

3-particle phase space in MRK

$$\mathrm{d}\mathcal{P}_3 = \prod_{i=0}^{2} \hat{\mathrm{d}}^D k_i\,\delta^+(k_i^2)\,\hat{\delta}^{(D)}\Big(p_1+p_2+\sum_{i=0}^{2} k_i\Big)$$

integrate out rapidities

get 2-pt functions in 2-2ϵ dimensions

$$2\mathrm{Im}\,\mathcal{M}^{(2)}_{2\to 2}(s,q^2) \simeq \frac{(8\pi G_N)^3 s^3}{8\pi}\log(s) H_1(q_\perp^2)$$

$$H_1(q_\perp^2) \equiv \varsigma^{2\epsilon} \int_{q_{1\perp},\,q_{2\perp}} \frac{\mathcal{H}_{\rm GR}(q_1,q_2;q)}{q_{1\perp}^2 q_{2\perp}^2 (q-q_1)_\perp^2 (q-q_2)_\perp^2}$$

$\varsigma = \mu^2 \exp(\gamma_E)$

- $$2\mathrm{Im}\,\mathcal{M}^{(2)}_{2\to 2}(s,q^2) = 8G_N^3 s^3 \log(s)\left(\frac{4\pi\mu^2}{q^2}\right)^{2\epsilon}\left(-\frac{1}{\epsilon^2}+\frac{2}{\epsilon}+\zeta_2+\mathcal{O}(\epsilon^0)\right)$$

- Fourier transforming to impact parameter space

$$\tilde{\mathcal{M}}_{2\to2}(s,b) \simeq \frac{1}{2s}\int \hat{d}^d q_\perp \, e^{ib\cdot q_\perp} \mathcal{M}_{2\to2}(s,-q_\perp^2) \equiv \mathrm{F.T.}[\mathcal{M}_{2\to2}](s,b)$$

$$2\mathrm{Im}\,\tilde{\mathcal{M}}^{(2)}_{2\to2}(s,b^2) = \frac{8G_N^3 s^2}{\pi b^2}\log(s)\left(\pi b^2 \varsigma e^{\gamma_E}\right)^{3\epsilon}\left(-\frac{1}{\epsilon}+2+\mathcal{O}(\epsilon^0)\right)$$

- in impact parameter space, amplitude eikonalizes

$$i\int_{q_\perp} e^{iq\cdot b}\frac{\mathcal{M}_{2\to2}(s,t=q^2)}{2s} = (1+\Delta_Q)e^{2i\delta_{\mathrm{Cl}}} - 1$$

with phase shift

$$\delta_{\mathrm{Cl}} = G_N s \sum_{j=0}^{\infty} \alpha_C^j \, \delta_{\mathrm{Cl}}^{(2j)}(b)$$

$\alpha_C = G_N\, st$

$$\Delta_Q = \sum_{n,\,k=0}^{\infty} \alpha_Q^n \alpha_C^{k+1} \, \Delta_Q^{(n,k)}(b)$$

$\alpha_Q = G_N\, t$

- expand phase shift in powers of log(s/t)

$$\delta_{\mathrm{Cl}}^{(2j)} = \sum_{k=0}^{j} \delta_{\mathrm{Cl}}^{(2j),k}(b)\log^k(s/|t|)$$

dispersion relation yields next-to-leading log term

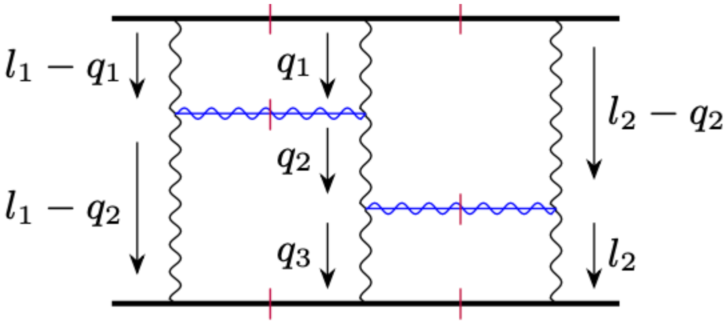
$$\mathrm{Re}\,\delta_{\mathrm{Cl}}^{(2),0} = \frac{\pi}{2}\mathrm{Im}\,\delta_{\mathrm{Cl}}^{(2),1} - 8\epsilon^2(\delta_{\mathrm{Cl}}^{(0)})^3$$

