New sum rules in the heavy quark limit of QCD

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Abstract

Using the OPE and the trace formalism, we have obtained a number of sum rules in the heavy quark limit of QCD that include the sum over all excited states for any value j^P of the light cloud. We show that these sum rules imply that the elastic Isgur-Wise function $\xi(w)$ is an alternate series in powers of $(w-1)$. Moreover, one gets that any n-th derivative of $\xi(w)$ at $w=1$ can be bounded by the $(n-1)$-th one. For the curvature $\sigma^2 = \xi''(1)$, this proves the already proposed bound $\sigma^2 \geq \frac{5}{4}\rho^2$. Moreover, we obtain the absolute lower bound for the n-th derivative $(-1)^n \xi^{(n)}(1) \geq \frac{(2n+1)!!}{2^{2n}}$, that generalizes the results $\rho^2 \geq \frac{3}{4}$ and $\sigma^2 \geq \frac{15}{16}$.

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Using the OPE and the trace formalism in the heavy quark limit, different initial and final four-velocities v_i and v_f for initial and final B-meson states, and arbitrary heavy quark currents

$$J_1 = \bar{h}_{v'}^{(c)}\ \Gamma_1\ h_{v_i}^{(b)} \quad , \qquad J_2 = \bar{h}_{v_f}^{(b)}\ \Gamma_2\ h_{v'}^{(c)} \tag{1}$$

the following sum rule can be written, as shown in ref. [1] :

$$\begin{aligned}&\Big\{ \sum_{D=P,V} \sum_n Tr\left[\bar{\mathcal{B}}_f(v_f)\bar{\Gamma}_2\mathcal{D}^{(n)}(v')\right] Tr\left[\bar{\mathcal{D}}^{(n)}(v')\Gamma_1\mathcal{B}_i(v_i)\right]\xi^{(n)}(w_i)\xi^{(n)}(w_f)\\ &+ \text{Other excited states}\Big\} = -2\xi(w_{if})Tr\left[\bar{\mathcal{B}}_f(v_f)\bar{\Gamma}_2 P'_+\Gamma_1\mathcal{B}_i(v_i)\right]\ .\end{aligned} \tag{2}$$

In this formula v' is the intermediate meson four-velocity, $P'_+ = \frac{1}{2}(1+\not{v}')$ comes from the residue of the positive energy part of the c-quark propagator, and $\xi(w_{if})$ is the elastic Isgur-Wise function that appears because one assumes $v_i \neq v_f$. $\mathcal{B}_i$ and $\mathcal{B}_f$ are the 4×4 matrices of the ground state B or B^* meson and $\mathcal{D}^{(n)}$ those of all possible ground state or excited state D mesons coupled to B_i and B_f through the currents. In formula (2) we have made explicit the $j = \frac{1}{2}^-$ D and D^* mesons and their radial excitations.

The variables w_i, w_f and w_{if} are defined as

$$w_i = v_i \cdot v' \qquad w_f = v_f \cdot v' \qquad w_{if} = v_i \cdot v_f\ . \tag{3}$$

The domain of $(w_i,\ w_f,\ w_{if})$ is

$$w_i, w_f \geq 1\ , \quad w_iw_f - \sqrt{(w_i^2-1)(w_f^2-1)} \leq w_{if} \leq w_iw_f + \sqrt{(w_i^2-1)(w_f^2-1)}\ . \tag{4}$$

There is a subdomain for $w_i = w_f = w$:

$$w \geq 1 \qquad 1 \leq w_{if} \leq 2w^2 - 1\ . \tag{5}$$

Calling now $L(w_i, w_f, w_{if})$ the l.h.s. and $R(w_i, w_f, w_{if})$ the r.h.s. of (2), this SR writes

$$L\left(w_i, w_f, w_{if}\right) = R\left(w_i, w_f, w_{if}\right) \tag{6}$$

where $L(w_i, w_f, w_{if})$ is the sum over the intermediate D states and $R(w_i, w_f, w_{if})$ the OPE side. Within the domain (4) one can derive relatively to any of the variables w_i, w_f and w_{if}

$$\frac{\partial^{p+q+r} L}{\partial w_i^p \partial w_f^q \partial w_{if}^r} = \frac{\partial^{p+q+r} R}{\partial w_i^p \partial w_f^q \partial w_{if}^r} \tag{7}$$

and obtain different SR taking different limits to the frontiers of the domain, e.g.

$$\begin{aligned} & w_{if} \to 1, w_i = w_f = w \\ \text{or} \qquad & w_i \to 1, w_{if} = w_f = w \\ \text{or} \qquad & w_f \to 1, w_{if} = w_i = w \ . \end{aligned} \tag{8}$$

The aim of the present paper is to present some results on the derivatives of the elastic Isgur-Wise function $\xi(w)$ at zero recoil $w = 1$.

