Finite volume effects for masses and decay constants

EURIDICE Midterm Collaboration Meeting

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Overview

- Introduction: ChPT in finite volume
- Lüscher's formula for masses
- Asymptotic formula for decay constants
- Numerics
- Summary

work done in collaboration with G. Colangelo and S. Dürr

Introduction: ChPT in finite volume

- Expansion in m_q/Λ and p/Λ
- Quantized momenta in finite volume: $p = \frac{2\pi}{L} n$
- Condition of applicability for ChPT:

$$m_q \ll \Lambda \qquad \text{and} \qquad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \qquad \Rightarrow \qquad 2LF_\pi \gg 1$$

- Once this condition is respected, we still have two different physical situations:

$$LM_\pi \lesssim 1 : \qquad \varepsilon\text{−regime}$$

$$\text{This talk :} \qquad LM_\pi \gg 1 : \qquad p\text{−regime}$$

p-regime: $M_\pi L \gg 1$

- Calculational rule in ChPT for isotropic finite box with periodic boundary conditions:

$$\text{Lagrangian}: \qquad \mathcal{L}_{\text{eff}}^{\text{L}} = \mathcal{L}_{\text{eff}}^{\infty}$$

$$\text{Propagator}: \qquad G_{\text{L}}(x^0, \vec{x}) = \sum_{\vec{n}\in\mathbb{Z}^3} G(x^0, \vec{x} + \vec{n}L)$$

J.Gasser, H.Leutwyler 87

- Implication for perturbative calculation:

$$\int \frac{d^4p}{(2\pi)^4} f(p) \overset{L<\infty}{\Longrightarrow} \int \frac{dp_0}{2\pi} \frac{1}{L^3} \sum_{\vec{p}} f(p) \overset{(*)}{=} \int \frac{d^4p}{(2\pi)^4} f(p) \sum_{\vec{n}\in\mathbb{Z}^3} e^{i\vec{p}\vec{n}L}$$

$(*)$: Poisson summation formula

p-regime: $M_\pi L \gg 1$

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- Examples:

$$M_\pi(L) = M_\pi\left(1+\tfrac{1}{4}\xi\, g(\lambda) + \mathcal{O}(\xi^2)\right) \;,$$

$$F_\pi(L) = F_\pi\left(1-\xi\, g(\lambda) + \mathcal{O}(\xi^2)\right) \;,$$

$$\xi = \frac{M_\pi^2}{(4\pi F_\pi)^2}\,, \qquad \lambda = M_\pi\, L\,, \qquad g(\lambda) = \sum_{\vec{n}\backslash\{0\}} \int_0^\infty dx\, e^{-\frac{1}{x}-\frac{x}{4}\vec{n}^2\lambda^2}\,.$$

p-regime: Recent applications in 2004

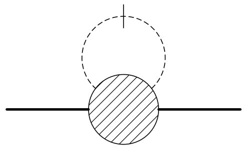
Foundations: Gasser and Leutwyler (87)

observables	authors	method
M_π	Colangelo and Dürr	asymptotic
F_π	Colangelo and C.H.	asymptotic
M_π, F_π and $\langle r_\pi^2 \rangle_V$	Borasoy and Lewis	Lattice ChPT
F_K and B_K	Becirevic and Villadoro	one-loop
m_P	QCDSF	one-loop
m_N, μ_N and g_A	Beane and Savage	one-loop
m_P	Koma and Koma	asymptotic
f_B and B_B	Arndt and Lin	one-loop

Lüscher's formula (applied for M_K)

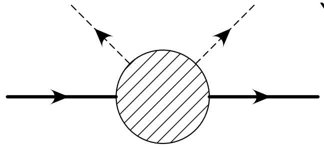
Leading corrections for $M_\pi L \gg 1$:

$$M_K(L) - M_K \; = $$



$$+ \mathcal{O}(e^{-\sqrt{2}M_\pi L})$$

$$= \sum_{|\vec{n}|=1} \int d^4\ell \left(\frac{e^{i\vec{\ell}\vec{n}L}}{M_\pi^2 + \ell^2} \right.$$



$$\left. \vphantom{\frac{e}{M}} \right) + \mathcal{O}(e^{-\sqrt{2}M_\pi L}) \; ,$$

