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Accuracies of K_{e4} Parameters at DAΦNE

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Abstract

We estimate the experimental accuracies to which the K_{e4} parameters will be measured at DAΦNE. We consider the accuracies obtainable using asymmetries, and the maximum likelihood method. We find the current determinations of the relevant parameters will be improved by a factor of five to ten after a year of running at the anticipated luminosity of DAΦNE.

1 Estimating experimental uncertainties: Generalities

In this contribution to the DAΦNE physics handbook, we estimate the accuracies achievable at DAΦNE in measuring the theoretical parameters describing the decay $K^+ \to \pi^+\pi^-e^+\nu_e$. We refer the reader to the contribution of Bijnens, Ecker, and Gasser [1] for a detailed discussion of the theory of this decay, and the definitions of the notation we use. We should note that we refer to this decay as K_{e4}; we consider only this particular decay and not the channels with neutral pions or kaons.

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We will consider two of the possible ways of extracting the parameters of a theory from a set of data. The first one is the classic technique of asymmetries. K_{e4} decays have a partial decay rate $\mathrm{d}^5\Gamma$ of the form

$$\mathrm{d}^5\Gamma = G_F^2|V_{us}|^2 N(s_\pi, s_l) J_5(s_\pi, s_l, \theta_\pi, \theta_l, \phi)\mathrm{d}s_\pi \mathrm{d}s_l\, \mathrm{d}(\cos\theta_\pi)\mathrm{d}(\cos\theta_l)\mathrm{d}\phi, \tag{1}$$

where s_π, s_l, θ_π, θ_l, and ϕ are the set of kinematic variables necessary to describe K_{e4}, defined in Ref. [1]. The quantity J_5 can be written as an expansion in simple functions of ϕ and θ_l multiplying nine intensities I_i, which can in turn be written in terms of the three remaining kinematical variables, and three form factors, F, G, and H, which are also dependent on these three kinematical variables. The explicit dependences may be found in Ref. [1]. As can be seen in the contribution of Colangelo, Knecht and Stern, [2] the tangent of the phase shift, $\tan(\delta_0 - \delta_1)$, can be neatly extracted from the ratio of the intensity functions, $\overline{I_7}/2\overline{I_4}$, or, equivalently, $\tilde{I_7}/2\tilde{I_4}$, where by the tilde we denote intensities integrated over s_π and s_l as well as over θ_π. All of the $\tilde{I}_i$, in turn, can be written as asymmetries; in particular we have

$$\frac{8}{3\pi}\tilde{I}_4 = \left(\int_0^{\pi/2}\int_{3\pi/2}^{5\pi/2} - \int_0^{\pi/2}\int_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{\pi}\int_{3\pi/2}^{5\pi/2} + \int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2}\right)\sin\theta_l \mathrm{d}\theta_l \mathrm{d}\phi \int J_5 \mathrm{d}s_\pi \mathrm{d}s_l \mathrm{d}\theta_\pi \tag{2}$$

and

$$\tilde{I}_7 = \left(\int_0^{\pi} - \int_{\pi}^{2\pi}\right)\mathrm{d}\phi \int J_5 \mathrm{d}s_\pi \mathrm{d}s_l \mathrm{d}\cos\theta_\pi \mathrm{d}\cos\theta_l. \tag{3}$$

The asymmetry presents a transparent, elegant and quick (computationally) way to determine a parameter.

The second, and main, method we consider is that of the maximum likelihood, which we shall refer to as the MLM. [3] Let $\vec{x}$ be the vector of phase space variables specifying an event in an experiment, and $f(\vec{x};\vec{p})$ be the probability distribution function predicted by a theory. $\vec{p}$ is the set of parameters in the theory, to be determined experimentally. The probability of observing an event at $\vec{x}$ in the interval $\mathrm{d}^n x$ is $f(\vec{x};\vec{p})\mathrm{d}^n x$. The function f is normalized to 1 over the whole $\vec{x}$ interval in which $\vec{x}$ is physical. In particular, if cuts are imposed on the phase space, f must be normalized over the reduced phase space; we must also be certain that the normalization is maintained even when the parameters are varied from their central value.

