Two-loop hadronic corrections to Bhabha scattering

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Important process to determine the luminosity:

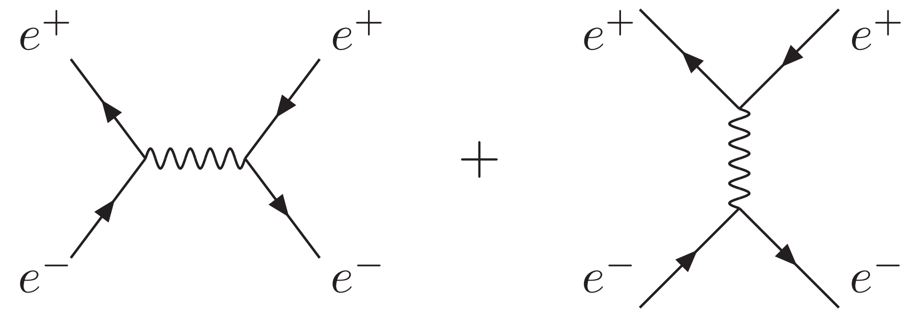
- at LEP and SLC in the small-angle region (to compete with Z-resonance)
- at flavor factories (BABAR, BELLE, DAΦNE,...) in the large-angle region
- at ILC in the large-angle region (luminosity spectrum)

Theoretical computations at two-loop in QED:

- Corrections with massless electrons (Bern-Dixon-Ghinculov '00)
- Electronic corrections with exact m_e^2/s dependence (Bonciani-Ferroglia-Mastrolia-Remiddi '04)
- Photonic corrections with small m_e^2/s (Penin '06, Becher-Melnikov '07)
- Corrections from light leptons with small m_e^2/s (Actis-Czakon-Gluza-Riemann '07, Becher-Melnikov '07)
- Corrections from heavy leptons with small m_e^2/s (Bonciani-Ferroglia-Penin '07)
- Hadronicnic corrections (Actis-Czakon-Gluza-Riemann '07, J.K.-Uccirati. '08)

The Born cross section

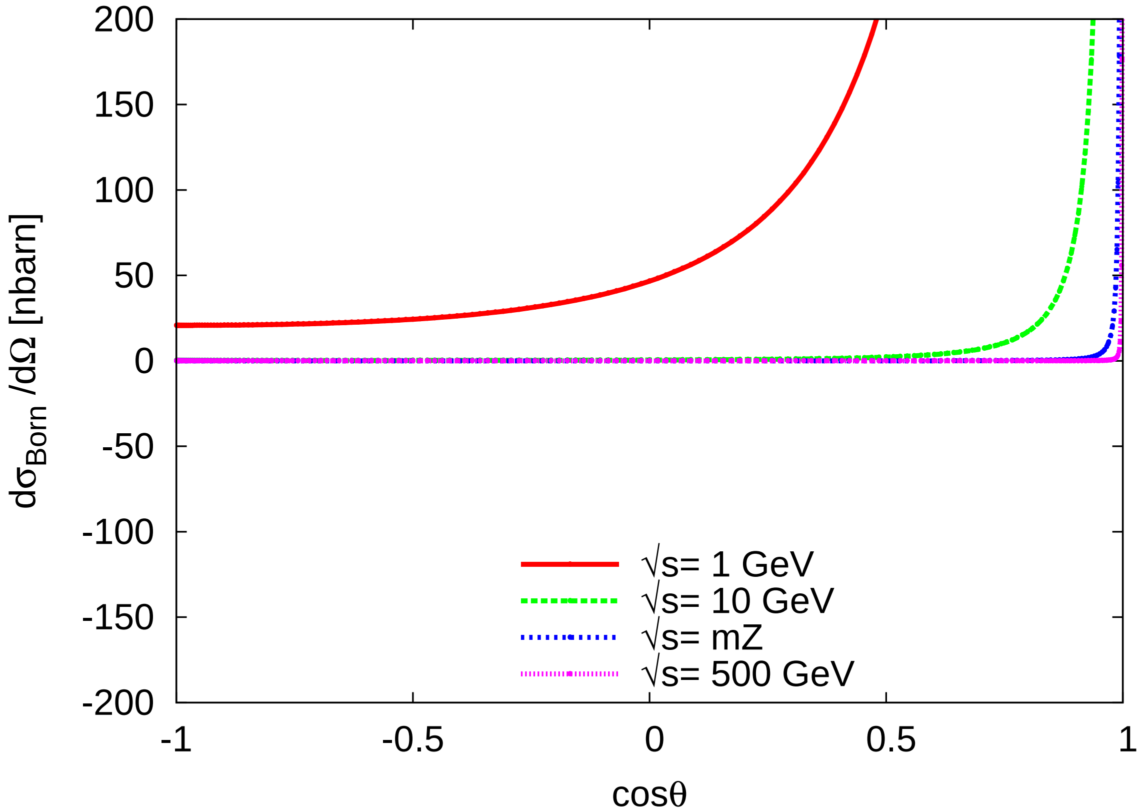
$\mathcal{A}_{\text{Born}} =$



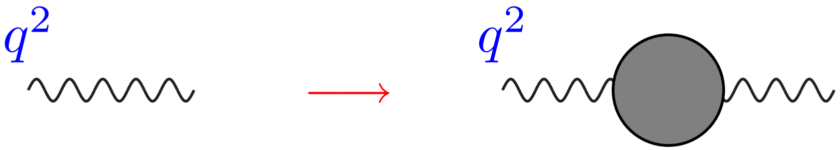
$$\frac{d\sigma_{\text{Born}}}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-x+x^2}{x}\right)^2$$

$$x = -\frac{t}{s} = \frac{1-\cos\theta}{2}$$

- Strong peaking in forward region $\sim \frac{1}{x^2}$



Hadronic corrections



$$\frac{-\,ig_{\alpha\beta}}{q^2+i\epsilon} \quad\longrightarrow\quad \frac{-\,ig_{\alpha\delta}}{q^2+i\epsilon}\, i\,(q^2 g^{\delta\epsilon} - q^\delta q^\epsilon)\,\Pi(q^2)\,\frac{-\,ig_{\epsilon\beta}}{q^2+i\epsilon} \quad\longrightarrow\quad \frac{-\,ig_{\alpha\beta}}{q^2+i\epsilon}\,\Pi(q^2)$$

Dispersion relation

$$\Pi(q^2) = -\,\frac{q^2}{\pi}\int_{4m^2}^{\infty}\frac{dz}{z}\,\frac{\mathrm{Im}\Pi(z)}{q^2-z+i\epsilon}, \qquad \mathrm{Im}\Pi(z) = -\,\frac{\alpha}{3}\,R(z)$$

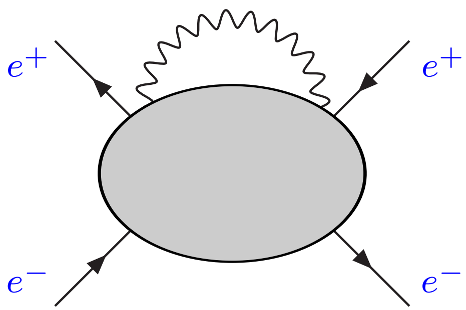
$$\frac{-\,ig_{\alpha\beta}}{q^2+i\epsilon} \quad\longrightarrow\quad \frac{\alpha}{3\pi}\int_{4m^2}^{\infty}\frac{dz}{z}\,R(z)\,\frac{-\,ig_{\alpha\beta}}{q^2-z+i\epsilon},$$

General recipe:

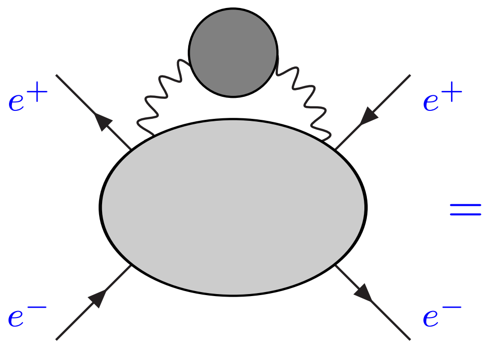
- Evaluate one-loop amplitude for massive vector boson

  Sum of all diagrams often more compact than individual diagram:

