Finite volume effects for masses and decay constants

Gilberto Colangelo

UNIVERSITÄT
BERN

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Outline

Introduction

CHPT: expansion in m_{q_l}/Λ and p/Λ

In finite volume the momentum is quantized: $p = \frac{2\pi}{L}n$

Condition of applicability of CHPT:

$$m_{q_l} \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1$$

Once this condition is respected we still have two different physical situations

$$LM_\pi \lesssim 1 \quad \Rightarrow \quad \epsilon\text{–regime} \quad M_\pi \sim \frac{1}{L^2} \sim O(\epsilon^2)$$

$$LM_\pi \gg 1 \quad \Rightarrow \quad p\text{–regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$

p– or ϵ–regime?

Two alternatives:

- Chiral limit on the lattice $\Rightarrow$ ϵ–regime

  (unless one can simulate enormous volumes)

  $\Rightarrow$ Rely on CHPT to relate unphysical observables to physical quantities

- $M_\pi > M_\pi^{\mathrm{phys}}$: choose $L \gg 1/M_\pi$, $\Rightarrow$ p–regime

  (e.g. $M_\pi = 300$ MeV, $L = 2$ fm, $M_\pi L \sim 3$)

  $\Rightarrow$ Rely on CHPT to make the chiral and the large volume extrapolation

p-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:

- the Lagrangian is the same as in infinite volume
- the propagators must be made periodic:

$$G_L(\vec{x}, t) = \sum_{\vec{n}} G_\infty(\vec{x} + \vec{n}L, t)$$

p-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:

Examples: Gasser and Leutwyler (88)

$$
\begin{aligned}
M_\pi(L) &= M_\pi\left[1+\frac{1}{2N_f}\xi\, g_1(\lambda)+O(\xi^2)\right] \\
F_\pi(L) &= F_\pi\left[1-\frac{N_f}{2}\xi\, g_1(\lambda)+O(\xi^2)\right]
\end{aligned}
$$

with

$$
\lambda = M_\pi L, \quad \xi = (M_\pi/4\pi F_\pi)^2
$$

$$
g_1(\lambda) = {\sum_{\vec{n}}}' \int_0^\infty dz\, e^{-\frac{1}{z}-\frac{z}{4}\vec{n}^2\lambda^2} = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)\big|_{t=\vec{x}=0}
$$

Finite volume effects in the p-regime

Foundations: Gasser and Leutwyler (87)

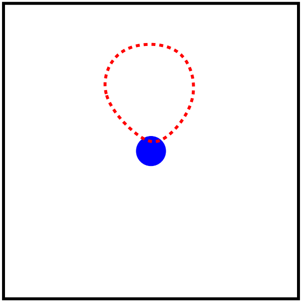
quenched CHPT: Sharpe, Bernard and Golterman (90's)

Recent applications:

- two–pion states — Lin, Martinelli, Pallante, Sachrajda and Villadoro (03)
- F_K and B_K — Becirevic and Villadoro (03)
- m_p — QCDSF (03)
- m_N, μ_N and g_A — Beane and Savage (03-04)
- f_B and B_B — Arndt and Lin (04)
- m_p — Koma and Koma (04)

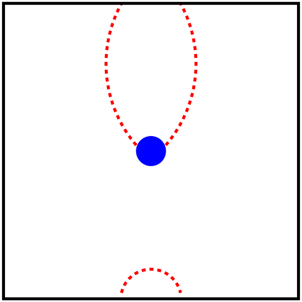
Outline

Masses in finite volume



Loop-diagram

Masses in finite volume



Loop diagram with
periodic
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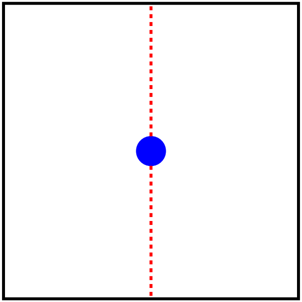
Masses in fi nite volume



