Radiative Corrections to K_{l2} Decays

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Abstract

We consider K_{l2} decays to the order $O(\alpha)$. We perform a matching calculation, using a specific model with vector meson dominance in the long distance part and the parton model in the short distance part. By considering the dependence on the matching scale and on the hadronic parameters, and by comparing with the leading model independent predictions, we scrutinize the model dependence of the results.

1 Lowest Order Prediction

From the theoretical point of view, K_{l2} decays

$$
\begin{aligned}
K^+ &\rightarrow \mu^+ \nu_\mu \\
K^+ &\rightarrow e^+ \nu_e
\end{aligned}
\tag{1}
$$

constitute the simplest semileptonic decay modes of the kaon. Their decay rates depend on the matrix element of the axial weak hadronic current

$$
< 0 | A^\mu(0) | K^+(p) > = i\sqrt{2} f_K p^\mu
\tag{2}
$$

and, neglecting electromagnetic corrections, the decay rates are given by

$$
\Gamma(K^+ \rightarrow l^+ \nu_l) = \frac{G_F^2 V_{us}^2}{4\pi} f_K^2 m_K m_l^2 \left(1 - \frac{m_l^2}{m_K^2} \right)^2
\tag{3}
$$

Using [1]

$$\Gamma(K \to \mu\nu_\mu(\gamma)) = (3.38 \pm 0.01) \cdot 10^{-14}\,\text{MeV}$$
$$|V_{us}| = 0.2205 \pm 0.0018 \tag{4}$$

we extract the kaon decay constant to the order $O(\alpha^0)$

$$f_K^{(0)} = (113.4 \pm 0.9)\,\text{MeV} \tag{5}$$

where the error is due to the uncertainty in V_{us}.

Of course, f_K and f_π, which is extracted from $\Gamma(\pi \to \mu\nu_\mu)$ in an analogous way, are important input parameters for chiral perturbation theory [2, 3]. It is therefore important to understand the effect of $O(\alpha)$ radiative corrections on their extraction [4, 5, 6].

Whereas f_π and f_K can not be predicted without being able to solve the non-perturbative sector of QCD, we can predict the ratio R_K of the electronic and muonic decay modes of the kaon. At the order $O(\alpha^0)$, we obtain

$$R_K^{(0)} := \frac{\Gamma(K \to e\nu_e)}{\Gamma(K \to \mu\nu_\mu)} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_K^2 - m_e^2}{m_K^2 - m_\mu^2}\right)^2 = 2.60 \cdot 10^{-5} \tag{6}$$

There are two points which should be emphasized. First, note that to the $O(\alpha^0)$, strong interaction effects, which we have parameterized by f_K, cancel completely in this ratio, and it is predicted in terms of the precisely measured particle masses only. As we will show, this cancelation of strong interaction effects occurs also to a large extent for the $O(\alpha)$ radiative corrections, allowing for a very precise prediction of R_K. Secondly, we observe the famous dynamical helicity suppression of the electronic mode, which is of course due to the fact that the weak interaction is mediated by a spin-1 particle. (Note that the helicity suppression does not only occur for a $V - A$ interaction, but for any (V, A) structure.) Therefore the ratio R_K allows for an important precision test of the standard model. Although at least at present [1] the experimental precision in pion decays is better than in kaon decays, and so the measurement of the corresponding ratio R_π is more precise, the study of K_{l2} is of equal importance, because due to the larger kaon mass, the effects of non-standard physics might be enhanced in R_K by a factor of m_K/m_π.

2 Structure of the Radiative Corrections

At the order $O(\alpha)$ one has to consider inclusive decay rates into final states with or without an additional photon,

$$\Gamma(K \to l\nu_l) + \Gamma(K \to l\nu_l\gamma) \tag{7}$$

because of the infra-red divergences in the virtual and soft photonic corrections, which cancel in the inclusive sum only. At this point, one can either to integrate over all photons, soft and hard ones, or to include only soft photons, using some type of an upper limit on the photon energy.

So we have to consider the Born amplitude $\mathcal{M}_0$, the amplitude of 1-loop virtual photonic corrections $\delta\mathcal{M}_v$, and the amplitude $\delta\mathcal{M}_r$ of the radiative decay:

$$\mathcal{M}_0(K \to l\nu_l)$$
$$\delta\mathcal{M}_v(K \to l\nu_l)$$
$$\delta\mathcal{M}_r(K \to l\nu_l\gamma) = \mathcal{M}_{IB} + \mathcal{M}_{SD} \tag{8}$$

The radiative decay $K \to l\nu_l\gamma$ is discussed in detail in this handbook in the section on semileptonic kaon decays [7]. Note that $\mathcal{M}_{IB}$ is of the $O(P^2)$, whereas $\mathcal{M}_{SD}$ starts at the order $O(P^4)$. The internal bremsstrahlung amplitude $\mathcal{M}_{IB}$ is identical to what is obtained assuming an effective point like kaon, and so $\mathcal{M}_{SD}$ includes all the hadronic structure dependent ("SD") effects. These amplitudes give rise to the Born level decay rate Γ_0, to the virtual correction $\delta\Gamma_v$ of the decay rate and to the contributions Γ_{IB}, Γ_{INT} and Γ_{SD} to the radiative decay:

$$\Gamma_0 \propto |\mathcal{M}_0|^2$$
$$\delta\Gamma_v \propto 2\text{Re}(\mathcal{M}_0\delta\mathcal{M}_v^\star)$$
$$\Gamma_{IB} \propto |\mathcal{M}_{IB}|^2$$
$$\Gamma_{INT} \propto 2\text{Re}(\mathcal{M}_{IB}\mathcal{M}_{SD}^\star)$$
$$\Gamma_{SD} \propto |\mathcal{M}_{SD}|^2 \tag{9}$$