The likelihood, or joint probability distribution, of an experiment yielding N events, each specified by a set of phase space variables $\vec{x_i}$, is then defined as

$$\mathcal{L} = \prod_{i=1}^{N} f(\vec{x_i};\vec{p}). \tag{4}$$

The best estimate for $\vec{p}$ is then simply the value $\overline{p}$ which maximizes $\mathcal{L}$ (or equivalently, of $W = \log\mathcal{L}$, which is easier to compute). It can be shown that the error matrix, in general non-diagonal for correlated parameters, is

$$\overline{(p_i - \overline{p}_i)(p_j - \overline{p}_j)} = \frac{1}{N}\int\left[\frac{1}{f}\left(\frac{\partial f(\vec{x};\vec{p})}{\partial p_i}\frac{\partial f(\vec{x};\vec{p})}{\partial p_j}\right)\mathrm{d}^n x\right]^{-1}. \tag{5}$$

Here the $[\ ]^{-1}$ denote matrix inversion. This integral is, in general, not possible to compute analytically, but is easily evaluated numerically. If we start with a non-normalized probability function $\mathcal{P}$, then the following equation is useful:

$$\int \frac{1}{f}\frac{\partial f}{\partial p_i}\frac{\partial f}{\partial p_j}\mathrm{d}^n x = -\frac{1}{a^2}\frac{\partial a}{\partial p_i}\frac{\partial a}{\partial p_j} + \int \frac{1}{a\mathcal{P}}\frac{\partial \mathcal{P}}{\partial p_i}\frac{\partial \mathcal{P}}{\partial p_j}\mathrm{d}^n x, \tag{6}$$

where $a = \int \mathcal{P}\mathrm{d}^n x$.

While this method is more complicated and time-consuming computationally, it has the decided advantage that it yields the absolute best possible determination of any given set of parameters. In today's age of fast computers, it is thus the method of choice. It also allows one to determine *any* parameters chosen, while the asymmetry is only good for certain parameters. Finally, the experimental *uncertainties* in the parameters can be determined reasonably easily, without need for an actual simulation of the determination of the parameters themselves using this method.

It is simple to see that the MLM should give a better determination of the parameters than an asymmetry: while the asymmetry only uses the information of whether an event is in one or another half of phase space, the MLM benefits from the information of the precise position of the event. For example, consider a process specified by the probability distribution

$$f(\theta; a) = (1 + a\cos\theta)/2. \tag{7}$$

The parameter a can be determined by the ratio $(N_1 - N_2)/(N_1 + N_2) = a/2$ where N_1 is the number of events with θ between 0 and $\pi/2$, and N_2 is the number of events with θ between $\pi/2$ and π. For a small compared to 1, the error on a in this determination is thus $2/\sqrt{N}$ where N is the total number of events. For the MLM, the integral in eq. 5 is

$$\lim_{a\Rightarrow 0}\int_{-1}^{1} \frac{x^2}{2(1+ax)}\mathrm{d}x = \frac{1}{3} \tag{8}$$

and the error on a in this determination is thus $\sqrt{3/N}$. While this improvement may not seem very impressive, larger improvements may be expected as the parametrization becomes more complicated. Using the MLM directly on the parameter that interests us, we are sure to use all possible information, and to take into account the effect of all possible cancelling uncertainties. We observe an improvement as above, of about fifteen percent, in the relative error $\Delta\tilde{I}_7/\tilde{I}_7$ when we go from the asymmetry method to the MLM. However, the relative error on $\delta = \delta_0 - \delta_1$ when δ is determined *directly* via the MLM is two-thirds of its error when determined *indirectly* by the ratio $\tilde{I}_7/2\tilde{I}_4$, if these $\tilde{I}_i$ are determined using the MLM, and their errors are then combined as uncorrelated. This appears to be due to the information lost in integrating the $\tilde{I}_i$ over s_π, s_l, and θ_l before using the MLM.