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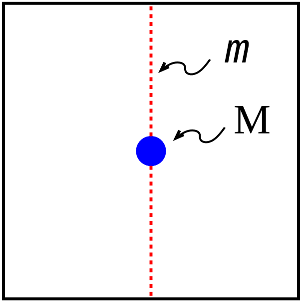
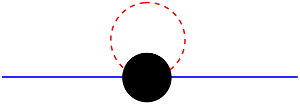


Loop diagram with periodic
boundary conditions

This diagram exists only for $L \neq \infty$

Its effect is of the order $\exp[-mL]$

Lüscher's Formula



$$= \int d\ell\, \Gamma(p, \ell, -\ell, -p) G_L(\ell)$$

$$G_L(\ell) = \sum_{\vec{n}} G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L} \qquad G_\infty(\ell) \sim \frac{1}{\ell^2 + m^2}$$

$$\begin{aligned} M_L - M_\infty &= \int d\ell\, \Gamma(p, \ell, -\ell, p) \left[G_L(\ell) - G_\infty(\ell) \right] \\ &= \sum_{\vec{n} \neq \vec{0}} \int d\ell\, \Gamma(p, \ell, -\ell, p) G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L} \end{aligned}$$

Lüscher's Formula

Leading correction for $mL \gg 1$: (Lüscher 86)

$$M_L - M_\infty = C \int_{-\infty}^{\infty} dy\; e^{-\sqrt{m^2+y^2}L} F(iy) + \ldots$$

$F(\nu)$ is the scattering amplitude between the red (m) and blue (M) particle, and C a constant that depends on $L,\ m$ and M

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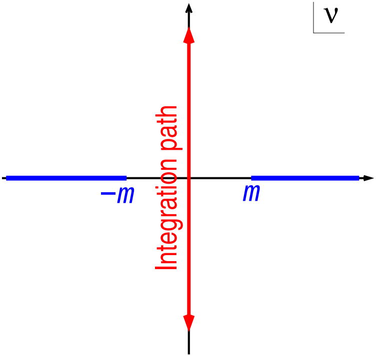

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- What matters for the behaviour of the corrections is not the mass of the particle itself, but rather
  the mass of the lightest particle to which it is coupled

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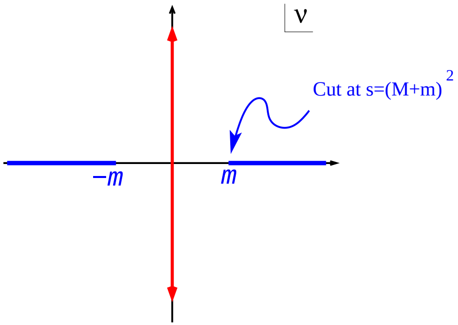
- The formula expresses the corrections as an integral over a physical amplitude (analytically continued)
- What matters for the behaviour of the corrections is not the mass of the particle itself, but rather the mass of the lightest particle to which it is coupled
- e.g. both the corrections for the pion as well as those for the proton mass depend exponentially on $M_\pi L$

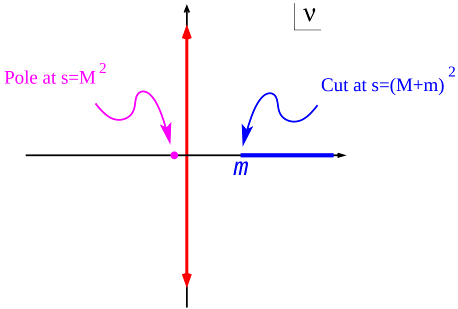
Cuts and poles in the scattering amplitude

Any scattering amplitude must have a cut at
$s, u = (M + m)^2 \Rightarrow \nu = \frac{s-u}{4M} = \pm m$



Cuts and poles in the scattering amplitude

In addition it may have poles,
e.g. at $s, u = M^2 \Rightarrow \nu = \frac{s-u}{4M} = \mp\frac{m^2}{2M}$

