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Electromagnetic η decays

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The Daphne ϕ–factory will hopefully represent a clean and copious source of η mesons. Indeed, some 2.8×10^8 $\phi \to \eta\gamma$ decays are expected per year with a clear signal consisting of an η recoiling against a monoenergetic photon. All the η decays detected so far [1] proceed through electromagnetic (e.m.) interactions. Some of them, like $\eta \to \gamma\gamma$, $\eta \to \pi^+\pi^-\gamma$, $\eta \to \pi^0\gamma\gamma$, contain external photon(s), whereas the remaining ones, $\eta \to \pi^+\pi^-\pi^0$, $\eta \to 3\pi^0$ are purely hadronic but violate G–parity (or isospin symmetry) and are therefore of e.m. origin too. We present here a survey on these e.m. η decays. A discussion on weak decays, with emphasis in C and/or CP violation can be found in Ref.[2]

Being the η mass, $m_\eta \simeq 550\ MeV$, just above the kaon mass, the variety of e.m. η decays represents, in principle, an excellent laboratory to test the Chiral Perturbation Theory (ChPT) approach, both in the anomalous and non–anomalous sectors. The main drawback is played, however, by the $\eta - \eta'$ mixing which requires the inclusion of the non Goldstone boson η_1 (a pure $SU(3)$ singlet) in the ChPT framework. Most of the theoretical predictions presented assume ChPT with resonance saturation of the needed counterterms. In some cases, however, definite ChPT predictions are not available and we present instead estimates based on phenomenological Vector Meson dominated models. We give in Section 1 a brief review on the treatment of the η–particle in ChPT. Section 2 is devoted to the anomalous (i.e. those proceeding through the Wess Zumino term) η decays, whereas the non–anomalous ones are presented in Section 3.

1 η–particle in Chiral Perturbation Theory

ChPT provides an accurate description of the strong and electroweak interactions of pseudoscalar mesons P at low energies [3]. In this sense and at lowest order, ChPT is essentially

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equivalent to Current Algebra (CA) [4]. At higher orders, loop effects appear giving rise to both finite corrections and divergencies. The former are known to improve the lowest order (or CA) predictions, while the latter require the introduction of counterterms and destroy the renormalizability of the theory.

The nonet of pseudoscalar mesons P is described in terms of the $SU(3)$ octet and singlet matrices

$$P_8 = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-2}{\sqrt{6}}\eta_8 \end{pmatrix}, \qquad P_1 = \frac{1}{\sqrt{3}}\eta_1 \mathrm{I}, \tag{1}$$

which appear in the ChPT lagrangian through the parametrization

$$\Sigma \equiv \Sigma_8 \Sigma_1 = \Sigma_1 \Sigma_8 = \exp\left(\frac{2i}{f}(P_8 + P_1)\right), \tag{2}$$

with $f = 132$ MeV, as can easily be deduced from the charged pion decay [3, 1]. The transformation properties of Σ under chiral $U(3)_L \times U(3)_R$ are given by $\Sigma \to U_L \Sigma U_R^\dagger$ and the physical η particle deviates from being a pure $SU(3)$–octet

$$\eta = \cos\theta \; \eta_8 - \sin\theta \; \eta_1 \simeq \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} - s\bar{s}), \quad \sin\theta \simeq -1/3, \tag{3}$$

as deduced from $\eta - \eta'$ phenomenology and its recent treatment in the ChPT context [5, 6, 7].

We assume that nonet symmetry gives a good description of the whole nonet except for the singlet mass which gets an extra term, correcting the $U_A(1)$ problem. Because of its heavy mass we neglect the singlet component in loops and use the octet component with the mass according to the Gell-Mann-Okubo formula. The mixing angle (3) is not negligible and can have rather large effects in η decays. This treatment also fits the mass matrix reasonably well if the loop corrections to the masses are taken into account [3].

The lowest order lagrangian of ChPT —order two in particle four–momenta or masses, $O(p^2)$, and ignoring the extra singlet mass term— is

$$\mathcal{L}_2 = \frac{f^2}{8} tr(D_\mu \Sigma D^\mu \Sigma^\dagger + \chi \Sigma^\dagger + \chi^\dagger \Sigma), \tag{4}$$

where the covariant derivative

$$D_\mu \equiv \partial_\mu + ieA_\mu[Q, \Sigma] \tag{5}$$

contains the photon field A_μ and the quark charge matrix Q [$Q = \mathrm{diag}(2/3, -1/3, -1/3)$] thus generating the e.m. couplings of charged P's through the commutator $[Q, \Sigma]$. The non–derivative terms in Eq. (4) with $\chi = \chi^\dagger = B\ \mathcal{M}$ contain the quark mass matrix $\mathcal{M}$ $[\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)]$ and lead to

$$\frac{1}{2}B = \frac{m_K^2}{m_u + m_s} = \frac{m_\pi^2}{m_u + m_d} = \frac{\sqrt{3}m_{\pi^0\eta}^2}{m_u - m_d} = \frac{\Delta m_K^2}{m_d - m_u}, \tag{6}$$

where $m_{\pi^0\eta}^2$ is the mass matrix element describing π^0–η mixing and

$$\Delta m_K^2 \equiv (m_{K^0}^2 - m_{K^+}^2)_{QCD} = 6.0 \times 10^{-3}\ \mathrm{GeV}^2 \tag{7}$$

is the QCD (or tadpole or non–photonic) $K^0 - K^+$ squared mass difference. Its numerical value is an average between the result [3]

$$\Delta m_K^2 = (m_{K^0}^2 - m_{K^+}^2 - m_{\pi^0}^2 + m_{\pi^+}^2) = 5.3 \times 10^{-3}\ \mathrm{GeV}^2,$$

following from Dashen's theorem [4], and independent estimates [8] (including improved versions of Dashen's theorem [9]) leading to Δm_K^2 in the range $(6.5 - 7.0) \times 10^{-3}$ GeV^2.