As in ref. [1], we choose as initial and final states the B meson,

$$\mathcal{B}_i(v_i) = P_{i+}(-\gamma_5) \qquad \mathcal{B}_f(v_f) = P_{f+}(-\gamma_5) \tag{9}$$

and vector or axial currents projected along the v_i and v_f four-velocities

$$J_1 = \bar{h}_{v'}^{(c)} \ \not{v}_i \ h_{v_i}^{(b)} \quad , \qquad J_2 = \bar{h}_{v_f}^{(b)} \ \not{v}_f \ h_{v'}^{(c)} \ . \tag{10}$$

Gathering the formulas (60) and (89)-(91) of ref. [1] we obtain the SR (2) with the sum of all excited states j^P in a compact form :

$$\begin{aligned} & (w_i+1)(w_f+1) \sum_{\ell \geq 0} \frac{\ell+1}{2\ell+1} S_\ell(w_i, w_f, w_{if}) \sum_n \tau_{\ell+1/2}^{(\ell)(n)}(w_i) \tau_{\ell+1/2}^{(\ell)(n)}(w_f) \\ & + \sum_{\ell \geq 1} S_\ell(w_i, w_f, w_{if}) \sum_n \tau_{\ell-1/2}^{(\ell)(n)}(w_i) \tau_{\ell-1/2}^{(\ell)(n)}(w_f) \\ & = (1 + w_i + w_f + w_{if}) \xi(w_{if}) \ . \end{aligned} \tag{11}$$

Choosing instead the axial currents

$$J_1 = \bar{h}_{v'}^{(c)} \ \not{v}_i \ \gamma_5 \ h_{v_i}^{(b)} \quad , \qquad J_2 = \bar{h}_{v_f}^{(b)} \ \not{v}_f \ \gamma_5 \ h_{v'}^{(c)} \tag{12}$$

the SR (2) writes, from the formulas (48) and (92)-(94) of ref. [1] :

$$\sum_{\ell \geq 0} S_{\ell+1}(w_i, w_f, w_{if}) \sum_n \tau_{\ell+1/2}^{(\ell)(n)}(w_i) \tau_{\ell+1/2}^{(\ell)(n)}(w_f)$$

$$+(w_i-1)(w_f-1)\sum_{\ell\geq 1}\frac{\ell}{2\ell-1}S_{\ell-1}(w_i,w_f,w_{if})\sum_n \tau_{\ell-1/2}^{(\ell)(n)}(w_i)\tau_{\ell-1/2}^{(\ell)(n)}(w_f)$$
$$= -(1-w_i-w_f+w_{if})\xi(w_{if}) \ . \qquad (13)$$

Following the formulation of heavy-light states for arbitrary j^P given by Falk [2], we have defined in ref. [1] the IW functions $\tau_{\ell+1/2}^{(\ell)(n)}(w)$ and $\tau_{\ell-1/2}^{(\ell)(n)}(w)$, that correspond to the orbital angular momentum ℓ of the light quark relative to the heavy quark, $j=\ell\pm\frac{1}{2}$ being the total angular momentum of the light cloud. We could made explicit the contribution of the states $\frac{1}{2}^-$, $\frac{1}{2}^+$, $\frac{3}{2}^+$, using the traditional notation of Isgur and Wise [3]. For the lowest values of ℓ, one has the identities

$$\tau_{1/2}^{(0)}(w)\equiv\xi(w) \quad , \quad \tau_{1/2}^{(1)}(w)\equiv 2\tau_{1/2}(w) \quad , \quad \tau_{3/2}^{(1)}(w)\equiv\sqrt{3}\ \tau_{3/2}(w) \qquad (14)$$

where a radial quantum number is implicit. Therefore, the functions $\tau_{1/2}^{(1)}(w)$ and $\tau_{3/2}^{(1)}(w)$ correspond, respectively, to the functions $\zeta(w)$ and $\tau(w)$ defined by Leibovich et al. [4]. But in all this paper we will keep the general notation $\tau_{\ell+1/2}^{(\ell)(n)}(w)$, $\tau_{\ell-1/2}^{(\ell)(n)}(w)$.