Γ_{IB} is proportional to Γ_0 and therefore vanishes for $m_l \to 0$. Γ_{SD}, on the other hand, is not helicity suppressed and becomes phase space enhanced for $m_l \to 0$. So in matter of fact, for the electronic decay mode of the kaon, where $m_e \ll m_K$, the amplitude for structure dependent radiation is of the same order of magnitude as the Born decay rate (see also [7]):

$$\Gamma_{SD}(K \to e\nu_e\gamma) \approx \Gamma_0(K \to e\nu_e) \tag{10}$$

Therefore we adopt the following convention, concerning the question of hard photons, which is more or less identical to the one used in the extraction of the experimental data. We include the full Γ_{IB} contribution, which is strongly dominated by very soft photons, but we exclude completely the structure dependent $\Gamma_{SD} + \Gamma_{INT}$ contribution, which is strongly dominated by hard photons.

Thus we calculate

$$\Gamma(K \to l\nu_l(\gamma)) = \Gamma_0 + \delta\Gamma \tag{11}$$

where the radiative correction $\delta\Gamma$ is defined by

$$\delta\Gamma := \delta\Gamma_v + \Gamma_{IB} \tag{12}$$

Note that while for the electronic mode the excluded structure dependent contribution is extremely large, it is completely negligible for $K \to \mu\nu_\mu\gamma$.

Of course it is not possible to tell whether a radiated photon is due to internal bremsstrahlung or to structure dependent radiation. However, if some small upper limit on the

photon energy is used, the measured rate of $K \to e\nu_e(\gamma)$ will include only a very small SD + INT background, and only very little of the IB part will have been discarded. Using the predicted differential distributions, which are given in detail in [7], the SD +INT background can be subtracted and the missing IB part added. Because of the smallness of this correction, it does not give rise to any important uncertainties.

The particle data book [1] quotes numbers for $K \to e\nu_e$, without stating clearly how much $K \to e\nu\gamma$ is included in these numbers. However, from reading the original papers such as [8], we believe that our convention comes close to the procedure applied in the extraction of the experimental data. With increasing experimental precision, it will become important that experimentalists state very clearly which corrections with respect to the radiative decay $K \to e\nu_e\gamma$ have been applied, and we suggest to use the approach described above.

Let us now discuss the virtual corrections is some more detail. Notice that we are discussing processes at a scale much smaller than m_W^2. Therefore we can use an effective local interaction, cut off at m_W^2 by the W propagator. In fact, it has been shown by Sirlin [9], that to the order αG_F, the radiative corrections within the full standard model are identical to the photonic corrections calculated with a local $V - A$ interaction and an ultra-violet cut-off equal to m_Z. In the proof, terms of the order $\alpha_s \alpha G_F$ and $G_F \alpha m^2/m_W^2$ are neglected, where m is the mass of a quark or of an external particle.

In order to be able to calculate the virtual corrections, one separates the loop integration $k_E^2 = 0 \cdots m_Z^2$ into two parts, using a matching scale μ_{cut} of the order of 1 GeV [10, 4, 11]. In the long distance part, $k_E^2 = 0 \cdots \mu_{cut}^2$, mesons are the relevant degrees of freedom. A reasonable first approximation for the long distance part is in fact a model using an effective point like kaon. This has then to be modified to account for hadronic structure effects. In the short distance part, $k_E^2 = \mu_{cut}^2 \cdots m_Z^2$, on the other hand, quarks are the relevant degrees of freedom, and one has to consider 1-loop photonic corrections to the effective four-point operator $\mathcal{A}_0 = [\bar{u}_\nu \gamma^\mu \gamma_- u_l][\bar{u}_s \gamma_\mu \gamma_- u_u]$.

Thus after adding the rate for internal bremsstrahlung, the total radiative correction can be written as the sum of three terms,

$$\frac{\delta\Gamma}{\Gamma_0}(K \to l\nu_l(\gamma)) = \left(\frac{\delta\Gamma}{\Gamma_0}\right)_{PM} + \left(\frac{\delta\Gamma}{\Gamma_0}\right)_{HSD} + \left(\frac{\delta\Gamma}{\Gamma_0}\right)_{s.d.} \tag{13}$$

the correction obtained with an effective point meson (PM), the hadronic structure dependent correction (HSD), and the short distance (s.d.) correction.