The next order lagrangian, $O(p^4)$, contains a series of ten counterterms identified and studied by Gasser and Leutwyler [3],

$$\begin{aligned}
\mathcal{L}_4 &= L_1(trD_\mu\Sigma D^\mu\Sigma^\dagger)^2 + L_2(trD_\mu\Sigma D_\nu\Sigma^\dagger)^2 + L_3tr(D_\mu\Sigma D^\mu\Sigma^\dagger D_\nu\Sigma D^\nu\Sigma^\dagger)\\
&+L_4tr(D_\mu\Sigma D^\mu\Sigma^\dagger)tr(\chi\Sigma^\dagger + \Sigma\chi^\dagger) + L_5tr\left[(D_\mu\Sigma D^\mu\Sigma^\dagger)(\chi\Sigma^\dagger + \Sigma\chi^\dagger)\right]\\
&+L_6\left[tr(\chi\Sigma^\dagger + \Sigma\chi^\dagger)\right]^2 + L_7\left[tr(\chi\Sigma^\dagger - \Sigma\chi^\dagger)\right]^2 + L_8tr(\chi\Sigma^\dagger\chi\Sigma^\dagger + \Sigma\chi^\dagger\Sigma\chi^\dagger)\\
&-ieL_9F_{\mu\nu}tr(QD^\mu\Sigma D^\nu\Sigma^\dagger + QD^\mu\Sigma^\dagger D^\nu\Sigma) + e^2L_{10}F_{\mu\nu}F^{\mu\nu}tr(\Sigma^\dagger Q\Sigma Q)
\end{aligned} \tag{8}$$

and the Wess–Zumino (WZ) term [10], responsible for the anomalous processes, whose action (ignoring external fields other than photons) is [6, 11]

$$\begin{aligned}
S_{WZ} = &\frac{-iN_c}{240\pi^2}\int d\Sigma^{ijklm}tr(\Sigma\partial_i\Sigma^\dagger\Sigma\partial_j\Sigma^\dagger\Sigma\partial_k\Sigma^\dagger\Sigma\partial_l\Sigma^\dagger\Sigma\partial_m\Sigma^\dagger)\\
&-\frac{eN_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}A_\mu tr\left[Q(\partial_\nu\Sigma\partial_\alpha\Sigma^\dagger\partial_\beta\Sigma\Sigma^\dagger - \partial_\nu\Sigma^\dagger\partial_\alpha\Sigma\partial_\beta\Sigma^\dagger\Sigma)\right]\\
&-\frac{iN_ce^2}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)A_\alpha tr\Big(Q^2\partial_\beta\Sigma\Sigma^\dagger + Q^2\Sigma^\dagger\partial_\beta\Sigma\\
&+\tfrac{1}{2}Q\Sigma Q\Sigma^\dagger\partial_\beta\Sigma\Sigma^\dagger - \tfrac{1}{2}Q\Sigma^\dagger Q\Sigma\partial_\beta\Sigma^\dagger\Sigma\Big).
\end{aligned} \tag{9}$$

In the above equations, $F_{\mu\nu}$ is the e.m. tensor and N_c the number of colors.

Accurate numerical predictions need the knowledge of the counterterm values. One can rely on experimental data to fix the finite part of their coefficients, as was originally done by Gasser and Leutwyler [3] or, alternatively, saturate them by the contributions of low–lying resonances (vector mesons usually providing the largest contributions) at $\mu \simeq M_\rho$ [12]. Vector mesons can be introduced using the hidden symmetry formulation of Bando et al.[13] assuming ideal mixing and nonet symmetry, $\rho_\mu = \rho_\mu^a \lambda^a/\sqrt{2} + \rho_\mu^1/\sqrt{3}$. The relevant lagrangian with vector mesons at $O(p^6)$ is

$$\begin{aligned} \mathcal{L}_\rho^\epsilon = \epsilon^{\mu\nu\alpha\beta}\Big(& tr[a_1 a_\mu a_\nu a_\alpha \sigma^\dagger D_\beta \sigma + ig a_2 \sigma^\dagger \rho_{\mu\nu} \sigma \{a_\alpha, \sigma^\dagger D_\beta \sigma\} \\ & + iea_3 F_{\mu\nu}(\xi^\dagger Q \xi + \xi Q \xi^\dagger)\{a_\alpha, \sigma^\dagger D_\beta \sigma\}]\Big) \end{aligned} \tag{10}$$

where σ is a 3×3 unitary matrix parametrized in terms of a set of unphysical scalar (compensating) fields that, when gauged away, generates a mass term for the vector mesons. The Lagrangian in Eq.(10) also contains the following hermitian operators

$$\begin{aligned} & i\sigma^\dagger D_\mu \sigma = i\sigma^\dagger \partial_\mu \sigma - g\sigma^\dagger \rho_\mu \sigma + v_\mu, && v_\mu = \frac{1}{2i}[\xi(\partial_\mu + ieQA_\mu)\xi^\dagger + \xi^\dagger(\partial_\mu + ieQA_\mu)\xi], \\ & \rho_{\mu\nu} = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu + ig[\rho_\mu, \rho_\nu], && a_\mu = \frac{1}{2i}[\xi(\partial_\mu + ieQA_\mu)\xi^\dagger - \xi^\dagger(\partial_\mu + ieQA_\mu)\xi], \end{aligned} \tag{11}$$

where $\xi = \exp[(i/f)P_8]$. In order to identify the contribution of the vector mesons to the finite part of the renormalized amplitudes one has to integrate out the ρ nonet fields assuming their masses to be much larger than all the momenta. Full expressions can be found in Ref. [6].

2 Anomalous η decays

2.1 $\eta \to \gamma\gamma$ and $\gamma\gamma^*$

Theoretically, $P\gamma\gamma$ transitions involving on–mass–shell photons, $k^2 = k^{*2} = 0$, contain valuable information on the mixing (quark–content) of the η, η' mesons. The lowest order amplitudes for $\pi^0, \eta_{8,1} \to \gamma\gamma$ in ChPT are

$$A(P \to \gamma\gamma^*) = \frac{-\sqrt{2} C_P\, \alpha}{\pi f_P} \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu k_\nu \epsilon_\alpha^* k_\beta^* \tag{12}$$

where $C_{\pi^0} = 1$, $C_{\eta_8} = 1/\sqrt{3}$ and $C_{\eta_1} = 2\sqrt{2}/\sqrt{3}$ and ϵ and ϵ^* are the polarization vectors of the photons with momenta k and k^*, respectively. The experimental widths are [1]

$$\begin{aligned} \Gamma(\pi^0 \to \gamma\gamma) &= 7.84 \pm 0.56\ eV \\ \Gamma(\eta \to \gamma\gamma) &= 0.46 \pm 0.04\ keV \\ \Gamma(\eta' \to \gamma\gamma) &= 4.30 \pm 0.53\ keV, \end{aligned} \tag{13}$$

where we have to remember that η and η' consist of a mixture of the η_8 and η_1, with a mixing angle $\theta \simeq -19.5°$, as quoted in Eq.(3). Fixing f_{η_8} through ChPT to one loop and using the η' decay width, one gets $f_{\eta_8} \simeq 1.3 f_\pi$ and $f_{\eta_1} \simeq 1.1 f_\pi$ [6].

For $k^{*2} \neq 0$ one defines a slope parameter of the $P\gamma\gamma^*$ transition form factor by

$$b_P \equiv \frac{1}{A(P\to\gamma\gamma)} \frac{d}{dk^{*2}} A(P\to\gamma\gamma^*)\Big|_{k^{*2}=0}, \tag{14}$$

related to the P–structure or dimensions. There is a single measurement using $\eta \to \gamma\mu^+\mu^-$ [14] leading to $b_\eta = (1.9 \pm 0.4)\ GeV^{-2}$. In addition, there is also a set of measurements of the process $\gamma\gamma^* \to P$ in electron–positron collisions for rather large values of $-k^{*2}$. When extrapolated to small values using a pole approximation, they globally average to $b_\eta = 1.69 \pm 0.18\ GeV^{-2}$[15].