2 Estimating experimental uncertainties in K_{e4}

For our first estimates, we consider only the first order terms in a partial wave expansion of the form factors F, G and H, i.e., we take

$$F = f_s e^{i\delta_0} \qquad G = g e^{i\delta_1} \qquad H = h e^{i\delta_1}. \tag{9}$$

This is consistent with the parametrization used by Pais and Treiman, [4] and Rosselet *et al.* [5] They consider higher order terms, but the coefficients of these terms are found to be consistent with zero by the experiment of Rosselet *et al.*, [5] so we do not consider further terms in our initial estimates.

All the ϕ and θ_l dependence in the problem is contained in the expression of J_5 as a function of these variables and the I_i; for this reason these two variables are referred to as "trivial." θ_π appears only in the equations for the I_i in terms of F, G, and H (and the phase space expansion, should we consider higher order terms). This leaves then f_s, g and h with a possible dependence on s_π and s_l. The $\pi\pi$ phase shifts δ_0 and δ_1 (which at this order appear only in the combination $\delta = \delta_0 - \delta_1$) depend only on s_π. For the moment we parametrize f_s, g and h by an expression of the form

$$y(s_\pi, s_l) = y_0(1 + \lambda q^2) \tag{10}$$

where $q^2 = (s_\pi - 4m_\pi^2)/4m_\pi^2$, and y stands for f, g, or h. We take the slope λ to be the same for f_s, g and h, and no slope in s_l at this stage, again consistent with Ref. [5]. For the dependence of δ on s_π we will consider average values in a set of 5 bins in s_π, and consider parametrizations of δ in a later section.

One last important detail remains to be mentioned. The MLM (or asymmetries, for that matter) says nothing about an *overall factor* in the intensity, as we require the probability density to be normalized to one. Thus we divide out f_{s0} (f_0 for short) from the amplitude, as it is the parameter with the most effect on the integrated intensity. Wherever g_0 and h_0 appear, they are divided by f_0, so we replace them by new parameters $g_0' = g_0/f_0$ and $h_0' = h_0/f_0$ (λ and δ are unaffected). We then apply the MLM to the set of parameters δ, g_0', h_0' and λ, and obtain the correlation matrix

$$(\mathrm{d}p\mathrm{d}p)_{ij} = \frac{1}{N}\begin{pmatrix} 6.5 & 0.8 & -0.7 & -0.1 \\ 0.8 & 2.8 & -0.6 & -0.8 \\ -0.7 & -0.6 & 187 & 0.4 \\ -0.1 & -0.8 & 0.4 & 3.8 \end{pmatrix}. \tag{11}$$

The diagonal entries of this matrix are variances of the four parameters, where N is the number of events. The off-diagonal elements represent correlations between the parameters; in this case they are small, but they can be significant, depending on the parametrization used. We do not report the full correlation matrix for each parametrization in this paper, but they are available from our programs if needed for further calculations. They can not be neglected in general if one wants to calculate functions of the parameters we use, and

Table 1: Central values and estimated errors for f_0, g_0, h_0 and λ

	f_0	g_0	h_0	λ
Central values	5.59	4.77	−2.68	0.08
Errors ($N = 30000$)	0.029	0.059	0.44	0.011
Errors ($N = 300000$)	0.009	0.019	0.14	0.004
Errors (Rosselet)	0.14	0.27	0.68	0.02

propagate the errors correctly. In the end of this paper, we consider the determination of some highly correlated parameters.