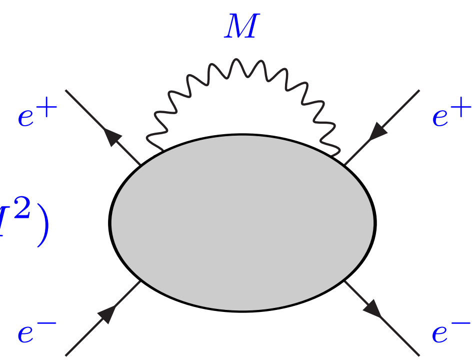
  e.g. box + crossed box



$\longrightarrow$



$$= \frac{\alpha}{3\pi}\int_{4m^2}^{\infty} \frac{dM^2}{M^2}\, R(M^2)$$

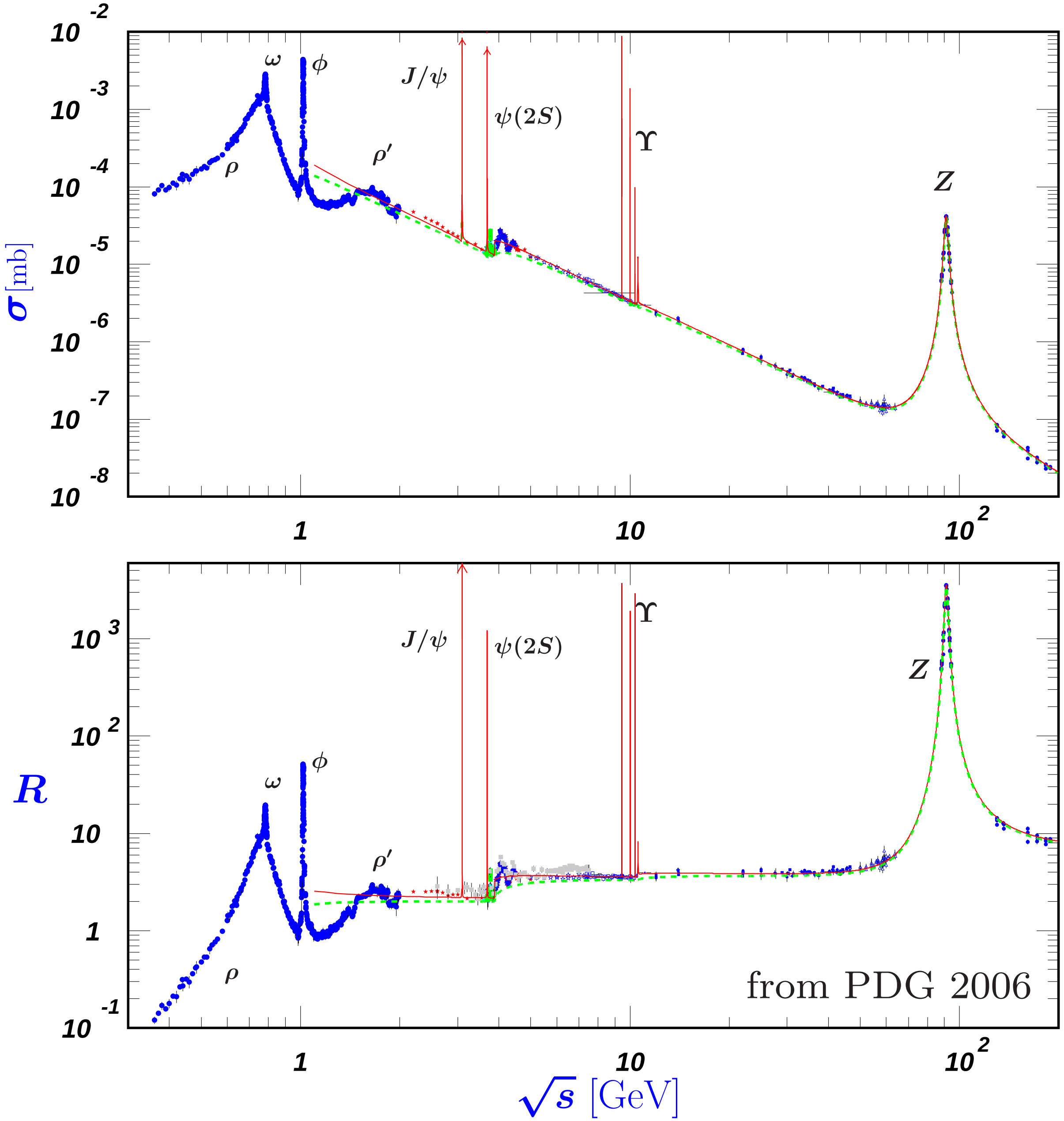


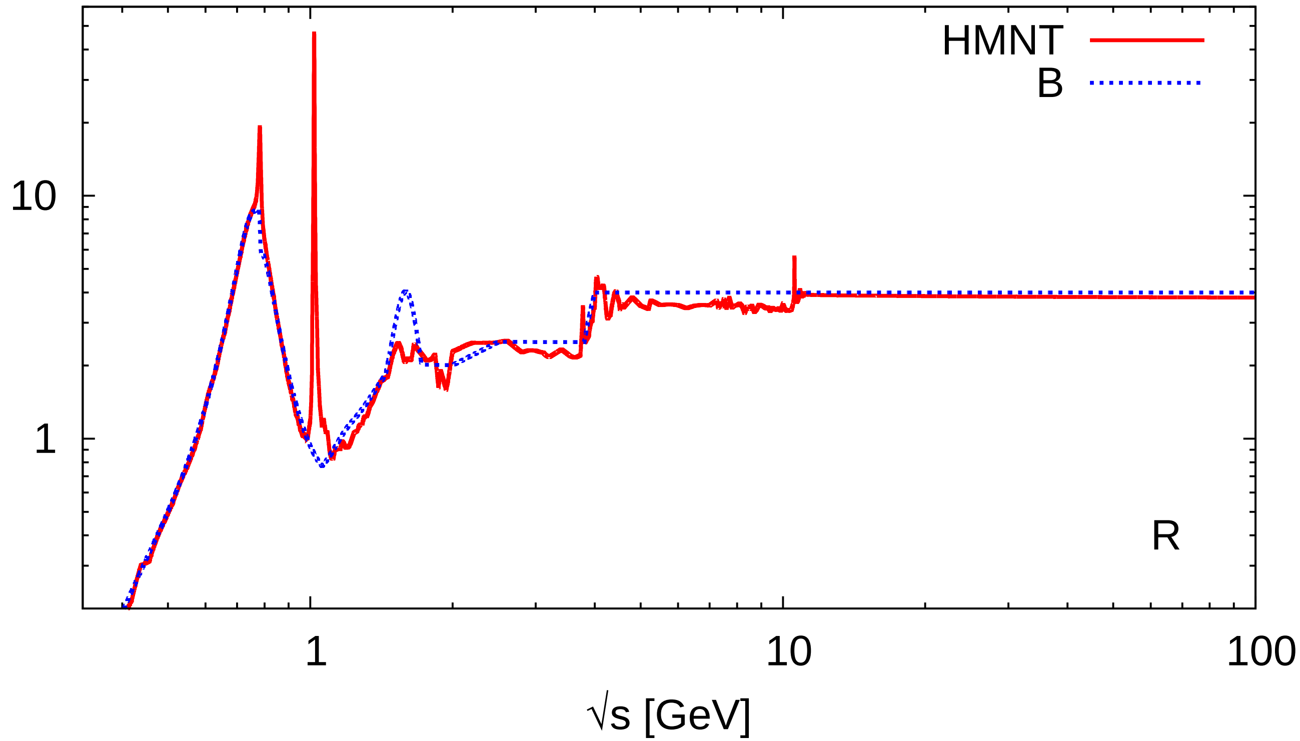
$$R(s) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-, \mathrm{QED}sp)}$$

$\downarrow$

leptonic case

$$R_l(s) = \left(1+\frac{4m_l^2}{2\,s}\right)\sqrt{1-\frac{4m_l^2}{s}}.$$





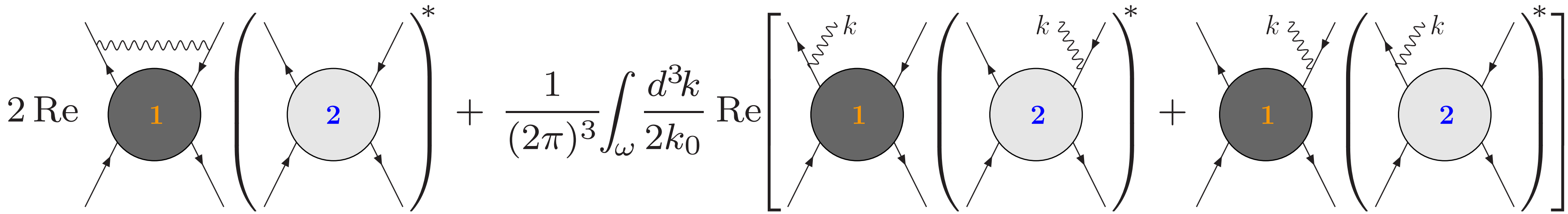
Parametrization by Burkhardt (B) and Hagiwara, Martin, Nomura, Teubner (HMNT)

	J/Ψ	$\Psi(2S)$	$\Upsilon(1S)$	$\Upsilon(2S)$	$\Upsilon(3S)$
$M(\mathrm{GeV})$	3.096916(11)	3.686093(34)	9.46030(26)	10.02326(31)	10.3552(5)
$\Gamma_{ee}(\mathrm{keV})$	5.55(14)	2.48(6)	1.340(18)	0.612(11)	0.443(8)
$(\alpha/\alpha(M))^2$	0.957785	0.95554	0.932069	0.93099	0.930811

Three classes of contributions:

1. Vacuum polarization insertion
2. Reducible vertex and box corrections
3. Irreducible vertex and box corrections

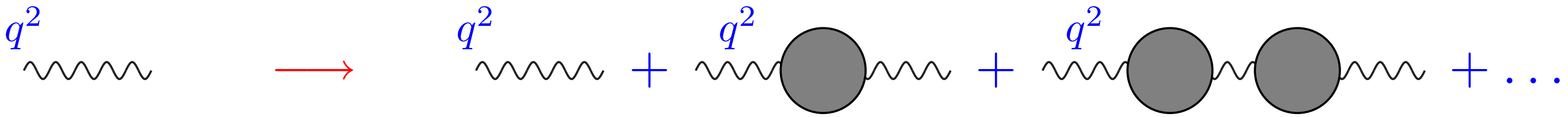
Inclusion of real soft photons



$$k = (k_0, \vec{k}) \qquad k_0^2 = \vec{k}^2 + \lambda^2 \qquad |\vec{k}| < \omega$$

1. Vacuum polarization insertion

- **Resummation:**



$$\frac{-ig_{\alpha\beta}}{q^2+i\epsilon} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2+i\epsilon}\left[1+\Pi(q^2)+\Pi(q^2)^2+\ldots\right] = \frac{-ig_{\alpha\delta}}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}$$