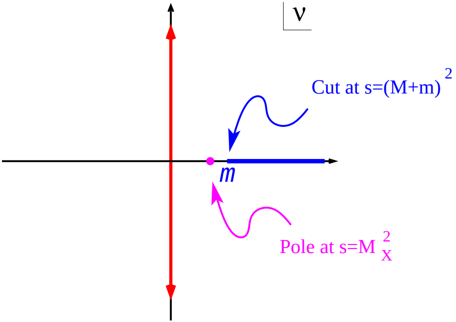


Poles on the lhs of the imaginary axis generate an extra term in the Lüscher's formula Nucleon mass formula

Cuts and poles in the scattering amplitude

In addition it may have poles,

or at $s, u = M_X^2 \Rightarrow \nu = \frac{s-u}{4M} = \mp\frac{m^2}{2M} + \Delta M \left(1 + \frac{\Delta M}{2M}\right)$ $\quad \Delta M = M_X - M$

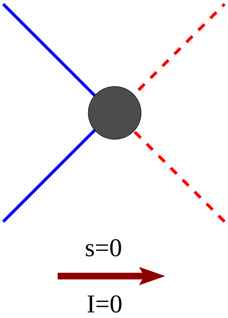


Poles on the rhs of the imaginary axis do not generate an extra term in the Lüscher's formula, cf. Arndt and Lin (04)

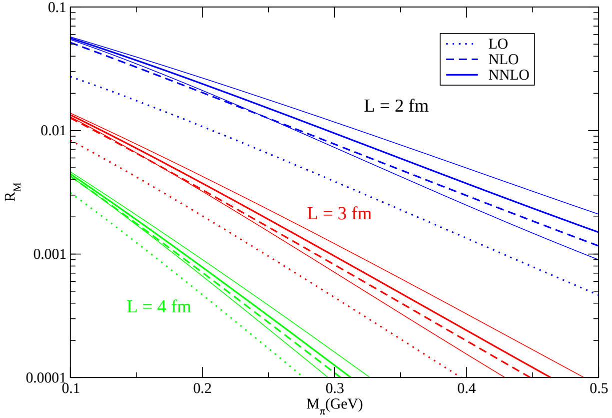
Corrections for M_π

$\pi\pi$ scattering amplitude with forward kinematics

$$F_{\pi\pi}(\nu) = T^{I=0}\left[0, 2M_\pi(M_\pi + \nu), 2M_\pi(M_\pi - \nu)\right] = -\frac{M_\pi^2}{F_\pi^2} + O(p^4)$$



Corrections for M_π



$$R_M = M_{\pi L}/M_\pi - 1$$

GC and S. Dürr 03

Lüscher's Formula or CHPT?

$$\Delta M^L_{\pi\ \text{Lüscher}} = \frac{-3}{16\pi^2\lambda} \int_{-\infty}^{\infty} dy\ F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

$$\Delta M^L_{\pi\ \text{CHPT}} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$

$$g_1(\lambda) = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L, t)\big|_{t=\vec{x}=0} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda} K_1\left(|\vec{n}|\,\lambda\right)$$

where $m(|\vec{n}|)$ is the multiplicity of a vector of length $|\vec{n}|$ in 3-dimensional discretized space

Lüscher's Formula or CHPT?

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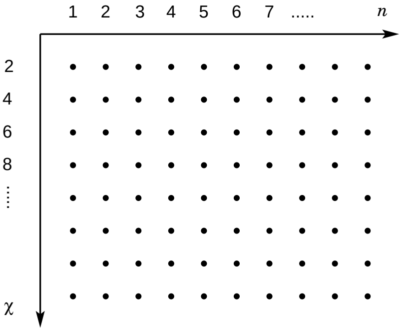
$$\Delta M^L_{\pi\ \mathrm{CHPT}} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$

These represent the leading term in two different expansions

Lüscher's Formula or CHPT?

