Heavy Quark Effective Theory

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Outline

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  - Extracting $V_{cb}$ and $V_{ub}$
QCD describes the dynamics of quarks, and has a non-perturbative scale $\Lambda_{QCD} \sim 200$ MeV.

Simplifications when $m_Q \gg \Lambda_{QCD}$

A single heavy quark interacting with light particles can be described by an effective field theory known as HQET.

Applied to $c$ and $b$ quarks. The $t$-quark decays via $t \rightarrow bW$ before it forms hadrons. The width in the standard model is $\Gamma_t \approx 1.5$ GeV
NRQCD

Systems with two heavy quarks (such as $J/\psi$, $\Upsilon$ or $\bar{t}t$ near threshold) are described by a completely different effective theory, NRQCD [non-relativistic QCD].
Velocity Superselection Rule

Consider a heavy quark $Q$ interacting with light degrees of freedom, such as light quarks and gluons.

\[
v^\mu = \frac{p^\mu}{m_Q}, \quad \delta v^\mu = \frac{\delta p^\mu}{m_Q} \sim \frac{\Lambda_{\text{QCD}}}{m_Q} \to 0
\]

Quark has a constant velocity.

\[
[x, v] = \frac{1}{m_Q} [x, p] = \frac{i\hbar}{m_Q} \to 0.
\]

Cannot simultaneously have a well-defined position and momentum, but can have a well-defined position and velocity.
Spin-Flavor Symmetry

In the $m_Q \to \infty$ limit: static color 3 source in the rest frame $v^\mu = (1, 0, 0, 0)$.

Strong interactions flavor blind
⇒ HQ flavor symmetry
Symmetry breaking $\propto 1/m_b - 1/m_c$

Color coupling is color electric charge. The magnetic interaction is $\propto 1/m_Q$ for a pointlike spin-1/2 fermion (not true for the proton). ⇒ HQ spin symmetry
Symmetry breaking $\propto 1/m_Q$.

Combining gives HQ $SU(4)$ spin-flavor symmetry:
$b \uparrow$, $b \downarrow$, $c \uparrow$ and $c \downarrow$ transform as a 4
The QCD Lagrangian is given by:

\[ \mathcal{L} = \sum_{i=c,b,t} \bar{Q}_i (i \slashed{D} - m_{Q_i}) Q_i + \mathcal{L}_{\text{light}} \]

The Lagrangian has a term \( m_{Q} \) and has no well-defined \( m_{Q} \to \infty \) limit. It describes the interactions of \( Q \) at all energies, including those greater than \( m_{Q} \).

Want to consider an effective theory valid for momenta smaller than \( m_{Q} \), which makes the simplifications of low momentum manifest.

Should have an expansion in \( 1/m_{Q} \).
HQET Lagrangian

HQET Lagrangian:

\[ \mathcal{L}_{\text{HQET}} = \mathcal{L}_0 + \frac{1}{m_Q} \mathcal{L}_1 + \frac{1}{m_Q^2} \mathcal{L}_2 + \ldots \]

\( \mathcal{L}_0 \) has spin-flavor symmetry,

\( 1/m_Q \) terms are symmetry breaking corrections.
Look at the quark propagator:

\[ i \frac{p + m_Q}{p^2 - m_Q^2 + i\epsilon} \]

\[ p = mv + k, \text{ where } k \text{ is called the residual momentum, and is of order } \Lambda_{QCD}. \]
HQET Propagator

\[ i \frac{m_Q \psi + \frac{\not{k}}{2} + m_Q}{(m_Q v + k)^2 - m_Q^2 + i \epsilon} \]

Expanding this in the limit \( k \ll m_Q \) gives

\[ i \frac{1 + \psi}{2 k \cdot v + i \epsilon} + \mathcal{O} \left( \frac{k}{m_Q} \right) = i \frac{P_+}{k \cdot v + i \epsilon} + \mathcal{O} \left( \frac{k}{m_Q} \right), \]

with a well defined limit.

\[ P_+ \equiv \frac{1 + \psi}{2} \]
Projectors

\[ P_+ = \frac{1 + \varphi}{2}, \quad P_- = \frac{1 - \varphi}{2}, \]

In the rest frame,

\[ P_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ P_+^2 = P_+, \quad P_-^2 = P_-, \quad P_+ + P_- = 1, \quad P_+P_- = 0, \quad P_-P_+ = 0. \]
Gluon Vertex

The quark-gluon vertex

\[-igT^a \gamma^\mu \rightarrow -igT^a P_+ \gamma^\mu P_+ = -igT^a v^\mu,\]

using the identity

\[P_+ \gamma^\mu = v^\mu + \gamma^\mu P_-\]

In the rest frame: the coupling is purely that of an electric charge.
HQET Lagrangian:

\[ \mathcal{L} = \bar{h}_v(x) (iD \cdot v) h_v(x) , \]

\( h_v(x) \) is the quark field in the effective theory and satisfies

\[ P_+ h_v(x) = h_v(x) . \]

\( h_v \) annihilates quarks with velocity \( v \), but does not create antiquarks
Dividing up momentum space

$v$ appears explicitly in the HQET Lagrangian. $h_v$ describes quarks with velocity $v$, and momenta within $\Lambda_{\text{QCD}}$ of $m_Qv$.