- Integrations over spatial momenta $\vec{\ell}$ can be performed within claimed accuracy. It remains the integral over ℓ_0.

$$M_K(L) - M_K = -\frac{3}{16\pi^2 M_K L} \int_{-\infty}^{\infty} d\ell_0 \; e^{-\sqrt{M_\pi^2 + \ell_0^2}\,L} \, T_{\pi K}^{I=0}(i\ell_0) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L}) \; ,$$

Lüscher 86

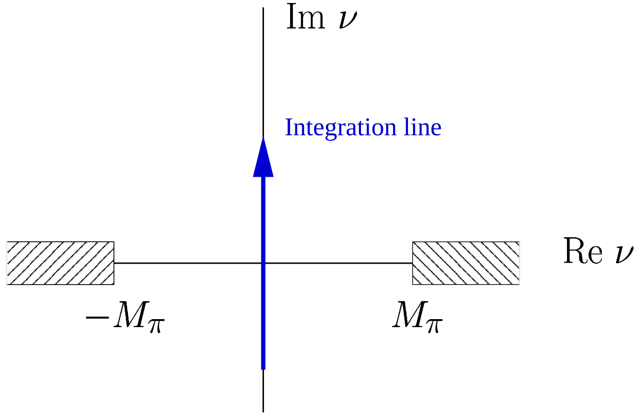
Lüscher's formula:

$P = \pi, K, \eta$:

$$M_P(L) - M_P = -\frac{3}{16\pi^2 M_P L} \int_{-\infty}^{\infty} dy \, e^{-\sqrt{M_\pi^2+y^2}\,L} T_{\pi P}^{I=0}(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L})\,.$$

- The coupling to the ligthest particle matters, i.e. the pions.
- Leading corrections of order $\exp(-M_\pi L)$.
- The formula expresses the corrections over a (analytically continued) physical amplitude.

Analyticity properties of $T_{\pi P}^{I=0}(\nu)$:

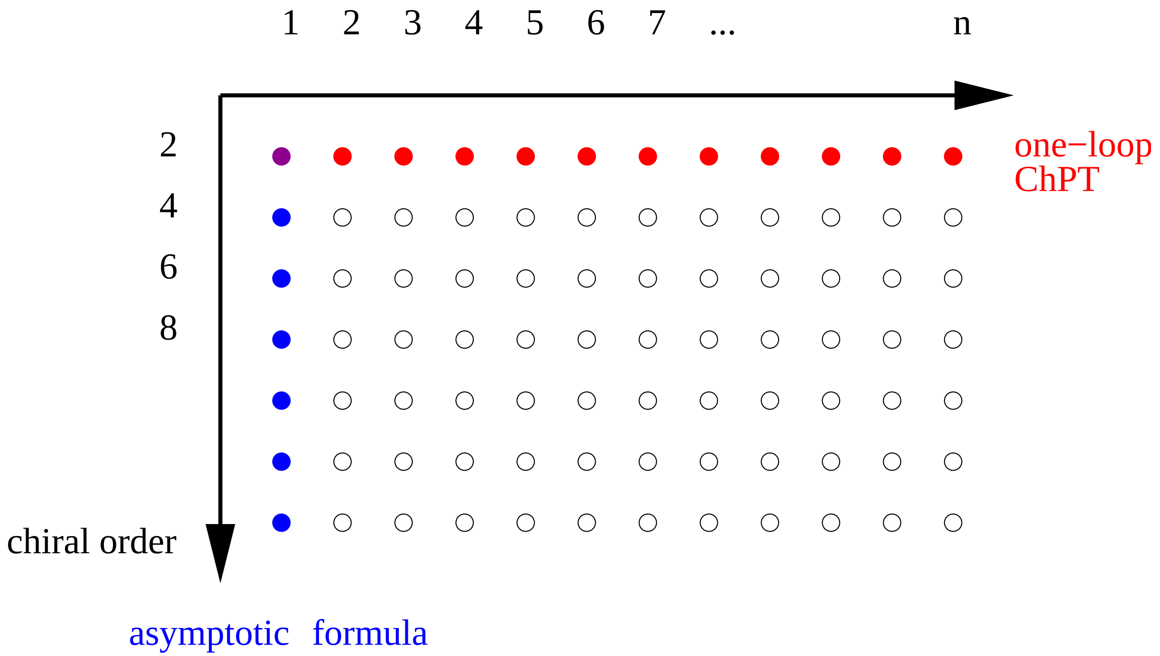


Lüscher's formula or ChPT?:

$$\Delta M_{\pi,\text{Lüscher}} = -\frac{3}{16\pi^2 M_\pi L}\int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} T_{\pi\pi}^{I=0}(iy) + \mathcal{O}(e^{-\sqrt{2}M_\pi L}),$$

$$\Delta M_{\pi,\text{ChPT}} = \xi \frac{M_\pi}{4} g(M_\pi L) + \mathcal{O}(\xi^2).$$

The two formulae give the leading term in two different expansions.