The next to leading order correction in ChPT was computed for on–shell–photons by Donoghue et al. [5] and in Ref.[16] and the expected cancellation of the divergencies was explicitly found. However, when one allows for (at least one) off–shell final photon(s), $k^{*2} \neq 0$, the cancellation no longer occurs. The amplitude reads [16]

$$\begin{aligned} A(P\to\gamma\gamma^*) = &-\alpha \frac{\sqrt{2}C_P}{f_P} \epsilon^{\mu\nu\alpha\beta}\epsilon_\mu k_\nu \epsilon^*_\alpha k^*_\beta \\ &\times \left(1 + \frac{1}{16\pi^2 f^2}\left\{\frac{2}{3}\lambda k^{*2} - \frac{1}{3}k^{*2}(\ln\frac{m_K^2}{\mu^2} + \ln\frac{m_\pi^2}{\mu^2}) \right.\right. \\ &\left.\left. + \frac{10}{9}k^{*2} + \frac{4}{3}[F(k^{*2}, m_\pi^2) + F(k^{*2}, m_K^2)]\right\}\right), \end{aligned} \tag{15}$$

where the first factor corresponds to the tree level contribution in terms of physical f_P, $\lambda \equiv \frac{1}{\epsilon} + 1 + \ln 4\pi - \gamma$ (γ is the Euler constant) contains the divergent terms, and

$$F(m^2, x) \equiv m^2\left(1 - \frac{x}{4}\right)\sqrt{\frac{x-4}{x}} \ln\frac{\sqrt{x}+\sqrt{x-4}}{-\sqrt{x}+\sqrt{x-4}} - 2m^2, \qquad x \equiv \frac{k^{*2}}{m^2}. \tag{16}$$

For small k^{*2}, $F(m^2, k^{*2}) \simeq -\frac{2}{3}k^{*2}$ and, in the case of real photons, $k^2 = k^{*2} = 0$, all the effects of loops and nonanomalous $\mathcal{L}_4$ terms reduce to change f into f_P for $P = \eta_8, \eta_1$. In order to give a finite result for virtual photons one has to include the counterterms of $O(p^6)$, which are of the type

$$C\epsilon_{\mu\nu\alpha\beta}\partial^\lambda F_{\lambda\nu}F_{\alpha\beta}\left(tr[Q^2\Sigma\partial_\mu\Sigma^\dagger - Q^2\Sigma^\dagger\partial_\mu\Sigma] + tr[Q\Sigma Q\partial_\mu\Sigma^\dagger - Q\Sigma^\dagger Q\partial_\mu\Sigma]\right). \tag{17}$$

The analysis was done by Bijnens et al. [16] assuming saturation by vector mesons in a hidden symmetry formulation [13]. They found the slope parameter

$$b_\eta = -\frac{1}{16\pi f^2}\frac{1}{3}(2 + \ln\frac{m_\pi^2}{M_\rho^2}\frac{m_K^2}{M_\rho^2}) + \frac{1}{M_\rho^2} \simeq (0.26 + 1.70)\ GeV^{-2} \simeq 1.96\ GeV^{-2} \tag{18}$$

where the first term corresponds to the chiral loops and the second, giving the major contribution, to the counterterms. The agreement with the experimental values above is rather good.

2.2 $\eta \to \mu^+\mu^-$, e^+e^-

New experimental data on the rare $\eta \to \mu^+\mu^-$ decay have been obtained very recently. The η tagging facility at Saturne reports [17] a branching ratio $BR(\eta \to \mu^+\mu^-) \equiv \Gamma(\eta \to \mu^+\mu^-)/\Gamma(\eta \to all) = (5.6^{+0.6}_{-0.7} \pm 0.5) \times 10^{-6}$, to be compared to the old result [18] $BR(\eta \to \mu^+\mu^-) = (6.5 \pm 2.1) \times 10^{-6}$. Normalizing the new Saturne's measurement to $\eta \to \gamma\gamma$ one gets

$$B(\eta \to \mu^+\mu^-) \equiv \frac{\Gamma(\eta \to \mu^+\mu^-)}{\Gamma(\eta \to \gamma\gamma)} = (1.4 \pm 0.2) \times 10^{-5} \ , \tag{19}$$

where the branching ratio [1] $BR(\eta \to \gamma\gamma) = 0.389 \pm 0.005$ has been used.

The "reduced" ratio (19) can be expressed in terms of a dimensionless "reduced" amplitude $R(\eta \to \mu^+\mu^-) \equiv R$, normalized to the intermediate $\eta \to \gamma\gamma$ amplitude, leading to

$$B(\eta \to \mu^+\mu^-) = 2\beta \left(\frac{\alpha}{\pi}\frac{m_l}{m_P}\right)^2 |R(\eta \to \mu^+\mu^-)|^2, \tag{20}$$

where $\beta = \sqrt{1 - 4m_\mu^2/m_\eta^2}$. The on-shell $\gamma\gamma$ intermediate state generates the model–independent imaginary part of R

$$\mathrm{Im}\, R(\eta \to \mu^+\mu^-) = \frac{\pi}{2\beta} \ln \frac{1-\beta}{1+\beta} = -5.47 \ . \tag{21}$$

The unitary bound on B, $B \geq B^{unit}$, is then obtained by setting $\mathrm{Re}\, R = 0$ in Eq.(20). It takes the value

$$B^{unit}(\eta \to \mu^+\mu^-) = 1.11 \times 10^{-5}. \tag{22}$$

In terms of B^{unit}, the Saturne's result (19) reads $B(\eta \to \mu^+\mu^-)/B^{unit} = 1 + (ReR/ImR)^2 = 1.3 \pm 0.2$ and can be used to extract ReR from experiment

$$\mathrm{Re}\, R(\eta \to \mu^+\mu^-) = \pm \left(3.0^{+0.9}_{-1.2}\right) . \tag{23}$$

It has been shown that one can fix the sign for ReR from theoretical considerations [19]. The imaginary part of R is finite, model-independent and dominant; by contrast, its real part contains an *a priori* divergent $\gamma\gamma$ loop, depends on the hadronic physics governing the $P \to \gamma^*\gamma^*$ transition (with off-shell photons) and amounts only to a fraction of ImR.

Calculations of ReR have been performed by many authors in, essentially, two different contexts. One consists in using traditional Vector Meson Dominance (VMD) ideas, thus introducing the corresponding VMD form factor to regularize the photon-photon loop. Hadronic couplings ($VV\eta$) cancel precisely in the "reduced" amplitude R, which turns out to depend essentially only on the vector meson mass M_V in the form factor. Alternatively, one can rely on (constituent) quark model ideas [20] to regularize the $\eta \to \gamma^*\gamma^*$ vertex, thus obtaining a finite and reasonable value for ReR. A recent paper by Margolis et al. [21] confirms the validity and the model-independency of this approach. In both contexts, one obtains rather stable results which are in reasonable agreement with the above data.