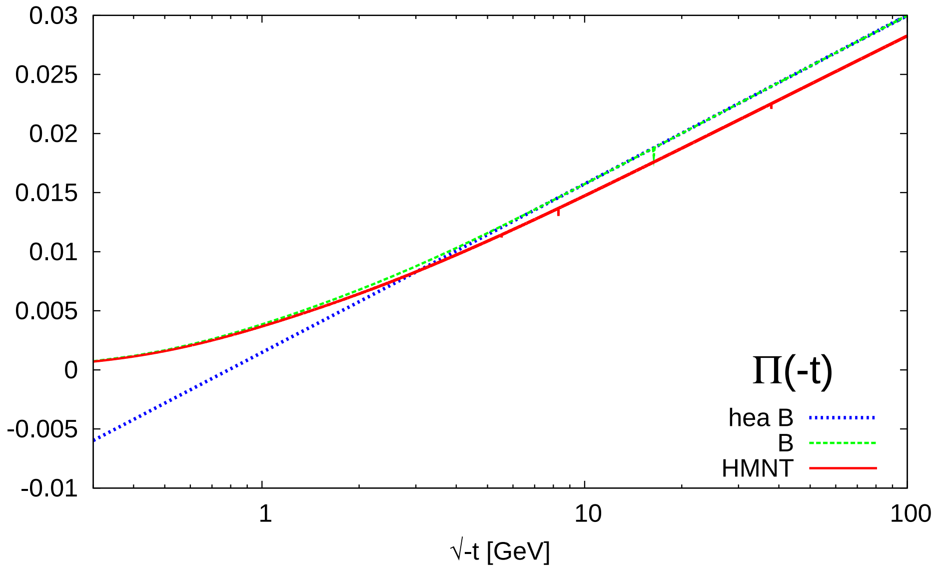
$$\frac{d\sigma_\Pi}{d\Omega} = \frac{\alpha^2}{s}\left\{ \frac{1-2x+2x^2}{2}\left|\frac{1}{1-\Pi(s)}\right|^2 + \frac{2-2x+x^2}{2\,x^2}\left|\frac{1}{1-\Pi(t)}\right|^2 - \frac{(1-x)^2}{x}\mathrm{Re}\,\frac{1}{[1-\Pi(s)]\,[1-\Pi(t)]}\right\}$$

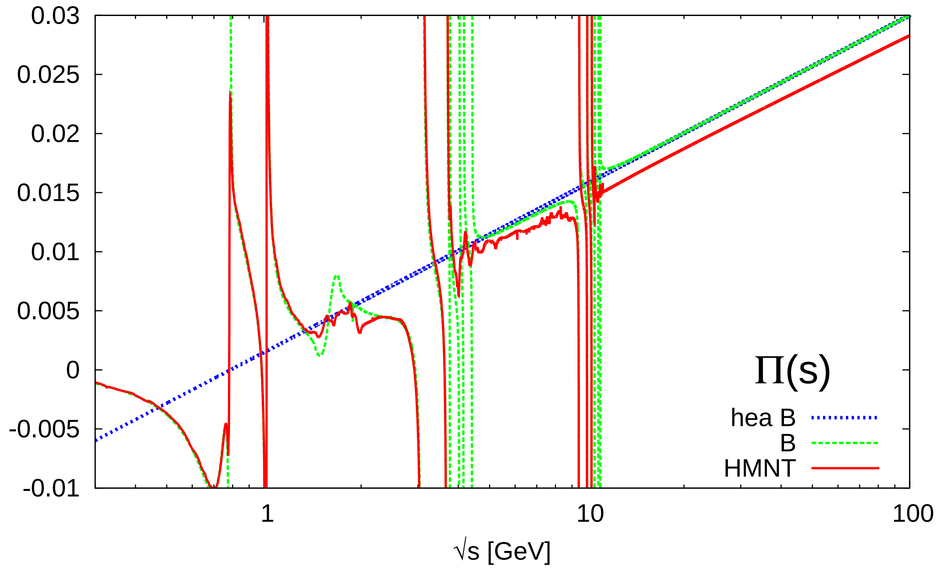
$$\Pi = \Pi_{\mathrm{had}} + \sum_l \Pi_l^{(1)} + \sum_l \Pi_l^{(2)} + \ldots$$

Evaluation of $\Pi(q^2)$

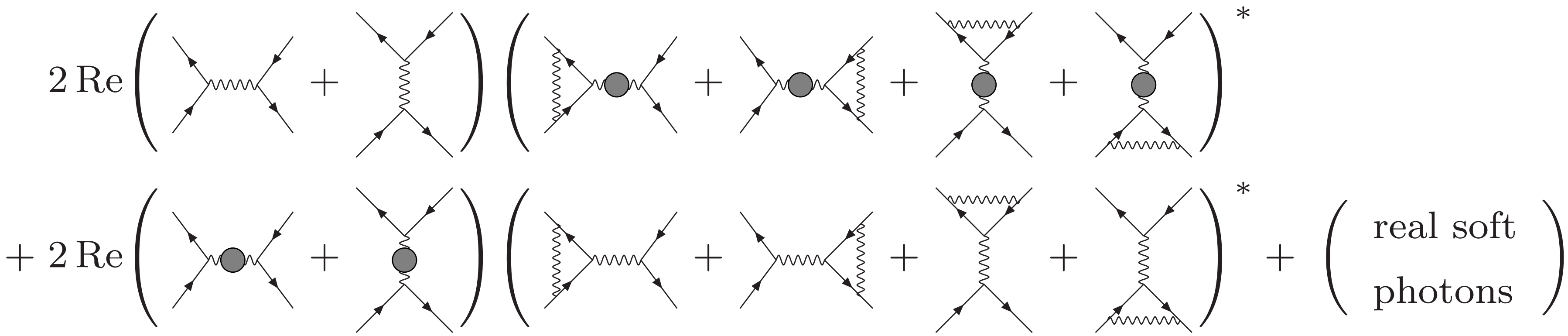
$$\Pi(t) = \frac{\alpha}{3\pi}\int_0^1 dy\,\frac{t}{yt-4m^2}\,R\left(\frac{4m^2}{y}\right),$$

$$\Pi(s) = \frac{\alpha}{3\pi}\left\{\ln\left(1-\frac{s}{4m^2-i\epsilon}\right)R(s) + \int_0^1 dy\,\frac{s}{ys-4m^2}\left[R\left(\frac{4m^2}{y}\right)-R(s)\right]\right\}$$





2. Reducible vertices ...



$$\frac{d\sigma_{\mathrm{red,V}}}{d\Omega} = \frac{\alpha^3}{s\pi}\left\{ \frac{1-2x+2x^2}{2}\left[4\,V_s^\gamma\,\mathrm{Re}\,\Pi(s)\right] + \frac{2-2x+x^2}{2\,x^2}\left[4\,V_t^\gamma\,\Pi(t)\right] - \frac{(1-x)^2}{x}\left[(V_s^\gamma + V_t^\gamma)\Big(\mathrm{Re}\,\Pi(s)+\Pi(t)\Big) + \pi\left(\ln\frac{\lambda^2}{s}+\frac{3}{2}\right)\mathrm{Im}\,\Pi(s)\right]\right\}$$

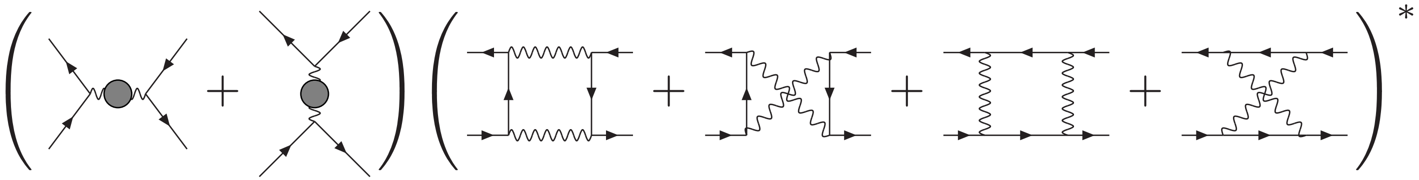
cancels with reducible boxes

$$V_s^\gamma = 2\ln\frac{2\omega}{\sqrt{s}}\left(\ln\frac{s}{m_e^2}-1\right) + \frac{3}{2}\ln\frac{s}{m_e^2} + 2\,\zeta(2) - 2$$

$$V_t^\gamma = 2\ln\frac{2\omega}{\sqrt{s}}\left(\ln\frac{-t}{m_e^2}-1\right) + \frac{3}{2}\ln\frac{-t}{m_e^2} - \ln\frac{-t}{s}\ln\frac{-u}{s} - \mathrm{Li}_2\left(\frac{-t}{s}\right) - 2$$

Enhancement $\sim \ln\frac{s}{m_e^2}$! dominant correction!

... and reducible boxes

2 Re  + (real soft photons)

$$\frac{d\sigma_{\rm red,B}}{d\Omega} = \frac{\alpha^3}{s\pi}\Bigg\{ \frac{1-2x+2x^2}{2}\left[2\,B_s^\gamma\,{\rm Re}\,\Pi(s) + 2\pi\ln\frac{t}{u}\,{\rm Im}\,\Pi(s)\right] + \frac{2-2x+x^2}{2\,x^2}\left[2\,B_t^\gamma\,\Pi(t)\right]$$

$$-\frac{(1-x)^2}{x}\left[B_t^\gamma{\rm Re}\Pi(s) + B_s^\gamma\Pi(t) - \pi\ln\frac{\lambda^2}{-t}\,{\rm Im}\Pi(s)\right]$$

$$-\,{\rm Re}\left[\Big(B^\gamma(s,t) - B^\gamma(s,u)\Big)\,\Pi^*(s)\right] - {\rm Re}\left[\Big(B^\gamma(t,s) - B^\gamma(t,u)\Big)\,\Pi(t)\right]$$

$$+{\rm Re}\left[xB^\gamma(t,s)\Pi^*(s) + \frac{1}{x}B^\gamma(s,t)\Pi(t)\right]\Bigg\}$$

$$B_s^\gamma = 2\ln\frac{2\omega}{\sqrt{s}}\ln\frac{t}{u} + \frac{1}{2}\ln^2\frac{-t}{s} - \frac{1}{2}\ln^2\frac{-u}{s} - \ln\frac{-t}{s}\ln\frac{-u}{s} - 2\,{\rm Li}_2\left(\frac{-t}{s}\right) + \zeta(2),$$

$$B_t^\gamma = -2\ln\frac{2\omega}{\sqrt{s}}\ln\frac{-u}{s} - \frac{1}{2}\ln^2\frac{-u}{s} + \ln\frac{-t}{s}\ln\frac{-u}{s} - {\rm Li}_2\left(\frac{-t}{s}\right),$$

$$B^\gamma(a,b) = -\frac{a+b}{2\,a}\ln\frac{b}{a+i\,\epsilon} + \frac{a+2b}{4\,a}\left(\ln^2\frac{b}{a+i\,\epsilon} + \pi^2\right).$$