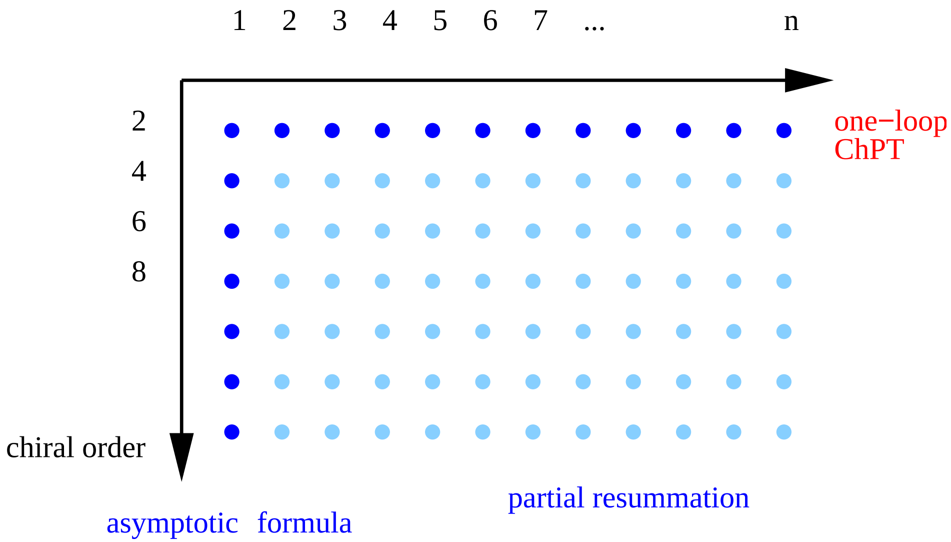


Extension of Lüscher's formula

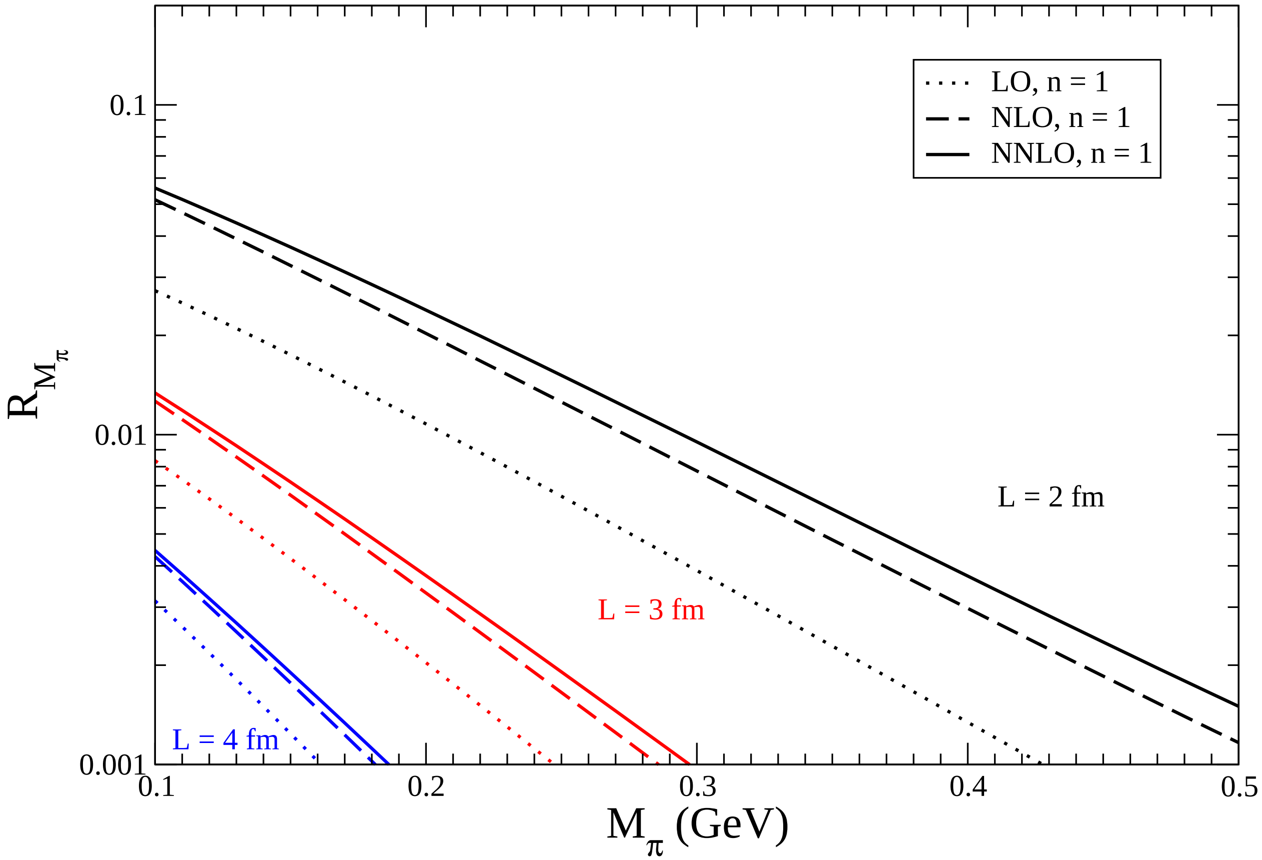
One can extend the formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

$$\Delta M_\pi = -\frac{1}{32\pi^2 M_\pi L}\sum_{|\vec{n}|=1}^{\infty}\frac{m(|\vec{n}|)}{|\vec{n}|}\int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}|\vec{n}|L} T_{\pi\pi}^{I=0}(iy) + \mathcal{O}(e^{-2M_\pi L})$$

The extension does not provide all exponentially subleading terms!



