Duality Violations and Spectral Sum Rules: the case of Π_{LR}

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(Based on work done in collaboration with **M. Golterman** *and* **S. Peris***)*

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Introduction

We define the two-point Green function

$$\Pi^{LR}_{\mu\nu}(q^2) = 2i \int dx^4\, e^{iq\cdot x} \left\langle 0 | L_\mu(x)\, R^\dagger_\nu(0) | 0 \right\rangle = (g_{\mu\nu} q^2 - q_\mu q_\nu)\Pi^{LR}(q^2) \tag{1}$$

$$L_\mu(x) = \bar{q}(x)\gamma_\mu \left(\frac{1-\gamma_5}{2}\right) q(x) \quad , \quad R_\nu(0) = \bar{q}(0)\gamma_\mu \left(\frac{1+\gamma_5}{2}\right) q(0) \tag{2}$$

that satisfies the (unsubstracted) dispersion relation

$$\Pi^{LR}(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{dt}{t + Q^2 - i\epsilon}\, Im\, \Pi^{LR}(t) \tag{3}$$

Common lore: Extraction of the OPE coefficients is performed by plugging in the OPE in the left-hand side. This is mathematically *wrong*, but the error is expected to reduce if s is sufficiently large. Before this asymptotic regime sets in, we see **quark-hadron duality violations**.

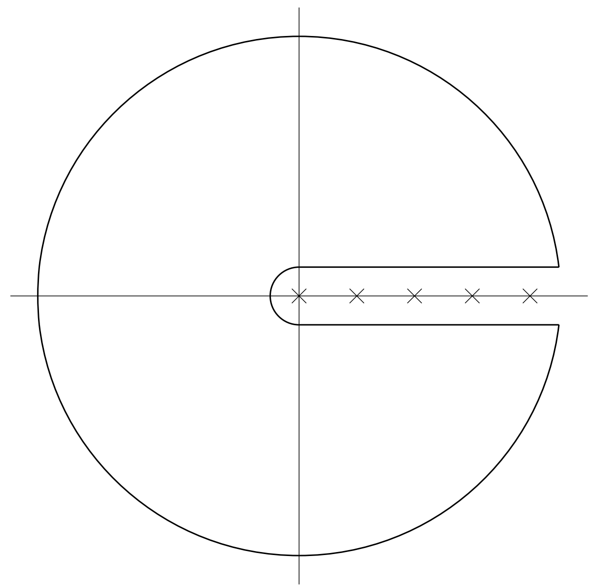

Figure 1:

Different approaches to extract the OPE coefficients, based on Cauchy's theorem:

$$\int_0^{s_0} dt\, t^n \, \frac{1}{\pi} \, Im\Pi_{LR}(t) = -\frac{1}{2\pi i} \oint_{|q^2|=s_0} dq^2 q^{2n} \, \Pi_{LR}(q^2) \quad \text{(FESR)} \tag{4}$$

and

$$\int_0^{s_0} dt\, w(t) \, \frac{1}{\pi} \, Im\Pi_{LR}(t) = -\frac{1}{2\pi i} \oint_{|q^2|=s_0} dq^2 w(q^2) \, \Pi_{LR}(q^2) \quad \text{(pFESR)} \tag{5}$$

where $w(t)$ is a polynomial that cancels at duality points s_0^*, and so kills contributions from the OPE near the resonances.

The Toy Model at large–N_C

We choose as our spectral function:

$$\frac{1}{\pi}\, Im\Pi_{LR}(t) = -f_\pi^2\delta(t) + f_\rho^2\delta(t-m_\rho^2) - \sum_{n=0}^{\infty} f^2\delta(t-m_A^2(n)) + \sum_{n=0}^{\infty} f^2\delta(t-m_V^2(n)) \qquad (6)$$

where resonances in the vector and axial towers are piled up as follows

$$m_V^2(n) = m_V^2 + n\,\Lambda^2 \qquad , \qquad m_A^2(n) = m_A^2 + n\,\Lambda^2 \qquad , \qquad f_{V,A}^2(n) = f^2 \qquad (7)$$

This leads to

$$\Pi_{LR}(Q^2) = -\frac{f_\pi^2}{Q^2} + \frac{f_\rho^2}{Q^2+m_\rho^2} + \sum_{n=0}^{\infty}\frac{f^2}{Q^2+m_V^2(n)} - \sum_{n=0}^{\infty}\frac{f^2}{Q^2+m_A^2(n)} = \sum_k \frac{c_{2k}}{Q^{2k}} \qquad (8)$$

We determine the parameters of the model by demanding that the vector and axial components reproduce the *parton-model* logarithm plus first and second *Weinberg sum rules*.