Assuming the obvious dominance of the two photon contribution, the reduced amplitude $R(q^2) = R(P \to l^+l^-)$ can be written as [19]

$$R(q^2) = \frac{2i}{\pi^2 q^2} \int d^4k \frac{q^2k^2 - (q \cdot k)^2}{k^2(q-k)^2[(p-k)^2 - m_l^2]} F(k^2, k^{*2}), \tag{24}$$

where $q^2 = m_\eta^2, p^2 = m_\mu^2$ and $k^* = q - k$, and F is a generic and model-dependent form factor, with $F(0,0) = 1$ for on-shell photons. The simplest and more transparent way to fix F is to invoke VMD, which essentially implies neglecting direct $P\gamma\gamma$ and (single V) $VP\gamma$ vertices, thus assuming the full dominance of the (double V) chain $P \to VV \to \gamma\gamma$. The form factor in this case is

$$F = F_{VV} = \frac{M_V^2}{M_V^2 - k^2} \frac{M_V^2}{M_V^2 - k'^2}. \tag{25}$$

Theoretical predictions for the real part of the amplitude in this naive and conventional VMD model have been worked out in Refs. [23, 19]. One gets $\mathrm{Re}\, R(\eta \to \mu^+\mu^-) = -1.3^{+0.7}_{-0.5}$ in the $SU(3)$-symmetric limit $M_V = M_{\rho,\omega}$, corresponding to

$$B(\eta \to \mu^+\mu^-) = (1.18^{+0.08}_{-0.06}) \times 10^{-5}, \tag{26}$$

in reasonable agreement with the corresponding experimental value (19). For completeness, we also quote the corresponding results for the $\eta \to e^+e^-$ decay amplitude: $\mathrm{Re}\, R(\eta \to e^+e^-) = 31.3 \pm 2.0$, $\mathrm{Im}\, R(\eta \to e^+e^-) = -21.9$ and $B(\eta \to e^+e^-) = (3.04 \pm 0.26) B^{unit} = (1.37 \pm 0.12) \times 10^{-8}$.

The above VMD calculations were essentially performed many years ago and appear now as old-fashioned and lacking of solid theoretical support. However, there are some more modern approaches, where the interactions among pseudoscalars, vector mesons and photons are dictated by well-defined and QCD-rooted lagrangians. In these contexts, the octet containing the lightest pseudoscalar mesons plays the role of the set of Goldstone bosons originated through the spontaneous symmetry breaking of the QCD lagrangian for vanishing u, d and s quark masses. We mention three types of models: the first refers to ChPT [24], and the other two refer to more recent attempts to include vector-mesons in these chiral lagrangians, particularly, the "massive Yang-Mills approach" [25] and the "hidden symmetry scheme" [13]. It has been argued that in the context of these three models the VMD form factor in (25) and the corresponding predictions are fully justified.

Let us only consider the previous results from the point of view of ChPT. Recently, Savage et al. [26] have discussed the $P \to l^+l^-$ decays using preliminary data from Saturne to fix the required local counterterms and then predicting the π^0, $\eta \to e^+e^-$ branching ratios. An alternative way to proceed consists in assuming that the relevant, finite part of the ChPT counterterms are saturated (dominated) by the contributions of meson resonances. In this sense, the previous VMD results on $P \to l^+l^-$ decays can also be considered as rather safe calculations in the context of ChPT with resonance saturation. To further illustrate this point the "reduced" amplitude R in terms of the local counterterms proposed in Ref. [26] within the same renormalization scheme reads [19]

$$
\begin{aligned}
\mathrm{Re}\, R(q^2 = m_P^2) = & -\frac{\chi_1(\Lambda) + \chi_2(\Lambda)}{4} - \frac{5}{2} + 3 \ln \frac{m_l}{\Lambda} \\
& + \frac{1}{4\beta} \ln^2 \frac{1-\beta}{1+\beta} + \frac{\pi^2}{12\beta} - \frac{1}{\beta} Li_2 \left(\frac{\beta - 1}{\beta + 1} \right),
\end{aligned}
\tag{27}
$$

where Λ is the subtraction point. The Saturne experiment should require a counterterm (for $\Lambda = M_\rho = 0.77$ GeV) given by

$$
\chi_1(M_\rho) + \chi_2(M_\rho) = \left\{ \begin{array}{c} -7^{+4}_{-5} \\ -31^{+5}_{-4} \end{array} \right. . \tag{28}
$$

In turn, the less precise $\pi^0 \to e^+e^-$ available experimental data [27] translate into

$$
\chi_1(M_\rho) + \chi_2(M_\rho) = \left\{ \begin{array}{c} -22^{+25}_{-16} \\ +81^{+16}_{-25} \end{array} \right. . \tag{29}
$$

The first (second) value in Eqs(28,29) corresponds to fixing the sign ambiguities for ReR according (contrary) to the VMD amplitudes. Notice that the first values are consistent with the existence of a unique counterterm, while the second ones are not and, consequently, have to be discarded. This further confirms that VMD amplitudes represent a good description for the $P \to l^+l^-$ processes. Thanks to the resonance saturation hypothesis, we can go one step further and predict the value of the finite part of these counterterms. This amounts to choose $M_{\rho,\omega} = 0.77$ GeV as both the subtraction point Λ and the mass M_V appearing in our F_{VV} form factor (25). The previous result for $\eta \to \mu^+\mu^-$ (26) can be taken also as the prediction from ChPT with resonance saturation. In this context, it leads to $\chi_1(M_\rho) + \chi_2(M_\rho) \simeq -14$, close to the experimental values displayed in the first row of (28,29). Notice that the VMD saturation hypothesis does not strictly lead to a constant counterterm, as explicitly required in Ref.[26], but to a function smoothly depending on M_V, m_P and m_l.

Both the experimental result [17] and reasonable models indicate $B(\eta \to \mu^+\mu^-) \simeq 10^{-5}$ well in the range of Daphne possibilities. The corresponding predictions for $B(\eta \to e^+e^-) \sim 10^{-8}$ indicate that this channel can hardly be detected at Daphne.