We then extract the error on f_0 from the equation

$$\Gamma_{K_{e4}} = C f_0^2 \int \mathcal{P} \mathrm{d}^n x, \tag{12}$$

where $\mathcal{P}$ is the unnormalized probability function inputted to the MLM calculation, and C represents all the constant factors (masses, two's, π's) needed to complete the equation. The relative error $\Delta\Gamma/\Gamma$ is given by $\approx 1/\sqrt{N}$ in an experiment like KLOE [6] where the statistical error will be dominant. The relative error on the integral (a) is given by the matrix product

$$\frac{\Delta a}{a} = \sqrt{\frac{1}{a^2} \frac{\partial a}{\partial p_i} (\mathrm{d}p\mathrm{d}p)_{ij} \frac{\partial a}{\partial p_j}}. \tag{13}$$

Combining this error in quadrature with the statistical error on Γ, we obtain the error we quote for f_0; combining the error on f_0 in quadrature with the errors on g_0' and h_0', we obtain the errors we quote for g_0 and h_0.

In Table 1 we display the results of this calculation. The central values (our input) are those found by the previous experiment. [5] We have used the program VEGAS [7] to do the necessary integrals in five-dimensional phase space. The normalization of the probability distribution is ensured automatically by the program, and the necessary derivatives also computed numerically. Estimated errors are shown for $N = 30000$ events, the statistics of the previous experiment) and $N = 300000$ events, the anticipated statistics [1] in one "year"$\equiv 10^7$ seconds of running with $\mathcal{L} = 5 \times 10^{32}$ cm^2 s^{-1}. All errors in this paper, unless otherwise noted, are statistical errors and can be simply scaled by $1/\sqrt{N}$ for different numbers of events. As a general rule, also, the *fractional* error on f_0 is roughly independent of its central value, while the *absolute* errors of the other parameters remain constant. The last line of Table 1 shows the errors on these parameters found by Rosselet *et al.*. We do not quote the error on δ because an error on δ averaged over the whole of phase space is not very meaningful. The errors shown are independent of the central value of δ used.

At this point, before going on to further discuss errors in KLOE, it is necessary to say some words about why our estimated errors at "Rosselet statistics" are so different from those that Rosselet quotes. The errors given above are purely statistical, but apply to a

Table 2: Estimated errors in five bins of 6000 events each.

$\sqrt{s_\pi}$ (GeV)	$0.279 - 0.3$	$0.3 - 0.316$	$0.316 - 0.334$	$0.334 - 0.357$	$0.357 - 0.494$
f	0.037	0.039	0.041	0.043	0.047
g	0.195	0.142	0.124	0.116	0.112
h	1.46	1.04	0.93	0.90	1.00
δ	0.062	0.041	0.034	0.029	0.025
δ (Rosselet)	0.13	0.07	0.05	0.04	0.04

"perfect" detector, i.e., one which covers the whole of phase space with unity efficiency everywhere. This is close to true for KLOE; we will attempt to illustrate this later in this paper, and will describe a more rigorous demonstration in a future paper. However, Rosselet's detector was far from "perfect." In the error we have quoted for f_0, the errors from $\Gamma_{K_{e4}}$ and the normalization a contribute about equally; the first is about 0.6% and the second about 0.9%. Rosselet, however, quotes a relative error $\Delta\Gamma_{K_{e4}}/\Gamma_{K_{e4}}$ of 4.5%, which completely accounts for their large error on f_0. Their fixed target experiment had a 10% overall efficiency for K_{e4}, and a highly variable efficiency as well, varying, for example, smoothly from $> 95\%$ in a very small portion of phase space with large s_l and small s_π, to near zero at large s_π and small s_l. KLOE is in contrast a hermetic detector, operating at a e^+e^- collider running at the ϕ resonance, producing self-tagging low momentum $K^\pm$ pairs. It will have a uniform near-100% efficiency over all of phase space, minus a few percent of phase space that will be cleanly cut and discarded. [8]