The point meson correction has been calculated long ago by Kinoshita [12],

$$\left(\frac{\delta\Gamma}{\Gamma_0}\right)_{PM} = \frac{\alpha}{2\pi}\left\{\frac{3}{2}\ln\frac{m_K^2}{\mu_{cut}^2} + 6\ln\frac{m_l}{m_K} + \frac{11}{4} - \frac{2}{3}\pi^2 + f\left(\frac{m_l}{m_K}\right)\right\} \tag{14}$$

where we have defined $f(r_l)$ in such a way that $f(0) = 0$:

$$f(r_l) = 4\left(\frac{1+r_l^2}{1-r_l^2}\ln r_l - 1\right)\ln(1-r_l^2) - \frac{r_l^2(8-5r_l^2)}{(1-r_l^2)^2}\ln r_l$$

$$+4\frac{1+r_l^2}{1-r_l^2}\mathrm{Li}_2(r_l^2) - \frac{r_l^2}{1-r_l^2}\left(\frac{3}{2} + \frac{4}{3}\pi^2\right) \tag{15}$$

Li_2 denotes the dilogarithmic function

$$\mathrm{Li}_2(x) = -\int_0^x dt\,\frac{\ln(1-t)}{t} \tag{16}$$

The leading contribution to the short distance correction in the limit $m_Z^2 \to \infty$ of a large Z boson mass has been given by Sirlin [13]

$$\left(\frac{\delta\Gamma}{\Gamma_0}\right)_{s.d} = \frac{2\alpha}{\pi}\ln\frac{m_Z}{\mu_{cut}} + \cdots \tag{17}$$

Finally, following [6], we classify the hadronic structure dependent corrections in terms of the dependence on the lepton mass,

$$\left(\frac{\delta\Gamma}{\Gamma_0}\right)_{HSD} = -\frac{\alpha}{\pi}\left\{C_1 + C_2\frac{m_l^2}{m_\rho^2}\ln\frac{m_\rho^2}{m_l^2} + C_{lms}\frac{m_K^2}{m_\rho^2}\ln\frac{m_\rho^2}{m_l^2} + C_3\frac{m_l^2}{m_\rho^2} + \cdots\right\} \tag{18}$$

where the coefficients C_i are independent on the lepton mass. The dots $\cdots$ indicate more suppressed lepton mass dependent terms. Note that in [6] the term proportional to C_{lms} is not considered. It diverges in the limit $m_l \to 0$ and therefore gives rise to a lepton mass singularity in the radiative correction. Note that such a mass singularity is allowed in spite of the Kinoshita-Lee-Nauenburg [12, 14] theorem, because the Born amplitude is proportional to m_l^2. According to a theorem by Marciano and Sirlin [16], however, the hadronic structure dependent effects on the coefficient of the lepton mass singularity cancel between the virtual $\delta\Gamma_v$ and the real photonic corrections $\delta\Gamma_r$, leaving the point meson result unchanged, if we integrate the real photons over the full phase space. However, this cancelation only works if we include the full $\Gamma_{SD} + \Gamma_{INT}$ contribution, which we do not do.

There is no way to calculate C_1 and C_3 in a model independent way. The coefficients C_2 and C_{lms}, on the other hand, are model independent and have been determined in [15]

It should be emphasized at this point that the extraction of the pseudoscalar decay constants f_M beyond the order $O(\alpha^0)$ is not unambiguous. One could by definition include part of the radiative correction into the f_M, e.g. the short distance part or the model dependent, lepton mass independent contribution from C_1. However, we choose to factor out all $O(\alpha)$ effects from the decay constants. This definition does not agree with the one used by Holstein [4], who absorbs process dependent terms proportional to $\ln(m_\pi/m_\mu)$ into f_π. See also the discussion in [5, 6] about this point. These ambiguities are of course due to the fact that f_π and f_K are no observables. The ambiguities cancel, however, in the ratios of the decay rates

$$R_M := \frac{\Gamma(M \to e\nu_e(\gamma))}{\Gamma(M \to \mu\nu_\mu(\gamma))} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_M^2 - m_e^2}{m_M^2 - m_\mu^2}\right)^2\left(1 + \delta R_M\right) \tag{19}$$

and the radiative correction δR_K is defined unambiguously and can in fact be predicted very precisely.

In the extraction of the decay constants f_π, f_K and in the calculation of the ratio R_K one has to decide which matching scale to use and what to do about the model dependent corrections.

In [6], the authors consider π_{l2} decays. Their approach is the following. They include only the leading model independent terms in the long distance part and use a rather low matching scale of $\mu_{cut} = m_\rho$. To extract f_π from $\Gamma(\pi \to \mu\nu_\mu(\gamma))$, they estimate C_1 very roughly by equating it with the effect of varying the cut-off by a factor of two. In the short distance correction, they include the also leading $O(\alpha^n)$ corrections by using the renormalization group and the leading logarithm of the order $O(\alpha\alpha_s)$.

In the ratio R_π, C_1 and all the short distance corrections, which the authors have considered, cancel because they are independent on the lepton mass. For the unknown coefficient C_3, which then is the main source of uncertainty, they consider a conservative range of $C_3 = 0 \pm 10$.

We will use a somewhat different approach. For the long distance part we use a phenomenological model, which includes vector (ρ, ω, Φ, $K^\star$, ...) and axial vector (K_1) resonances as explicit degrees of freedom. This allows us to push the matching scale up to $\mu_{cut} = 1 \ldots 2\,\mathrm{GeV}$, rendering the calculation of the short distance part more reliable. In the short distance correction, which we calculate using the parton model, we include small lepton mass dependent corrections which do not cancel in R_K. Within our model, we obtain an error estimate based on the dependence on the matching scale and on the hadronic parameters. We then compare the predictions from our model with the leading model independent corrections. This allows us to obtain an essentially model independent prediction by taking the results from our model as central values and by taking the full difference between our model and the model independent corrections as uncertainty. We will be able to show in this way that the ratio R_K can be predicted with an uncertainty of the order of $4 \cdot 10^{-4}$. So indeed K_{l2} decays offer the possibility of an important low energy precision test of the standard model.

