The $a_0 - a_2$ pion scattering length from $K^+ \to \pi^+\pi^0\pi^0$ decay

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Summary

- Pion scattering lengths and the cusp.
- Unitarity and analiticity.
- $\pi\pi$ scattering.
- $K \to 3\pi$ amplitudes at $O(a_i^2)$.

N. Cabibbo, *Phys. Rev. Lett.*, 93:121801, 2004.   N. Cabibbo and G.Isidori, pre-preprint

Pion scattering lengths and chiral dynamics

Weinberg 1966 :

$$a_0\, m_{\pi^+} = \frac{7\, m_{\pi^+}^2}{16\pi f_\pi^2} = 0.159$$

$$a_2\, m_{\pi^+} = \frac{-m_{\pi^+}^2}{8\pi f_\pi^2} = -0.045$$

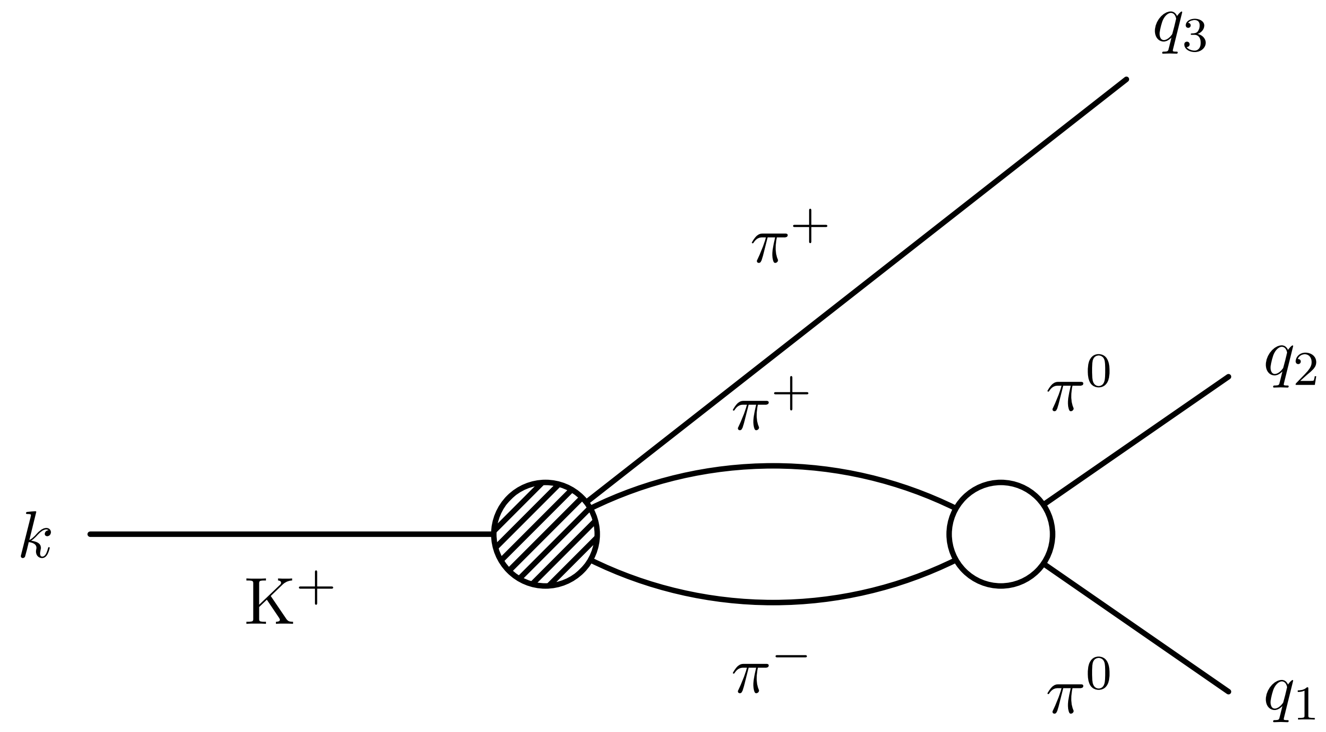
Colangelo et al. 2001 :