Finite Energy Sum Rules

OPE coefficients can be computed and yield

$$
\begin{aligned}
c_2 &= +F_\rho^2 - \; F_\pi^2 - \; F^2 \left\{ B_1\left(\frac{m_V^2}{\Lambda^2}\right) - B_1\left(\frac{m_A^2}{\Lambda^2}\right) \right\} \;, \\
c_4 &= -F_\rho^2 M_\rho^2 + \frac{1}{2} F^2 \Lambda^2 \left\{ B_2\left(\frac{m_V^2}{\Lambda^2}\right) - B_2\left(\frac{m_A^2}{\Lambda^2}\right) \right\} \;, \\
c_6 &= +F_\rho^2 M_\rho^4 - \frac{1}{3} F^2 \Lambda^4 \left\{ B_3\left(\frac{m_V^2}{\Lambda^2}\right) - B_3\left(\frac{m_A^2}{\Lambda^2}\right) \right\} \;, \\
c_8 &= -F_\rho^2 M_\rho^6 + \frac{1}{4} F^2 \Lambda^6 \left\{ B_4\left(\frac{m_V^2}{\Lambda^2}\right) - B_4\left(\frac{m_A^2}{\Lambda^2}\right) \right\} \;.
\end{aligned} \tag{9}
$$

On the other hand, moments are defined as

$$
M_{2n}(s_0) = (-1)^{n-1} \int_0^{s_0} \; dt \; t^{n-1} \rho(t) \tag{10}
$$

The first ones read

$$
\begin{aligned}
M_2(s_0) &= c_2 - F^2 \Big[B_1(x_V) - B_1(x_A) \Big] \;, \\
M_4(s_0) &= c_4 + F^2 \Big[B_1(x_V) - B_1(x_A) \Big] s_0 + \cdots \;, \\
M_6(s_0) &= c_6 - F^2 \Big[B_1(x_V) - B_1(x_A) \Big] s_0^2 + \cdots \;, \\
M_8(s_0) &= c_8 + F^2 \Big[B_1(x_V) - B_1(x_A) \Big] s_0^3 + \cdots \;.
\end{aligned} \tag{11}
$$