Quarks with velocity $v' \neq v$ are far away in the EFT.
Feynman rules

\[ \mathcal{L} = \bar{h}_v (iD \cdot v) h_v \]

\[ D_\mu = \partial_\mu + igT^a A^a_\mu \]

The \( \partial \cdot v \) term gives a propagator

\[ \frac{iP_+}{k \cdot v} \]

The \( A \cdot v \) term gives a vertex

\[ -igT^a v^\mu \]
\[ \mathcal{L}_0 = \sum_{f=c,b,t} \bar{h}_f v (iD \cdot v) h_f, \]

has manifest spin-flavor symmetry, since \( D \cdot v \) does not depend on the spin or the flavor of the heavy quark.
Light degrees of freedom

Hadrons containing a single heavy quark contain $Q$, and light quarks and gluons [light degrees of freedom $\ell$].

$D^+$ meson has a $c$ quark, $\bar{d}$ quark, plus $\bar{q}q$ pairs and gluons. Quantum numbers of $\ell$ are the same as the $\bar{d}$.

Total angular momentum $J$ is conserved

$S_Q$ is conserved as $m_Q \to \infty$

Define $S_\ell \equiv J - S_Q$

Spin of the light degrees of freedom
\[ J^2 = j(j + 1), \quad S_Q^2 = s_Q(s_Q + 1), \quad S_\ell^2 = s_\ell(s_\ell + 1) \]

\( s_Q = 1/2 \), so heavy hadrons are in degenerate multiplets with \( j = s_\ell \pm 1/2 \), unless \( s_\ell = 0 \), in which case there is a single \( j = 1/2 \) multiplet.

Ground state mesons: \( Q \) and a light antiquark \( \bar{q} \), so \( s_\ell = 1/2 \).

\( j = 0 \oplus 1 \) and negative parity, since quarks and antiquarks have opposite parity

Degenerate \( 0^- \) and \( 1^- \) mesons which form a flavor \( \bar{3} \)

Called \( H^{(Q)} \)
Ground State Mesons

$D_s^+, D_s^{*+}$

$c\bar{s}$

$D^0, D^{*0}$

$c\bar{u}$

$D^+, D^{*+}$

$c\bar{d}$

$\bar{B}_s^0, \bar{B}^-, \bar{B}^0$ (spin-0)

$\bar{B}_s^{*0}, \bar{B}^{*-}, \bar{B}^{*0}$ (spin-1)

NOTE: $c \in D, b \in \bar{B}$
Excited Mesons

In the quark model, the first excitation has $L = 1$, with $s_\ell = 1/2$ and $s_\ell = 3/2$

$s_\ell = 1/2 \Rightarrow 0^+ \text{ and } 1^+$ states $D_0^*$ and $D_1^*$

$s_\ell = 3/2 \Rightarrow 1^+ \text{ and } 2^+$ states $D_1$ and $D_2^*$

The $s_\ell = 1/2$ and $s_\ell = 3/2$ multiplets are not related by HQ symmetry, though they are related in a NR quark model
Write the quark field as

\[ Q(x) = e^{-im_{Qv} \cdot x} \left[ h_v(x) + Q_v(x) \right] \]

\[ = e^{-imQt} \begin{pmatrix} h_v(x) \\ Q_v(x) \end{pmatrix} \]

\[ p = m_{Qv} + k, \text{ so the } x \text{ dependence of } h_v(x) \text{ is } k. \]

\[ \phi h_v(x) = h_v(x), \]

\[ \phi Q_v(x) = -Q_v(x) \]
At tree-level, one does not have to worry about renormalization effects:

\[ \mathcal{L} = \bar{Q} (i \not\!\! D - m_Q) Q \]

\[ = (\bar{h}_v + \bar{Q}_v) e^{im_Q v \cdot x} (i \not\!\! D - m_Q) e^{-im_Q v \cdot x} (h_v + Q_v) \]

\[ = (\bar{h}_v + \bar{Q}_v) (i \not\!\! D - m_Q + m_Q \not\!\! v) (h_v + Q_v) \]

\[ = (\bar{h}_v + \bar{Q}_v) (i \not\!\! h_v + [i \not\!\! D - 2m_Q] Q_v) \]

The result can be simplified using

\[ P_+ \gamma^\mu P_+ = v^\mu, \quad P_- \gamma^\mu P_- = -v^\mu, \quad P_+ \gamma^\mu P_+ = \gamma_\perp^\mu, \quad P_- \gamma^\mu P_- = \gamma_\perp^\mu, \]

where the \( \perp \) projector is defined for any vector \( A \) by

\[ A_\perp^\mu \equiv A^\mu - v^\mu v \cdot A. \]
\[
\mathcal{L} = \bar{h}_v (iv \cdot D) h_v - Q_v (iv \cdot D + 2m_Q) Q_v + \bar{h}_v i \bar{\Phi}_\perp Q_v + \bar{Q}_v i \bar{\Phi}_\perp h_v
\]

Quadratic in \(Q_v\):

\[
\mathcal{L} = \bar{h}_v (iv \cdot D) h_v + \bar{h}_v i \bar{\Phi}_\perp \frac{1}{2m_Q + iv \cdot D} i \bar{\Phi}_\perp h_v.
\]

The last term can be expanded in a power series in \(1/m_Q\),

\[
\frac{1}{2m_Q + iv \cdot D} = \frac{1}{2m_Q} - \frac{1}{4m_Q^2} iv \cdot D + \ldots
\]
The effective Lagrangian to order $1/m_Q$ is thus

$$\mathcal{L} = \bar{h}_v (iv \cdot D) h_v + \frac{1}{2m_Q} \bar{h}_v i \not{D} \not{i} \not{h} v.$$ 