The next step in our analysis was to drop the slope parameter λ and determine the errors on the parameters in five bins in s_π, chosen so as to have equal numbers of events. Such an analysis with real data would have the advantages of studying the s_π dependence in a more parametrization independent way. If, however, the s_π dependence is correctly given by eq. 10, this method will not determine λ as accurately, so in general both types of approach are necessary. For our purposes, displaying the error in bins is also important to illustrate the possible accuracies with which $\delta(s_\pi)$ may be measured, before we implement a possible parametrization of δ. In Table 2, we give the estimated errors, taking an average of $y_0(1 + \lambda q^2)$ in each bin as our inputs for $y = f$, g, and h, with Rosselet values for the y_0 and λ. The errors on δ are essentially independent of the inputs of its central value. In the last line, for comparison, we display the errors on δ as measured by Rosselet *et al.* The improvement is not as drastic as that of f_0 was, but is nonetheless a factor of 1.5 to 2. This should be further multiplied by a factor of $\sqrt{10}$ to $\sqrt{20}$ per DAΦNE running year. The accuracy on f in bins is even better than we might have expected from the error on f_0 multiplied by $\sqrt{5}$. This is because the error on λ gives most of the contribution to the error on the normalization a, and thus a significant contribution to the error on f_0.

We have examined in some detail the question of what accuracy δ can be measured to. We have first of all determined that while δ appears in I_1, I_2, I_4, I_5, I_7, and I_8, it is only the dependence of I_7 that gives us the above accuracy on δ. This can be seen by replacing δ in all the I_i, except I_7, by a dummy variable δ_0, set equal to the central value of δ. When we proceed to apply the MLM to the new probability function, we find the same error on δ as before, within a few percent. If, however, we apply the MLM to the $\tilde{I}_i$ as parameters in their own right (we cannot use the $\overline{I}_i$ as parameters, because they are functions of the phase space variables) and then take the ratio to determine δ, we find that the error on δ increases by 50%. (If we use the asymmetry method to determine the I_i, the error increases another 15%.) We have not taken care to cancel correlated errors in $\tilde{I}_4$ and $\tilde{I}_7$, but we have checked that the correlated parts of the errors are small relative to the uncorrelated parts. So, this 50% increase appears to be mainly due to the information lost in integrating the I_i over three out of five of the phase space variables before applying the MLM. Equivalently, the better error on δ can be attributed to applying a more detailed parametrization (therefore more information) from the beginning of the calculation.

Nonetheless, it may be interesting to determine the $\tilde{I}_i$ and their errors as a parameterization independent way to present the data. We have estimated that the combination $I_1 - I_2/3$ can be determined to 0.4% in five bins of 60,000 events each. The other I_i can be determined with absolute errors of one to two times this error.

3 Other experimental uncertainties

So far we have estimated the statistical experimental uncertainties in a perfect detector. In this section we would like to say a few words about other experimental uncertainties. Errors (other than those already estimated) in a detector like KLOE, where the efficiency is essentially 100% outside of a small region of phase space to be removed by cuts, fall into two categories: those associated with cuts, and those associated with resolution. We intend to write an event generator, that would allow us to impose a proper cut on true kinematical variables such as the angles of the particles themselves, or their momenta, but for the moment are only calculating the MLM error estimating integral over s_l, s_π, θ_l, θ_π and ϕ. We expect that [8] appropriate cuts will be applied to reject charged particles that have momenta less than 20 MeV, and those that are within 9° of being parallel to the beam pipe (because they will not cross enough wires). We expect that both of these cuts will (from the point of view of our problem) reject events fairly randomly, and thus only have the effect of reducing the number of events by a couple of percent, having a negligible effect on our error estimates. Just to check, we have considered cutting s_π in the range $4m_\pi^2$ to $4m_\pi^2 + 4*(20\mathrm{MeV})^2$, and s_l in the range 0 to $(20\ \mathrm{MeV})^2$. The s_π cut has negligible effect and the s_l cut increases our errors by about 10 percent or less. Eventually we would also like to investigate the effect of inaccurate cuts (for example a cut that is believed to be at 20 MeV and is really at 15 MeV), which can be an important source of systematic errors.