Gravity scattering: H^2 diagram

- early work by Amati Ciafaloni Veneziano 0712.1209
  Ciafaloni Colferai Coradeschi Veneziano 1512.00281

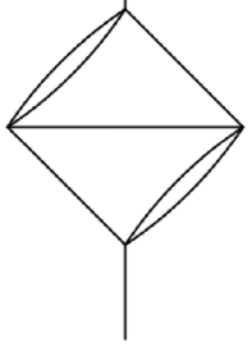
- through iterated *s*-channel 3-particle cuts

$$2\mathrm{Re}\,\mathcal{M}^{(4)}_{2\to 2}(s,q^2) \simeq -\frac{(8\pi G_N)^5 s^4}{64\pi^2}\log^2(s) H_2(q_\perp^2)$$



with H_2 the convolution of two gravity kernels

the result is reduced to bubbles and a kite integral



- in momentum space

$$2\mathrm{Re}\mathcal{M}^{(4)}_{2\to 2}(s,q^2) \simeq -4\,G_N^5 s^4\log^2(s)\frac{q^2}{\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^{4\epsilon}\left[-\frac{1}{\epsilon^3}-\frac{1}{\epsilon^2}-\frac{9}{\epsilon}+\frac{2}{\epsilon}\zeta_2+\frac{2}{\epsilon}\zeta_3+O(1)\right]$$

i.e. an $O(G^5\ s^4 \log^2(s/|t|))$ term

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in impact parameter space

$$2\mathrm{Re}\tilde{\mathcal{M}}^{(4)}_{2\to 2}(s,b^2) \simeq -256\,G_N^5\frac{s^3\log^2(s)}{(\pi b^2)^2}\left(\pi b^2 \varsigma e^{\gamma_E}\right)^{5\epsilon}\left[\frac{1}{8\epsilon^2}-\frac{1}{\epsilon}+\frac{5}{2}+\frac{5}{16}\zeta_2-\frac{1}{4}\zeta_3+O(\epsilon)\right]$$

- dispersion relation yields

$$\mathrm{Im}\,\delta^{(4),1}_{\mathrm{Cl}} = 16\epsilon(1-4\epsilon)(\delta^{(0)}_{\mathrm{Cl}})^2\mathrm{Im}\,\delta^{(2),1}_{\mathrm{Cl}} - \pi\mathrm{Re}\,\delta^{(4),2}_{\mathrm{Cl}}$$

Discussion

- in Regge theory, 2→2 amplitudes are given by a *t* channel ladder convoluted with impact factors

$$\mathcal{M}_{2\to 2} = i \sum_{M} J_{(M)} \otimes S_{(M)} \otimes \bar{J}_{(M)}$$

- for a *t* channel ladder at leading logarithmic accuracy, we need impact factors at leading order (LO)

- the LO impact factors for a massless graviton and for a massive scalar differ only by a phase

- since the *t* channel ladder is universal and the LO impact factors for a (massless) graviton and for a massive scalar are the same up to a phase, due to the spin and not to the mass, this implies that the scattering of massive scalars and the scattering of gravitons share the same leading logarithms!

- this verifies *a posteriori* that at leading log accuracy the massless limit of the Regge limit is smooth

- … and implies that in the scattering of massive scalars we have determined the leading logarithmic contribution to the 5PM-2SF amplitude

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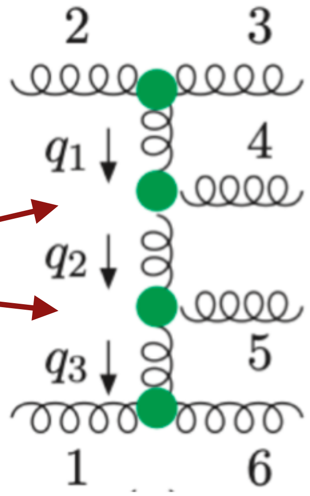
Going to next-to-leading log