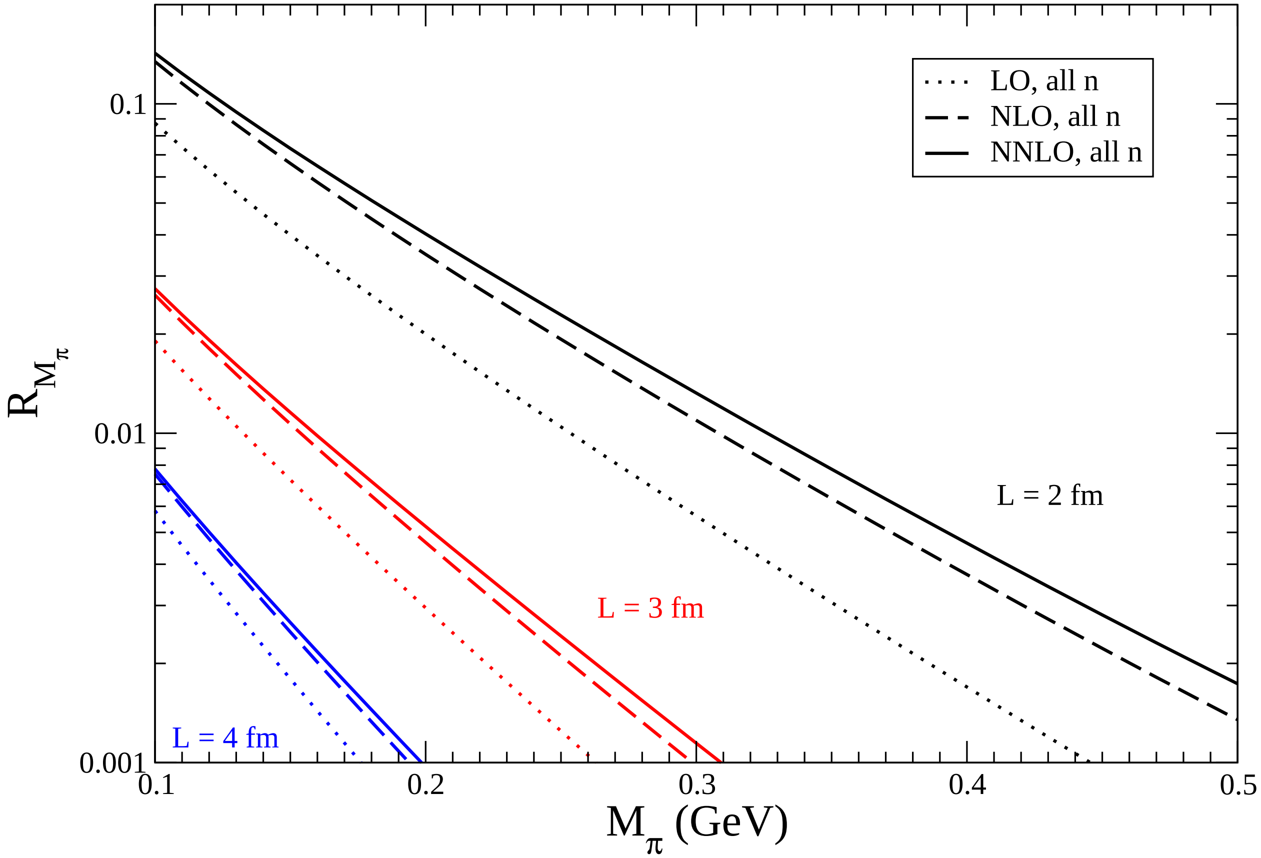
Non-leading exponential terms in $M_\pi(L)$



$$R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi}$$

taken from G.Colangelo and S.Dürr 04

Non-leading exponential terms in $M_\pi(L)$



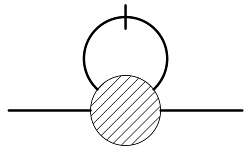
$$R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi}$$

Asymptotic formula for decay constants

$$\Delta P_\pi \equiv P_\pi(L) - P_\pi \ , \quad P_\pi = M_\pi, F_\pi \ .$$

$$\Delta M_\pi = -\frac{3}{16\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}\,L} T_{\pi\pi}(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L}) \,,$$

$$=$$



$$+\,\mathcal{O}(e^{-\sqrt{2}M_\pi L}) \,,$$

$$T_{\pi\pi}(\nu) \longleftrightarrow \langle \pi\pi | \pi\pi \rangle \ .$$

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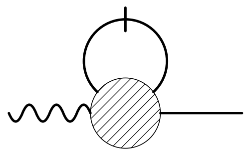
$$\Delta F_\pi = \frac{3}{8\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}\,L} N_F(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L}) \,,$$

$$=$$



$$+\,\mathcal{O}(e^{-\sqrt{2}M_\pi L}) \,,$$

$$N_F(\nu) \longleftrightarrow \langle 3\pi | A_\mu | 0 \rangle \sim A(\tau \rightarrow 3\pi\nu_\tau) \ .$$