This can be rewritten using the identities

$$\gamma^\alpha \gamma^\beta = g^{\alpha\beta} - ig^{\alpha\beta}, \quad [D^\alpha, D^\beta] = igG^{\alpha\beta}$$

$$\mathcal{L} = \bar{h}_v (iv \cdot D) h_v + \frac{1}{2m_Q} \bar{h}_v (iD_\perp)^2 h_v - \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v$$

$$+ \mathcal{O} \left( \frac{1}{m_Q^2} \right)$$
\[ \mathcal{L} = \bar{h}_v (i v \cdot D) h_v + \frac{1}{2m_Q} \bar{h}_v (i D_\perp)^2 h_v - c_F \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v \]

The \((i D_\perp)^2\) term violates flavor symmetry at order \(1/m_Q\)

\(g \sigma_{\alpha\beta} G^{\alpha\beta}\) term violates spin and flavor symmetry at order \(1/m_Q\)

One can carry out the expansion to higher order in \(1/m_Q\) to obtain the tree level HQET Lagrangian.
Field Redefinitions

Field redefinition

\[ h_v \rightarrow \left[ 1 + \frac{a}{m_Q} iv \cdot D \right] h_v \]

changes the effective Lagrangian to

\[ \mathcal{L} = \bar{h}_v (iv \cdot D) h_v + \frac{1}{2m_Q} \bar{h}_v (iD_\perp)^2 h_v - \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v \]

\[ + \frac{2a}{m_Q} h_v (iv \cdot D)^2 h_v + \mathcal{O} \left( \frac{1}{m_Q^2} \right) \]

For \( a = 1/2 \), one can replace \( D_\perp^2 \rightarrow D_\perp^2 + (v \cdot D)^2 = D^2 \)
Field redefinition change off-shell amplitudes, but not $S$-matrix elements.

The only thing that the effective theory and full theory have to agree on are $S$-matrix elements.

Convenient to eliminate $t$ derivatives, i.e. $v \cdot D$ terms in $\mathcal{L}$

[Use the lowest order equation of motion $(v \cdot D) h_v = 0$ to remove time derivatives]
Hadron mass mass in effective theory is $M_H - m_Q$.

- **Lowest order**: all hadrons degenerate, mass $m_Q$
- **Order one**: Hadron mass

$$\langle H_Q | \mathcal{H}_0 | H_Q \rangle \equiv \bar{\Lambda}$$

where $H_0 = $ Hamiltonian from lowest order Lagrangian (including the light degrees of freedom).

$\bar{\Lambda}$ has different values for each multiplet:
- $\bar{\Lambda}$ for mesons, $\bar{\Lambda}_\Lambda$, $\bar{\Lambda}_\Sigma$
\( \lambda_1, \lambda_2 \)

- **Order 1/\( m_Q \):**

\[
\frac{\mathcal{H}_1}{m_Q} = -\frac{L_1}{m_Q} = -\frac{1}{2m_Q} \bar{h}_v (iD_\perp)^2 h_v + c_F \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v.
\]

Define two non-perturbative parameters

\[
\lambda_1 = \langle H_Q | \bar{h}_v (iD_\perp)^2 h_v | H_Q \rangle,
\]
\[
8 (S_Q \cdot S_\ell) \lambda_2 = \langle H_Q | \bar{h}_v g \sigma_{\alpha\beta} G^{\alpha\beta} h_v | H_Q \rangle,
\]

In the rest frame:

\[
(iD_\perp)^2 = -p^2
\]
\[
\sigma_{\alpha\beta} G^{\alpha\beta} = -2\sigma \cdot B
\]
Meson Masses

\[ m_H = m_Q + \bar{\Lambda} - \frac{\lambda_1}{2m_Q} + c_F \frac{2\lambda_2 S_Q \cdot S_\ell}{m_Q} \]

\[ S_Q \cdot S_\ell = (J^2 - S_Q^2 - S_\ell^2)/2, \text{ so} \]

\[ m_B = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2}{2m_b} \]

\[ m_{B^*} = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} + \frac{\lambda_2}{2m_b} \]

\[ m_D = m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} - \frac{3\lambda_2}{2m_c} \]

\[ m_{D^*} = m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} + \frac{\lambda_2}{2m_c} \]
Note that heavy quark symmetry implies that $\bar{\Lambda}$, $\lambda_1$ and $\lambda_2$ have the same value in the $b$ and $c$ systems (upto renormalization)

$$0.49 \text{ GeV}^2 = m_{B^*}^2 - m_B^2 = 4\lambda_2 = m_{D^*}^2 - m_D^2 = 0.55 \text{ GeV}^2,$$

up to corrections of order $1/m_{b,c}$.

$$90 \pm 3 \text{MeV} = m_{B_s} - m_{B_d} = \bar{\Lambda}_s - \bar{\Lambda}_d = m_{D_s} - m_{D_d} = 99 \pm 1 \text{MeV}$$
$$345 \pm 9 \text{MeV} = m_{\Lambda_b} - m_B = \bar{\Lambda}_\Lambda - \bar{\Lambda}_d = m_{\Lambda_c} - m_D = 416 \pm 1 \text{MeV}$$

$\bar{\Lambda}$, $\lambda_1$ and $\lambda_2$ will occur elsewhere
Meson Field

$Q\bar{q}$ mesons $\rightarrow$ field $H_{v}^{(Q)}$ ($4 \times 4$ matrix, bispinor)

$$H_{v}^{(Q)}(x) \rightarrow D(\Lambda) H_{\Lambda^{-1}v}^{(Q)}(\Lambda^{-1}x) D(\Lambda)^{-1}$$

Pseudoscalar $P_{v}^{(Q)}(x)$ and vector $P_{v\mu}^{*(Q)}(x)$

Vector particles have a polarization vector $\epsilon_{\mu}$, with $\epsilon \cdot \epsilon = -1$, and $v \cdot \epsilon = 0$.