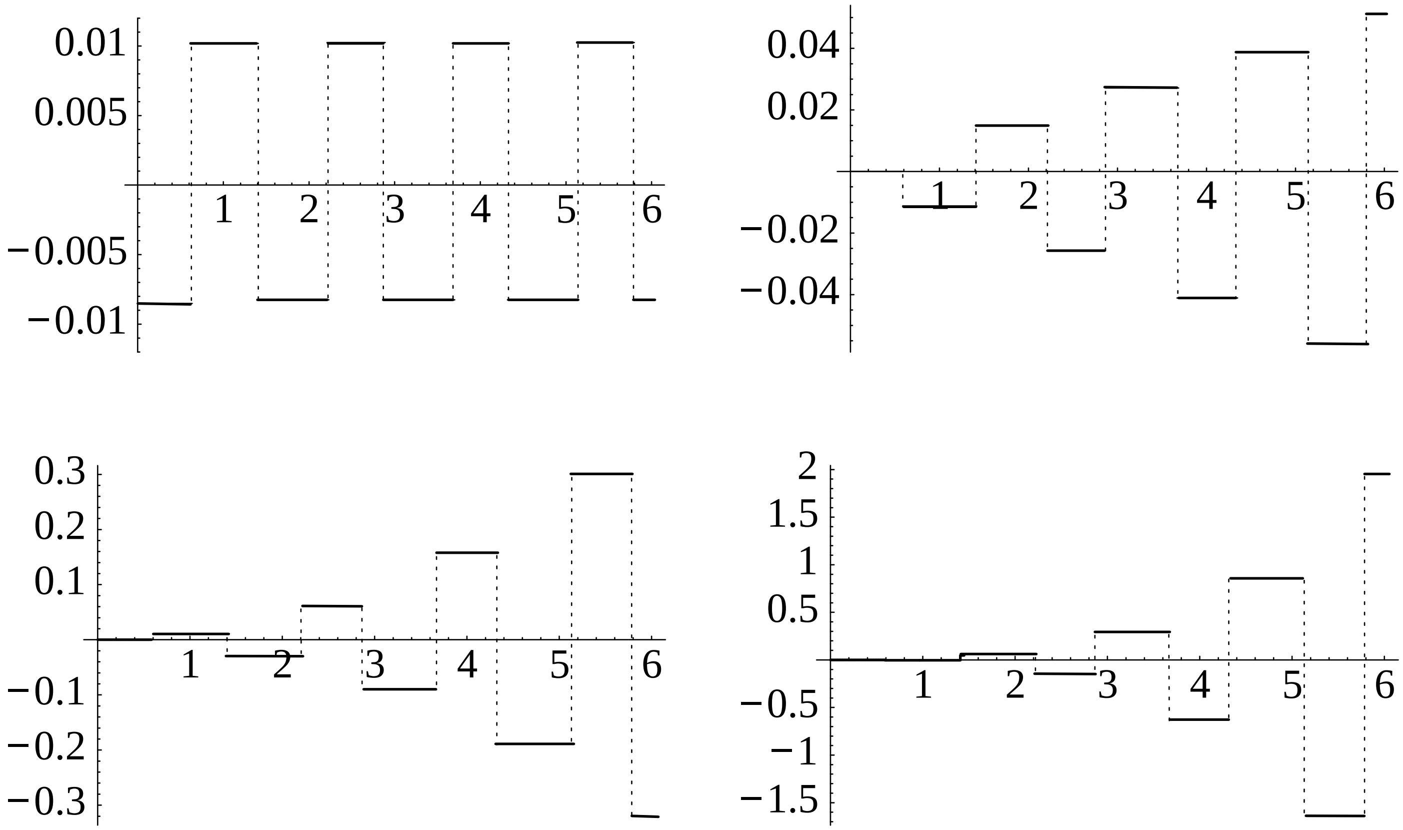

Figure 2: First moments of the spectral function.

Pinched-weight FESR

We define the following combination of moments

$$J_{w_1}(s_0) \equiv \int_0^{s_0} dt \left(1 - 3\frac{t}{s_0}\right)\left(1 - \frac{t}{s_0}\right)^2 \rho(t) \quad = 7\ \frac{\widetilde{c}_6}{s_0^2} + 3\ \frac{\widetilde{c}_8}{s_0^3}$$

$$J_{w_2}(s_0) \equiv \int_0^{s_0} dt\ \frac{t}{s_0}\left(1 - \frac{t}{s_0}\right)^2 \rho(t) \quad = -2\ \frac{\widetilde{c}_6}{s_0^2} - \frac{\widetilde{c}_8}{s_0^3}\ , \tag{12}$$

which can be solved for $\tilde{c}_6$ and $\tilde{c}_8$

$$\widetilde{c}_6 = c_6 - \frac{1}{3}F^2\Lambda^4\Big[B_3(x_V) - B_3(x_A)\Big]$$

$$\widetilde{c}_8 = c_8 + F^2\Lambda^2\Big[B_3(x_V) - B_3(x_A)\Big]s_0 - \frac{1}{4}F^2\Lambda^4\Big[B_4(x_V) - B_4(x_A)\Big]\ , \tag{13}$$