$$a_0\, m_{\pi^+} = 0.220 \pm 0.005$$

$$a_2\, m_{\pi^+} = -0.0444 \pm 0.0010$$

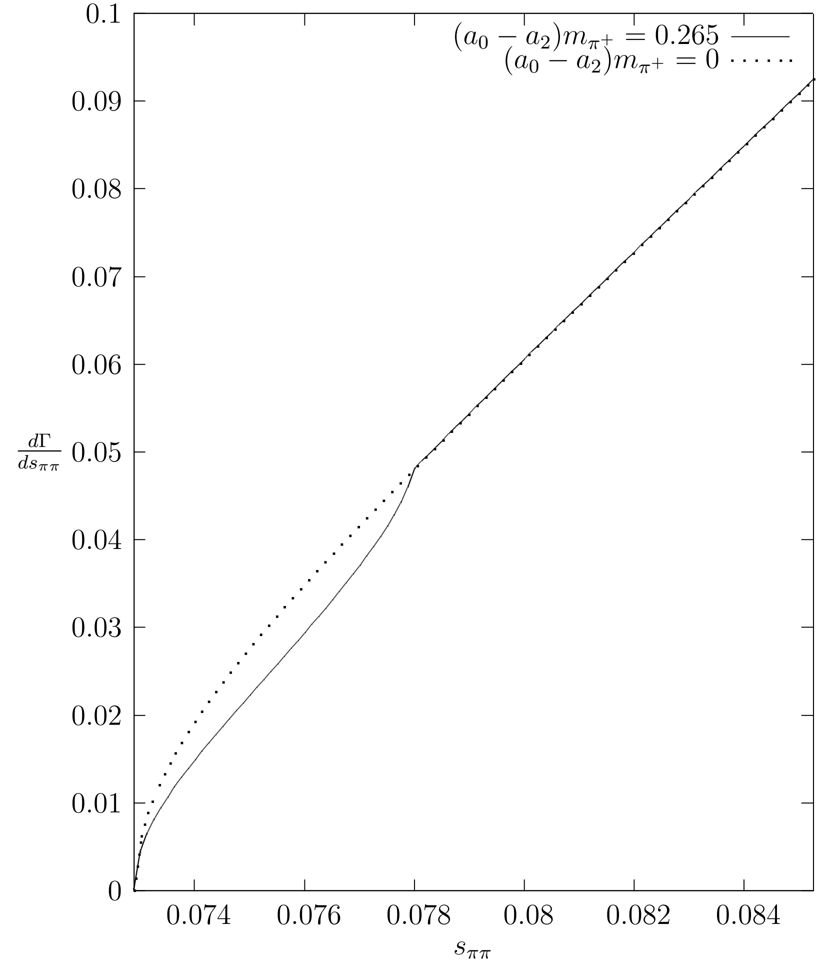
$$(a_0 - a_2)\, m_{\pi^+} = 0.265 \pm 0.004$$

Can we match this precision?



$\pi^+\pi^- \to \pi^0\pi^0$ re-scattering from $K^+ \to \pi^+\pi^+\pi^-$ causes a cusp singularity in the $\pi^0\pi^0$ spectrum at the $\pi^+\pi^-$ threshold. The cusp is proportional to $(a_0 - a_2)$

N. Cabibbo, Phys. Rev. Lett. **93** (2004) 121801.



The $\pi^0\pi^0$ spectrum in $K^+ \to \pi^+\pi^0\pi^0$ in the $\pi^+\pi^-$ threshold region, at the first order in an expansion in powers of a_i

How does the cusp arise

Let us write:

$$\mathcal{M}(K^+ \to \pi^+\pi^0\pi^0) = \mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$$

where $\mathcal{M}_1$ is the contribution of the re-scattering graph.