3   Parametrization of the Amplitudes

We will be brief here an refer to [11, 17] for details.

We separate the loop integration into a long and a short distance part by splitting the photon propagator

$$\frac{1}{k^2 - \lambda^2} = \underbrace{\frac{1}{k^2 - \lambda^2}\frac{\mu_{cut}^2}{\mu_{cut}^2 - k^2}}_{\text{``long distance''}} + \underbrace{\frac{1}{k^2 - \mu_{cut}^2}}_{\text{``short distance''}} \tag{20}$$

using a matching scale $\mu_{cut} = (0.75 \ldots 3)\,\mathrm{GeV}$.

The long distance part, involving a regulated photon propagator, is calculated using a phenomenological model where mesons are the relevant degrees of freedom. The short distance part, involving a massive photon propagator is calculated using the parton model.

To calculate the long distance corrections, we start from the amplitudes obtained with an effective point like meson. These amplitudes are good approximations for very small momentum transfers only. Consider for example the amplitude V^μ for the coupling of a photon to two pions. In the point meson (P.M.) approximation it is given by

$$V^\mu(\pi^+(p)\pi^-(p') \to \gamma)^{(P.M.)} = ie(p - p')^\mu \tag{21}$$

However, this coupling *defines* the electromagnetic form factor F_π of the pion via

$$V^\mu(\pi^+(p)\pi^-(p') \to \gamma) =: ie F_\pi[(p + p')^2](p - p')^\mu \tag{22}$$

Therefore we should modify the effective point pion diagrams by multiplying this coupling by F_π. This modification in turn determines by gauge invariance the appropriate modification of the weak-electromagnetic seagull coupling $\pi\gamma W$.

Analogously in the kaonic case, we modify the point kaon coupling by multiplying it by F_K:

$$V^\mu(K^+(p)K^-(p') \to \gamma) \longrightarrow ie F_K[(p + p')^2](p - p')^\mu \tag{23}$$

In [11] a parameterization of F_K with a simple ρ dominance

$$F_K(t) = \mathrm{BW}_\rho(t) \tag{24}$$

was used. However, this assumes exact $SU(3)$ flavour symmetry, $m_\rho = m_\omega = m_\Phi$. We will now drop this assumption. Thus we have to consider the relative contributions of the ρ, the ω and the Φ to the form factor F_K. Assuming ideal mixing, $\Phi = (s\bar{s})$, we obtain

$$F_K(t) = \frac{1}{2}\mathrm{BW}_\rho(t) + \frac{1}{6}\mathrm{BW}_\omega(t) + \frac{1}{3}\mathrm{BW}_\Phi(t) \tag{25}$$

which we will use in the present paper.

In addition to the modified effective point meson diagrams, there are loop diagrams which are obtained from the hadronic structure dependent radiation (SD) by contracting the emitted photon with the lepton. If k^2 is small, where k is the momentum of the virtual photon, the relevant form factors H_V and H_A in these hadronic structure dependent loops will obviously be identical to F_V and F_A determining the radiative decay. However, they can additionally depend on k^2.

For $k^2 = 0$ (on-shell photon), we use the following ansatz

$$F_V^{(K)}(t) = F_V^{(K)}(0)\mathrm{BW}_{K^\star}(t)$$
$$F_A^{(K)}(t) = F_A^{(K)}(0)\mathrm{BW}_{K_1}(t) \tag{26}$$

where

$$\begin{aligned} F_V^{(K)}(0) &= 0.0955 \\ F_A^{(K)}(0) &= 0.0525 \pm 0.010 \end{aligned} \tag{27}$$

Here $F_V^{(K)}(0)$ has been obtained from flavour symmetry and the anomaly and $F_A^{(K)}(0)$ from this value for $F_V^{(K)}(0)$ and the measurement of the sum [1]. This value is slightly higher than $F_A^{(K)}(0) = 0.0410$ which corresponds to the $O(P^4)$ prediction of chiral perturbation theory [7], which will, however, be subject to higher order flavour symmetry breaking corrections.

$\mathrm{BW}_X(t)$ denotes a normalized Breit-Wigner amplitude

$$\mathrm{BW}_X(t) = \frac{m_X^2}{m_X^2 - t} \tag{28}$$

The virtual corrections are calculated in the Euclidean k^2 region, and so we will not include an imaginary part of the mass.

For $k^2 \neq 0$, we adopt the following ansatz for the momentum dependence, which is based on a double vector meson dominance:

$$H_V^{(K)}(k, p) = \mathrm{BW}_V(k^2) F_V^{(K)}[(k - p)^2]$$
$$H_A^{(K)}(k, p) = \mathrm{BW}_V(k^2) F_A^{(K)}[(k - p)^2] \tag{29}$$

where p is the momentum of the decaying kaon. Note that here we assume $m_V = m_\rho = m_\omega = m_\Phi$. We do not calculate the $SU(3)$ flavour symmetry breaking effects. This will be justified below by the observation that the dependence of the result on m_V is very small.