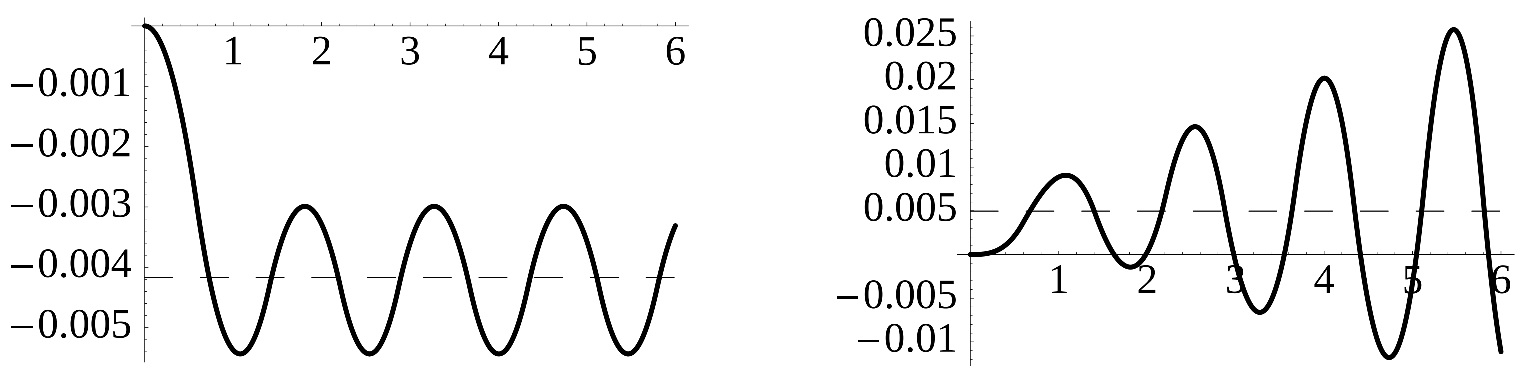

Figure 3:

$\frac{1}{N_C}$ corrections

We want to add widths while respecting the analiticity of the Green function. One such possibility is

$$\Pi_{LR}(z) = -\frac{f_\pi^2}{z} + \frac{f_\rho^2}{z+m_\rho^2} + \sum_{n=0}^{\infty} \frac{f^2}{z+m_V^2(n)} - \sum_{n=0}^{\infty} \frac{f^2}{z+m_A^2(n)} \tag{14}$$

where the variable z is defined as

$$z = Q^2 \left(\frac{Q^2}{\Lambda^2}\right)^{-\frac{a}{\pi N_C}} \sim Q^2 \left(1 - \frac{a}{\pi N_C} \log \frac{Q^2}{\Lambda^2}\right) \tag{15}$$

such that $\Gamma_i \sim \frac{a}{\pi N_C}\, m_i(n)$. Again, we determine the parameters of the model by imposing

$$\frac{2}{3}\frac{N_c}{16\pi^2} = \frac{F^2}{\Lambda^2}\left(1 - \frac{a}{\pi N_C}\right) , \tag{16}$$

$$\begin{aligned}\left[F_\rho^2 - F^2\left(\frac{m_V^2}{\Lambda^2} - \frac{1}{2}\right)\right]\left(1 + \frac{a}{N_C}\log\left(\frac{Q^2}{\Lambda^2}\right)\right) &= 0 \\ \left[F_\pi^2 - F^2\left(\frac{m_A^2}{\Lambda^2} - \frac{1}{2}\right)\right]\left(1 + \frac{a}{N_C}\log\left(\frac{Q^2}{\Lambda^2}\right)\right) &= 0 ,\end{aligned} \tag{17}$$

$$\left[2F_\rho^2 M_\rho^2 - F^2\Lambda^2\left(\frac{m_V^4}{\Lambda^4} - \frac{m_V^2}{\Lambda^2} + \frac{1}{6}\right) + F^2\Lambda^2\left(\frac{m_A^4}{\Lambda^4} - \frac{m_A^2}{\Lambda^2} + \frac{1}{6}\right)\right]\left(1 + \frac{2a}{N_C}\log\left(\frac{Q^2}{\Lambda^2}\right)\right) = 0 , \tag{18}$$

We choose F_π, M_ρ, m_A and a as our free parameters and determine them to fit to a reasonable extend the experimental curve. The full set of parameters of the model then reads

$$\boxed{\begin{aligned}&F_\pi = 92.5 \text{ MeV}, \quad F_\rho = 141 \text{ MeV}, \quad F = 146.5 \text{ MeV}\,,\\ &m_\rho = 767 \text{ MeV}, \quad m_A = 1.186 \text{ GeV}, \quad m_V = 1.49 \text{ GeV}\,,\\ &\Lambda = 1.25 \text{ GeV}, \quad a = 0.72\end{aligned} \qquad (19)}$$