$$
\begin{aligned}
\Delta M_{\pi\ \text{Lüscher}}^{L} &= \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\ F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L}) \\
\Delta M_{\pi\ \mathrm{CHPT}}^{L} &= \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)
\end{aligned}
$$



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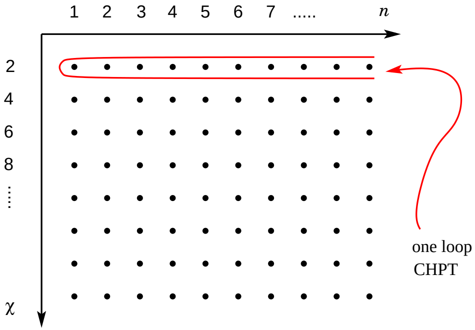

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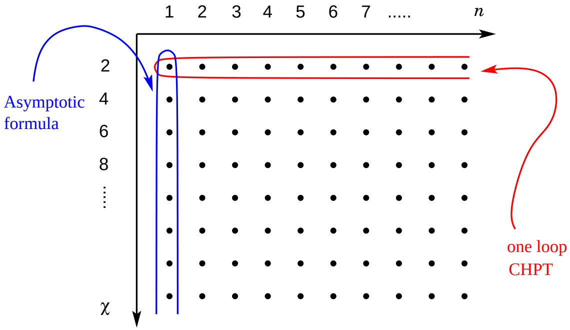

Extension of the Lüscher's Formula

One-loop CHPT corrections

$$g_1(\lambda) = \sum_{\vec{n} \neq \vec{0}} G_\infty(\vec{x} + \vec{n}L, t)|_{t=\vec{x}=0} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda} K_1\left(|\vec{n}|\,\lambda\right)$$

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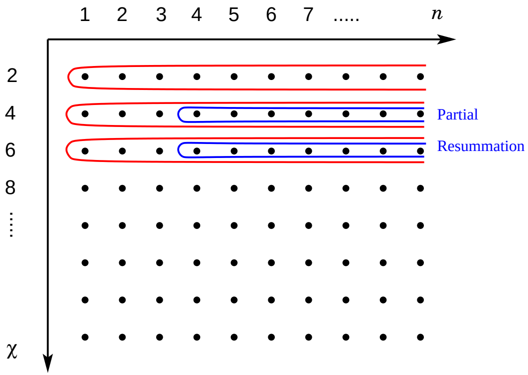
Analogously one can extend the Lüscher's Formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

$$M_{\pi,L} - M_\pi = -\frac{1}{32\pi^2\lambda} \sum_{|\vec{n}|=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$

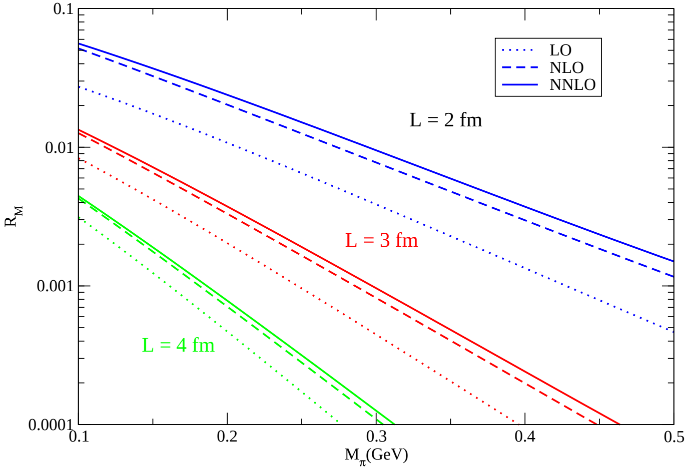
The extension does not give all exponentially subleading terms!