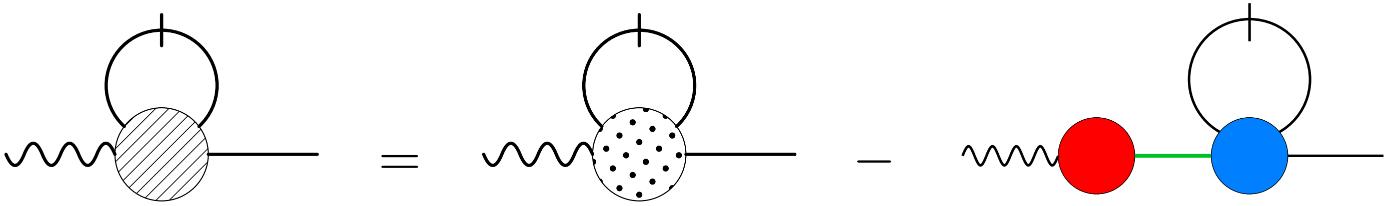
G.Colangelo, C.H. 04

Definition of the amplitude $N_F(\nu)$

$$\Delta F_\pi = \frac{3}{8\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}\,L} N_F(iy) + \mathcal{O}(e^{-\sqrt{2}\,M_\pi L}),$$

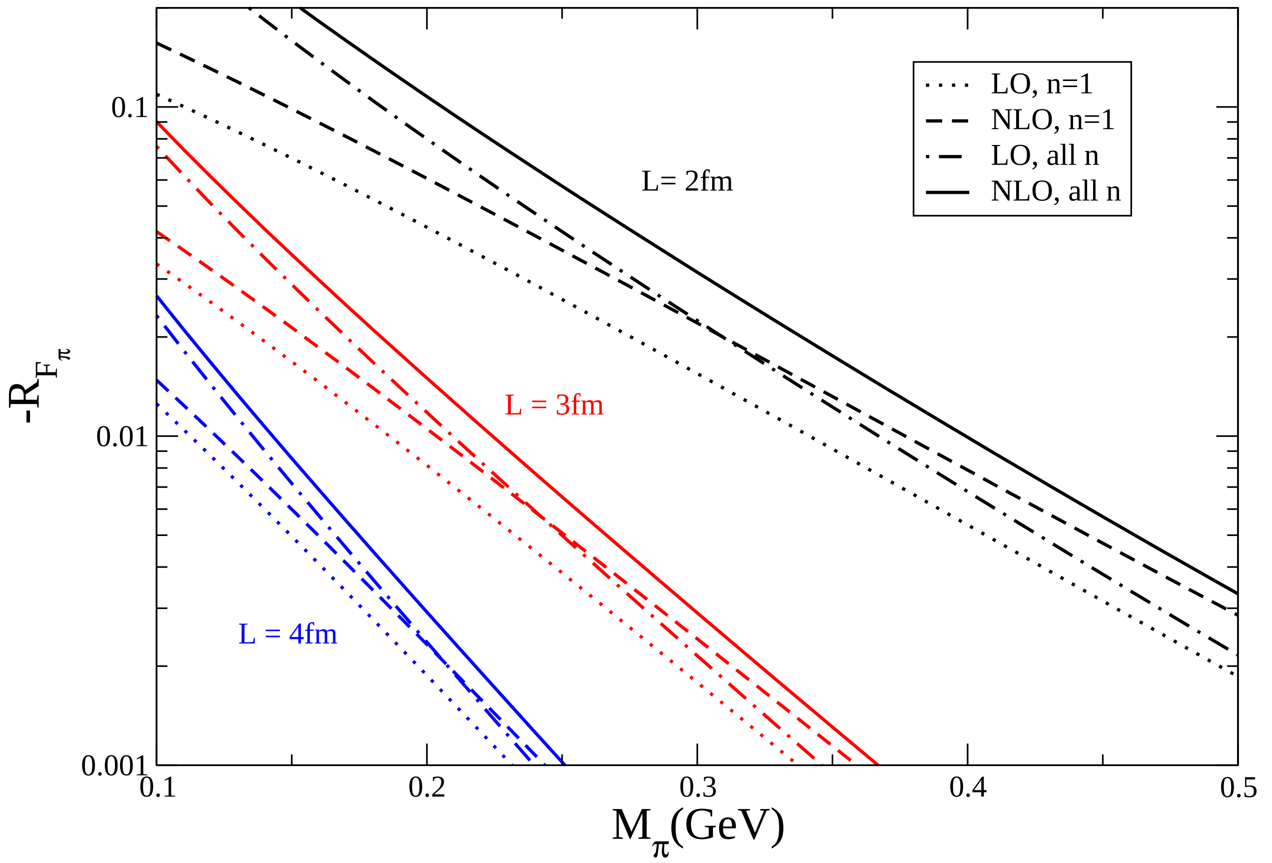
$N_F(\nu)$ is the subtracted $\langle 3\pi|A_\mu|0\rangle$ matrix element in the forward kinematic region.