We must consider two cases, with $s_{\pi\pi}$ above or below the $\pi^+\pi^-$ threshold.

$$s_{\pi\pi} > 4m_{\pi^+}^2 \; : \qquad \mathcal{M}_1 = i2\frac{(a_0 - a_2)\, m_{\pi^+}}{3}\mathcal{M}_{+,\mathrm{thr}}\sqrt{(s_{\pi\pi} - 4m_{\pi^+}^2)/s_{\pi\pi}}$$

$$|\mathcal{M}|^2 = (\mathcal{M}_0)^2 + |\mathcal{M}_1|^2$$

$$s_{\pi\pi} < 4m_{\pi^+}^2 \; : \qquad \mathcal{M}_1 = -2\frac{(a_0 - a_2)\, m_{\pi^+}}{3}\mathcal{M}_{+,\mathrm{thr}}\sqrt{(4m_{\pi^+}^2 - s_{\pi\pi})/s_{\pi\pi}}$$

$$|\mathcal{M}|^2 = (\mathcal{M}_0)^2 + (\mathcal{M}_1)^2 + 2\mathcal{M}_0\mathcal{M}_1$$

where $\mathcal{M}_{+,\mathrm{thr}}$ is the value of the $K^+ \to \pi^+\pi^+\pi^-$ amplitude at the $\pi^+\pi^-$ threshold.

- Above threshold: $\mathcal{M}_1$ imaginary — No interference
- Below threshold: $\mathcal{M}_1$ real and negative — Interferes destructively with $\mathcal{M}_0$

NA48 data are very precise.

With 10^8 $K^+ \to \pi^+\pi^0\pi^0$ events, a 1÷2 % measurement of $(a_0 - a_2)$ seems possible, but since the cusp is a 10% effect, this requires a theory good to $1 \div 2$ parts in 10^{-3}, and implicates higher order rescattering effects, and radiative corrections.

It is possible to set up a systematic computation of the singular parts of an amplitude in terms of its non-singular parts. This leads to an expansion of the $K \to 3\pi$ amplitudes in powers of the $\pi\pi$ scattering lengths a_0, a_2.

The development is useful because the scattering lengths are small, which is a general consequence of the fact that pions act as “pseudo Goldstone Bosons” for chiral symmetry breaking.

The development to the second order in powers of a_0, a_2 reveals a second cusp above the $\pi^+\pi^-$ threshold.

New cusp results.

