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The $\pi\pi$ scattering amplitude in chiral perturbation theory

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Abstract

We discuss $\pi\pi$ scattering in the framework of chiral perturbation theory. In particular, we recall the predictions [1] for the threshold parameters and for the phase shift difference $\delta_0^0 - \delta_1^1$.

1 Notation

The scattering amplitude for $\pi\pi$ scattering,

$$\pi^i(p_1)\pi^k(p_2) \to \pi^l(p_3)\pi^m(p_4) \quad ,$$

reads

$$\begin{aligned}\langle \pi^m(p_4)\pi^l(p_3)\text{out}|\pi^i(p_1)\pi^k(p_2)\text{in}\rangle &= \langle \pi^m(p_4)\pi^l(p_3)\text{in}|\pi^i(p_1)\pi^k(p_2)\text{in}\rangle \\ &+ i(2\pi)^4\delta^{(4)}(P_f - P_i)T^{ik;lm}(s,t,u) \quad ,\end{aligned}$$

where $T^{ik;lm}(s,t,u)$ is a Lorentz invariant function of the standard Mandelstam variables

$$\begin{aligned} s &= (p_1+p_2)^2 = 4(M_\pi^2 + q^2) \quad , \\ t &= (p_3-p_1)^2 = -2q^2(1-\cos\theta) \quad , \\ u &= (p_4-p_1)^2 = -2q^2(1+\cos\theta) \quad . \end{aligned}$$

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q (θ) is the center-of-mass momentum (center-of-mass scattering angle). On account of isospin symmetry, the amplitude $T^{ik;lm}(s,t,u)$ may be expressed in terms of a single amplitude $A(s,t,u) = A(s,u,t)$,

$$T^{ik;lm}(s,t,u) = \delta^{ik}\delta^{lm}A(s,t,u) + \delta^{il}\delta^{km}A(t,s,u) + \delta^{im}\delta^{kl}A(u,t,s) \ .$$

To compare the calculated amplitude with data on $\pi\pi$ scattering [2], one expands the combinations with definite isospin in the s-channel

$$\begin{aligned} T^0(s,t) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t) \\ T^1(s,t) &= A(t,u,s) - A(u,s,t) \\ T^2(s,t) &= A(t,u,s) + A(u,s,t) \end{aligned}$$

into partial waves,

$$T^I(s,t) = 32\pi \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)t_l^I(s) \ .$$

Unitarity implies that in the elastic region $4M_\pi^2 < s < 16M_\pi^2$ the partial wave amplitudes t_l^I are described by real phase shifts δ_l^I,

$$t_l^I(s) = \left(\frac{s}{s-4M_\pi^2}\right)^{1/2} \frac{1}{2i}\{e^{2i\delta_l^I(s)} - 1\} \ .$$

The behaviour of the partial waves near threshold is of the form

$$\text{Re}\ t_l^I(s) = q^{2l}\{a_l^I + q^2 b_l^I + O(q^4)\} \ .$$

The quantities a_l^I are referred to as the $\pi\pi$ scattering lengths.

2 Data

At low energies, the difference $\delta_0^0 - \delta_1^1$ may be extracted in a theoretically clean manner from data on K_{e4} decays [3]. In the high-statistics CERN-Saclay experiment [4], it has been measured in five energy bins – the values are displayed in table III in Ref. [4] and plotted in figure 1 below. For earlier determinations of $\delta_0^0 - \delta_1^1$ from data on K_{e4} decays, see table IV in Ref. [4]. Furthermore, the measurement of the lifetime of pionic atoms allows one to determine the difference $|a_0^0 - a_0^2|$. A corresponding experiment has been proposed at CERN [5].

3 Theory

In Ref. [1], the amplitude $A(s,t,u)$ has been evaluated in the framework of chiral $SU(2)\times SU(2)$ to one-loop accuracy,

$$
\begin{aligned}
A(s,t,u) &= \frac{s-M_\pi^2}{F_\pi^2} + B(s,t,u) + C(s,t,u) + O(E^6) \ , \\
B(s,t,u) &= (6F_\pi^4)^{-1}\{3(s^2-M_\pi^4)\bar{J}(s) \\
&+ [t(t-u) - 2M_\pi^2 t + 4M_\pi^2 u - 2M_\pi^4]\bar{J}(t) + (t\leftrightarrow u)\} \ , \\
C(s,t,u) &= (96\pi^2F_\pi^4)^{-1}\left\{2(\bar{l}_1 - \frac{4}{3})(s-2M_\pi^2)^2\right. \\
&+ \left.(\bar{l}_2 - \frac{5}{6})[s^2 + (t-u)^2] + 12sM_\pi^2(\bar{l}_4 - 1) + 3M_\pi^4(5 - 4\bar{l}_4 - \bar{l}_3)\right\} \ , \qquad (1)
\end{aligned}
$$

where F_π is the pion decay constant, and $\bar{l}_1,\ldots,\bar{l}_4$ are four of the ten low-energy parameters that parametrize the effective lagrangian at next-to-leading order [1]. The constants $\bar{l}_{1,2}$ can e.g. be determined by measuring the D-wave scattering lengths a_2^0 and a_2^2 [1],

$$
\begin{aligned}
\bar{l}_1 &= 480\pi^3F_\pi^4(-a_2^0 + 4a_2^2) + 49/40 + O(M_\pi^2) \ , \\
\bar{l}_2 &= 480\pi^3F_\pi^4(a_2^0 - a_2^2) + 27/20 + O(M_\pi^2) \ ,
\end{aligned}
$$

whereas the constant $\bar{l}_4$ is related to the scalar radius of the pion [1],

$$
\bar{l}_4 = \frac{13}{12} + \frac{8\pi^2F_\pi^2}{3}\langle r^2\rangle_S^\pi + O(M_\pi^2) \ .
$$