Smearing, also known as convolution with a resolution function, can be imposed on our variables with more confidence, being a random effect. We assume a gaussian resolution function, i.e., we replace the probability function $f(x;p)$ by a new function $g(x;p)$

$$\begin{aligned} g(x;p) &= \int_{x_0}^{x_1} f(x'';p)\frac{e^{-(x''-x)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\mathrm{d}x'' \approx \int_{-\infty}^{\infty} f(x-x';p)\frac{e^{-x'^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\mathrm{d}x' \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x';p)e^{-x'^2/2\sigma^2}e^{-y^2/2\sigma^2}\frac{1}{2\pi\sigma^2}\mathrm{d}x'\mathrm{d}y = \int_0^1\int_0^1 f(x-x';p)\mathrm{d}a\mathrm{d}b \end{aligned} \tag{14}$$

where y is a dummy variable introduced to make the gaussian integrable, $a = e^{-(x'^2+y^2)/2\sigma^2}$ and $\tan(2\pi b) = x'/y$. x_0 and x_1 are the lower and upper bounds on x; $x_1 - x_0$ is assumed to be large compared to σ. Thus we have

$$g(x;p) = \int_0^1\int_0^1 f\left(x - \sigma\sqrt{-2\log a}\,\sin(2\pi b);p\right)\mathrm{d}a\mathrm{d}b, \tag{15}$$

showing the equivalence of convolution with a gaussian and integrating after a gaussian *smearing* of the independent variable. Note that the quantity $\sqrt{-2\log a}\,\sin(2\pi b)$, for a and b uniformly distributed in the interval 0 to 1, is gaussian distributed with variance 1.

KLOE expects a resolution of about half a percent to one percent in angles and momenta. We make the assumption that smearing by one percent (of 2π or π in the case of our angles) in our five variables is a reasonable and generous approximation to smearing in the actual kinematic variables. We observe negligible effect on our errors; in fact smearing of as much as 5% of the maximum of our variables has no effect except in s_π, where it yields a ten percent (fractional!) increase in our errors. This is not surprising. If one *knows* one's resolution function, one can compensate for it, and the accuracy is unaffected, at least if the scale of the smearing is small compared to the scale of the effect (this scale in our case is of the order of the whole of phase space). If however there are *unknown* parts to the resolution function, i.e., *systematic* errors, these can result in systematic errors in the result, that the MLM error estimating technique above will never find, simply because we have no way to input an *unknown* error. These errors could in principle be estimated by applying the MLM itself to simulated data, but this is beyond the scope of our investigation. The accurate estimation of systematic errors is at any rate something that will have to be done by KLOE. Meanwhile, however, we have made one attempt to estimate the effect of such systematic errors by returning to the determination of δ from the asymmetry method. Here we can easily calculate what δ we would "measure" from an imaginary set of data described by a set of input parameters; normally of course, we get back the value of δ we input. If, however, we replace the probability distribution $f(x)$ by a convoluted distribution $g(x)$, then, by the shift in the recovered δ, we can see what would be the effect of a resolution function that we did not know about, and therefore did not compensate for.

We have first verified that gaussian smearing still has no significant effect, which means that it is not neccessary to know the exact form of the resolution function, as long as it is symmetric and not too broad in the appropriate scale. We have next examined the effect of an actual shift in each of our phase space variables. The effect is tiny compared to the statistical errors for shifts in θ_l, θ_π, s_l and s_π. For a 1% shift in ϕ, δ goes down by 0.01, or about half of the statistical error with 300,000 K_{e4} events. We remark that a 1% systematic shift in ϕ is already so large as to be inconcievable, as ϕ is a difference of observable angles.