In order to obtain the short distance corrections, we calculate the one-loop corrections $\delta \mathcal{A}$ to the operator $\mathcal{A}_0 = [\bar{u}_\nu \gamma^\mu \gamma_- u_l] [\bar{u}_s \gamma_\mu \gamma_- u_u]$. Neglecting all masses except for m_l and μ_{cut}, we obtain

$$\left(\frac{\delta\Gamma}{\Gamma_0}\right)_{short\ dist.} \approx \frac{2\alpha}{\pi} \frac{1}{m_l^2 - \mu_{cut}^2} \left(m_l^2 \ln \frac{m_Z}{m_l} - \mu_{cut}^2 \ln \frac{m_Z}{\mu_{cut}}\right) \tag{30}$$

which has to be compared to Sirlin's logarithm $2\alpha/\pi \ln(m_Z/\mu_{cut})$ [13]. Of course, if we do not neglect m_μ, we should also take the meson mass m_M into account. However, the contributions depending on m_M cancel in the ratios R_M, whereas in the radiative correction to the decay rates themselves, μ_{cut} dominates anyway.

Note that for this leading logarithm, the correction to the quark level short distance amplitude $\delta \mathcal{A}$ is proportional to the Born amplitude $\mathcal{A}_0$. Thus the same logarithm is involved in the corrections to the hadronic amplitude without any model dependence resulting from hadronization.

While in the correction to the individual decay rates $M \to l\nu_l(\gamma)$ this leading logarithm dominates the short distance correction, it depends only very little on the lepton mass and thus cancels almost completely in the ratios δR_M. Therefore in the case of these ratios we go beyond the leading logarithm and calculate the full one-loop short distance correction.

The complete result for $\delta \mathcal{A}$ is no longer proportional to the Born amplitude $\mathcal{A}_0$, and furthermore it depends on the relative momentum of the two quarks. Therefore we project onto the $J^P = 0^-$ component and integrate over the relative momentum $u \times p$ of the quarks in the infinite momentum frame ($u = -1 \ldots +1$). The result can be written in the form

$$\left(\delta R_K\right)_{short\ dist.} = \frac{3}{2f_K} \int_{-1}^{+1} du\ \Phi_K(u) r_K(u) \tag{31}$$

Figure 1: Radiative correction to $\Gamma(K \to \mu\nu_\mu)$, using different choices for the hadronic parameters: Standard choice (I, solid) and variations (II, dashed and III, dotted)
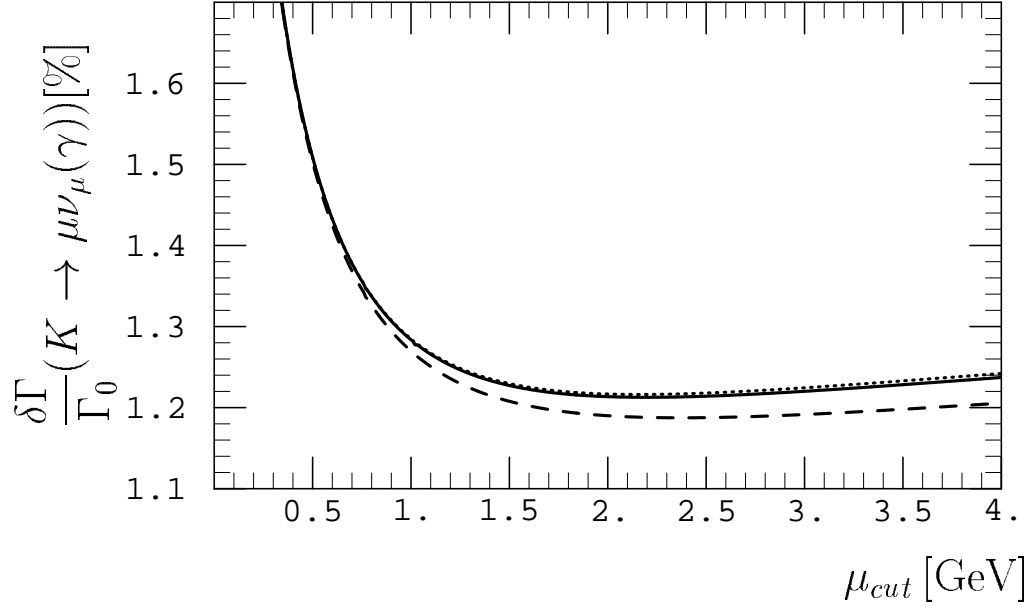
Here $\Phi_K(u)$ is an unknown parton distribution function (kaon wave function), whereas $r_K(u)$ is calculated from the short distance diagrams for arbitrary u. We find, however, that $r_K(u)$ depends only very little on u, and we can approximate it by its values at $u = 0$, where the wave is presumably peaked:

$$\left(\delta R_K\right)_{short\ dist.} \approx r_K(u = 0)\frac{3}{2f_K} \int_{-1}^{+1} du\ \Phi(u) = r_K(0) \tag{32}$$

where the last equation follows from the Brand-Preparata sum rule [18].

4   Numerical Results

Adding up long and short distance corrections, we obtain the full radiative correction. This depends on the choice of the matching scale μ_{cut} and on the hadronic parameters.

In Fig. 1 we display the correction to the decay rate $\Gamma(K \to \mu\nu_\mu(\gamma))$ in variation with μ_{cut}, using three different choices for the hadronic parameters. The solid line (I) corresponds to the central values given above. The dashed (II) and the dotted (III) lines are obtained by varying the hadronic parameters, viz. $F_A(0)$ and the relative contributions of higher radial excitations in F_K and H_V. Note that the matching scale dependence above $\mu_{cut} = 1\,\mathrm{GeV}$ is very moderate, indicating that our phenomenological model for the long distance part is indeed rather reasonable.