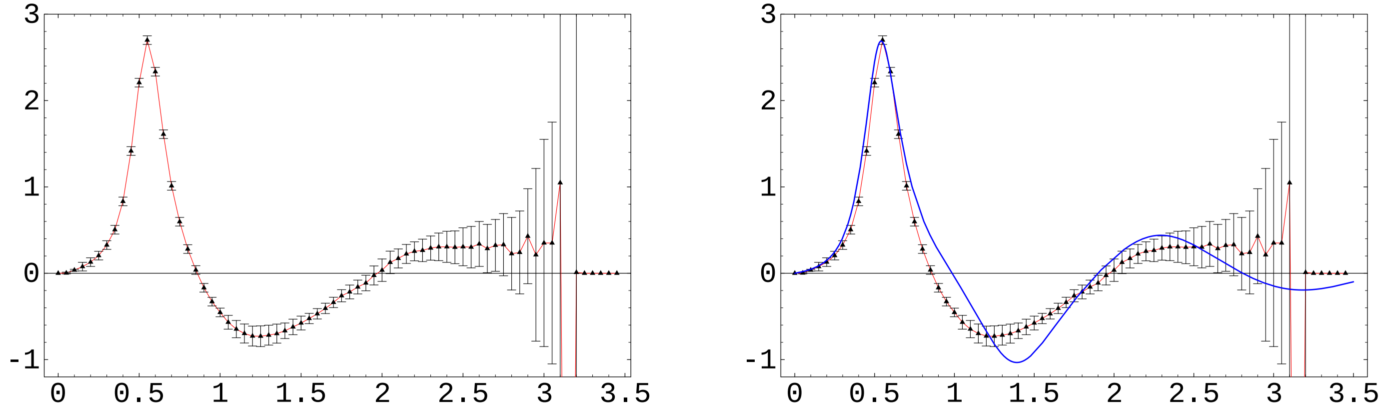

Figure 4:

and OPE coefficients are determined to be

$$\begin{aligned}
C_2 &= 0 \quad , \quad C_4 = 0 \\
C_6 &= \left\{-3.1\cdot 10^{-3} + 8.1\cdot 10^{-4}\log\left(\frac{1}{s_0}\right)\right\}\ \mathrm{GeV}^8 \\
C_8 &= \left\{2.8\cdot 10^{-3} - 1.0\cdot 10^{-3}\log\left(\frac{1}{s_0}\right)\right\}\ \mathrm{GeV}^{10}
\end{aligned} \tag{20}$$

It is interesting to know the asymptotic behaviour of the spectral function. In the Minkowskian, $z = t\left(1 - \frac{a}{\pi N_C}\log\left(\frac{t}{\Lambda^2}\right) + i\frac{a}{N_C}\right)$, and

$$\begin{aligned}
\mathrm{Im}\Pi_{LR}(t) &\sim -\frac{3aC_6}{N_C}\frac{1}{t^3} + \\
&+ 4\pi e^{-\frac{2\pi a t}{\Lambda^2 N_C}}\left[\sin\left(\frac{2\pi}{\Lambda^2}\left(\frac{2t + m_A^2 + m_V^2}{2}\right)\right)\sin\left(\frac{2\pi}{\Lambda^2}\left(\frac{m_V^2 - m_A^2}{2}\right)\right)\right].
\end{aligned} \tag{21}$$

The first piece is just the analytical continuation of the OPE and the second one is a damped oscillation **not** seen by the OPE. At some t the oscillation is killed and the power fall-off, dictated by the OPE, takes over.