$$H_{v}^{(Q)} = \frac{1 + \psi}{2} \left[ P_{v}^{*(Q)} + iP_{v}^{(Q)}\gamma_{5} \right] .$$
\[ H_v^{(Q)} = \frac{1 + \slashed{v}}{2} \left[ \slashed{H}_v^{(*)(Q)} + iP_v^{(Q)}\gamma_5 \right]. \]

Parity:

\[ H_v^{(Q)}(x) \rightarrow \gamma^0 H_{v_P}^{(Q)}(x_P) \gamma^0, \]

where

\[ x_P = (x^0, -x), \quad \slashed{v}_P = (\slashed{v}^0, -\slashed{v}). \]
\[ H_{v}^{(Q)} = \frac{1 + \psi}{2} \left[ \Phi_{v}^{*(Q)} + iP_{v}^{(Q)} \gamma_{5} \right]. \]

\[ \psi H_{v}^{(Q)} = H_{v}^{(Q)} , \quad H_{v}^{(Q)} \psi = -H_{v}^{(Q)}. \]

Using \( v \cdot P_{v}^{*(Q)} = 0 \)
Conjugate field:

\[
\bar{H}_v^{(Q)} = \gamma^0 H_v^{(Q)\dagger} \gamma^0 = \left[ P_v^{(Q)\dagger} + iP_v^{(Q)\dagger} \gamma_5 \right] \frac{1 + \phi}{2},
\]

which also transforms as a bispinor,

\[
\bar{H}_v^{(Q)} \rightarrow D(\Lambda) \bar{H}_v^{(Q')} D(\Lambda)^{-1}.
\]

since

\[
\gamma^0 D(\Lambda)^\dagger \gamma^0 = D(\Lambda)^{-1}.
\]
Normalization of States

\[ \langle H(p', \varepsilon') | H(p, \varepsilon) \rangle = 2E_p (2\pi)^3 \delta^3(p - p') \delta_{\varepsilon \varepsilon'}, \]

Mass dimension \(-1\)

HQET states eigenstates of the \( m_Q \to \infty \) theory and labelled by \( v \) and \( k \), with \( v \cdot k = 0 \). They differ from full QCD states.

\[ \langle H(v', k', \varepsilon') | H(v, k, \varepsilon) \rangle = 2v^0 (2\pi)^3 \delta_{vv'} \delta^3(k - k') \delta_{\varepsilon \varepsilon'}. \]

Usually take \( k = 0 \). States have mass dimension \(-3/2\)
\[ |H(p)\rangle = \sqrt{m_H} \left[ |H(v)\rangle + \mathcal{O}\left(\frac{1}{m_Q}\right) \right] \]

Similarly

\[
\overline{u}(p, s) \gamma^\mu u(p, s) = 2p^\mu \\
\overline{u}(v, s) \gamma^\mu u(v, s) = 2v^\mu \\
u(p, s) = \sqrt{m_H} u(v, s)
\]
Meson Decay Constants

\[ \langle 0 | \bar{q} \gamma^\mu \gamma_5 Q | P (p) \rangle = -i f_P p^\mu, \]

where \( f_P \) has mass dimension one. \( (f_\pi = 131 \text{ MeV}) \)

\[ \langle 0 | \bar{q} \gamma^\mu Q | P^* (p, \epsilon) \rangle = f_{P^*} \epsilon^\mu, \]

\( f_{P^*} \) has mass dimension two.

\[ \bar{q} \Gamma Q = \bar{q} \Gamma Q_v + \mathcal{O} (\alpha_s) + \mathcal{O} \left( \frac{1}{m_Q} \right) \]

so we need

\[ \langle 0 | \bar{q} \Gamma Q_v | H(v) \rangle , \]
$\bar{q} \Gamma Q_v \rightarrow \bar{q} \Gamma D(R)_Q Q_v$

- Pretend that $\Gamma$ transforms as $\Gamma \rightarrow \Gamma D(R)_Q^{-1}$
- Write down operators which are invariant when $Q_v \rightarrow D(R)_Q Q_v$, $\Gamma \rightarrow \Gamma D(R)_Q^{-1}$, $H_v^{(Q)} \rightarrow D(R)_Q H_v^{(Q)}$.
- Set $\Gamma$ to its fixed value $\gamma^\mu$ or $\gamma^\mu \gamma_5$ to obtain the operator with the correct transformation properties.
Decay Matrix Element

\[ \langle 0 | \bar{q} \Gamma Q_v | H(v) \rangle = \text{Tr} \left( \frac{a}{2} \Gamma H_v^{(Q)} \right) \]

\[ a = a_0 (v^2) + a_1 (v^2) \rho \]

\[ a \times \begin{cases} 
- i v^\mu P_v^{(Q)} & \text{if } \Gamma = \gamma^\mu \gamma_5, \\
 P_v^{*(Q)} \mu & \text{if } \Gamma = \gamma^\mu,
\end{cases} \]

\[ \langle 0 | \bar{q} \gamma^\mu \gamma_5 Q_v | P(v) \rangle = - i a v^\mu, \]

\[ \langle 0 | \bar{q} \gamma^\mu Q_v | P^*(v) \rangle = a \epsilon^\mu. \]
\[ f_P = \frac{a}{\sqrt{m_P}}, \quad f_{P^*} = a\sqrt{m_{P^*}}. \]

\[ f_P = \frac{a}{\sqrt{m_P}}, \quad f_{P^*} = m_P f_P, \]

so \( f_P \propto m_P^{-1/2}, \quad f_{P^*} \propto m_P^{1/2} \).