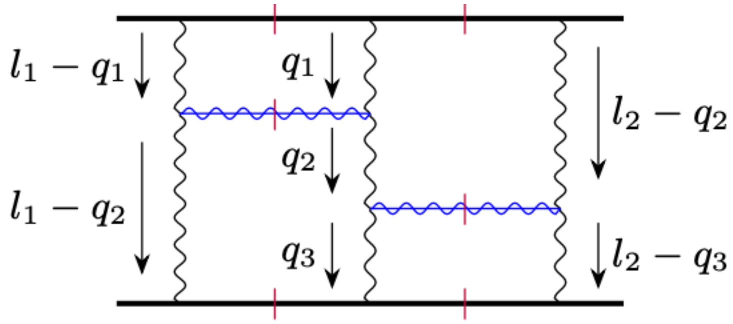
anatomy of gravity amplitudes in the Regge limit

- leading log

one-graviton CEV

Lipatov 1982

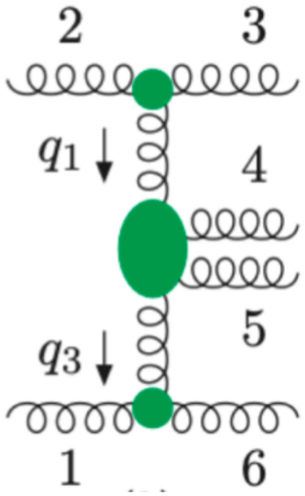




- next-to-leading log

two-graviton CEV

Barcaro VDD 2506.11822



?

requires NMHV graviton amplitudes

Cachazo Svrcek 2005
Hodges 1108.2227

we broke ground… but more work is required for a next-to-leading log ladder

Conclusions

- Using an EFT of gravity scattering of massless particles in the Regge limit, Rothstein & Saavedra showed that, at leading logarithmic accuracy, 2→2 amplitudes exhibit an *s*-channel sequence of classical terms, which generalise the *H* diagram through corrections of $O((G^2 s \log(s/|t|))^n)$

- Each of those terms, of (2n+3)-PM order, is the sole classical term of a tower of (quantum) terms associated to the exchange of (n+2) Glauber (Reggeised) gravitons

- In gravity scattering of massless particles, the same s-channel sequence may be obtained through a gravity version of the shock-wave formalism, which is equivalent to Rothstein-Saavedra's EFT, and through iterated *s*-channel unitarity cuts

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- The unitarity-cut formalism may be extended to the scattering of massive particles, and yields, at leading logarithmic accuracy, the same *s*-channel sequence of classical terms, which are of (2n+3)-PM and (n+1)-SF order, i.e. maximal SF within a given PM

- Using Rothstein-Saavedra’s EFT, the shock-wave formalism, and iterated *s*-channel unitarity cuts, we computed the H^2 diagram, which is an $O(G^5\, s^4 \log(s/|t|)^2)$ term, i.e. a 5 PM term, and represents the first correction to the *H* diagram