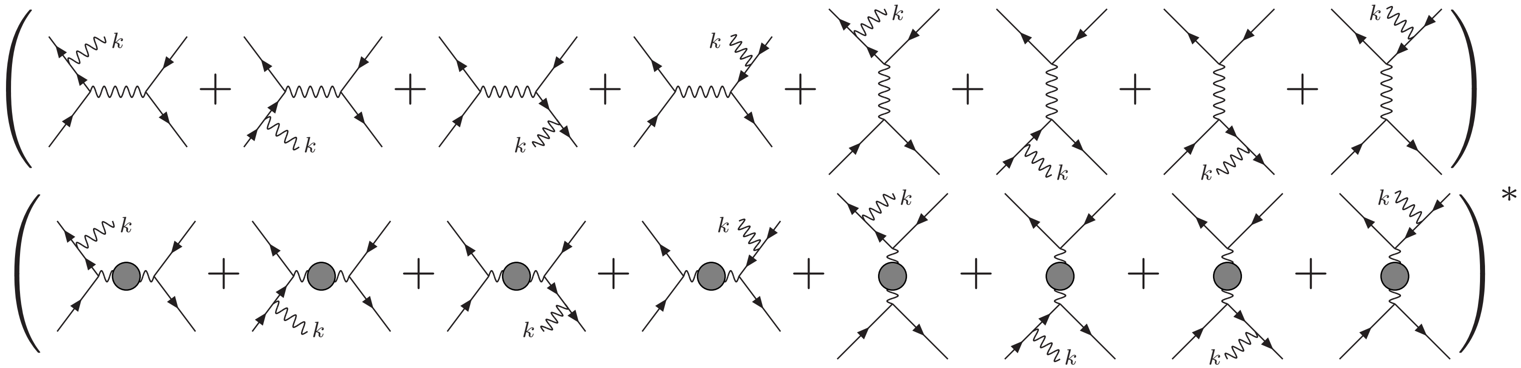
- Evaluation of reducible vertices and boxes ⇝ trivial

$$\text{(well known one-loop)} \times (\ \Pi(s) \text{ or } \Pi(t)\ )$$

$\Pi(s)$ and $\Pi(t)$, evaluated once for ever, should not be mixed with irreducible two-loop kernel.

- Real soft radiation

$$\frac{1}{(2\pi)^3}\int_\omega \frac{d^3k}{2k_0}\, 2\mathrm{Re}$$



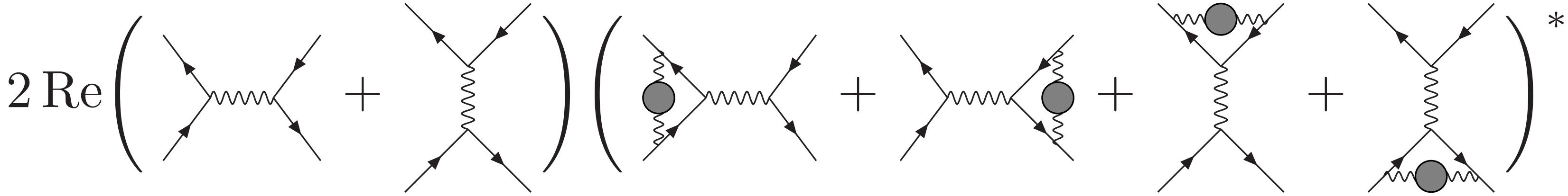
⇝ Always $\sim \Pi(s)$ or $\Pi(t)$

⇝ Compensates terms $\sim \Pi(s)$ or $\Pi(t)$ in the two-loop terms

- Reducible contributions can be combined with one-loop corrections:

$$1 + \Pi \ \rightarrow\ \frac{1}{1-\Pi}$$

3. Irreducible vertices ...



$$\frac{d\sigma_{\rm V}}{d\Omega} = \frac{\alpha^3}{s\,\pi}\left\{\frac{1-2x+2x^2}{2}\left[4{\rm Re}V(s)\right] + \frac{2-2x+x^2}{2\,x^2}\left[4V(t)\right] - \frac{(1-x)^2}{x}\left[2V(t)+2{\rm Re}V(s)\right]\right\}$$

- **Second dispersion integral:**

$$V(q^2) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty}\frac{dz}{z}R(z)\,\rho\,(q^2, z-i\epsilon)$$

$$\rho\,(q^2,z) = -\frac{7}{8} - \frac{z}{2q^2} + \frac{1}{2}\left(\frac{3}{2}+\frac{z}{q^2}\right)\ln\frac{-z}{q^2} + \frac{1}{2}\left(1+\frac{z}{q^2}\right)^2\left[\zeta(2) - {\rm Li}_2\left(1+\frac{z}{q^2}\right)\right].$$

  - Infrared and UV finite
  - Monte Carlo: $V(q^2)$ can be combined with Born !

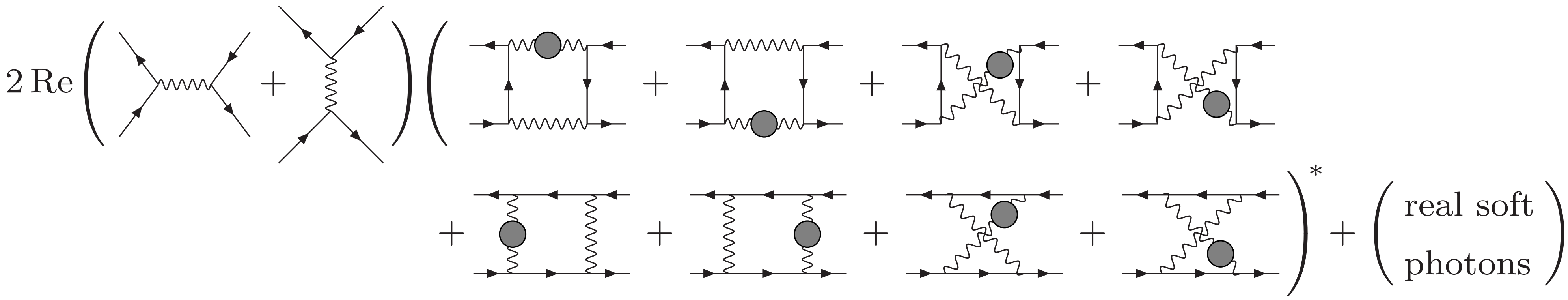
Evaluation of $V(q^2)$ (Kniehl, Krawczyk, J.K., Stuart, 1988)

$$V(q^2) = \frac{\alpha}{3\pi}\left\{R(\infty)\int_0^1 \frac{dy}{y}\rho\left(q^2, \frac{4m^2}{y}\right) + \int_0^1 \frac{dy}{y}\rho\left(q^2, \frac{4m^2}{y}\right)\left[R\left(\frac{4m^2}{y}\right) - R(\infty)\right]\right\}$$

$$\int_0^1 \frac{dy}{y}\rho\left(q^2, \frac{4m^2}{y}\right) = -\frac{1}{12}\ln^3(-r) - \ln(-r)\left[\zeta(2) + \frac{7}{8} + \frac{1}{4r} + \frac{1}{2}\mathrm{Li}_2\left(-\frac{1}{r}\right)\right]$$
$$+\left(\frac{3}{4} + \frac{1}{r} + \frac{1}{4r^2}\right)\left[\zeta(2) - \mathrm{Li}_2(1+r)\right] + \frac{15}{16} + \frac{1}{4r} - \mathrm{Li}_3\left(-\frac{1}{r}\right) \qquad r = \frac{q^2}{4m^2}$$

This transformation leads to better convergence and allows to extract in a simple way the high energy behaviour of $V(q^2)$

... and irreducible boxes



$$\frac{d\sigma_{\rm B}}{d\Omega} = \frac{\alpha^3}{s\pi}\left\{ \frac{1-2x+2x^2}{2}\Big[2B_s{\rm Re}\Pi(s)\Big] + \frac{2-2x+x^2}{2\,x^2}\Big[2B_t\Pi(t)\Big] - \frac{(1-x)^2}{x}\Big[B_t{\rm Re}\Pi(s)+B_s\Pi(t)\Big] \right.$$

$$\left. -\,{\rm Re}\Big[B(s,t,u)-B(s,u,t)\Big] - {\rm Re}\Big[B(t,s,u)-B(t,u,s)\Big] + {\rm Re}\Big[xB(t,s,u)+\frac{1}{x}B(s,t,u)\Big]\right\}$$

$$B_s = 2\ln\frac{2\omega}{\sqrt{s}}\,\ln\frac{t}{u} - \ln\frac{-t}{s}\,\ln\frac{-u}{s} - 2\,{\rm Li}_2\left(\frac{-t}{s}\right) + \zeta(2) \qquad B_t = -2\ln\frac{2\omega}{\sqrt{s}}\,\ln\frac{-u}{s} - {\rm Li}_2\left(\frac{-t}{s}\right) + 3\,\zeta(2)$$