Extension of the Lüscher's Formula

$$M_{\pi,L} - M_{\pi} = -\frac{1}{32\pi^2\lambda}\sum_{|\vec{n}|=1}^{\infty}\frac{m(|\vec{n}|)}{|\vec{n}|}\int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$



Nonleading exp. terms in $M_{\pi,L}$



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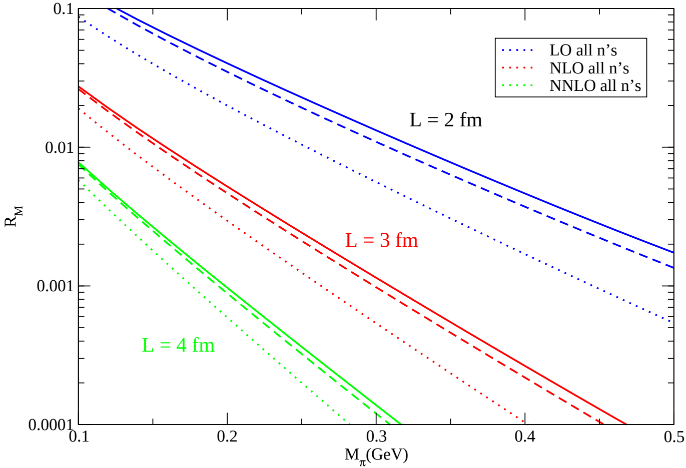
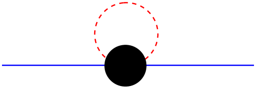
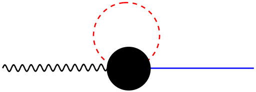


Extension to decay constants



$$\Rightarrow \quad \Delta M \propto \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \dots$$

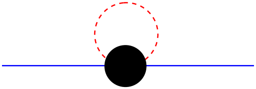
$$F(\nu) \Leftrightarrow \langle \pi\pi | T | \pi\pi \rangle$$



$$\Rightarrow \quad \Delta F \propto \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} N(iy) + \dots$$

$$N(\nu) \Leftrightarrow \langle 0 | A_\mu | \pi\pi\pi \rangle \sim A(\tau \to 3\pi\nu_\tau)$$

Extension to decay constants



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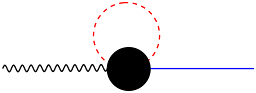

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$$N(\nu) \Leftrightarrow \langle 0 | A_\mu | \pi\pi\pi \rangle \sim A(\tau \to 3\pi\nu_\tau)$$

The $\langle 0 | A_\mu | \pi\pi\pi \rangle$ amplitude must be subtracted:

$$N(\nu) = \langle (2\pi)_{I=0}\pi | A_\mu(0) | 0 \rangle - iQ_\mu \frac{F_\pi F(\nu)}{M_\pi^2 - Q^2}$$

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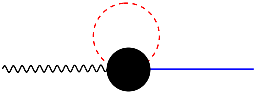
Extension to decay constants



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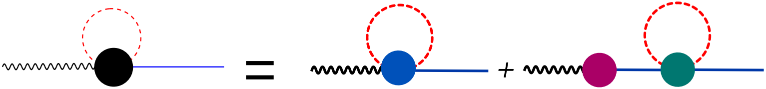

Ward identity in fi nite volume

$$\langle 0|A^i_\mu(0)|\pi^k(p)\rangle = i\delta^{ik}F_\pi p_\mu \qquad \langle 0|P^i(0)|\pi^k(p)\rangle = i\delta^{ik}G_\pi$$

Ward identity

$$F_\pi M_\pi^2 = \hat{m} G_\pi$$

The identity is valid also in finite volume

$$R_G = R_F + 2R_M \qquad (R_X := \Delta X_\pi^L / X_\pi)$$

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and must be satisfied by the asymptotic formulae :

$$C\int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2}L}\left[\frac{1}{G_\pi}N_G(iy) - \frac{1}{F_\pi}N_F(iy) + \frac{1}{M_\pi}F(iy)\right] = 0$$