2.3 $\eta \to \pi^+\pi^-\gamma$

The lowest order amplitudes in ChPT for $\eta \to \pi^+\pi^-\gamma$ can be easily calculated from the WZ action (9). One obtains

$$
A = \frac{-eC'_P}{\sqrt{2}\pi^2 f^3} \epsilon^{\mu\nu\alpha\beta} A_\mu p_{1_\nu} p_{2_\alpha} p_{0_\beta}, \tag{30}
$$

where $C'_{\eta_8} = 1/\sqrt{3}$ and $C'_{\eta_1} = \sqrt{2}/\sqrt{3}$. The reliability of this approach can be tested observing that the related $\gamma\pi\pi\pi$ amplitude can be expressed in terms of the coupling constant $F^{3\pi}$ ($C'_\pi = 1$ in Eq.(30)), which at this lowest order is related to the $\pi^0\gamma\gamma$ coupling, F^π (see Eq.(12), $C_{\pi^0} = 1$), by

$$F^{3\pi} = \frac{e}{\sqrt{2}\pi^2 f^3} = \frac{2F^\pi}{ef^2}. \tag{31}$$

From the first equality one obtains $F^{3\pi} = 9.4\ GeV^{-3}$, not far from the experimental value [28]: $F^{3\pi}_{exp} = 12.9 \pm 0.9 \pm 0.5\ GeV^{-3}$.

The use of the tree level amplitude in Eq.(30) leads to $\Gamma(\eta \to \pi^+\pi^-\gamma) = 35\ eV$, somewhat lower than the experimental value, $\Gamma_{exp}(\eta \to \pi^+\pi^-\gamma) = 58 \pm 6\ eV$ [1], and to a photon energy spectrum less softer than the observed one [29]. Moreover, in the calculation we have used $f_{\eta_8} = f_{\eta_1} = f_\pi$, as predicted in ChPT at this level. Next to leading order corrections break this $U(3)$ symmetry and have been estimated to be $f_{\eta_8}/f_\pi = 1.3$, $f_{\eta_1}/f_\pi = 1.1$ [6] as discussed in subsection 2.1 where a good description of $\pi^0, \eta, \eta' \to \gamma\gamma$ was achieved. The introduction of the $SU(3)$ breaking in the decay constants requires one loop contributions.

The one loop diagrams contributing to the processes $\eta, \eta' \to \pi^+\pi^-\gamma$ have been calculated using dimensional regularization in Ref.[30]. The results can be conveniently written in terms of divergent parts, containing $\lambda = 1/\epsilon + 1 + \ln 4\pi - \gamma$, and the function F in Eq.(16). The amplitudes for the processes involving the η particle turn out to be

$$\begin{aligned} A(\eta_8 &\to \pi^+\pi^-\gamma) = \frac{-e}{\sqrt{6}\pi^2 f_\pi^2 f_{\eta_8}} \epsilon^{\mu\nu\alpha\beta} A_\mu p_{1_\nu} p_{2_\alpha} p_{0_\beta} \\ &\{1 + \frac{1}{16\pi^2 f^2}[\lambda(4m_\pi^2 - 4m_K^2 + p_{12}^2 + k^2) - (4m_\pi^2 + \frac{1}{3}p_{12}^2)\ln\frac{m_\pi^2}{\mu^2} - k^2 \ln\frac{m_K^2}{\mu^2} \\ &+ (4m_K^2 - \frac{2}{3}p_{12}^2)\ln\frac{m_K^2}{\mu^2} + \frac{5}{3}p_{12}^2 + \frac{5}{3}k^2 + \frac{4}{3}F(m_\pi^2, p_{12}^2) + \frac{8}{3}F(m_K^2, p_{12}^2)] \\ &+ 4F(m_K^2, k^2)\}, \end{aligned} \tag{32}$$

$$\begin{aligned} A(\eta_1 &\to \pi^+\pi^-\gamma) = \frac{-e}{\sqrt{3}\pi^2 f_\pi^2 f_{\eta_1}} \epsilon^{\mu\nu\alpha\beta} A_\mu p_{1_\nu} p_{2_\alpha} p_{0_\beta} \\ &\{1 + \frac{1}{16\pi^2 f^2}[\lambda(4m_\pi^2 + 2m_K^2 + \frac{1}{2}p_{12}^2) - (4m_\pi^2 + \frac{1}{3}p_{12}^2)\ln\frac{m_\pi^2}{\mu^2} \\ &- (2m_K^2 + \frac{1}{6}p_{12}^2)\ln\frac{m_K^2}{\mu^2} + \frac{5}{6}p_{12}^2 + \frac{4}{3}F(m_\pi^2, p_{12}^2) + \frac{2}{3}F(m_K^2, p_{12}^2)]\}. \end{aligned}$$

valid for both real ($k^2 = 0$) and virtual ($k^2 > 0$) photons.

The divergences appearing in the loop calculation of the amplitudes are eliminated redefining the coefficients of the $O(p^6)$ lagrangian. The terms contributing to the $\eta_8\pi^+\pi^-\gamma$

vertex can be easily read from the general expression obtained in Ref. [31, 32]. The relevant terms are

$$
\begin{aligned}
\mathcal{L}_6^{\epsilon} &= C_1\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}tr[Q(\partial_\alpha\Sigma^\dagger\Sigma+\partial_\alpha\Sigma\Sigma^\dagger)]tr(\chi\partial_\beta\Sigma^\dagger+\chi^\dagger\partial_\beta\Sigma)\\
&+C_2\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}tr[(\Sigma Q\partial_\alpha\Sigma^\dagger+Q\Sigma\partial_\alpha\Sigma^\dagger-\partial_\alpha\Sigma Q\Sigma^\dagger+\Sigma\partial_\alpha\Sigma^\dagger Q)\\
&(\partial_\lambda\partial_\beta\Sigma\partial^\lambda\Sigma^\dagger-\partial_\lambda\Sigma\partial_\beta\partial^\lambda\Sigma^\dagger+\partial_\beta\Sigma\partial^2\Sigma^\dagger-\partial^2\Sigma\partial_\beta\Sigma^\dagger)]\\
&+C_3\epsilon^{\mu\nu\alpha\beta}\partial^\lambda F_{\lambda\mu}tr[Q\Sigma\partial_\nu\Sigma^\dagger\Sigma\partial_\alpha\Sigma^\dagger\Sigma\partial_\beta\Sigma^\dagger],
\end{aligned}
\tag{33}
$$

where the first and the second terms take care of the divergences proportional to the pseudoscalar masses and invariant momenta, respectively. The terms of the lagrangian in Eq. (33) are not enough to eliminate the divergences for $\eta_1\to\pi^+\pi^-\gamma$. This is because the calculation in Ref. [31, 32] was performed for $SU(3)_L\times SU(3)_R$ broken to $SU(3)_V$. When extending the symmetry to include the singlet, new terms such as

$$
\epsilon^{\mu\nu\alpha\beta}A_{\mu\nu}tr[(\chi\Sigma^\dagger+\Sigma\chi^\dagger)(\Sigma Q\Sigma^\dagger-Q)\Sigma\partial_\alpha\Sigma^\dagger]tr(\Sigma\partial_\beta\Sigma^\dagger),
\tag{34}
$$

appear contributing only to this singlet part.