Choosing $\mu_{cut} = 1.5\,\mathrm{GeV}$ as a central value for we find from Fig. 1 the following $O(\alpha)$ correction to the decay rate

$$\frac{\delta\Gamma}{\Gamma_0}(K \to \mu\nu_\mu(\gamma)) = (1.23 \pm 0.13 \pm 0.02)\% + O(\alpha^2) + O(\alpha\alpha_s)$$

$$= (1.23 \pm 0.13)\% + \cdots \qquad (33)$$

The first error given (0.13%) is the matching uncertainty, obtained by varying μ_{cut} by a factor of two $(0.75\ldots3\,\mathrm{GeV})$, the second one (0.02%) is the uncertainty from the hadronic parameters.

A few comments are in order:

1. As we have already states above, this radiative correction is not a physical observable and therefore is not defined unambiguously. The definition we have adopted is to include *all* $O(\alpha)$ corrections in the number 1.23% given. Part of this might be absorbed into f_K by definition, but we choose not to do so.

2. We employ m_Z as an ultra-violet cut-off for the short distance corrections, according to the general theorems of Sirlin [9] on short-distance electroweak corrections to semileptonic processes. But this implies that there is an arbitrariness of the order of $\alpha/(2\pi) \times O(1) \approx 0.1\%$ in the definition of the radiative correction, because a change of the cut-off scheme would induce a change of the result of this order. Note that the error of the $O(\alpha)$ correction which we have determined is of the same order of magnitude as this inherent ambiguity.

3. The error $\pm0.13\%$ quoted above is the uncertainty of the $O(\alpha)$ correction only. In [6] the authors have summed up the leading $O(\alpha^n)$ corrections for the dominant contribution $2\alpha/\pi\, ln(m_Z/\mu_{cut})$ in the short distance part, using the renormalization group. This leads to an enhancement of the short distance correction of 0.13%. Furthermore they considered the leading QCD short distance correction, which decreases the short distance part by -0.03%. Similar $O(\alpha^n)$ effects should be considered in the long distance part.

Taking into account these higher order short distance corrections, and considering the uncertainties discussed above, we will use the following value in order to extract f_K:
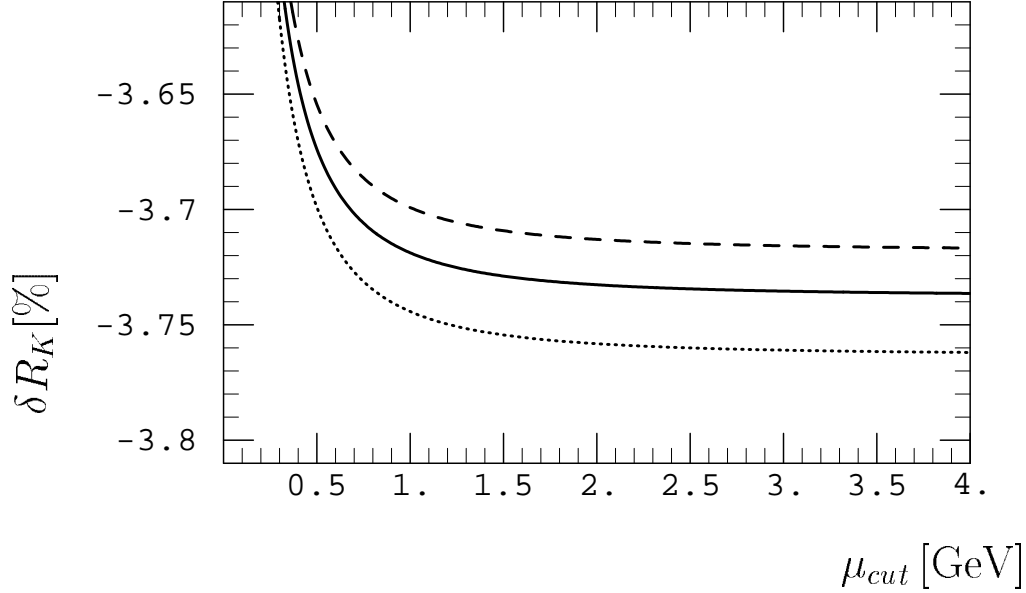
$$\frac{\delta\Gamma}{\Gamma_0}(\pi \to \mu\nu_\mu(\gamma)) = (1.3 \pm 0.2)\% \qquad (34)$$

Using the experimental input parameters given in the introduction, we obtain

$$f_K = (112.4 \pm 0.9 \pm 0.1)\,\mathrm{MeV} = (112.4 \pm 0.9)\,\mathrm{MeV} \qquad (35)$$

where the first error is due to V_{us} and the second one to the radiative correction.

Figure 2: Radiative correction to the ratio R_K, using different choices for the hadronic parameters: Standard choice (I, solid) and variations (II, dashed and III, dotted)

From a similar analysis of π_{l2} decays, we obtain [17]

$$f_\pi = (92.14 \pm 0.09 \pm 0.09)\,\text{MeV} = (92.1 \pm 0.1)\,\text{MeV} \tag{36}$$

(The first error, ± 0.09, is due to V_{ud}, and the second one to the radiative correction.)