$$N_F(\nu) = -i\frac{p_\mu}{M_\pi}\left(\langle (2\pi)_{I=0}\pi|A_\mu|0\rangle - iQ_\mu F_\pi \frac{T_{\pi\pi}^{I=0}}{M_\pi^2 - Q^2}\right)$$



G.Colangelo and C.H. 04

Finite volume corrections for F_π



$$R_{F_\pi} = \frac{F_\pi(L) - F_\pi}{F_\pi}$$

Universality of asymptotic formula

Masses (Lüscher):

ΔM	scattering	theory status
M_π	$\pi\pi \to \pi\pi$	$\mathcal{O}(p^6)$ (Bijnens et al.)
M_K	$\pi K \to \pi K$	$\mathcal{O}(p^6)$ (Bijnens et al.)
M_η	$\pi\eta \to \pi\eta$	$\mathcal{O}(p^4)$ (Bernard et al.)
M_N	$\pi N \to \pi N$	$\mathcal{O}(p^4)$ (various authors)
M_B	$\pi B \to \pi B$	?

Decay constants:

ΔF	decays	theory status
F_π	$\tau \to 3\pi$	$\mathcal{O}(p^4)$ Colangelo et al.
F_K	$K_{\ell 4}$	$\mathcal{O}(p^4)$ Bijnens et al.
F_η	η_{ℓ_4}	?
F_B	$B_{\ell 4}$	?

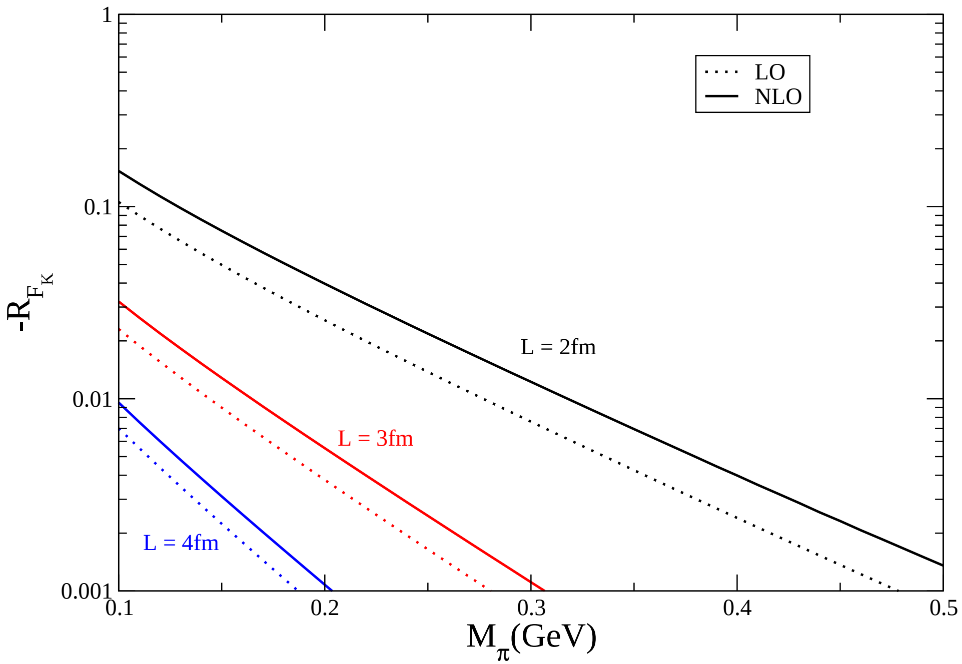
Work in progress: G.Colangelo, S.Dürr, A.Fuhrer, C.H.