Third dispersion integral

$$B(a,b,c) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty}\frac{dz}{z}R(z)\left[\xi_A(a,b,c,z-i\epsilon)+\xi_B(a,b,c,z-i\epsilon)\right]$$

$$\xi_A(a,b,c,z) = \frac{c^2}{a\,(z-a)}\left[2\ln\frac{c}{b+i\epsilon}\ln\left(1-\frac{a}{z}\right)-\mathrm{Li}_2\left(1+\frac{b}{z}\right)+\mathrm{Li}_2\left(1+\frac{c}{z}\right)\right]$$

$$\begin{aligned}\xi_B(a,b,c,z) &= \frac{c}{a}\left[\left(\frac{z}{a}-1\right)\ln\left(1-\frac{a}{z}\right)+\ln\frac{-b}{z}\right]\\ &+\frac{c-b-z}{a}\left[\ln\frac{b+i\epsilon}{-a}\ln\left(1-\frac{a}{z}\right)-\mathrm{Li}_2\left(1-\frac{a}{z}\right)+\mathrm{Li}_2\left(1+\frac{b}{z}\right)\right]\end{aligned}$$

- Can be taken directly from the literature ($Z-\gamma$ box) !
- No need to keep $m_e \neq 0$!
- Compact expression !
- Optimally suited for numerical evaluation

Evaluation of $B(a,b,c)$

Direct numerical integration of $R\times$ kernel possible. Improvement by reshuffling:

- Terms involving $\int dz \frac{R(z)}{z-s}\ldots$ are replaced by:

$$\int dz\,\frac{R(z)}{z-s}\ldots \;\Rightarrow\; R(s)\int dz\,\frac{1}{z-s}\ldots + \int dz\,\frac{R(z)-R(s)}{z-s}\ldots$$

The first term is evaluated analytically, the second is a smooth integral

$$B_A(s,b,c) = \frac{\alpha}{3\pi}\left\{R(s)\int_0^1 \frac{dy}{y}\xi_A\left(s,b,c,\frac{4m^2}{y}\right) + \int_0^1 \frac{dy}{y}\xi_A\left(s,b,c,\frac{4m^2}{y}\right)\left[R\left(\frac{4m^2}{y}\right) - R(s)\right]\right\}$$

$$\int_0^1 \frac{dy}{y}\xi_A\left(s,b,c,\frac{4m^2}{y}\right) = \frac{c^2}{s^2}\left[\ln\frac{c}{b+i\epsilon}\ln^2\frac{4m^2-s}{4m^2} + J_A\left(-\frac{b}{s}\right) - J_A\left(-\frac{c}{s}\right)\right]$$

$$\begin{aligned}J_A(x) = {}& \frac{1}{6}\ln^3(-xr) + \frac{1}{6}\ln^3\frac{xr}{-\bar{r}} - \frac{1}{2}\ln x\ln^2(-xr) - \frac{1}{2}\ln(\bar{x}r)\ln^2(1-\bar{x}r) + \frac{1}{2}\ln\frac{1-\bar{x}r}{x}\ln^2\frac{xr}{-\bar{r}}\\ & - \ln\bar{r}\,\mathrm{Li}_2(1-\bar{x}r) + \ln\frac{xr}{-\bar{r}}\mathrm{Li}_2\left(\frac{\bar{r}\bar{x}}{-x}\right) + \ln(\bar{x}r)\left[\mathrm{Li}_2(r) - \mathrm{Li}_2(\bar{x}r) - \mathrm{Li}_2\left(\frac{xr}{1-\bar{x}r}\right)\right]\\ & + \ln(-xr)\left[\mathrm{Li}_2(r) - \mathrm{Li}_2(\bar{x}r) - \mathrm{Li}_2\left(\frac{-\bar{x}}{x}\right)\right] - \mathrm{Li}_3\left(\frac{-\bar{x}}{x}\right) + \mathrm{Li}_3(\bar{x}r) + \mathrm{Li}_3\left(\frac{\bar{r}\bar{x}}{-x}\right) + S_{12}\left(\frac{1-\bar{x}r}{\bar{r}}\right).\end{aligned}$$

$$x = -\frac{t}{s} \qquad \bar{x} = 1-x \qquad r = \frac{s}{4m^2} \qquad \bar{r} = 1-r$$

- $R \times \text{kernel} = [R - R(\infty)] \times \text{kernel} + R(\infty) \times \text{kernel}$

  Evaluate const$\times$kernel analytically $\Rightarrow$ Improved convergence for large z

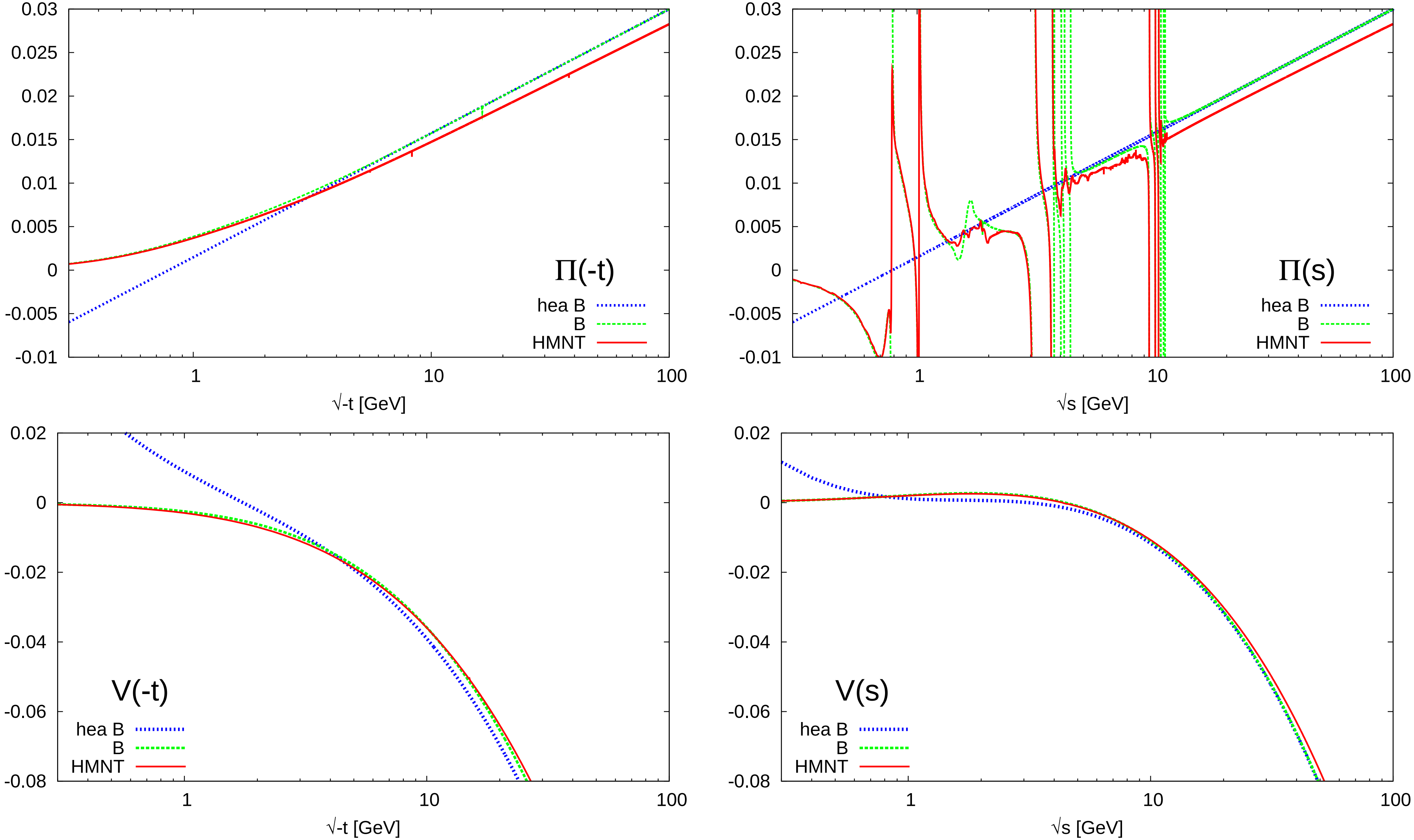
$$B_B(a,b,c) = \frac{\alpha}{3\pi}\left\{R(\infty)\int_0^1 \frac{dy}{y}\xi_B\left(a,b,c,\frac{4m^2}{y}\right) + \int_0^1 \frac{dy}{y}\xi_B\left(a,b,c,\frac{4m^2}{y}\right)\left[R\left(\frac{4m^2}{y}\right) - R(\infty)\right]\right\}$$