This implies

$$\frac{f_K}{f_\pi} = 1.22 \pm 0.01 \tag{37}$$

which agrees with the conventional value [2]. Note, however, that in our result, the error is entirely dominated by the uncertainty of V_{us}.

Let us now come to the prediction for the ratio R_K of the electronic and muonic decay modes of the kaon. In contrast to f_K, this is a physical observable and therefore free from ambiguities in its definition. In Fig. 2, we display the radiative correction δR_K of the ratio using the same standard parameter set (I) and variations (II) and (III) we used above for $\Gamma(K \to \mu\nu_\mu(\gamma))$. From this, we obtain the $O(\alpha)$ correction

$$\delta R_K = -(3.729 \pm 0.023 \pm 0.025)\% + O(\alpha^2) \tag{38}$$

where the first error (0.023%) gives the matching uncertainty, estimated by varying μ_{cut} from 0.75 up to 3 GeV, and the second error (0.025%) arises from the uncertainties in the hadronic parameters.

The error estimate, taking from the dependence on the matching scale and on the hadronic parameters, actually gives only a lower limit of the true error, although the

Table 1: Contributions from photons with momenta within a given range to the various radiative corrections

Contribution from photons with $\lvert k_E \rvert$ in the range [MeV]:	Radiative correction to: (all numbers in units of %)		
	$K \to e\nu_e$	$K \to \mu\nu_\mu$	R_K
$0 \ldots 125$	-3.926	-0.515	3.411
$125 \ldots 250$	-0.450	-0.246	-0.204
$250 \ldots 500$	-0.201	-0.115	-0.086
$500 \ldots 750$	-0.000	0.020	-0.020
$750 \ldots 1000$	0.047	0.055	-0.008
$1000 \ldots 1500$	0.121	0.126	-0.005
$1500 \ldots 3000$	0.322	0.318	0.004
$3000 \ldots m_Z$	1.586	1.584	0.002

very small dependence of the result on the matching scale supports some confidence in our results. We will, nevertheless, scrutinize the model dependence further in two ways: Firstly we examine, from which scales the contributions to δR_K actually come. Secondly we compare the results from our model with the leading model independent contributions.

Consider Tab. 1. We display the contribution to the radiative corrections from photons with given Euclidean momenta $\lvert k_E \rvert$. We find that the contributions to the individual decay rates at large $\lvert k_E^2 \rvert$ are quite sizeable. However, the contributions to the electronic and the muonic mode approach each other for large momenta, such that the contribution to the correction to the ratio R_K comes predominantly from very small scales. Uncertainties from the hadronics in the long distance regime and from QCD and wave function corrections in the short distance regime are large in the intermediate energy range of about $\lvert k_E \rvert = 500 \ldots 3000$ MeV only. The total contribution within this range is given by 0.037%, where we have added the absolute values in order to take care of cancellations. So by far the largest part of the radiative correction comes from the region below 500 MeV, where the model dependence is very small.

Next we will compare our model with model independent estimates. In [15] the author calculates the leading logarithmic corrections to the ratio R_π, which arise from hadronic structure dependent effects. He proves that this leading contribution is model independent, viz. independent on the form of the hadronic form factors. The only assumption needed is that the scale over which the form factor vary is given by a large hadronic scale of the order of m_ρ.

The leading correction $\delta R^{(HSD)}$ can be separated into three contributions

$$\delta R_K^{(HSD)} = \delta R_K^{(VMD)} + \delta R_K^{(v)} + \delta R_K^{(a)} \tag{39}$$

Table 2: The different contributions adding up to the total radiative corrections (the numbers in brackets are obtained assuming exact $SU(3)$ flavour symmetry

contribution	$\delta R_\pi \, [\%]$
(1) effective point meson	-3.786
(2) vector meson dominance in the point meson loops	0.048 (0.055)
(3) hadronic structure dependent loops proportional to $F_V(0)$	0.135
(4) hadronic structure dependent loops proportional to $F_A(0)$	-0.134
(5) cutting off the long distance part at $\mu_{cut} = 1.5\,\mathrm{GeV}$	0.003
(6) short distance corrections	0.006
(7) total	-3.729 (-3.723)

where $R_K^{(VMD)}$ is due to the vector meson dominance of the kaon electromagnetic form factor, and $R_K^{(v/a)}$ correspond to virtual corrections proportional to the form factors $F_{V/A}(0)$. Translating the results to the kaon case, we find

$$\delta R_K^{(VMD)} = \frac{3\alpha}{\pi}\frac{m_\mu^2}{m_\rho^2}\ln\frac{m_\rho^2}{m_\mu^2} = 5.2 \cdot 10^{-4}$$

$$\delta R_K^{(v)} = \frac{\alpha}{6\pi}\frac{F_V(0)}{\sqrt{2}m_K f_K}\left[m_K^2 \ln\frac{m_\mu^2}{m_e^2} + 4m_\mu^2 \ln\frac{m_\rho^2}{m_\mu^2}\right]$$
$$= (14.8 + 1.0)\cdot 10^{-4} = 15.8 \cdot 10^{-4}$$

$$\delta R_K^{(a)} = -\frac{\alpha}{6\pi}\frac{F_A(0)}{\sqrt{2}m_K f_K}\left[m_K^2 \ln\frac{m_\mu^2}{m_e^2} + 7m_\mu^2 \ln\frac{m_\rho^2}{m_\mu^2}\right]$$
$$= -(8.1 + 1.0)\cdot 10^{-4} = -9.1 \cdot 10^{-4} \qquad (40)$$

Note that $\delta R_K^{(v)}$ and $\delta R_K^{(a)}$ consist of two parts. The first one, being proportional to $m_K^2 \ln(m_\mu^2/m_e^2)$ and giving rise to a lepton mass singularity, results from contributions to C_{lms} (see Eqn. 18), and the second one, proportional to $m_\mu^2 \ln(m_\rho^2/m_\mu^2)$, results from the contributions to C_2.