\( a \) has the same value for \( c \) and \( b \):

\[ \frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}}, \quad f_{D^*} = m_D f_D, \quad f_{B^*} = m_B f_B. \]
Measure from the decays $D \rightarrow \ell \nu \ell$ and $B \rightarrow \ell \bar{\nu} \ell$

\[
\Gamma = \frac{G_F^2 |V_{Qq}|^2}{8\pi} f_P^2 m_\ell m_P \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2.
\]


<table>
<thead>
<tr>
<th>Decay Constant</th>
<th>Value in MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_D$</td>
<td>197 ± 2</td>
</tr>
<tr>
<td>$f_{Ds}$</td>
<td>224 ± 2</td>
</tr>
<tr>
<td>$f_B$</td>
<td>173 ± 4</td>
</tr>
<tr>
<td>$f_{Bs}$</td>
<td>199 ± 3</td>
</tr>
</tbody>
</table>

Note that this simulation suggests that there is a substantial correction to the heavy quark symmetry prediction $f_B/f_D = \sqrt{m_D/m_B} \simeq 0.6$. 

LNF School, Frascati, May 17–18, 2004 – p.45
Semileptonic $b \to c$ decays via the weak current $\bar{c} \gamma_\mu P_L b$

Decay form-factors are defined by:

\[
\begin{align*}
\langle D(p') | V^\mu | \bar{B}(p) \rangle &= f_+(q^2) (p + p')^\mu + f_-(q^2) (p - p')^\mu, \\
\langle D^*(p', \epsilon) | V^\mu | \bar{B}(p) \rangle &= g(q^2) \epsilon^{\mu\nu\alpha\tau} \epsilon^*_\nu (p + p')_\alpha (p - p')_\tau, \\
\langle D^*(p', \epsilon) | A^\mu | \bar{B}(p) \rangle &= -if(q^2) \epsilon^{*\mu} \\
&\quad -i\epsilon^* \cdot p \left[ a_+(q^2) (p + p')^\mu + a_-(q^2) (p - p')^\mu \right],
\end{align*}
\]

where $q = p - p'$

Six form-factors
Label states by $v$ and $v'$, and use

$$w = v \cdot v' = \frac{m_B^2 + m_{D(*)}^2 - q^2}{2m_B m_{D(*)}}$$

The allowed kinematic range for $w$ is

$$0 \leq w - 1 \leq \frac{(m_B - m_{D(*)})^2}{2m_B m_{D(*)}}$$

The zero-recoil point, at which $D(*)$ is at rest in the $\bar{B}$ rest frame, is $w = 1$ (maximum $q^2$)
Better to use:

\[
\begin{align*}
\langle D(p')|V^\mu|\bar{B}(p)\rangle &= \frac{h_+(w)(v + v')^\mu + h_-(w)(v - v')^\mu}{\sqrt{m_Bm_D}}, \\
\langle D^*(p', \epsilon)|V^\mu|\bar{B}(p)\rangle &= \frac{h_V(w)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu_v^\alpha v^\beta}{\sqrt{m_Bm_{D^*}}} \\
\langle D^*(p', \epsilon)|A^\mu|\bar{B}(p)\rangle &= -ih_{A_1}(w)(w + 1)\epsilon^{*\mu} + ih_{A_2}(w)(\epsilon^* \cdot v)v^\mu \\
&\quad + ih_{A_3}(w)(\epsilon^* \cdot v)v'^\mu.
\end{align*}
\]
\[ q_{\text{light}}^2 \sim (\Lambda_{\text{QCD}} v - \Lambda_{\text{QCD}} v')^2 = 2\Lambda_{\text{QCD}}^2 (1 - w) \]

HQ symmetry should hold if:

\[ 2\Lambda_{\text{QCD}}^2 (w - 1) \ll m_{b,c}^2. \]

The heavy meson form factors are expected to vary on the scale \( q_{\text{light}}^2 \sim \Lambda_{\text{QCD}}^2 \), i.e. on the scale \( w \sim 1 \).
QCD matrix elements are of the form:

\[ \langle H^{(c)}(p') | \bar{c} \Gamma b | H^{(b)}(p) \rangle \]

At leading order in \(1/m_{c,b}\) and \(\alpha_s(m_{c,b})\):

\[ \langle H^{(c)}(v') | \bar{c} v' \Gamma b v | H^{(b)}(v) \rangle \]

use trick as before \(\Gamma \rightarrow D(R)_c \Gamma D(R)_b^{-1}\)

\[ \bar{c} v' \Gamma b v = \text{Tr} \ X \bar{H}^{(c)}_{v'} \Gamma H^{(b)}_v, \]

\[ X = X_0 + X_1 \phi + X_2 \phi' + X_3 \phi \phi', \]

where the coefficients are functions of \(w = v \cdot v'\).
Isgur-Wise Function

Use $X = -\xi(w)$:

$$\langle D(v') | \bar{c}_{v'} \gamma_\mu b_v | \bar{B}(v) \rangle = \xi(w) [v_\mu + v'_\mu],$$

$$\langle D^*(v', \epsilon) | \bar{c}_{v'} \gamma_\mu \gamma_5 b_v | \bar{B}(v) \rangle = -i \xi(w) [(1 + w) \epsilon^*_\mu - (\epsilon^* \cdot v) v'_\mu],$$

$$\langle D^*(v', \epsilon) | \bar{c}_{v'} \gamma_\mu b_v | \bar{B}(v) \rangle = \xi(w) \epsilon_{\mu\nu\alpha\beta} \epsilon^{*\nu} v'^\alpha v^\beta.$$

Six form-factors in terms of one Isgur-Wise function

$$h_+(w) = h_V(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w),$$

$$h_-(w) = h_{A_2}(w) = 0$$
Consider the forward matrix element of the vector current $\bar{b} \gamma^\mu b$ between $\bar{B}$ meson states. Setting $v' = v$, and letting $c \rightarrow b$, $D \rightarrow \bar{B}$,

$$\frac{\langle \bar{B}(p)|\bar{b} \gamma_\mu b|\bar{B}(p)\rangle}{m_B} = \langle \bar{B}(v)|\bar{b}_{v} \gamma_\mu b_{v}|\bar{B}(v)\rangle = 2 \xi(w = 1)v_\mu.$$ 

where $\xi$ for $b \rightarrow b$ is the same as for $b \rightarrow c$.