4 Other parametrizations

In this section we would like to present results for some extensions of the minimal parametrization we have used so far: (1) we include a p-wave term in F, and simultaneously adopt a notation consistent with the partial wave expansion of Ref. [1]; (2) we consider an s_l dependence in f, g, etc.; (3) we consider two parametrizations of δ in terms of s-wave scattering length a_0^0 to give a first indication on what errors can be expected on this parameter.

F thus becomes

$$F = f_s e^{i\delta_0} + f_p e^{i\delta_1}\cos\theta_\pi - \frac{\sigma_\pi\, P\cdot L}{X}\cos\theta_\pi g e^{i\delta_1}, \tag{16}$$

where σ_π, $P\cdot L$ and X are defined in Ref. [1]. Our previous parametrization is thus equivalent to this one for $f_p = \sigma_\pi\, P\cdot L\, g/X$. For the s_l dependence, chiral perturbation theory to one loop predicts [9] approximately the same slope for s_π and s_l, or, more precisely,

$$y(s_\pi, s_l) \approx y_0\left(1+\lambda\left(\frac{s_\pi - 4m_\pi^2 - s_l}{4m_\pi^2}\right)\right). \tag{17}$$

To be more model-independent, and to have a separate determination of the s_π and s_l slopes, we choose the parametrization

$$y(s_\pi, s_l) = y_0\left(1+\lambda q^2\right)\left(1-\lambda_2 s_l/4m_\pi^2\right). \tag{18}$$

We then find, for 300000 events,

$$\Delta f_{s0} = 0.014 \quad \Delta g_0 = 0.038 \quad \Delta h_0 = 0.14 \quad \Delta f_{p0} = 0.014 \tag{19}$$

$$\Delta\lambda = 0.004 \quad \Delta\lambda_2 = 0.011 \tag{20}$$

to be compared with table 1. We have displayed the errors for the old parameters as well as the added ones, as some of them have increased slightly. We have taken central values $\lambda_2 = 0.08$ and $f_{p0} = 3.3$ (which maintains the normalization unchanged); if other parameters are preferred it should be remembered that the absolute errors of f_{p0} and the slope remain essentially constant.

For δ we have first considered the parametrization used [10] by the previous experiment, to compare our estimated errors with the ones they determined. They use

$$\sin 2\delta = 2\sqrt{\frac{s_\pi - 4m_\pi^2}{s_\pi}}\left(a_0^0 + bq^2\right), \tag{21}$$

where $b = b_0^0 - a_1^1$, the difference between the s-wave slope and the p-wave scattering length. There is moreover a possible relation between b and a_0^0

$$b = 0.19 - (a_0^0 - 0.15)^2. \tag{22}$$

For 30000 events (to compare with Ref. [5]) we find $\Delta a_0^0 = 0.029$, to be compared with 0.05 in Ref. [5], if we use both eqs. 21 and 22. If we use only eq. 21 we find $\Delta a_0^0 = 0.06$ and $\Delta b = 0.07$, to be compared with 0.11 and 0.16 in Ref. [5]. We have used the central values $a_0^0 = 0.28$ in the first case, and $a_0^0 = 0.31$, $b = 0.11$ in the second case, as found in Ref. [5]. So, in one year running at DAΦNE at a luminosity of $5 \times 10^{32}\mathrm{cm}^2\ \mathrm{s}^{-1}$, we expect a factor of five improvement in the error on the $\pi\pi$ scattering length, meaning that DAΦNE should be able to determine if the existing discrepancy between measurements and predictions for a_0^0 is statistically significant.