Now let us compare these numbers with the corresponding results from our model, see Tab. 2. In the first row (1) we give the results obtained with an effective point meson. In the second row (2) we display the change of the result, when switching on the vector meson dominance in the meson electromagnetic form factor. In rows (3) and (4) we give the contributions from those loop diagrams which correspond to the SD part in the real radiation. In (2)–(4), we have extended the loop integration up to m_Z, and so in row (5) we display the change when cutting off the long distance correction at $\mu_{cut} = 1.5\,\mathrm{GeV}$, and in row (6) we give the short distance correction.

Now comparing the model independent numbers 5.2, 15.8 and -9.1 with our numbers 4.8, 13.5 and -13.4 (hadronic structure dependent corrections in units of 10^{-4}), we find that the model dependent contribution in the long distance give rise to an uncertainty of the order of $\pm 3.9 \cdot 10^{-4}$. This number has been obtained by adding up quadratically the differences between the model independent estimates and the results from our model.

Remember that we have included the $SU(3)$ flavour symmetry breaking in the electromagnetic form factor of the kaon, i.e. in row (2) of Tab. 2). But in the vector meson dominance of the photon coupling in the hadronic structure dependent loops, row (3) and (4), we used $m_\rho = m_\omega = m_\Phi = m_V = 768\,\mathrm{MeV}$. However, as can be seen from (40), the contribution which depends on the vector meson mass m_V is very small, $O(1\cdot 10^{-4})$, in both cases, and the correction is strongly dominated by the modification of the ratio of lepton mass singularities. In fact we have checked that the result from our model in rows (3) and (4) changes only by 0.001% if we use increase m_V up to $1\,\mathrm{GeV}$. So this approximation of flavour symmetry does not induce a significant uncertainty.

Let us now consider our result for the short distance correction, $0.6 \cdot 10^{-4}$, which includes contributions which depend on the kaon wave function. However, we find (compare Eqns. (31–32))

$$
\begin{aligned}
r_K(u = 0) &= 0.6 \cdot 10^{-4} \\
r_K(u = 1) &= 1.0 \cdot 10^{-4}
\end{aligned}
\tag{41}
$$

So for any choice of the pion wave function, the resulting short distance correction will be within the range $(0.6 \cdots 1.0) \cdot 10^{-4}$. This should also be compared with contribution from the leading lepton mass dependent logarithm,

$$
\frac{2\alpha}{\pi} \frac{m_\mu^2}{m_\mu^2 - \mu_{cut}^2} \ln \frac{m_\mu}{\mu_{cut}} = 0.6 \cdot 10^{-4}
\tag{42}
$$

Thus in the short distance part, the model dependent contribution is within the range $(0.0 \cdots 0.4) \cdot 10^{-4}$.

And so by adding up all quadratically the model dependent corrections, we obtain $3.9 \cdot 10^{-4}$, which we consider as an upper limit for the uncertainty of our result for the $O(\alpha)$ radiative correction.

However, we have to worry about corrections of higher order, $O(\alpha^n)$. Given the fact that the $O(\alpha)$ correction is dominated by the contribution from the lepton mass singularity, $-\frac{3\alpha}{\pi} \ln \frac{m_\mu}{m_e}$, in [6] the leading higher order corrections are estimated by summing up all such logs via the renormalization group, yielding a correction to R_K of

$$
1 - \frac{\left(1 - \frac{2\alpha}{3\pi} \ln \frac{m_\mu}{m_e}\right)^{9/2}}{1 - \frac{3\alpha}{\pi} \ln \frac{m_\mu}{m_e}} = 5.5 \cdot 10^{-4}
\tag{43}
$$

And so our final prediction for δR_K is

$$
\delta R_K = -(3.729 \pm 0.039 + 0.055 \pm 0.01)\% = (3.78 \pm 0.04)\%
\tag{44}
$$

In the sum, the first number (-3.729) is the central value and the second number (0.039) the uncertainty of the $O(\alpha)$ correction. The third number $(+0.055)$ is the leading higher order correction and ± 0.01 our estimate of the next-to-leading correction.

For the ratio R_K this implies

$$
R_K = R_K^{(0)}\left(1 + \delta R_K\right) = 2.569 \cdot 10^{-5} \times \left(1 - 0.0378 \pm 0.0004\right) = (2.472 \pm 0.001) \cdot 10^{-5}
\tag{45}
$$

in good agreement with the particle data group value $R_K = (2.45 \pm 0.11) \cdot 10^{-5}$ [1]. It should be noted, however, that this number is based on experiments published in the the period 1972–1976. Thus there has been no experimental progress on this number within almost 20 years, a situation not unusual in kaon physics.

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