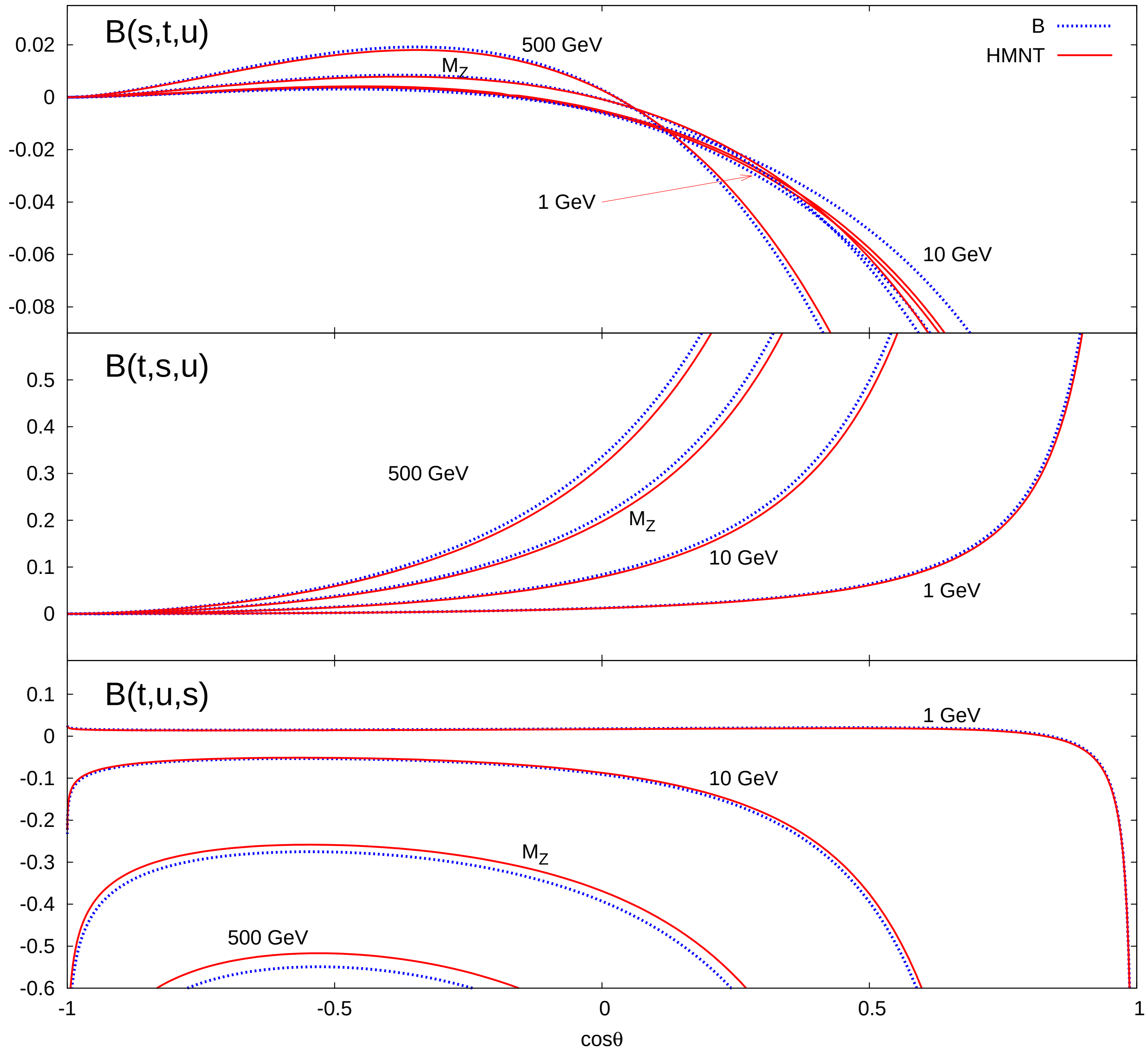
$$\begin{aligned}\int_0^1 \frac{dy}{y}\xi_B(a,b,c,\frac{4m^2}{y}) = &\frac{4m^2-a}{a}\left[\left(\ln\frac{-b}{4m^2} - \frac{c}{a}\right)\ln\frac{4m^2-a}{4m^2} + \mathrm{Li}_2\left(\frac{a}{4m^2}\right)\right] \\ &- \frac{4m^2+b}{a}\left[\ln\frac{-b}{4m^2}\ln\frac{4m^2+b}{4m^2} + \mathrm{Li}_2\left(\frac{-b}{4m^2}\right)\right] + \frac{c}{a}\left[\mathrm{Li}_2\left(\frac{a}{4m^2}\right) - \ln\frac{-b}{4m^2}\right] \\ &+ \frac{c-b}{a}\left\{\ln\frac{-b}{4m^2}\left[\mathrm{Li}_2\left(\frac{-b}{4m^2}\right) - \mathrm{Li}_2\left(\frac{a}{4m^2}\right)\right] + 2\,\mathrm{Li}_3\left(\frac{a}{4m^2}\right) - 2\,\mathrm{Li}_3\left(\frac{-b}{4m^2}\right)\right\}\end{aligned}$$

- The separation of reducible terms ($\sim \Pi(s), \Pi(t)$) and irreducible terms (B, V) allows a fast, stable and efficient evaluation also for more complicated R-functions.

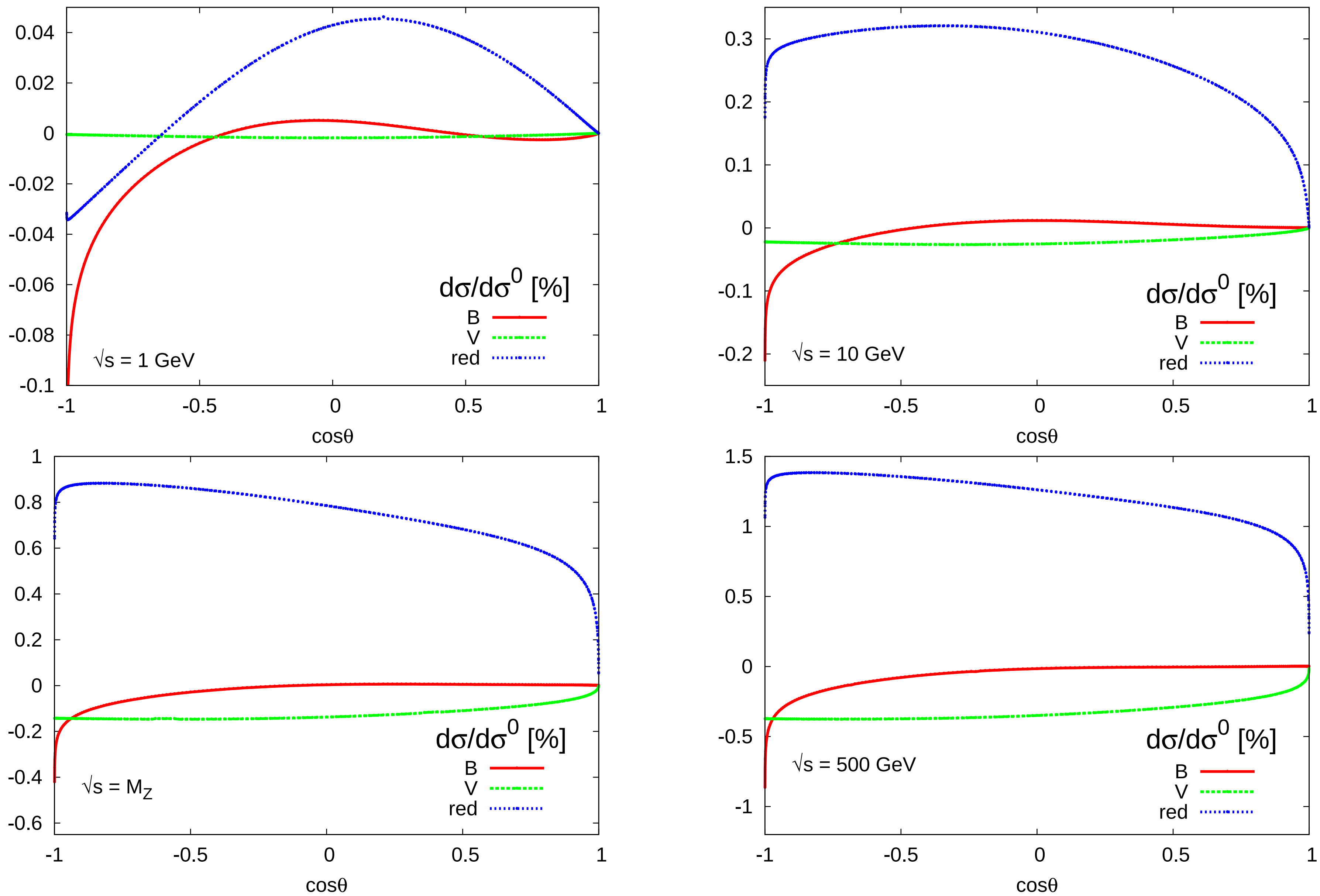
Note: all these expressions are just a consequence of the optimization of numerics. The original kernel is a three-line expression

Recall **building blocks**, which can be studied individually (excellent convergence, stable numerics)




B(s,t,u)
B
HMNT
500 GeV
M_Z
1 GeV
10 GeV
0.02
0
-0.02
-0.04
-0.06
-0.08
B(t,s,u)
500 GeV
M_Z
10 GeV
1 GeV
0.5
0.4
0.3
0.2
0.1
0
B(t,u,s)
1 GeV
10 GeV
M_Z
500 GeV
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-1
-0.5
0
0.5
1
cosθ

Relative importance of reducible terms vs. irreducible vertices vs. irreducible boxes



Light leptons contribution $(m_l^2 \ll s, |t|, |u|)$

$$m \quad \rightarrow \quad m_l \qquad\qquad R \quad \rightarrow \quad R_l(s) = \left(1 + \frac{4m_l^2}{2\,s}\right)\sqrt{1 - \frac{4m_l^2}{s}}$$

The dispersion integrals are computed analytically:

$$\Pi_l(q^2) = -\frac{\alpha}{3\pi}\left(\ln\frac{-q^2}{m_l^2} - \frac{5}{3}\right)$$

$$V_l(q^2) = -\frac{\alpha}{3\pi}\left[\frac{1}{12}\ln^3\frac{-q^2}{m_l^2} - \frac{19}{24}\ln^2\frac{-q^2}{m_l^2} + \frac{1}{2}\left(\zeta(2) + \frac{265}{72}\right)\ln\frac{-q^2}{m_l^2} + \zeta(3) - \frac{19}{12}\zeta(2) - \frac{3355}{432}\right]$$

$$B_l(a,b,c) = -\frac{\alpha}{3\pi}\left\{\frac{c-b}{2a}\left[L_b^2 + 6\zeta(2)\right]\left(L_{am} - \frac{8}{3} + \frac{L_b}{3}\right) - \frac{c}{a}\left[L_b\left(L_{am} - \frac{8}{3}\right) - \frac{L_b^2}{2} - 5\zeta(2)\right]\right.$$
$$\left. - \frac{c^2}{a^2}\left[\frac{L_{bc}}{2}\left(L_{am}^2 - L_{am}\left(L_b + L_c + \frac{10}{3}\right) + \frac{5}{3}\left(L_b + L_c\right) + \frac{56}{9}\right) - J(b,c) + J(c,b)\right]\right\}$$

$$L_{am} = \ln\frac{-a}{m_l^2} \quad L_b = \ln\frac{b+i\epsilon}{a} \quad L_c = \ln\frac{c+i\epsilon}{a} \quad L_{bc} = \ln\frac{b+i\epsilon}{c} \quad J(x,y) = S_{12}\left(\frac{-x}{y-i\epsilon}\right) + i\pi \mathrm{Li}_2\left(\frac{-x}{y-i\epsilon}\right)$$