We have also considered a more recent parametrization, due to Schenk. [11] He gives

$$\tan \delta_l^I(s_\pi) = \sqrt{\frac{s_\pi - 4m_\pi^2}{s_\pi}} q^{2l} \left(a_l^I + \tilde{b}_l^I q^2 + c_l^I q^4\right) \left(\frac{4m_\pi^2 - s_l^I}{s_\pi - s_l^I}\right) \tag{23}$$

where $q^2 = (s_\pi - 4m_\pi^2)/4m_\pi^2$ and

$$\tilde{b}_l^I = b_l^I - a_l^I \frac{4m_\pi^2}{s_l^I - 4m_\pi^2} + \left(a_l^I\right)^3 \delta_{l0}. \tag{24}$$

Here I denotes isospin, l angular momentum, and δ_{ij} is the Kronecker delta. We need to calculate $\delta_0 \equiv \delta_0^0$ and $\delta_1 \equiv \delta_1^1$; s_0^0 and s_1^1 (the values at which the phase shifts should pass through 90°) are the squares of the σ and ρ meson masses. There are far too many parameters here to be able to determine them by K_{e4} measurements alone; to begin with, we fix the higher order coefficients c_0^0 and c_1^1 to zero. The remaining set a_0^0, a_1^1, b_0^0 and b_1^1 is still too large: if we plot δ versus s_π, we find that while changing a_1^1 from its central value of 0.037 to a new value of 0.087 changes δ essentially uniformly by 15 percent, changing instead b_0^0 from its central value of 0.24 to 0.19 has the same effect, to within about 1 percent. Moreover, with appropriate variations in a_0^0 and b_1^1, one can quite well mock up a variation in a_1^1 (or b_0^0, naturally). Thus, one can only hope to determine two independent parameters without recourse to other experiments or other theoretical constraints.

The MLM quantifies these conclusions. Assuming Schenk's central values ($a_0^0 = 0.2$, $b_1^1 = 0.005$, and above), a_0^0 or b_1^1 can be determined to 0.03 (for Rosselet statistics of 30,000 events), or b_0^0 or a_1^1 to 0.04, if only one parameter is considered, and the rest held fixed. If one determines a_0^0 and b_1^1 simultaneously, holding the rest of the parameters fixed, the correlation is 0.71 (defined as the covariance of the two parameters divided by the σ's for each, 1 for a maximally correlated pair of variables), and the errors are 0.04 and 0.05 respectively. If one determines instead a_0^0 and $b_0^0 - a_1^1$ simultaneously, the correlation is -0.9 and the errors are 0.06 and 0.08; for b_1^1 and $b_0^0 - a_1^1$ the correlation is 0.94 and the errors are 0.09 and 0.11. To determine more than two parameters simultaneously is not possible.

5 Conclusion

We have considered asymmetries and the maximum likelihood method, and shown how the latter is more powerful and yields more precise determinations. We have found that the current measurements of these parameters will be improved by a factor of five to ten in one effective year (10^7 seconds) of running with $\mathcal{L} = 5 \times 10^{32}$ cm^2 s^{-1}; in particular we expect that the $\pi\pi$ scattering length a_0^0 should be measureable to an accuracy of about 0.01.

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References

[1] J. Bijnens, G. Ecker and J. Gasser, in this Handbook.

[2] G. Colangelo, M. Knecht and J. Stern, in this Handbook

[3] For more details see for example P. Franzini in the *DAΦNE Physics Handbook*, 1st ed., vol. 1, p. 15, 1992.

[4] A. Pais and S. B. Treiman, Phys. Rev. **168** (1968) 1858.

[5] L. Rosselet *et al.*, Phys. Rev. **D15** (1976) 574.

[6] See for example *The KLOE detector technical proposal*, by the KLOE collaboration, A. Aloisio et al., LNF-93/002.

[7] G. P. Lepage, CLNS-80/447.

[8] P. Franzini, private communication.

[9] G. Colangelo, private communication.

[10] J. L. Basdevant, C. D. Frogatt and J. L. Petersen, Nucl. Phys. **B72** (1974) 413.

[11] A. Schenk, Nucl. Phys. **B363** (1991) 97.