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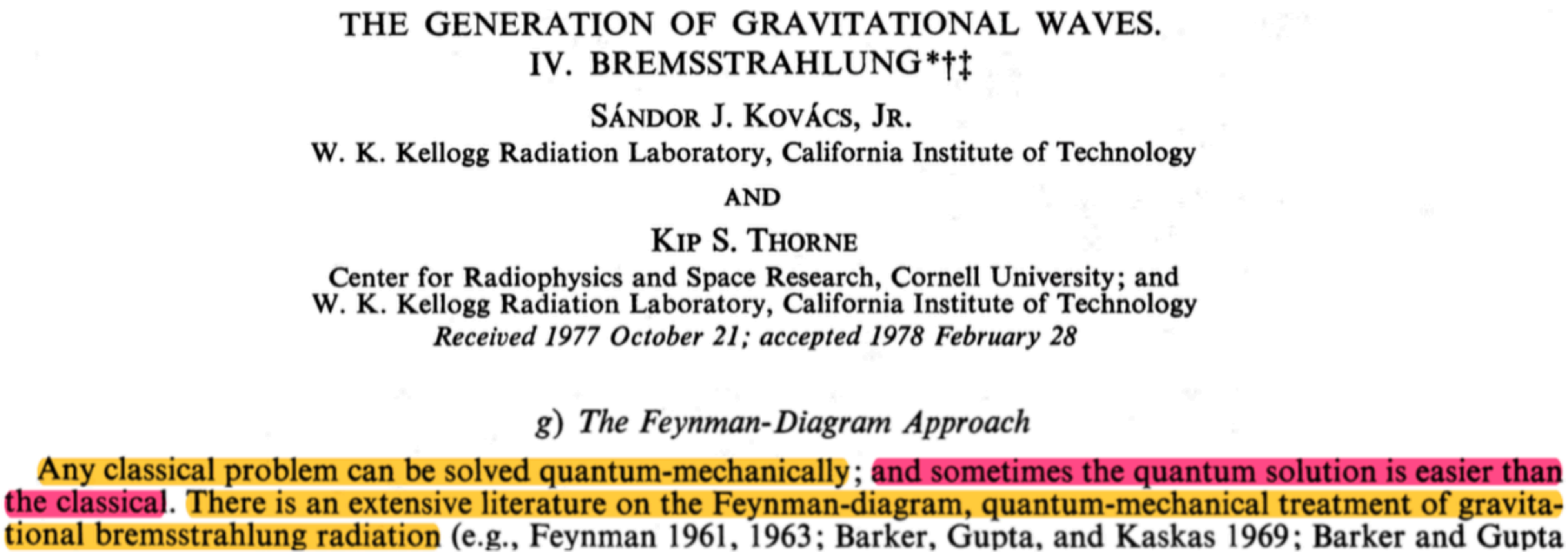
- Through the unitarity-cut formalism, that computation is extended to the scattering of massive scalars, and yields the 5 PM, 2 SF term at leading logarithmic accuracy

so the take away is…

- For gravity scattering of massless particles in the Regge limit, we have 3 different but equivalent solutions:
  - Rothstein-Saavedra’s EFT
  - shock-wave formalism
  - iterated *s*-channel unitarity cuts

- at leading logarithmic accuracy, unitarity-cut outcome can be ported to the scattering of massive particles for free

Back-up slides



THE GENERATION OF GRAVITATIONAL WAVES.
IV. BREMSSTRAHLUNG*†‡

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g) *The Feynman-Diagram Approach*

Any classical problem can be solved quantum-mechanically; and sometimes the quantum solution is easier than the classical. There is an extensive literature on the Feynman-diagram, quantum-mechanical treatment of gravitational bremsstrahlung radiation (e.g., Feynman 1961, 1963; Barker, Gupta, and Kaskas 1969; Barker and Gupta

Kovacs Thorne, Astrophysics. J. 224 (1978) 62

How classical is a quantum loop?

- put $\hbar$ back
- $\hbar$ counting   1/$\hbar$ from vertex $\exp\left(\frac{i}{\hbar}\int d^4x\mathcal{L}_{int}(\varphi)\right)$

  $\hbar$ from (massless) propagator $[\varphi(\vec{x}), \pi(\vec{y})] = i\hbar\delta^3(\vec{x}-\vec{y})$

  get $\hbar^{\mathrm{I-V+1}} = \hbar^{\mathrm{L}}$

  $$\langle 0|T\varphi(x)\varphi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i\hbar\, e^{ik(x-y)}}{k^2 - i\varepsilon}$$

  Note that *k* is wavenumber, with *p*=$\hbar$k

- with masses   Klein-Gordon is $\left(\Box + \frac{m^2}{\hbar^2}\right)\varphi(x) = 0$

  $$\langle 0|T\varphi(x)\varphi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i\hbar\, e^{ik(x-y)}}{k^2 - \frac{m^2}{\hbar^2} - i\varepsilon}$$

  so, effectively

  $e \to e/\sqrt{\hbar}$ $\kappa \to \kappa/\sqrt{\hbar}$

  massless momenta $p \to \hbar p$

Boulware Deser 1975
Gupta Radford 1980
….
Donoghue Holstein hep-th/0405239
Kosower Maybee O’Connell 1811.10950