So $\xi(1) = 1$.

This fixes the absolute normalization and allows one to determine $V_{cb}$.
Consider heavy quark wavefunction renormalization:

In Feynman gauge ($n = 4 - \epsilon$)

\[
\int \frac{d^n q}{(2\pi)^n} (-igT^A \mu^{\epsilon/2}) v_\lambda \frac{i}{(q + p) \cdot v} (-igT^A \mu^{\epsilon/2}) v^\lambda \frac{(-i)}{q^2}
\]

\[
= - \left( \frac{4}{3} \right) g^2 \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2} v \cdot (q + p)
\]

Add a gluon mass to regulate the IR divergence, so that one can isolate the UV divergence from the $1/\epsilon$ pole
\[- \left( \frac{4}{3} \right) g^2 \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m^2) v \cdot (q + p)} \]

Use the identity

\[
\frac{1}{a^r b^s} = 2^s \frac{\Gamma(r + s)}{\Gamma(r) \Gamma(s)} \int_0^\infty d\lambda \frac{\lambda^{s-1}}{[a + 2b\lambda]^{r+s}},
\]

to get

\[- \left( \frac{8}{3} \right) g^2 \mu^\epsilon \int_0^\infty d\lambda \int \frac{d^n q}{(2\pi)^n} \frac{1}{[q^2 - m^2 + 2\lambda v \cdot (q + p)]^2}.\]

Let \( q \rightarrow q - \lambda v \)
\[- \left( \frac{8}{3} \right) g^2 \mu^\epsilon \int_0^\infty d\lambda \int \frac{d^n q}{(2\pi)^n} \frac{1}{[q^2 - m^2 - \lambda^2 + 2\lambda \nu \cdot p]^2} \cdot \]

Use the standard dim reg formula:

\[
\int \frac{d^n q}{(2\pi)^n} \frac{(q^2)^\alpha}{(q^2 - M^2)^\beta} = \frac{i}{2^{n-1} \pi^{n/2}} (-1)^{\alpha + \beta} (M^2)^{\alpha - \beta + n/2} \frac{\Gamma(\alpha + n/2) \Gamma(\beta - \alpha - n/2)}{\Gamma(n/2) \Gamma(\beta)} ,
\]

\[- \frac{i}{(4\pi)^{2-\epsilon/2}} \left( \frac{8}{3} \right) g^2 \mu^\epsilon \Gamma(\epsilon/2) \int_0^\infty d\lambda \left[ \lambda^2 - 2\lambda \nu \cdot p + m^2 \right]^{-\epsilon/2} . \]
Evaluate $\lambda$ integral using the recursion relation

\[
I (a, b, c) \equiv \int_0^\infty d\lambda \left[ \lambda^2 + 2b\lambda + c \right]^a
\]

\[
= \frac{1}{1 + 2a} \left[ \left( \lambda^2 + 2b\lambda + c \right)^a (\lambda + b) \right]_0^\infty + 2a (c - b^2) I (a - 1, b, c)
\]

to convert it to one that is convergent when $\epsilon = 0$,

\[
\int_0^\infty d\lambda \left[ \lambda^2 - 2\lambda v \cdot p + m^2 \right]^{-\epsilon/2}
\]

\[
= \frac{1}{1 - \epsilon} \left[ \left( \lambda^2 - 2\lambda v \cdot p + m^2 \right)^{-\epsilon/2} (\lambda - v \cdot p) \right]_0^\infty
\]

\[
- \epsilon (m^2 - (v \cdot p)^2) \int_0^\infty d\lambda \left[ \lambda^2 - 2\lambda v \cdot p + m^2 \right]^{-1-\epsilon/2}
\]
Can set $\epsilon = 0$ in the last term. Also use

$$\lim_{\lambda \to \infty} \lambda^z = 0,$$

to get

$$-i \frac{g^2}{3\pi^2 \epsilon} \mathbf{v} \cdot \mathbf{p} + \text{finite}$$

This gives

$$Z_h = 1 + \frac{g^2}{3\pi^2 \epsilon}, \quad \gamma_h = \frac{1}{2} \frac{\mu}{Z_h} \frac{dZ_h}{d\mu} = -\frac{g^2}{6\pi^2}.$$ 

Note that $Z_q = 1 - \frac{g^2}{6\pi^2 \epsilon}$
Compute the anomalous dimension for

\[ O_\Gamma = \bar{q}_\Gamma Q_v \]

The light quark vertex is \( \gamma^\mu \) and the heavy quark vertex is \( v^\mu \). Find that

\[ \gamma_O = -\frac{g^2}{4\pi^2}. \]

independent of \( \Gamma \).
Operator Mixing

Use the convention that

\[ \mu \frac{d}{d\mu} O_j = -\gamma_{ji} O_i \]

\[ \mu \frac{d}{d\mu} C_i = \gamma_{ji} C_j \]

with

\[ L = C_i O_i \]
Heavy-Heavy Current

\[ \bar{Q}_v \Gamma Q_v \]

\[
\begin{align*}
\gamma &= \frac{g^2}{3\pi^2} [w r(w) - 1] \\
r(w) &= \frac{1}{\sqrt{w^2 - 1}} \ln \left( w + \sqrt{w^2 - 1} \right) \\
\gamma &= \frac{g^2}{\pi^2} \left[ \frac{2}{9} (w - 1) - \frac{1}{15} (w - 1)^2 + \ldots \right],
\end{align*}
\]

that vanishes at \( w = 1 \).
Matching heavy-light currents

\[ \bar{q} \gamma^{\lambda} Q = C_1^{(V)} \bar{q} \gamma^{\lambda} Q_v + C_2^{(V)} \bar{q} v^{\lambda} Q_v \]

where to lowest order \( C_1 = 1, \ C_2 = 0 \).