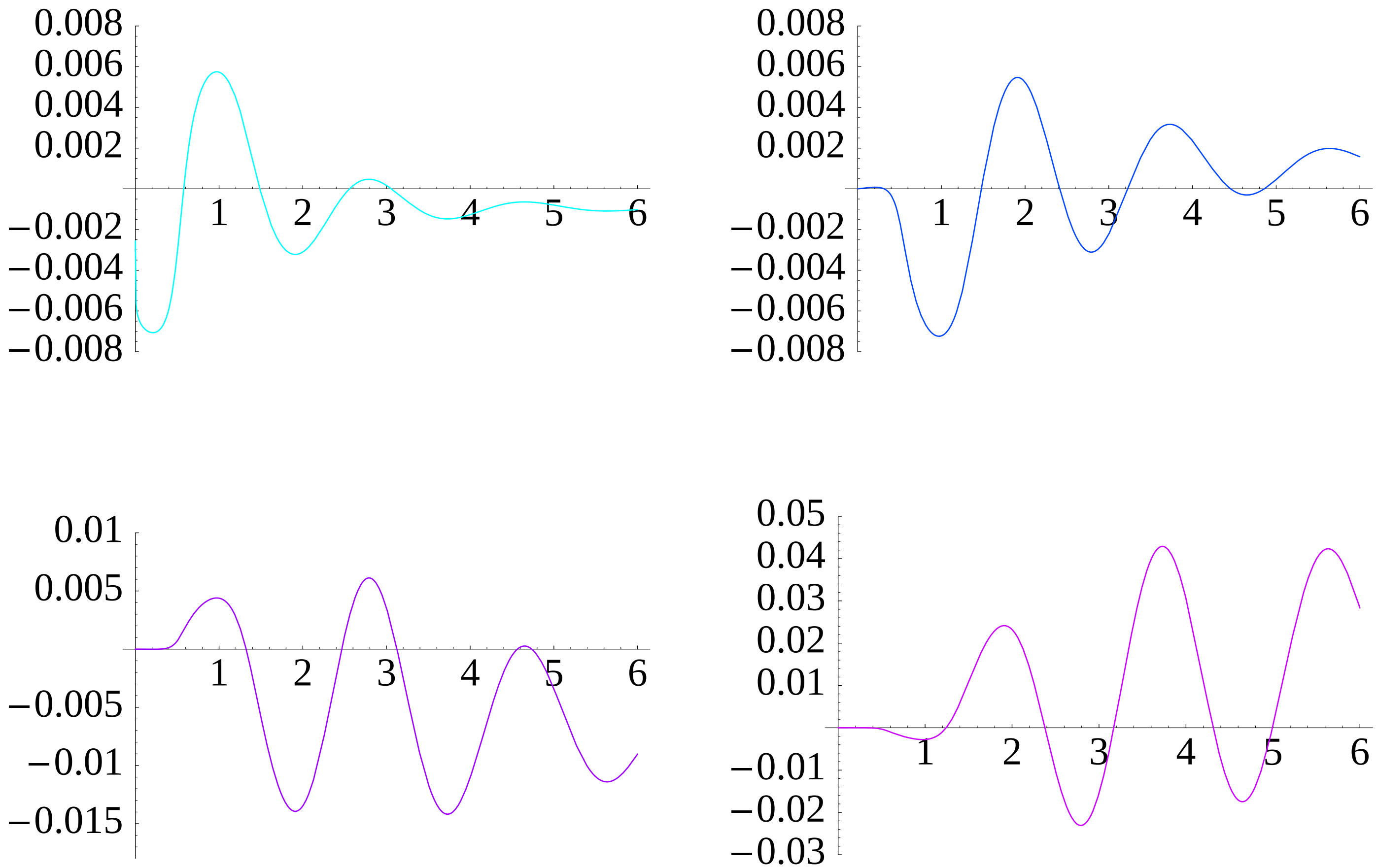

Figure 5: First moments of the spectral function

Duality points at $s_0^{(1)} = 1.5\, GeV^2$ and $s_0^{(2)} = 2.4\, GeV^2$

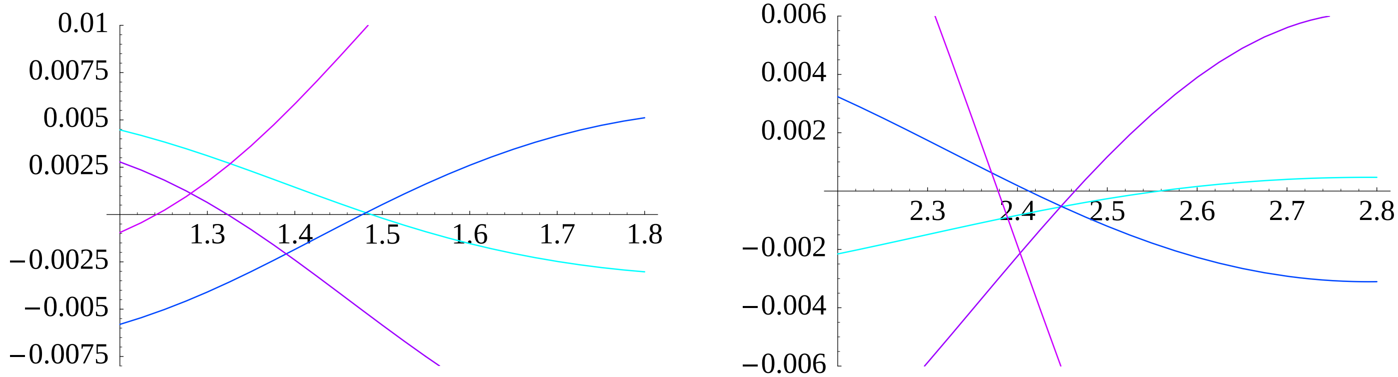

Figure 6: Blow-up regions near the first two duality points

	$c_6\ (GeV^6)$	$c_8\ (GeV^8)$
OPE($s_0^{(1)}$)	-3.5 $\cdot 10^{-3}$	3.2 $\cdot 10^{-3}$
FESR ($s_0^{(1)}$)	-5 $\cdot 10^{-3}$	9.7 $\cdot 10^{-3}$
OPE ($s_0^{(2)}$)	-3.9 $\cdot 10^{-3}$	3.7 $\cdot 10^{-3}$
FESR ($s_0^{(2)}$)	-1.8 $\cdot 10^{-3}$	-3.0 $\cdot 10^{-3}$