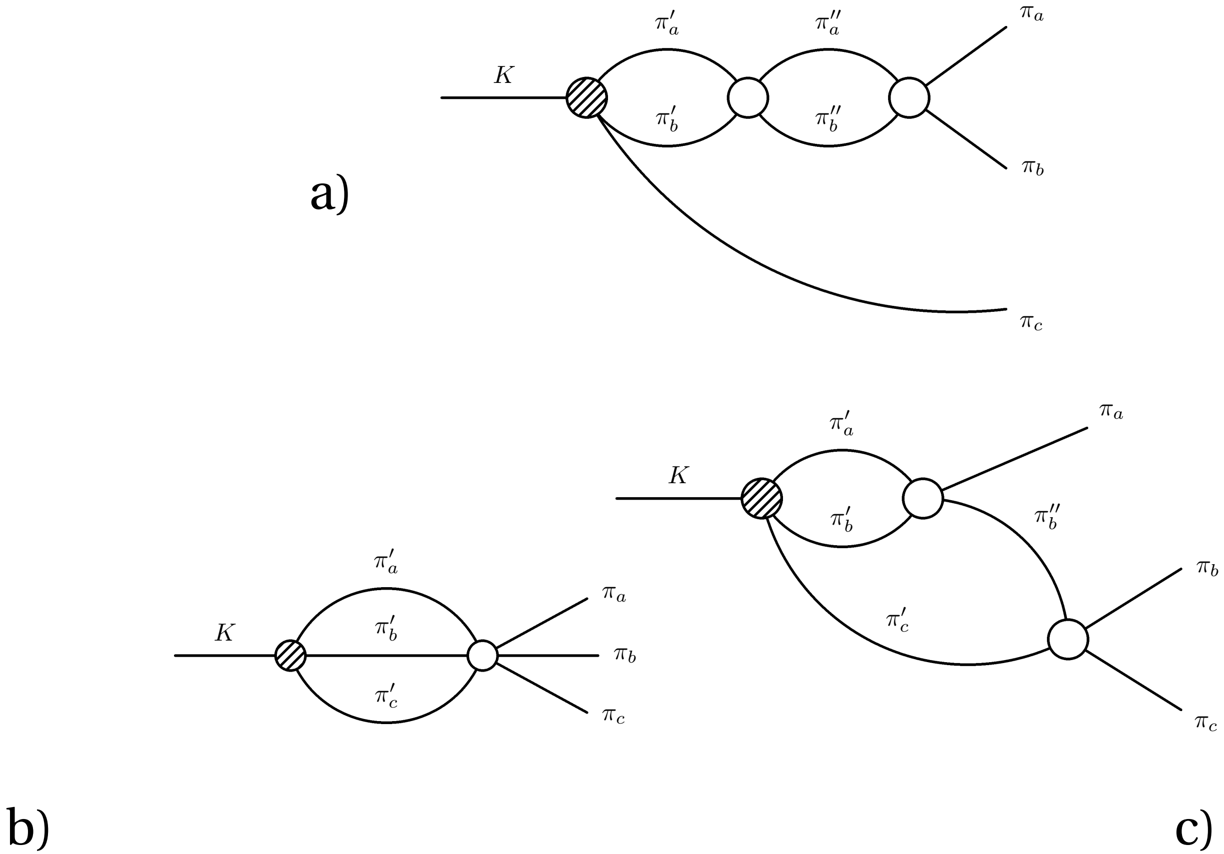
With Gino Isidori we have completed a computation of the $O(a_i^2)$ corrections to the $K \to 3\pi$ amplitudes. This work includes:

- Other rescattering corrections: $\pi^0\pi^0 \to \pi^0\pi^0$, $\pi^+\pi^0 \to \pi^+\pi^0$, etc.
- Corrections from "two-loops" graphs.
- Cover $K^+ \to \pi^+\pi^0\pi^0$, $K_L \to \pi^0\pi^0\pi^0$, which have cusps, and $K^+ \to \pi^+\pi^+\pi^-$, $K_L \to \pi^+\pi^-\pi^0$. which do not.

This work allows for a determination of $a_0 - a_2$ with 5% theoretical uncertainty. To reach a 1% theoretical uncertainty we need:

- An evaluation of $O(a_i^2)$ corrections.
- An evaluation of radiative corrections.

The accuracy of NA48 data merits this effort! Statistical errors in the 1÷2% range are foreseen, and systematic errors should also be small.



$K \to 3\pi$ rescattering topologies at the two-loop level:

a) single-channel $\pi\pi$ scattering;

b) irreducible $3\pi \to 3\pi$ contributions;

c) $3\pi \to 3\pi$ amplitude due to multi-channel $\pi\pi$ scattering.