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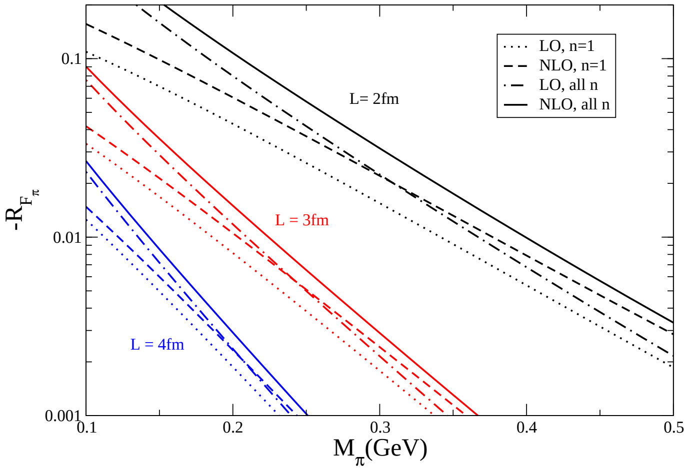
and must be satisfied by the asymptotic formulae
– in particular for the integrands:

$$\frac{1}{G_\pi} N_G(\nu) - \frac{1}{F_\pi} N_F(\nu) + \frac{1}{M_\pi} F(\nu) = 0$$

This is a Ward identity for 4-point functions

Corrections for F_π



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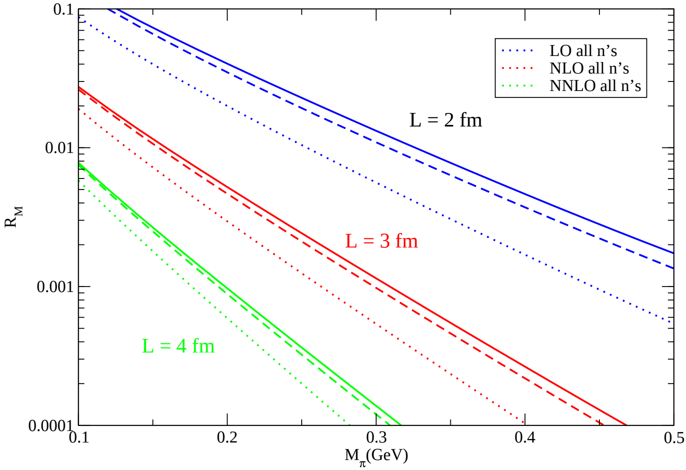
Other applications

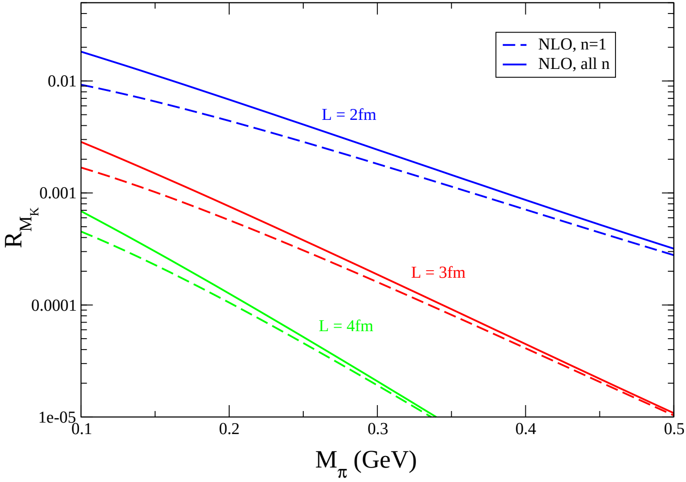
Quantity	Amplitude	Theory status
M_K	$A(\pi K \to \pi K)$	$O(p^6)$ (Bijnens et al.)
F_K	$A(K_{l4})$	$O(p^6)$ (Bijnens et al.)
M_η	$A(\pi\eta \to \pi\eta)$	$O(p^4)$ (Bernard et al.)
F_η	$A(\eta_{l4})$	?
M_N	$A(\pi N \to \pi N)$	$O(p^4)$ various Authors
M_B	$A(\pi B \to \pi B)$	?
F_B	$A(B_{l4})$	?