Numerical predictions cannot be automatically made at this point because the finite part of the coefficients in $\mathcal{L}_6^{\epsilon}$ are free constants. Here we will assume again that at $\mu\sim M_\rho$ resonance saturation works and we fix the value of the relevant coefficients by the vector meson contributions à la Bando [13]. Then, the complete expressions for the amplitudes of the processes we are studying are

$$
\begin{aligned}
A(\eta_8\to\pi^+\pi^-\gamma) &= \frac{-e}{\sqrt{6}\pi^2f_\pi^2f_{\eta_8}}\epsilon^{\mu\nu\alpha\beta}A_\mu p_{1_\nu}p_{2_\alpha}p_{0_\beta}\{1+C^{\eta_8}_{loops}+\frac{3}{2m_\rho^2}p_{12}^2\},\\
A(\eta_1\to\pi^+\pi^-\gamma) &= \frac{-e}{\sqrt{3}\pi^2f_\pi^2f_{\eta_1}}\epsilon^{\mu\nu\alpha\beta}A_\mu p_{1_\nu}p_{2_\alpha}p_{0_\beta}\{1+C^{\eta_1}_{loops}+\frac{3}{2m_\rho^2}p_{12}^2\},
\end{aligned}
\tag{35}
$$

where the last pieces come from resonance saturated counterterms and $C^{\eta_8}_{loops}$ and $C^{\eta_1}_{loops}$ are the finite part of the loop corrections quoted in Eqs.(32). The k^2 term in these Eqs. follows from the (phenomenologically reasonable) assumption that there is no direct (contact) coupling between vector–mesons and PPP–states and that the $P\gamma\gamma$ and $PV\gamma$ vertices proceed exclusively through PVV.

The predicted decay width for $\eta\to\pi^+\pi^-\gamma$ at $O(p^6)$ in our ChPT approach turns out to be $\Gamma(\eta\to\pi^+\pi^-\gamma)=47\ eV$, which is in good agreement with the experimental value $\Gamma_{exp}(\eta\to\pi^+\pi^-\gamma)=58\pm6\ eV$. The total correction is now dominated by the contribution of the $O(p^6)$ terms of the lagrangian. Setting their coefficients to be zero, the prediction for the decay width would be $\Gamma_{loops}(\eta\to\pi^+\pi^-\gamma)=27\pm3\ eV$, lower than the tree level prediction and very different from the experimental value. Daphne could provide accurate values of this branching ratio as well as for the corresponding photonic spectrum (details on the later can be found in Ref.[7]).

3 Non–anomalous η decays

3.1 $\eta \to 3\pi$

The $\eta \to 3\pi$ transition violates G–parity. This makes the amplitude proportional to $m_d - m_u$, and therefore vanishes in the limit of good isospin symmetry. Using the ChPT lagrangian (4) one finds the well known tree level amplitude (CA result)

$$A(\eta_8 \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)) = -\frac{B(m_d - m_u)}{3\sqrt{3}f^2}\left[1 + 3\frac{s - s_0}{M_\eta^2 - m_\pi^2}\right], \tag{36}$$

where $s = (p_+ + p_-)^2$, $s_0 = (s + t + u)/3 = m_\pi^2 + M_\eta^2/3$ and $B(m_d - m_u)/2 = \Delta m_K^2$ is the QCD part of the kaonic squared mass difference (see eqs.(6,7)). Ten years ago, Gasser and Leutwyler [33] used the Dashen's theorem [4] to fix

$$\Delta m_K^2|_{DT} = m_{K^0}^2 - m_{K^+}^2 - m_{\pi^0}^2 + m_{\pi^2}^2 = 5.3 \times 10^{-3}\ GeV^2, \tag{37}$$

which leads to the (lowest order) rate through (36) (where the η appears as a pure $SU(3)$ octet).

$$\Gamma_{DT}^{tree}(\eta_8 \to \pi^+\pi^-\pi^0) = 66\ eV, \tag{38}$$

well below the experimental result [1]

$$\Gamma^{exp}(\eta \to \pi^+\pi^-\pi^0) = 281 \pm 27\ eV. \tag{39}$$

This prompted the same authors to introduce the one loop corrections plus the corresponding counterterms. They found an enhancement by a factor 2.4 respect to the lowest order prediction [33]

$$\Gamma_{DT}^{1\ loop}(\eta_8 \to \pi^+\pi^-\pi^0) = 160 \pm 50\ eV, \tag{40}$$

where the error accounts for higher order uncertainties. This result was in qualitative agreement with the analysis of Roiesnel an Truong [34] showing that the unitarity corrections generated by $\pi\pi$ final state interactions are large.

The above results can be reanalysed, ten years later, with the inclusion of two new ingredients. First, it is now rather clear that Dashen's theorem should be corrected [9] resulting in $\Delta m_K^2 \simeq 1.2\Delta m_K^2|_{DT}$, which represents an enhancement of $\sim 40\%$ to the previous values. (Conservatively, we fix $\Delta m_K^2 = 1.13\Delta m_K^2|_{DT}$ for the following estimates, as discussed in the comment after Eq.(7)). Independently, when one considers $\eta - \eta'$ mixing as in Eq.(3), the tree level amplitude increases by a factor $\sqrt{2}$ and the one loop correction is amplified by a similar factor. When both factors are naïvely taken into account, the Gasser and Leutwyler one loop result turns out to be

$$\Gamma^{1\ loop}(\eta \to \pi^+\pi^-\pi^0) \simeq 410\ eV, \tag{41}$$

somewhat above the experimental result (39).

On the other hand, some discrepancies with ChPT values seem to appear when expanding the decay amplitude around the center of the Dalitz plot. Moreover, this not completely satisfactory situation extrapolates to the η decay into neutral pions. In terms of the branching ratio

$$r = \frac{\Gamma(\eta \to 3\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)} \tag{42}$$

one gets $r^{tree} = 1.51$, $r^{1\ loop} = 1.43$ [33], whereas the experimental value is $r^{exp} = 1.35 \pm 0.05$ [1]. The above discussion indicates that our understanding of the $\eta \to 3\pi$ processes claims for new and more accurate data, as well as for further theoretical considerations.

3.2 $\eta \to \pi^0\gamma\gamma$

At lowest order in ChPT, one predicts a vanishing $\eta \to \pi^0\gamma\gamma$ decay width contrasting with the measured value [1]

$$\Gamma(\eta \to \pi^0\gamma\gamma) = (7.1 \pm 1.4) \times 10^{-4}\ \Gamma(\eta \to \text{all}) = 0.84 \pm 0.18 \text{ eV}. \tag{43}$$

Our main purpose is to show that the corresponding ChPT prediction is not far from this value thanks to a rather complicated interplay among loop corrections and counterterms.