The method

1. Time reversal: In the sectors of interest the **S** matrix is symmetric,

$$\langle B|\mathbf{S}|A\rangle = \langle A|\mathbf{S}|B\rangle \qquad K \to 3\pi \text{ and } \pi\pi \to \pi\pi$$

2. Unitarity:

$$\mathbf{S} = \mathbf{1} + i(\mathbf{R} + i\mathbf{I}) \qquad \text{where } \mathbf{R} \text{ and } \mathbf{I} \text{ are hermitian.}$$

From time reversal: **R** and **I** are symmetric, their matrix elements are the real and imaginary parts of the matrix elements of **S**.

$$2\mathbf{I} = \mathbf{R}^2 + \mathbf{I}^2 \qquad \text{or, solving for } \mathbf{I},$$

$$\mathbf{I} = \mathbf{1} - \sqrt{\mathbf{1} - \mathbf{R}^2} \qquad = \qquad \frac{1}{2}\mathbf{R}^2 + \frac{1}{8}\mathbf{R}^4 + \frac{1}{16}\mathbf{R}^6 + \frac{5}{128}\mathbf{R}^8 \ldots$$

3. Analiticity, illustrated by an example: $\pi^0\pi^0 \to \pi^0\pi^0$ scattering.

$\pi^0\pi^0 \to \pi^0\pi^0$ scattering at $O(a_i^3)$ — 1

Define the "velocities",

$$v_\pm(s) = \sqrt{\frac{|s - 4m_{\pi^+}^2|}{s}} \qquad v_{00}(s) = \sqrt{\frac{|s - 4m_{\pi^0}^2|}{s}}$$

We can then write

$$\mathcal{M}_{00} = A_{00} + B_{00}\, v_\pm(s) \qquad s > 4m_{\pi^+}^2$$

$$\mathcal{M}_{00} = A_{00} + i B_{00}\, v_\pm(s) \qquad s < 4m_{\pi^+}^2,$$

and, for $\pi^+\pi^- \to \pi^0\pi^0$,

$$\mathcal{M}_x = A_x + B_x\, v_\pm(s) \qquad s > 4m_{\pi^+}^2$$

where A_{00}, B_{00}, A_x, B_x are regular (analytic) at the $\pi^+\pi^-$ threshold.

$\pi^0\pi^0 \to \pi^0\pi^0$ scattering at $O(a_i^3)$ — 2

We can express Re(A) as a polynomial in s. Write:

$$\mathrm{Re}(A_{00}) = \frac{8a_{00}(s)}{\pi}; \qquad a_{00}(s) = a_{00}\left[1 + r_{00}\frac{(s-4m_{\pi^+}^2)}{4m_{\pi^+}^2} + \ldots\right]$$

and similarly for $\pi^+\pi^- \to \pi^0\pi^0$,

$$\mathrm{Re}(A_x) = \frac{8a_x(s)}{\pi}; \qquad a_x(s) = a_x\left[1 + r_x\frac{(s-4m_{\pi^+}^2)}{4m_{\pi^+}^2} + \ldots\right]$$

In the limit of exact SU(2), $a_{00} = \frac{a_0+2a_2}{3}$, $a_x = \frac{a_0-a_2}{3}$

The $\pi^+\pi^-$ intermediate state contributes to Im$\mathscr{M}_{00}$ only above the $\pi^+\pi^-$ threshold, while the $\pi^0\pi^0$ state contributes both above and below, so that, at $O(\mathbf{R}^2)$,

$$\mathrm{Im}\,\mathscr{M}_{00} = \frac{\pi}{4}v_\pm(s)\,(\mathrm{Re}\,\mathscr{M}_x)^2\,\Theta(s-4m_{\pi^+}^2) + \frac{\pi}{8}v_{00}(s)\,(\mathrm{Re}\,\mathscr{M}_{00})^2 + O(\mathbf{R}^4)$$

$\pi^0\pi^0 \to \pi^0\pi^0$ scattering at $O(a_i^3)$ — 3

Note that $\quad \operatorname{Im}\mathscr{M}_{00} = \begin{cases} \operatorname{Im} A_{00} + \operatorname{Im} B_{00}\, v_\pm(s) & : \quad s > 4m_{\pi^+}^2 \\ \operatorname{Im} A_{00} + \operatorname{Re} B_{00}\, v_\pm(s) & : \quad s < 4m_{\pi^+}^2 \end{cases}$