Outline

Corrections for M_π

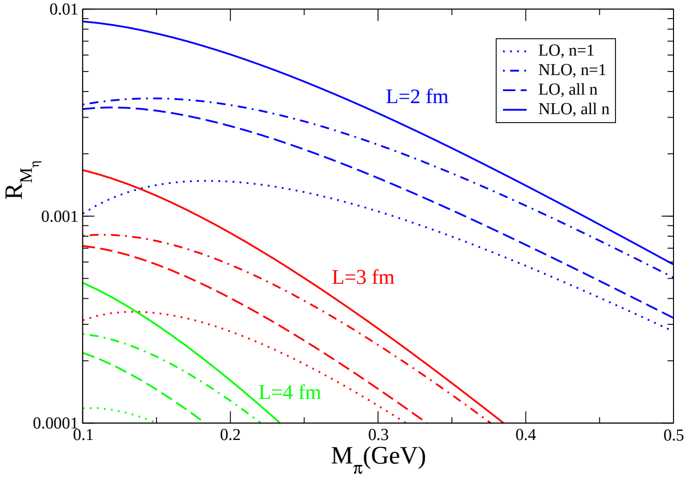


Corrections for M_K



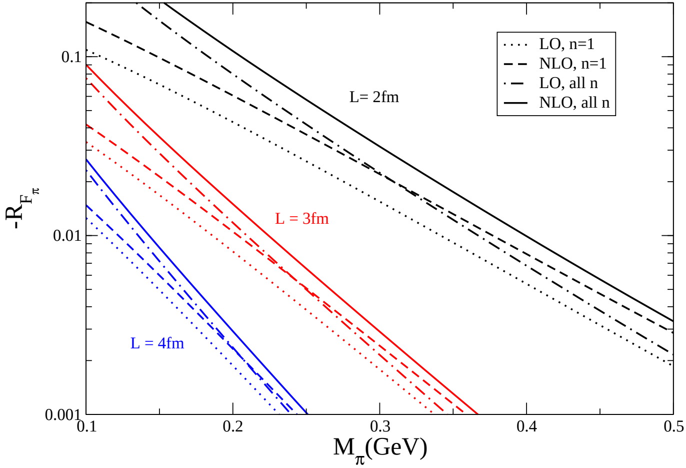
GC, S. Dürr and C. Haefeli (05)

Corrections for M_η



GC, S. Dürr and C. Haefeli (05)

Corrections for F_π



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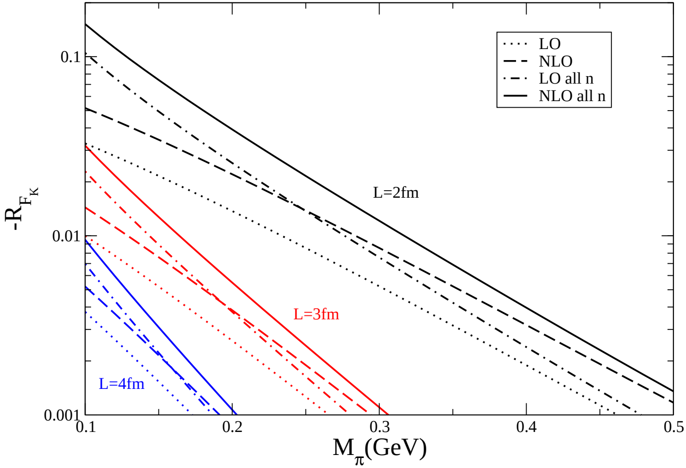
Corrections for F_K



LO agrees with Becirevic and Villadoro (04)

GC, S. Dürr and C. Haefeli (05)

V_{us} and F_K/F_π

A lattice determination of F_K/F_π can be used to pin down V_{us}

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obtained from simulations at, e.g.