The amplitude for the decay $\eta(P) \to \pi^0(p)\gamma(q_1)\gamma(q_2)$ can be written in the following form

$$M = \sum_i (a_i^t + a_i^P)A + \sum_i (b_i^t + b_i^P)B, \tag{44}$$

where A and B are the two kinematically possible amplitudes

$$\begin{aligned} A &= (\epsilon_1 \cdot \epsilon_2)(q_1 \cdot q_2) - (\epsilon_1 \cdot q_2)(\epsilon_2 \cdot q_1) \\ B &= -(\epsilon_1 \cdot \epsilon_2)(P \cdot q_1)(P \cdot q_2) - (\epsilon_1 \cdot P)(\epsilon_2 \cdot P)(q_1 \cdot q_2) \\ &\quad +(\epsilon_1 \cdot q_2)(\epsilon_2 \cdot P)(P \cdot q_1) + (\epsilon_1 \cdot P)(\epsilon_2 \cdot q_1)(P \cdot q_2). \end{aligned} \tag{45}$$

Their coefficients a_i^t, b_i^t refer to the tree–level and counter–term contributions, whereas a_i^P, b_i^P, with $P = \pi$ or K, refer to the contributions of pion or kaon loops. The sum extends from the lowest order terms, $i = 2$, to higher ones $i = 4, 6, ...$ We also define $s = (q_1 + q_2)^2 = 2q_1 \cdot q_2$, $t = (P - q_2)^2$, $u = (P - q_1)^2$ and $s + t + u = m_\eta^2 + m_\pi^2$. With these conventions fixed we now proceed to discuss the different contributions to the amplitude M.

As previously stated, there is no lowest order $O(p^2)$ contribution from $\mathcal{L}_2$ to the $\eta \to \pi^0\gamma\gamma$ amplitude, i.e. $a_2^t = b_2^t = 0$, as can be immediately seen from Eqs.(4) and (5) with $[Q, \Sigma] = 0$ for neutral pseudoscalars.

At next order, $O(p^4)$, the tree–level amplitude coming from Eq.(8) is also zero, $a_4^t = b_4^t = 0$. This is clearly the case for the WZ–term contributing only to anomalous processes and also for the ten counterterms. Indeed, photon couplings are generated either by covariant derivatives or by $F_{\mu\nu}$ terms as the proportional to L_9 (for one photon) or to L_{10} (for both) in (8), but the vanishing of $[Q, \Sigma]$ when only neutral π^0 and η are involved is enough to guarantee a null contribution in all cases [37]. This absence of counterterms implies that loop contributions at this order cannot be divergent. They contain two vertices of the lagrangian $\mathcal{L}_2$ and a (charged) pion or kaon loop. One obtains $b_4^{\pi,K} = 0$ and the finite expressions [35]

$$
\begin{aligned}
a_4^{\pi} &= \frac{-4\sqrt{2}\alpha}{3\sqrt{3}\pi f^2}\Delta m_K^2 \left(1 + \frac{3s - m_\eta^2 - 3m_\pi^2}{m_\eta^2 - m_\pi^2}\right) H(s, m_\pi^2) \\
a_4^{K} &= \frac{2\sqrt{2}\alpha}{3\sqrt{3}\pi f^2}\left(3s - m_\eta^2 - \frac{1}{3}m_\pi^2 - \frac{8}{3}m_K^2\right) H(s, m_K^2)
\end{aligned}
\tag{46}
$$

with ($x \equiv s/m^2$)

$$
\begin{aligned}
sH(s,m^2) &\equiv s\int_0^1 dz \int_0^{1-z} dy \frac{zy}{m^2 - szy} = -\frac{1}{2} - \frac{1}{2x}\ln^2 \frac{\sqrt{x} + \sqrt{x-4}}{-\sqrt{x} + \sqrt{x-4}} \\
sH(s, m_K^2) &\simeq \frac{s}{24 m_K^2} \qquad \text{for } s << m_K^2.
\end{aligned}
\tag{47}
$$

The small correction to a_4^K proportional to Δm_K^2 has been ignored.

Eqs.(46) correspond to the physical η. If one further introduces the numerical values $f = 132$ MeV, Δm_K^2 as in Eq.(7) and the η mass, one obtains the loop contributions up to $O(p^4)$ to the $\eta \to \pi^0\gamma\gamma$ width

$$
\Gamma^{(4)}(\eta \to \pi^0\gamma\gamma) = 7.18 \times 10^{-3} \text{ eV} \tag{48}
$$

more than two orders of magnitude below the measured width (43). The pion loop contribution is small due to the G–parity violation in the vertex $\eta\pi^+\pi^-\pi^0$, which makes the amplitude proportional to $m_d - m_u$ or Δm_K^2 as seen in Eq.(46). The contribution of the G–parity conserving kaon loops is also small because the loop–integration introduces the function $H(s, m_K^2) \simeq 1/24m_K^2$ with a large denominator in our case with $m_K^2 >> s$. Further contributions are required. This was noted independently by Ecker et al., Ref.[36].

At order six, $O(p^6)$, one has the contributions coming from the tree level (or counterterms), from one–loop and from two–loops. For $\eta \to \pi^0\gamma\gamma$ (involving only neutral particles), the former reduce to two types of counterterms in $\mathcal{L}_6$. We restrict ourselves to terms with only a single flavour trace, which are the leading ones in $1/N_c$:

$$
L_6 = d_1 F_{\mu\nu}F^{\mu\nu} tr(Q^2\partial^\lambda\Sigma\partial_\lambda\Sigma^\dagger) + d_2 F_{\mu\alpha}F^{\mu\beta} tr(Q^2\partial^\alpha\Sigma^\dagger\partial_\beta\Sigma), \tag{49}
$$

with low–energy constants d_1 and d_2 to be fixed from experiment or from their saturation with meson resonances R. In order to predict the $\eta \to \pi^0\gamma\gamma$ decay width we adopt this latter approach and, consequently, write

$$
d_{1,2} \equiv \sum_R d_{1,2}^R = d_{1,2}^{\omega+\rho} + d_{1,2}^{a_0} + d_{1,2}^{a_2} \tag{50}
$$

since, apart from the Vector Meson contribution (which is expected to be the dominant one, VMD), also the $a_0(980)$ and $a_2(1310)$ mesons contribute. In terms of the amplitudes (44) and (45), Eq.(49) implies

$$a_6^t = \frac{-4\sqrt{2}}{3\sqrt{3}f^2}[4d_1 m_\eta^2 - (4d_1 + d_2)P\cdot(q_1+q_2)], \qquad b_6^t = \frac{8\sqrt{2}}{3\sqrt{3}f^2}d_2 \tag{51}$$

and allows for the identification of the $d_{1,2}$ constants through resonance saturation. This program was pushed forward in [35] using the experimental data $\Gamma(\omega \to \pi^0\gamma) = 717\ keV$ [1] and the measured cross sections [38] $\sigma(\gamma\gamma \to \pi^0\eta) \simeq 30\ nb$ and $35\ nb$ at the $a_0(980)$ and $a_2(1310)$ peaks after a reasonable background subtraction. The tree level $O(p^6)$ rate was found to be $\Gamma^{(6)}_{\rm tree}(\eta \to \pi^0\gamma\gamma) = 0.18 \pm 0.02$ eV But, apart from these tree–level (or counterterm) resonance–dominated contributions from L_6, loop–corrections are also possible at this same order $O(p^6)$. One–loop corrections can be generated taking vertices from $\mathcal{L}_2$ and one from the non–anomalous part of $\mathcal{L}_4$ such as that in (8). Similarly, taking three vertices from $\mathcal{L}_2$ one generates two–loop corrections of order $O(p^6)$. But all these loop corrections are suppressed because G–parity is violated for pion loops and the smallness of $H(s, m_K^2) = 1/24m_K^2$ (47) suppresses the G-parity conserving kaon loops. In the framework of ChPT all these corrections are smaller than those from previous order, $O(p^4)$, quoted in Eqs.(48).