To compute the \( \alpha_s \) corrections, compute on-shell matrix elements of both sides at one loop.

Computation of a matching coefficient. Need the difference between the full and effective theory results.

Full and EFT results can have IR divergences, but the difference is IR finite. [UV divergences taken care of by renormalization]
Full theory: evaluate on-shell with \( p = m_Q v \)

\[
C_1 = - \frac{2\alpha_s}{3\pi} \left[ \frac{1}{\epsilon_{IR}} + 1 \right], \quad C_2 = \frac{2\alpha_s}{3\pi}.
\]

Wavefunction renormalization:

\[
\Sigma(p) = A(p^2)m + B(p^2)\not{\phi}
\]
\[ \delta Z \equiv B + 2m^2 \frac{d(A + B)}{dp^2} \bigg|_{p^2=m^2} \]

Gives \( C_1 = \frac{\delta Z_Q}{2} \),

\[ C_1 = -\frac{\alpha_s}{3\pi} \left[ \frac{2}{\epsilon_{IR}} + 2 + 3 \ln \frac{\mu}{m_Q} \right], \quad C_2 = 0 \]

(including UV counterterm)

Neglect \( Z_q \) as it will cancel out.
Effective theory:

\[ C_1 = -\frac{2\alpha_s}{3\pi\epsilon_{IR}} \quad C_2 = 0. \]

Light quark wavefunction cancels between full and EFT
Scaleless Integrals

EFT integrals: have no scale in them when evaluated on-shell. e.g. wavefunction graph

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2} \frac{1}{v \cdot (q + p)} \rightarrow \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2} \frac{1}{(v \cdot q)^2}$$

and is zero in dim-reg. Adding the UV counterterm gives the $1/\epsilon$ terms.

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^4} = \frac{i}{8\pi^2 \epsilon_{UV}} - \frac{i}{8\pi^2 \epsilon_{IR}},$$

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^4} + \text{counterterm} = -\frac{i}{8\pi^2 \epsilon_{IR}},$$
Matching UV and IR

Full theory integrals can have finite parts, since they depend on $m_Q$.

IR divergence match between full and effective theory. Since the EFT integrals are scaleless, UV$=$IR in the EFT. Thus one finds

$$\left. \frac{1}{\epsilon_{IR}} \right|_{\text{FULL}} = - \left. \frac{1}{\epsilon_{UV}} \right|_{\text{EFT}}$$

The anomalous dimensions in the EFT are related to the IR behavior in the full theory.
Matching Correction

Full theory:

\[ C_1 = -\frac{\alpha_s}{3\pi} \left[ \frac{4}{\epsilon} + 4 + 3 \ln \frac{\mu}{m_Q} \right], \quad C_2 = \frac{2\alpha_s}{3\pi}. \]

EFT:

\[ C_1 = -\frac{\alpha_s}{3\pi} \left[ \frac{4}{\epsilon} \right], \quad C_2 = 0 \]

Difference

\[ C_1' = -\frac{\alpha_s}{3\pi} \left[ 4 + 3 \ln \frac{\mu}{m_Q} \right], \quad C_2' = \frac{2\alpha_s}{3\pi}. \]

Or compute in full theory and drop $1/\epsilon$ terms.
Log in matching related to difference in anomalous dimensions in Full and EFT.

Usually choose matching scale $\mu = m_Q$, so one finds:

$$C^{(V)}_1(m) = 1 - \frac{4\alpha_s}{3\pi} \quad C^{(V)}_2(m) = \frac{2\alpha_s}{3\pi}.$$ 

One can show that the matching for the axial current is $C_1 \rightarrow C'_1$, $C'_2 \rightarrow -C_2$

$$C^{(A)}_1(m) = 1 - \frac{4\alpha_s}{3\pi} \quad C^{(A)}_2(m) = -\frac{2\alpha_s}{3\pi}.$$
Meson Decay Constants

Compute the radiative corrections to the meson decay constants:

Match at the scale $m$ to the EFT:

$$\bar{q}\gamma^\mu\gamma_5 Q \rightarrow C_1^{(A)}(m) \bar{q}\gamma^\mu\gamma_5 Q_v + C_2^{(A)}(m) \bar{v}\gamma^\mu\gamma_5 Q_v$$

Run in the EFT to $\mu$:

$$\bar{q}\gamma^\mu\gamma_5 Q \rightarrow C_1^{(A)}(\mu) \bar{q}\gamma^\mu\gamma_5 Q_v + C_2^{(A)}(\mu) \bar{v}\gamma^\mu\gamma_5 Q_v$$

$\mu$ dependence given by the anomalous dimension in the EFT
Can integrate a one-loop anomalous dimension:

\[ \mu \frac{dC}{d\mu} = \gamma_0 C = \frac{\alpha_s}{\pi} C \]

\[ \mu \frac{dg}{d\mu} = -\frac{g^3}{16\pi^2} b_0 \]