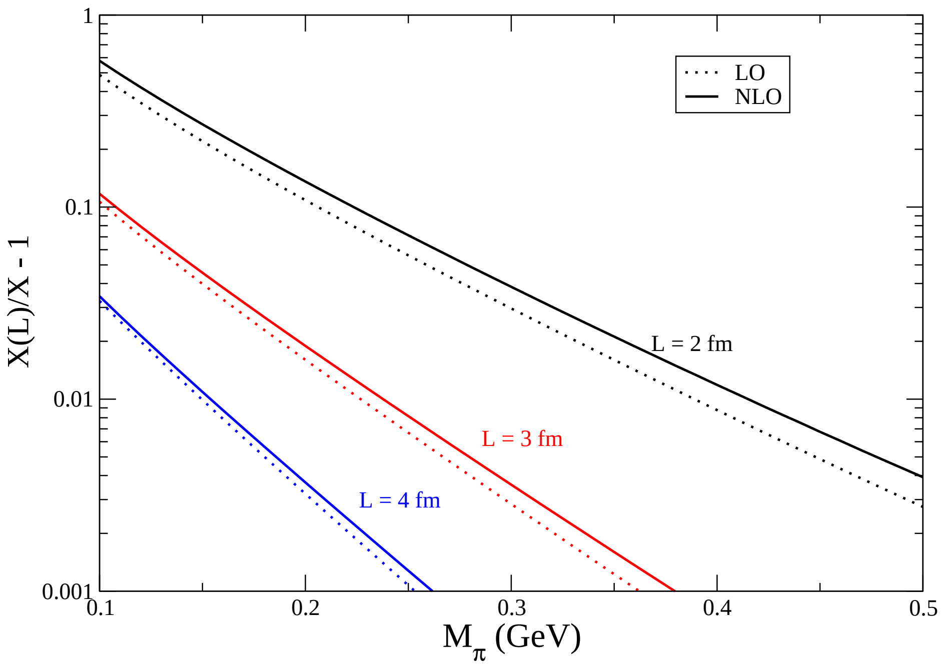
CKM matrix element: V_{us}

Lattice simulations can provide us with the ratio F_K/F_π,

$$X \equiv F_K^2/F_\pi^2 \Rightarrow |V_{us}|$$

W.J.Marciano 04





Work in progress: G. Colangelo, S. Dürr and C.H.

Summary

- For large volumes $(2LF_\pi \gg 1)$, finite volume effects can be calculated analytically within ChPT.
- Several one-loop ChPT calculations in the p-regime $(M_\pi L \gg 1)$ have appeared in the recent literature.
- The combined use of ChPT and asymptotic formulae à la Lüscher offers the most efficient way to get to higher orders in ChPT.
- I have presented numerical evaluations of these finite volume corrections for pseudoscalar masses and decay constants.
- The extrapolation $L \longrightarrow \infty$ can be made analytically.