For pinched-weights the delicate cancellation we encounter in the zero-width spectrum no longer takes place. Instead, **each** moment is polluted with the highest divergence entering $w(t)$.

$$\begin{aligned} J_{w_1}(s_0) &\equiv \int_0^{s_0} dt\ \left(1-3\frac{t}{s_0}\right)\left(1-\frac{t}{s_0}\right)^2 \rho(t) \quad = 7\ \frac{\widetilde{c}_6}{s_0^2} + 3\ \frac{\widetilde{c}_8}{s_0^3} \\ J_{w_2}(s_0) &\equiv \int_0^{s_0} dt\ \frac{t}{s_0}\left(1-\frac{t}{s_0}\right)^2 \rho(t) \quad = -2\ \frac{\widetilde{c}_6}{s_0^2} - \frac{\widetilde{c}_8}{s_0^3}\ , \end{aligned} \tag{22}$$

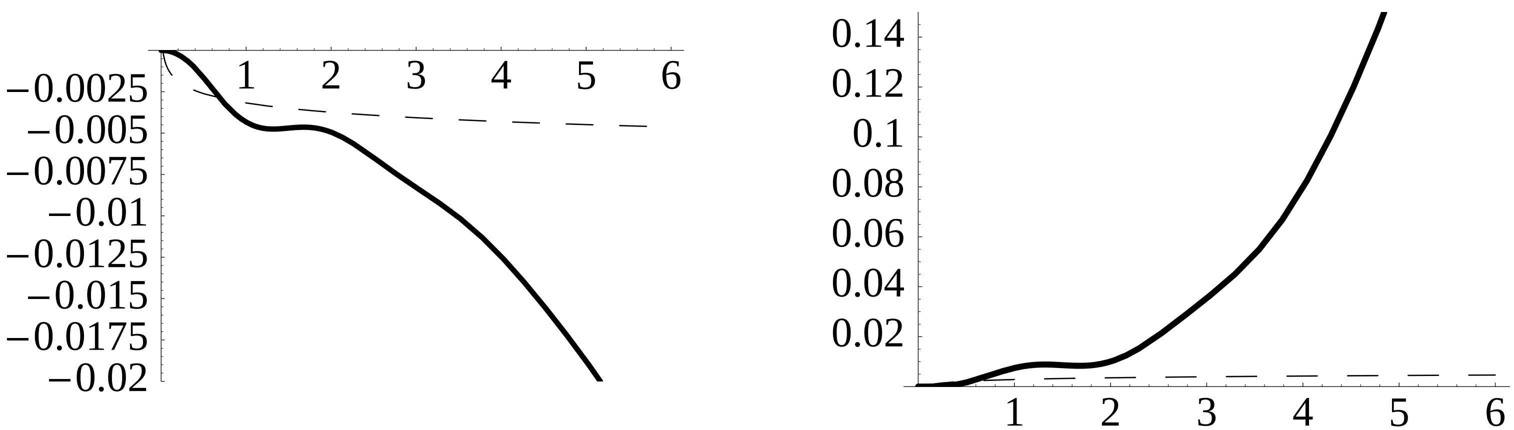

Figure 7:

Conclusions

• In the **zero-width model** duality points

$$M_{2k}(s_0^*) = c_{2k}$$

sit on the mass poles of the resonances.

• They are common to all moments, but for FESR there is no predictability at all (infinite slope).

• On the contrary, pinched-weights are indeed effective if one averages over a full oscillation.

• As a general rule, moments diverge and it is safer to stay to low values of s.

• When **adding widths**, duality points disalign but some memory remains and they sit close to the mass poles of resonances.

• Pinched-weights do not seem to help much.

• Surprisingly enough, **moments still diverge** and so it is not a good strategy to do the analysis at high duality points. This has to happen also in QCD, for there the OPE coefficients also have logarithmic corrections. Therefore, it is not true that the higher the duality point where the analysis is done, the closer the results to the asymptotic values of the OPE.