In equations (10) and (12) the quantity S_n is defined by

$$S_n = v_{i\nu_1}\cdots v_{i\nu_n}\ v_{f\mu_1}\cdots v_{f\mu_n}\ T^{\nu_1\cdots\nu_k,\mu_1\cdots\mu_n} \qquad (15)$$

and the polarisation tensor $T^{\nu_1\cdots\nu_k,\mu_1\cdots\mu_n}$, given by

$$T^{\nu_1\cdots\nu_k,\mu_1\cdots\mu_n}=\sum_\lambda \varepsilon'^{(\lambda)*\nu_1\cdots\nu_n}\ \varepsilon'^{(\lambda)\mu_1\cdots\mu_n} \qquad (16)$$

depends only on the four-velocity v'. The polarization tensor $\varepsilon'^{\mu_1\cdots\mu_n}$ is a traceless symmetric tensor, i.e. a symmetric tensor with vanishing contractions. Moreover, as demonstrated in the Appendix A of ref. [1], S_n is given by the following expression :

$$S_n(w_i,w_f,w_{if})=\sum_{0\leq k\leq \frac{n}{2}} C_{n,k}(w_i^2-1)^k(w_f^2-1)^k(w_iw_f-w_{if})^{n-2k} \qquad (17)$$

with

$$C_{n,k}=(-1)^k\frac{(n!)^2}{(2n)!}\ \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} \ . \qquad (18)$$

Making the sum of both equations (11) and (13), one obtains

$$
\begin{aligned}
&2(w_i + w_f)\xi(w_{if}) \\
&= \sum_{\ell\geq 0}\left[\frac{\ell+1}{2\ell+1}(w_i+1)(w_f+1)S_\ell(w_i,w_f,w_{if}) + S_{\ell+1}(w_i,w_f,w_{if})\right] \\
&\sum_n \tau_{\ell+1/2}^{(\ell)(n)}(w_i)\tau_{\ell+1/2}^{(\ell)(n)}(w_f) \\
&+ \sum_{\ell\geq 1}\left[S_\ell(w_i,w_f,w_{if}) + \frac{\ell}{2\ell-1}(w_i-1)(w_f-1)S_{\ell-1}(w_i,w_f,w_{if})\right] \\
&\sum_n \tau_{\ell-1/2}^{(\ell)(n)}(w_i)\tau_{\ell-1/2}^{(\ell)(n)}(w_f)\ . \qquad (19)
\end{aligned}
$$

Taking the m-th derivative of this relation relatively to w_{if}, making $w_i = w_f = w_{if} = 1$, and using

$$
\frac{\partial^m}{\partial w_{if}^m}\ S_n(w_i, w_f w_{if})\bigg|_{w_i=w_f=w_{if}=1} = (-1)^n\ n!\ \delta_{m,n} \qquad (20)
$$

one readily obtains the derivatives at zero recoil

$$
\begin{aligned}
\xi^{(\ell)}(1) = \frac{1}{4}\ (-1)^\ell\ \ell! &\left\{\frac{\ell+1}{2\ell+1}4\sum_n\left[\tau_{\ell+1/2}^{(\ell)(n)}(1)\right]^2\right. \\
&\left.+\sum_n\left[\tau_{\ell-1/2}^{(\ell-1)(n)}(1)\right]^2 + \sum_n\left[\tau_{\ell-1/2}^{(\ell)(n)}(1)\right]^2\right\} \quad (\ell \geq 0)\ . \qquad (21)
\end{aligned}
$$

This relation shows that $\xi(w)$ is an alternate series in powers of $(w-1)$. Equation (21) gives $1 = 1$ for $\ell = 0$, and reduces to Bjorken SR [5], [3] for $\ell = 1$ (with the notation of Isgur and Wise (14)) :

$$
\rho^2 = \frac{1}{4} + \sum_n\left[\tau_{1/2}^{(n)}(1)\right]^2 + 2\sum_n\left[\tau_{3/2}^{(n)}(1)\right]^2\ . \qquad (22)
$$

We consider now the SR obtained from the axial current (13), taking the m-th derivative relatively to w_{if} and making $w_i = w_f = w_{if} = 1$ one obtains from (20) the simple formula :

$$
\xi^{(\ell)}(1) = \ell!\ (-1)^\ell\sum_n\left[\tau_{\ell+1/2}^{(\ell)(n)}(1)\right]^2 \quad (\ell\geq 0)\ . \qquad (23)
$$

This expression is the generalization to the ℓ-th derivative of the relation obtained in ref. [1] for the slope (with the notation (14), formula (55)) :

$$\rho^2 = 3 \sum_n \left[\tau_{3/2}^{(n)}(1) \right]^2 \ . \tag{24}$$

Combining (21) and (23) one obtains the relation among the IW functions :

$$\frac{\ell}{2\ell+1} \sum_n \left[\tau_{\ell+1/2}^{(\ell)(n)}(1) \right]^2 - \frac{1}{4} \sum_n \left[\tau_{\ell-1/2}^{(\ell)(n)}(1) \right]^2 = \frac{1}{4} \sum_n \left[\tau_{\ell-1/2}^{(\ell-1)(n)}(1) \right]^2 \tag{25}$$

that gives $0 = 0$ for $\ell = 0$ and reduces to Uraltsev SR [6] for $\ell = 1$ (with the notation (14)) :

$$\sum_n \left[\tau_{3/2}^{(n)}(1) \right]^2 - \sum_n \left[\tau_{1/2}^{(n)}(1) \right]^2 = \frac{1}{4} \tag{26}$$

and generalizes it for all ℓ.