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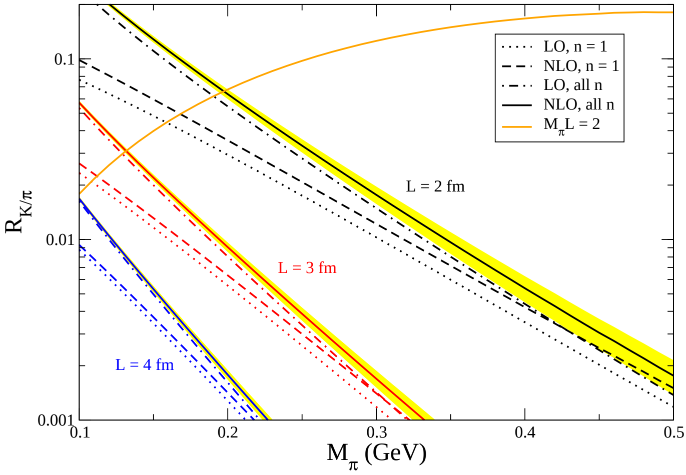
Finite volume effects?

V_{us} and F_K/F_π



GC, S. Dürr and C. Haefeli (05)

The nucleon mass

$$R_N = \frac{3}{4\pi^2}\sum_{n=1}^{\infty}\frac{m(n)}{\sqrt{n}\lambda_\pi}\Big[2\pi\varepsilon_\pi g_{\pi N} e^{-\sqrt{n(1-\varepsilon_\pi^2)}\lambda_\pi} - \int_{-\infty}^{\infty} dy e^{-\sqrt{n(1+y^2)}\lambda_\pi}\tilde{D}^+(y)\Big]$$

Go back

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$$\varepsilon_\pi = M_\pi/(2m_N) \quad \tilde{D}^+(y) = m_N D^+(iM_\pi y, 0)$$

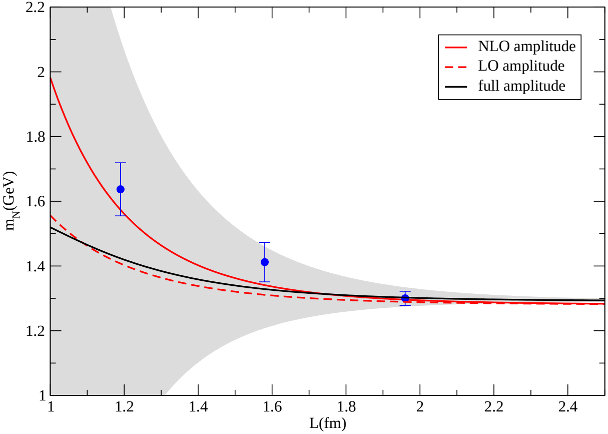
D^+ is a component of the elastic πN scattering amplitude

$$T_{a'a} = \delta_{a'a}T^+ + \frac{1}{2}[\tau_{a'}, \tau_a]T^-$$

$$T^+ = \bar{u}'\left[D^{\pm}(\nu,t) - \frac{1}{4m_N}[\not{q}', \not{q}]B^{\pm}(\nu,t)\right]u$$

$$s = (p+q)^2 \quad t = (q-q')^2 \quad u = (p-q')^2 \quad \nu = \frac{s-u}{4m_N}$$

The nucleon mass



NLO analysis from QCDSF (04) – full amplitude and error band from GC and A. Fuhrer

Outline

Summary

- For large volumes ($2LF_\pi \gg 1$), finite–volume effects can be calculated analytically within CHPT
- several one-loop CHPT calculations in the *p*–regime ($M_\pi L \gg 1$) have appeared in the recent literature
- the combined use of CHPT and asymptotic formulae *à la* Lüscher offers the most efficient way to get to higher orders in CHPT
- I have presented numerical evaluations of these finite volume corrections for all pseudoscalar masses and decay constants, as well as for the nucleon mass

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