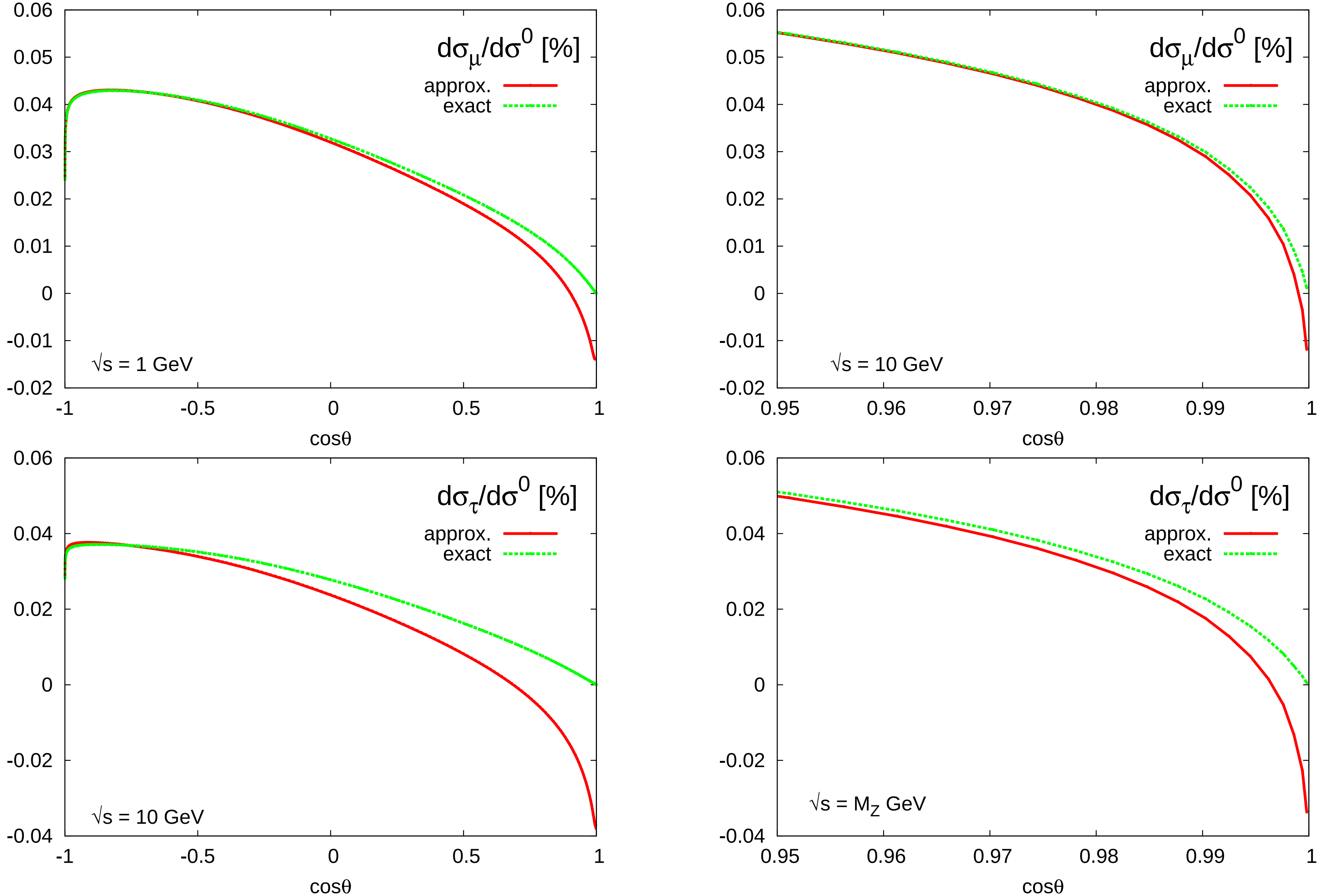
- In agreement with earlier results (Actis-Czakon-Gluza-Riemann, Becher-Melnikov)
- $\Pi_e = \Pi_l|_{m_l \to m_e}\,,\qquad B_e = B_l|_{m_l \to m_e}\,,\qquad V_e = V_l|_{m_l \to m_e} - \frac{\alpha}{3\pi}\left[-\zeta(3) + \frac{7}{3}\zeta(2) - \frac{1241}{432}\right]$

- **High energy approximation** can also be formulated for **hadrons**
- Result expressed in terms of simple analytic functions (similar to lepton case) multiplied by moments $R(\infty)$ and R_n $(n = 0, 1, 2)$

$$R(\infty) = R(s \to \infty) \qquad\qquad R_n = \int_0^1 \frac{dx}{x}\frac{\ln^n x}{n!}\left[R\left(\frac{4m^2}{x}\right) - R(\infty)\right].$$

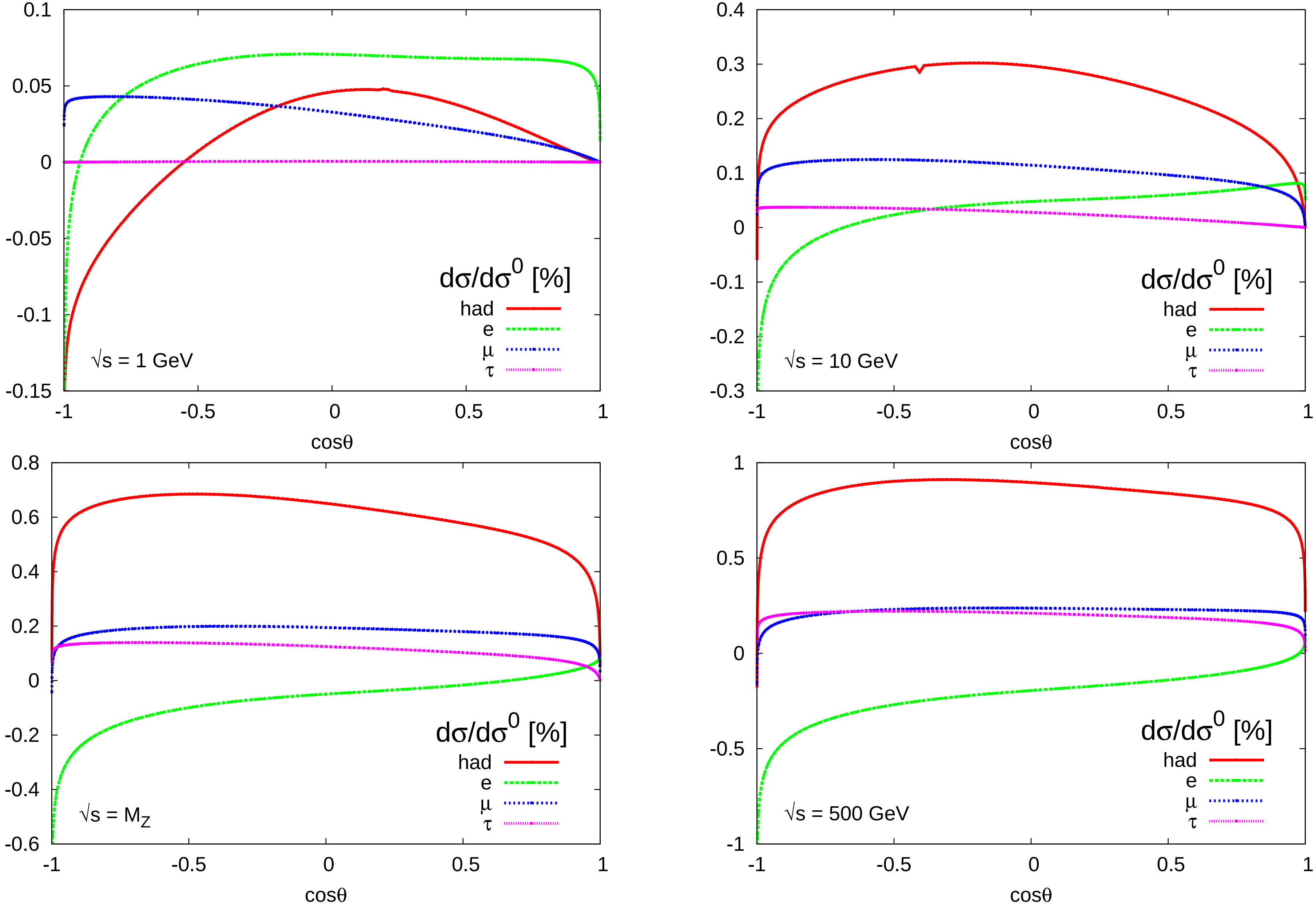
$\rightsquigarrow$ Only useful for $s, |t|, |u| \;\geq\; 2$ GeV

Leptons



Deviations between exact and high energy approximation for small angles (small t)