At $O(p^8)$ more counterterms appear and a new type of loop–correction becomes potentially important. Since the counterterm contributions are assumed to be resonance dominated one can simply take the full VMD amplitude,

$$M_{VMD} = \frac{2\sqrt{2}}{3\sqrt{3}}g^2_{\omega\pi^0\gamma}\left\{\left[\frac{P\cdot q_2 - m_\eta^2}{M_V^2 - t} + \frac{P\cdot q_1 - m_\eta^2}{M_V^2 - u}\right]A - \left[\frac{1}{M_V^2 - t} + \frac{1}{M_V^2 - u}\right]B\right\}, \tag{52}$$

as an "all–order" estimate, i.e., at $O(p^6)$ and higher. This represents a substantial increase to the rate, namely,

$$\Gamma_{VMD}(\eta \to \pi^0\gamma\gamma) = 0.31 \text{ eV}, \tag{53}$$

in reasonable agreement with older VMD estimates [39]. Analogous a_0 and a_2 contributions are expected to slightly modify this result as discussed above. The new type of loop effects looks *a priori* more interesting. Taking two vertices from the anomalous L_{WZ} implies a non–anomalous one–loop correction of $O(p^8)$. Pion loops are no longer suppressed because there is no G–parity violation in the $\pi\pi\pi\gamma$ and $\eta\pi\pi\gamma$ vertices and moreover the kaon loop suppression due to the function H in Eq.(47) does not occur. Indeed, one easily obtains

$$\Gamma^{(8)}(\eta \to \pi^0\gamma\gamma) = 4.8 \times 10^{-3} \text{ eV} \tag{54}$$

to be compared with $\Gamma^{(4)}$ in Eq.(48). The dominant contribution to $\Gamma^{(8)}$ comes from the absorptive part of the amplitude, which is much larger than that of the pion loop in $\Gamma^{(4)}$ as expected. In our opinion, this does not represent a violation of the perturbative character of ChPT. The structure of the anomalous vertices is such that it removes the suppression that was present in the $O(p^4)$ loops, hence the $O(p^8)$ "doubly–anomalous" loop can be of the same order as the $O(p^4)$ non–anomalous loops.

A convenient way of presenting the ChPT predictions for $\Gamma(\eta \to \pi^0\gamma\gamma)$ consists in summing up all the contributions which are not negligible and present no sign ambiguities, i.e., the non–anomalous pion and kaon loops at $O(p^4)$, Eq.(46), the corresponding loops at $O(p^8)$ with two WZ vertices, and the "all–order" sum of VMD counterterms in Eq.(52). This implies [35]

$$\Gamma(\eta \to \pi^0\gamma\gamma) = 0.42 \pm 0.20 \text{ eV} \tag{55}$$

where the error comes essentially from the estimated contribution of the a_0 and a_2 resonances and the lack of knowledge of higher order contributions. This ChPT result is in agreement with related work by Picciotto [40] (performed also in the same ChPT context) and Ng and Peters [41] (who estimated the dominant VMD contribution). The estimation of the error in Eq.(55) has been critiziced by Ivanov et al. in Ref.[42].

3.3 $\eta \to \pi^0 l\bar{l}$

The rare decays $\eta \to \pi^0e^+e^-, \pi^0\mu^+\mu^-$ have not yet been observed. They can occur at second order of the e.m. interaction without C violation, through the chain $\eta \to \pi^0\gamma\gamma \to \pi^0e^+e^-$. The imaginary part of the amplitude is given by forcing the two intermediate photons to be on–shell and requires good knowledge of the $\eta\pi^0\gamma\gamma$ decay which unfortunately is not the case at the moment, as we discussed in the preceding subsection. In addition, the real part of the amplitude diverges unless one introduces suitable form factors for the off–shell photons that regularizes the loop integration at expenses of making it model dependent.

Some years ago, Cheng [43] computed the $\eta \to \pi^0e^+e^-$ decay in the limit $m_e = 0$ using a VMD model. He found $\Gamma(\eta \to \pi^0e^+e^-) = 13\ \mu eV$. Recently, Ng and Peters [41] have addressed the $\eta \to \pi^0\mu^+\mu^-$ channel, where the muon mass can not be neglected. Making also use of VMD (and also considering the a_0 exchange in the two photon intermediate state) they give approximate analytical expressions for the imaginary part of the $\eta \to \pi^0 l\bar{l}$ amplitudes and for the energy spectrum of the pion. The following unitarity bounds were obtained

$$\begin{aligned} \Gamma^{unit}_{2\gamma}(\eta \to \pi^0e^+e^-)\Big|_{VMD} &= 1.1^{+0.6}_{-0.5}\ \mu eV \\ \Gamma^{unit}_{2\gamma}(\eta \to \pi^0\mu^+\mu^-)\Big|_{VMD} &= 0.5^{+0.3}_{-0.2}\ \mu eV. \end{aligned} \tag{56}$$

The inclusion of the a_0 exchange affects only the muon channel, leading to $0.9^{+0.6}_{-0.5}$ $(0.3^{+0.4}_{-0.2})$ μeV for constructive (destructive) interference. They estimated the dispersive part of the amplitude by means of an *ad hoc* cutoff of 1 GeV to be $|ReA_{\mu\bar{\mu}}/ImA_{\mu\bar{\mu}}| \leq 0.5$, suggesting that the rate can not be much greater than the unitarity bound. This conclusion is consistent with the results obtained by Heiliger and Sehgal [44] using a dispersive treatment for the similar processes $K_L \to \pi^0e^+e^-$ and $K_l \to \pi^0\mu^+\mu^-$, $\left|ReA^{K_L}_{\mu\bar{\mu}}/ImA^{K_L}_{\mu\bar{\mu}}\right| \leq 0.7$, $\left|ReA^{K_L}_{e\bar{e}}/ImA^{K_L}_{e\bar{e}}\right| \leq 1.25$.

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