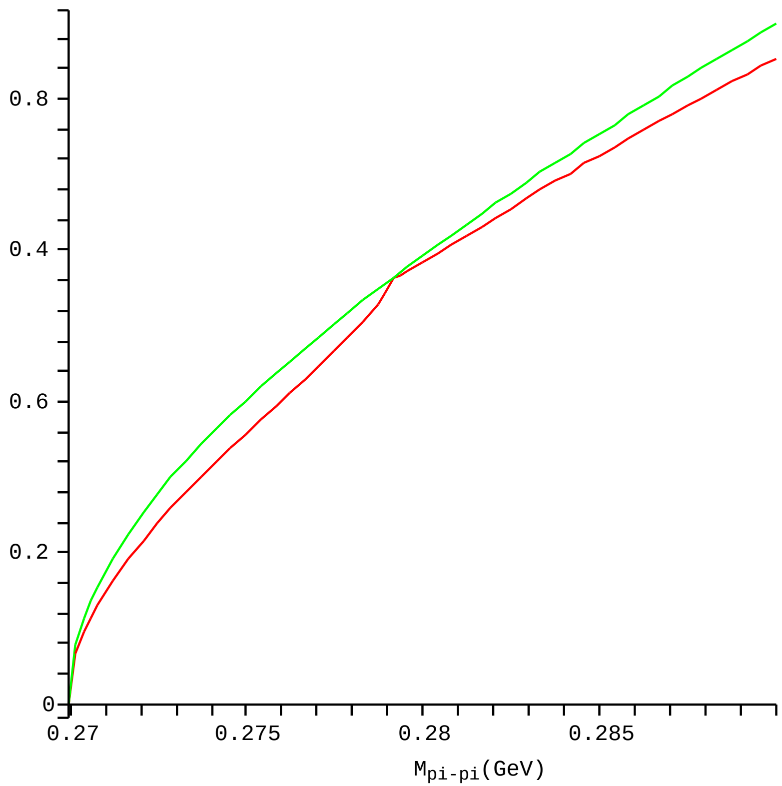
Applying unitarity, $\mathbf{I} = \mathbf{R}^2/2$ at $O(\mathbf{R}^2)$ both above and below the $\pi^+\pi^-$ threshold we then obtain

$$\operatorname{Im} B_{00} = \frac{\pi}{4}(\operatorname{Re} A_x)^2 = \frac{16}{\pi}(a_x(s))^2$$

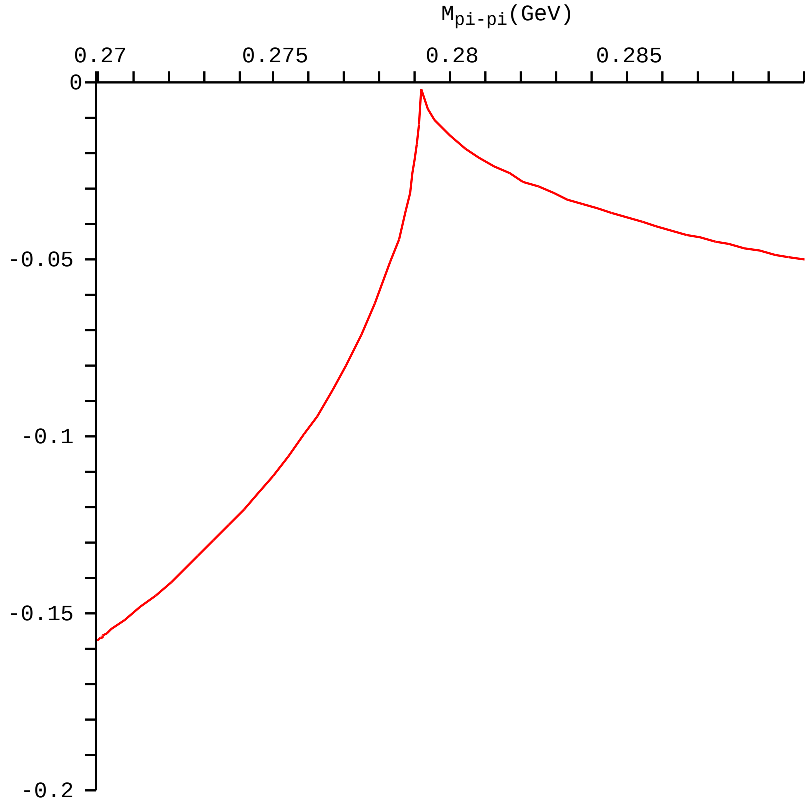
$$\operatorname{Re} B_{00} = -\frac{\pi v_{00}(s)}{4} \operatorname{Re} A_{00} \operatorname{Im} B_{00} = -\frac{32 v_{00}(s)}{\pi} a_{00}(s)(a_x(s))^2$$