Relative importance of hadrons vs. e vs. μ vs. τ



Relative size of corrections

$\sqrt{s}$	1 GeV			10 GeV		
θ	3^o	90^o	177^o	3^o	90^o	177^o
$d\sigma^0/d\Omega$ [pb]	$440994\cdot10^5$	46653.7	20735.0	$440994\cdot10^3$	466.537	207.350
$d\sigma^B/d\sigma^0$ $[10^{-3}]$	$-8.626\cdot10^{-4}$ $-8.182\cdot10^{-4}$	0.05267 0.05024	−1.295 −1.230	$1.235\cdot10^{-4}$ $-0.076(1)\cdot10^{-4}$	0.1267 0.1168	−1.421 −1.532
$d\sigma^V/d\sigma^0$ $[10^{-3}]$	$-1.234\cdot10^{-4}$ $-1.191\cdot10^{-4}$	−0.01877 −0.01796	−0.005167 −0.004983	−0.004261 −0.004084	−0.2695 −0.2560	−0.2352 −0.2246
$d\sigma^{\rm red}/d\sigma^0$ $[10^{-3}]$	$8.934\cdot10^{-4}$ $8.169\cdot10^{-4}$	0.4461 0.4286	−0.3666 −0.3388	0.08860 0.08529	3.317 3.098	2.644 2.290
$d\sigma_{\rm had}/d\sigma^0$ $[10^{-3}]$	$-0.9259\cdot10^{-4}$ $-1.204\cdot10^{-4}$	0.4800 0.4609	−1.667 −1.575	0.08446 0.08120	3.175 2.959	0.9880 0.5341
$d\sigma_e/d\sigma^0$ $[10^{-3}]$	0.3114	0.7070	−1.460	0.6862	0.4773	−3.516
$d\sigma_\mu/d\sigma^0$ $[10^{-3}]$	$6.623\cdot10^{-4}$	0.3273	0.3275	0.09040	1.143	0.6128
$d\sigma_\tau/d\sigma^0$ $[10^{-3}]$	$4.100\cdot10^{-6}$	0.004869	$-7.525\cdot10^{-4}$	$3.926\cdot10^{-4}$	0.2776	0.3265

upper/lower lines → B/HMNT

Relative size of corrections

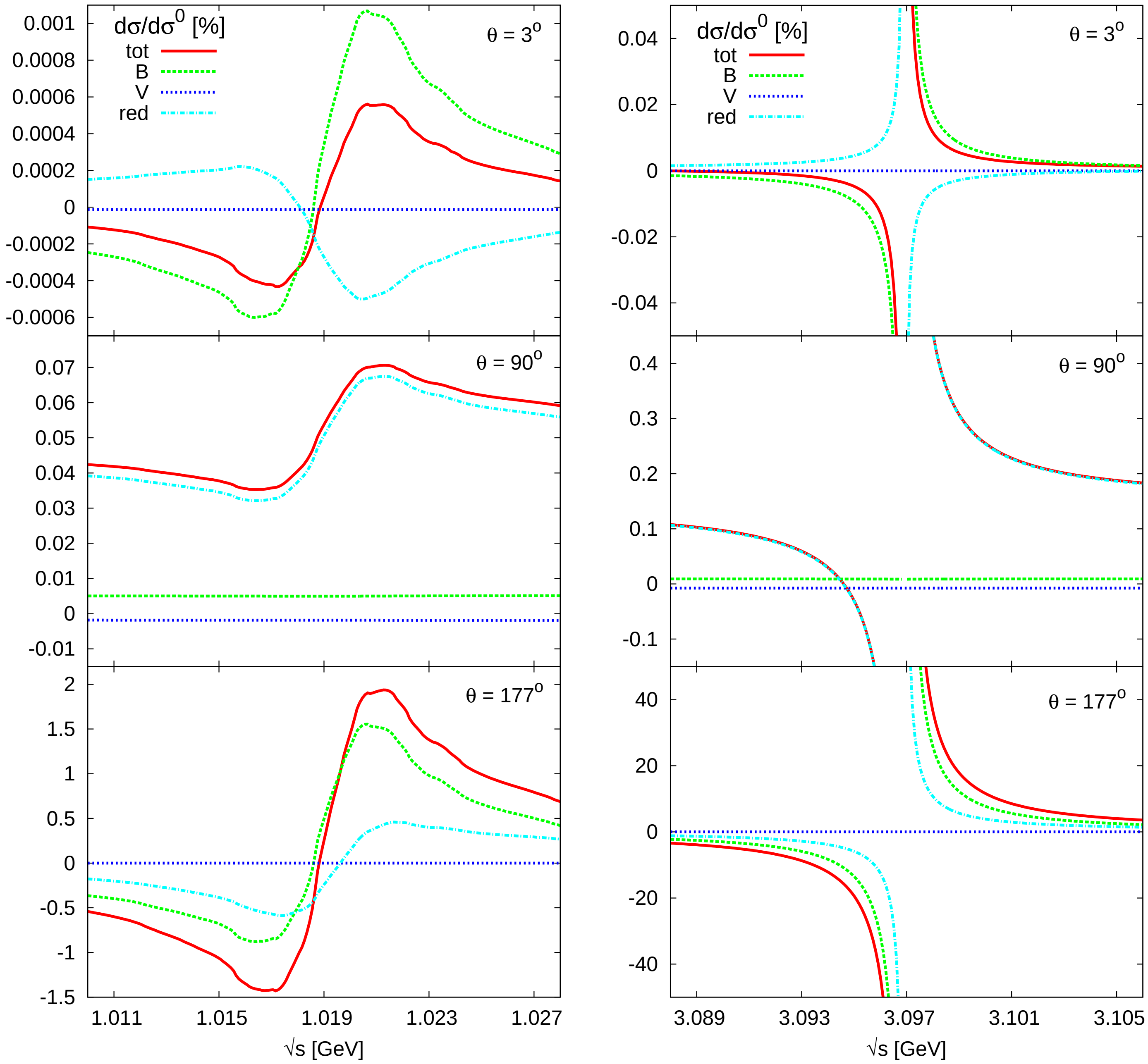
$\sqrt{s}$	M_Z			500 GeV		
θ	3^o	90^o	177^o	3^o	90^o	177^o
$d\sigma^0/d\Omega$ [pb]	5303480	5.61067	2.49363	176398	0.186615	0.0829400
$d\sigma^B/d\sigma^0$ $[10^{-3}]$	0.001685 0.001579	0.03648 0.03537(1)	-3.418 -3.200	0.002188 0.002055	-0.1682 -0.1557	-7.017 -6.589
$d\sigma^V/d\sigma^0$ $[10^{-3}]$	-0.08749 -0.08347	-1.458 -1.375	-1.516 -1.430	-0.4614 -0.4373	-3.715 -3.495	-3.970 -3.734
$d\sigma^{\rm red}/d\sigma^0$ $[10^{-3}]$	1.650 1.562	8.340 7.835	8.001 7.526	4.594 4.289	13.35 12.60	12.84 12.12
$d\sigma_{\rm had}/d\sigma^0$ $[10^{-3}]$	1.565 1.480	6.918 6.495	3.066 2.895	4.135 3.854	9.469 8.944	1.855 1.795
$d\sigma_e/d\sigma^0$ $[10^{-3}]$	0.8128	-0.4993	-6.769	0.5561	-1.954	-10.33
$d\sigma_\mu/d\sigma^0$ $[10^{-3}]$	0.7078	1.943	0.3714	1.378	2.366	-0.3557
$d\sigma_\tau/d\sigma^0$ $[10^{-3}]$	0.04932	1.246	0.9548	0.4787	2.100	1.162

upper/lower lines $\rightarrow$ B/HMNT

Comparison with Actis-Czakon-Gluza-Riemann 0807.0174 (lower lines)

$\theta = 3^o$	$\sqrt{s}$	1 GeV	10 GeV	M_Z	500 GeV
had $[10^2$ nb]	$d\sigma^B/d\Omega$	-0.380382	0.000544704	0.0000893354	0.00000386014
	$d\sigma^{B+\text{red}}/d\Omega$	0.0136085 < 1	0.391247 0.39	0.0876187 0.0877	0.00810786 0.0081
μ $[10^2$ nb]	$d\sigma^B/d\Omega$	0.0419870	0.00132934	0.0000283258	0.00000100529
	$d\sigma^{B+\text{red}}/d\Omega$	0.339976 < 1	0.417217 0.42	0.0407916 0.0408	0.00287809 0.00288
τ $[10^2$ nb]	$d\sigma^B/d\Omega$	-0.000277434	0.000350300	0.0000118080	0.000000773826
	$d\sigma^{B+\text{red}}/d\Omega$	0.00227893 < 1	0.00193150 $< 10^{-2}$	0.00270529 0.0027	0.000876352 0.00088
$\theta = 90^o$	$\sqrt{s}$	1 GeV	10 GeV	M_Z	500 GeV
had $[10^{-4}$ nb]	$d\sigma^B/d\Omega$	24.5724	0.591300	0.00204702	-0.000313808
	$d\sigma^{B+\text{red}}/d\Omega$	232.674 234	16.0671 16.07	0.469946 0.4701	0.0246035 0.02461
μ $[10^{-4}$ nb]	$d\sigma^B/d\Omega$	12.8008	0.133680	-0.00115553	-0.000171183
	$d\sigma^{B+\text{red}}/d\Omega$	160.197 160	6.08187 6.08	0.147046 0.1470	0.00725789 0.00726
τ $[10^{-4}$ nb]	$d\sigma^B/d\Omega$	0.465857	0.0939460	0.00188681	0.0000195543
	$d\sigma^{B+\text{red}}/d\Omega$	2.38272 2	1.33347 1.33	0.0752669 0.0752	0.00457124 0.00457

- **Strong energy dependence close to resonances**
- Detailled study would require beam-energiy spread
- Relative importance of box, vertex, reducible varies drastically



Summary

- Full QED corrections under control
    - hadronic: two independent calculations
    - leptonic: four independent calculations
    - photonic: two independent calculations
- Hadronic corrections are of the same order of the leptonic ones or bigger ($0.1\% - 1\%$)
- Dispersion relation is the simplest tool to deal with fermionic corrections (also for lepton loops)

- Modular construction of our result:
  - Reducible terms ($\sim \Pi(s),\ \Pi(t)$)
    $\rightsquigarrow$ can be combined with one-loop corrections: $1 + \Pi = \frac{1}{1-\Pi}$
  - Irreducible vertices ($\sim V(s),\ V(t)$)
    $\rightsquigarrow$ can be combined with Born: $1 + V(s)$
  - Irreducible boxes (small!)
    $\rightsquigarrow$ efficient and precise integration, works well also for complicated $R(s)$

$\Rightarrow$ Ideal building blocks for Monte Carlo generator