Then

\[ \frac{dC}{dg} = -\frac{\gamma_0 C}{b_0 g} \]
Solution to one-loop RGE

\[
\frac{C(\mu_1)}{C(\mu_2)} = \left[ \frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)} \right]^{-\gamma_0/(2b_0)}
\]

\[
b_0 = 11 - \frac{2}{3} n_f
\]

\[
\gamma_0 = -\frac{g^2}{4\pi^2}
\]

so \( \gamma_0 = -4, \ b_0 = 25/3 \) below \( m_b \), and

\[
\frac{C(\mu)}{C(m_b)} = \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_b)} \right]^{6/25}
\]
Compute matrix elements at $\mu$:

$$a(\mu) \times \begin{cases} 
-i\nu^\mu P_v^{(Q)} & \text{if } \Gamma^\mu = \gamma^\mu \gamma_5, \\
 i\nu^\mu P_v^{(Q)} & \text{if } \Gamma^\mu = \nu^\mu \gamma_5, \\
 P_v^{*(Q)\mu} & \text{if } \Gamma^\mu = \gamma^\mu, \\
 0 & \text{if } \Gamma^\mu = \nu^\mu.
\end{cases}$$

so that

$$f_{P^*} = \sqrt{m_{P^*}} a(\mu) C_1^{(V)}(\mu),$$

$$f_P = \frac{1}{\sqrt{m_P}} a(\mu) \left( C_1^{(A)}(\mu) - C_2^{(A)}(\mu) \right).$$
\( a(\mu)C(\mu) \) is \( \mu \) independent.

\[
\frac{f_{P^*}}{f_P} = \sqrt{m_{P^*}m_P} \left\{ \frac{C_1^{(V)}}{C_1^{(A)} - C_2^{(A)}} \right\} = \sqrt{m_{P^*}m_P} \left\{ 1 - \frac{2\alpha_s(m_Q)}{3\pi} \right\}.
\]

\[
f_B\sqrt{m_B} = a(\mu) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_b)} \right]^{6/25} \left\{ 1 - \frac{2\alpha_s(m_b)}{3\pi} \right\}
\]

\[
f_D\sqrt{m_D} = a(m_c) \left\{ 1 - \frac{2\alpha_s(m_c)}{3\pi} \right\}
\]

\[
\frac{f_B\sqrt{m_B}}{f_D\sqrt{m_D}} = \left[ \frac{\alpha_s(m_c)}{\alpha_s(m_b)} \right]^{6/25}
\]
One-loop running and tree-level matching

two-loop running and one-loop matching

Matching and running must be computed in the same scheme.
\[ \mathcal{L} = \bar{h}_v (i \nu \cdot D) h_v + \frac{1}{2m_Q} \bar{h}_v (i D_\perp)^2 h_v - c_F \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v \]

\[ c_F(\mu) = \left[ \frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right]^{9/(33-2n_f)} \]
Reparameterization Invariance

\[ p_Q = m_Q v + k, \]
\[ v \rightarrow v + \varepsilon/m_Q, \]
\[ k \rightarrow k - \varepsilon. \]

Since \( v^2 = 1, \) \( v \cdot \varepsilon = 0. \)
Also \( \psi Q_v = Q_v \) so the change in the field:

\[ Q_v \rightarrow Q_v + \delta Q_v, \]

\( \delta Q_v \) satisfies

\[ \left( \psi + \frac{\xi}{m_Q} \right) (Q_v + \delta Q_v) = Q_v + \delta Q_v. \]
so that

$$(1 - \psi)\delta Q_v = \frac{\mathcal{J}}{m_Q} Q_v.$$ 

One can choose:

$$\delta Q_v = \frac{\mathcal{J}}{2m_Q} Q_v.$$

[Not unique, one can always make field redefinitions] $L$ invariant under

$$v \rightarrow v + \varepsilon/m_Q,$$

$$Q_v \rightarrow e^{i\varepsilon x} \left(1 + \frac{\mathcal{J}}{2m_Q}\right) Q_v,$$
\[ L_0 \rightarrow L_0 + \frac{1}{m_Q} \bar{Q}_v (i \varepsilon \cdot D) Q_v, \]
\[ L_1 \rightarrow L_1 - \frac{1}{m_Q} \bar{Q}_v (i \varepsilon \cdot D) Q_v. \]

so that the kinetic energy is not renormalized.

Other connections that follow form reparameterization invariance:

\[ c_S = 2c_F - 1, \quad \sigma \cdot \nabla \times E \]

E.g. relates matching coefficients of leading order and \(1/m\) operators, and their anomalous dimensions.
Luke’s Theorem

Can compute $1/m$ corrections to meson form-factors. Two sources of $1/m$ corrections, those from the Lagrangian, and from the current. So one has

$$T(L_1, J_0), \quad J_1$$

where

$$L = L_0 + \frac{1}{m}L_1 + \ldots, \quad J = J_0 + \frac{1}{m}J_1 + \ldots$$

Can apply the same spurion analysis as before, and work out the form-factors. Complicated expressions involving more Isgur-Wise functions for the matrix elements that enter.
Luke’s theorem: no $1/m$ corrections to the form-factor at zero recoil.
Experimentally, measure $\bar{B} \rightarrow D^*$ which determines $|V_{cb} \mathcal{F}(1)|$.

$$\mathcal{F}(1) = \eta_A + 0 + \mathcal{O} \left( \frac{1}{m^2} \right)$$

$\eta_A = 0.96$, and $1/m^2 \approx -0.05$, so

$$\mathcal{F}(1) = 0.91 \pm 0.05$$

and from this one finds

$$|V_{cb}| = \left[ 38.6 \pm 1.5(\text{exp}) \pm 2.0(\text{th}) \right] \times 10^{-3},$$