$$\operatorname{Im} A_{00} = \frac{\pi v_{00}(s)}{8}(\operatorname{Re} A_{00})^2 = \frac{8 v_{00}(s)}{\pi}(a_{00}(s))^2$$

These results (and similar ones for other channels) are then fed into the unitarity relations for $K \to 3\pi$ to obtain all corrections at $O(a_i^2)$.



The theoretical $\pi^0\pi^0$ spectrum in $K^+ \to \pi^+\pi^0\pi^0$ (lower) computed at second order in powers of the scattering lengths.



Theoretical $\pi^0\pi^0$ spectrum in $K^+ \to \pi^+\pi^0\pi^0$ (lower) at $O(a_i{}^2)$ normalized to the "unperturbed" amplitude.

The picture will be complicated by the existence of a $\pi^+\pi^-$ pionium atom signal at the $\pi^+\pi^-$ threshold!.

Space for improvement

- Stopping at $O(a_i^2)$ level and neglecting radiative corrections implies a $\sim 5\%$ theoretical error on $(a_0 - a_2)$. This may be sufficent at the present state of NA48 systematics.
- $O(a_i^3)$ terms and radiative corrections can be computed with a finite, but lengthy effort.
- $\rightarrow$ Compute $O(a_i^3)$ terms.
- $\rightarrow$ Compute radiative corrections
- $\rightarrow$ These computations will to a large extent be independent from Chiral Perturbation Theory which we wish to test!
- $\rightarrow$ The theoretical error on $(a_0 - a_2)$ can be reduced to the 1% level.

Conclusions

- The experimental study of $K \to 3\pi$ decays is a powerful tool for gathering information on $\pi\pi$ scattering in the low energy region.
- We have outlined a method that allows to systematically evaluate rescattering effects in $K \to 3\pi$ decays by means of an expansion in powers of the $\pi\pi$ scattering lengths. This approach is less ambitious than the ordinary loop expansion performed in effective field theories, such as CHPT: the scope is not a dynamical calculation of the entire decay amplitudes, but a systematic evaluation of the singular terms due to rescattering effects. In particular, our main goal has been a systematical description of the cusp effect in $K^+ \to \pi^+\pi^0\pi^0$ in terms of the $\pi\pi$ scattering lengths. From this point of view, the approach we have proposed is more efficient and substantially simpler than ordinary CHPT.
- Using this method we have explicitly computed all the $O(a_i^2)$ corrections to the leading cusp effect in $K^+ \to \pi^+\pi^0\pi^0$, extending the results of my previos paper. The extra terms produce a small square-root behavior also above the $\pi^+\pi^-$ singularity.
- The 5% level of precision is probably not sufficient to fully exploit the potentially very accurate data of NA48, and is also larger than the error on the CHPT predictions of $a_0 - a_2$. To improve the situation we need a complete evaluation of the $O(a_i^3)$ corrections and of the effects due to radiative corrections.