Replacing now $\sum\limits_n [\tau_{\ell+1/2}^{(\ell)(n)}(1)]^2$ from the expression (23) into the generalization of Bjorken SR (21) one obtains :

$$(-1)^\ell \ \xi^{(\ell)}(1) = \frac{1}{4} \ \frac{2\ell+1}{\ell} \ell! \left\{ \sum_n \left[\tau_{\ell-1/2}^{(\ell-1)(n)}(1) \right]^2 + \sum_n \left[\tau_{\ell-1/2}^{(\ell)(n)}(1) \right]^2 \right\} \ . \tag{27}$$

For $\ell = 1$ one obtains :

$$\rho^2 = -\xi'(1) = \frac{3}{4} \left\{ 1 + \sum_n \left[\tau_{1/2}^{(1)(n)}(1) \right]^2 \right\} \tag{28}$$

and therefore :

$$\rho^2 \geq \frac{3}{4} \tag{29}$$

i.e., one obtains the lower bound for the slope obtained from the combination of Bjorken and Uraltsev SR (22) and (26).

For $\ell = 2$ one gets :

$$\xi''(1) = \frac{1}{4} \ \frac{5}{2} \ 2! \left\{ \sum_n \left[\tau_{3/2}^{(1)(n)}(1) \right]^2 + \sum_n \left[\tau_{3/2}^{(2)(n)}(1) \right]^2 \right\} \tag{30}$$

and

$$\sigma^2 = \xi''(1) \geq \frac{1}{4} \ \frac{5}{2} \ 2! \ \sum_n \left[\tau_{3/2}^{(1)(n)}(1) \right]^2 \ . \tag{31}$$

Using equation (24), with the notation (14), we have therefore demonstrated on rigorous grounds the bound proposed, modulo a phenomenological hypothesis, in ref. [1] :

$$\sigma^2 \geq \frac{5}{4}\rho^2 \tag{32}$$

that implies, from (29), the absolute bound

$$\sigma^2 \geq \frac{15}{16} \ . \tag{33}$$

One can similarly bound any derivative $(-1)^\ell \xi^{(\ell)}(1)$ in terms of the precedent one $(-1)^{\ell-1}\xi^{(\ell-1)}(1)$:

$$(-1)^\ell \xi^{(\ell)}(1) \geq \frac{2\ell+1}{4}\left[(-1)^{\ell-1}\xi^{(\ell-1)}(1)\right] \tag{34}$$

implying the following absolute bound for the ℓ-th derivative of the Isgur-Wise function :

$$(-1)^\ell \xi^{(\ell)}(1) \geq \frac{(2\ell+1)!!}{2^{2\ell}} \tag{35}$$

that gives, in particular, (29) and (33) for the lower cases. Notice that this bound is satisfied by the phenomenological dipole formula

$$\xi_{Dipole}(w) = \left(\frac{2}{w+1}\right)^2 \tag{36}$$

obtained approximately within the Bakamjian-Thomas relativistic quark models [7] and from QCD Sum Rules at large recoil [8]. Indeed, one can easily show that the derivatives

$$(-1)^\ell \ \xi^{(\ell)}_{Dipole}(1) = \frac{(\ell+1)!}{2^\ell} \tag{37}$$

satisfy (34) and (35).

An interesting phenomenological remark is that a parametrization of the IW function of the form

$$\xi(w) = \left(\frac{2}{w+1}\right)^{2\rho^2} \tag{38}$$

with $\rho^2 \geq \frac{3}{4}$ satisfies also the inequalities (34) and (35). At the lower bound $\rho^2 = \frac{3}{4}$, (38) satisfies (34) and (35) with the equality.

The result (35), that shows that all derivatives at zero recoil are large, should have important phenomenological implications for the empirical fit needed for the extraction of $|V_{cb}|$ in $B \to D^* \ell \nu$ [9]. In the case of the slope and the curvature, the lower bounds (29) and (33) are complementary to the upper bounds obtained from unitarity constraints [10], [11].

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