The S and P wave threshold parameters are

$$
\begin{aligned}
a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2}\left\{1 + \frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi + \frac{200\pi F_\pi^2M_\pi^2}{7}(a_2^0 + 2a_2^2) - \frac{M_\pi^2}{672\pi^2F_\pi^2}(15\bar{l}_3 - 353) + O(M_\pi^4)\right\}, \\
b_0^0 &= \frac{1}{4\pi F_\pi^2}\left\{1 + \frac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi + 40\pi F_\pi^2M_\pi^2(a_2^0 + 5a_2^2) + \frac{39M_\pi^2}{64\pi^2F_\pi^2} + O(M_\pi^4)\right\} \ , \\
a_1^1 &= \frac{1}{24\pi F_\pi^2}\left\{1 + \frac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi + 80\pi F_\pi^2M_\pi^2(a_2^0 - \frac{5}{2}a_2^2) + \frac{19M_\pi^2}{576\pi^2F_\pi^2} + O(M_\pi^4)\right\} \ , \\
b_1^1 &= \frac{7}{2160\pi^3F_\pi^4} + \frac{10}{3}(a_2^0 - \frac{5}{2}a_2^2) + O(M_\pi^2) \ . \qquad (2)
\end{aligned}
$$

The numerical values obtained by evaluating these improved low energy theorems are given in table 1 (in units of M_{π^+}). In column 2 we give the soft pion predictions of Weinberg [11], obtained from the terms proportional to F_π^{-2} in Eq. (2). The third column contains the results of an analysis of the data as reported by Petersen in the compilation of coupling constants and low-energy parameters [7]. The entries in the fourth column correspond to

Table 1: Threshold parameters that are relevant in K_{e4} experiments, in units of M_{π^+}.

	Soft pions	Experiment	Improved low energy theorems	size of correction 3:1
a_0^0	0.16	0.26 ± 0.05	0.20 ± 0.005	1.28
b_0^0	0.18	0.25 ± 0.03	0.25 ± 0.02	1.37
a_1^1	0.030	0.038 ± 0.002	0.038 ± 0.003	1.26
b_1^1			$(5 \pm 3) \times 10^{-3}$	

the representation (2). Here, we have used the experimental D-wave scattering lengths and the scalar radius of the pions as an input, together with the value for $\bar{l}_3$ determined in [1][2].

Remark: The errors quoted in column 4 are obtained by adding the uncertainties in $\langle r^2\rangle_S^\pi, a_2^0, a_2^2$ and in $\bar{l}_3$ in quadrature. They measure the accuracy, to which the first order corrections can be calculated, and do not include an estimate of the contributions due to higher order terms. Work to determine those reliably is in progress [10]. Note also, that the S-wave scattering lengths vanish in the chiral limit and we therefore have to expect relatively large electromagnetic corrections to these quantities. To illustrate: if we use the mass of the neutral pion rather than M_{π^+}, the prediction for a_0^0 is lowered by 0.016 (at a fixed value of $\bar{l}_{1,2,3,4}$). End of remark.

Turning now to the energy dependence of the phase shifts, we note that these may be worked out from the explicit expression for the scattering amplitude given above by use of [12]

$$\delta_l^I(s) = (1 - 4M_\pi^2/s)^{1/2} \mathrm{Re}\, t_l^I(s) + O(E^6) \quad .$$

In the following, we concentrate on the phase shift difference

$$\Delta = \delta_0^0 - \delta_1^1 \quad ,$$

and obtain

$$\begin{aligned}
\Delta &= \Delta^{(2)} + \Delta^{(4)} + O(E^6) \quad , \\
\Delta^{(2)} &= \frac{\rho M_\pi^2}{96\pi F_\pi^2}(5x+1) \quad , \\
\Delta^{(4)} &= \rho M_\pi^4 \left\{ \frac{h(x)}{55296\rho^4 x^2 \pi^3 F_\pi^4} + \frac{(5x+1)\langle r^2\rangle_S^\pi}{288\pi F_\pi^2} + \frac{5}{48}(x^2+8x+12)a_2^0 \right. \\
&\quad \left. + \frac{25}{48}(7x^2 - 28x + 24)a_2^2 - \frac{5\bar{l}_3}{1024\pi^3 F_\pi^4} \right\} \quad , \qquad (3)
\end{aligned}$$

[2] To be more specific, we use $\langle r^2\rangle_S^\pi = 0.60 \pm 0.05\,\mathrm{fm}^2$ [6], $\bar{l}_3 = 2.9 \pm 2.4$ [1], $a_2^0 = (17 \pm 3) \cdot 10^{-4} M_\pi^{-4}$ [7], $a_2^2 = (1.3 \pm 3) \cdot 10^{-4} M_\pi^{-4}$[7], $M_\pi = 139.57$ MeV [9], $F_\pi = 92.4$ MeV [9].

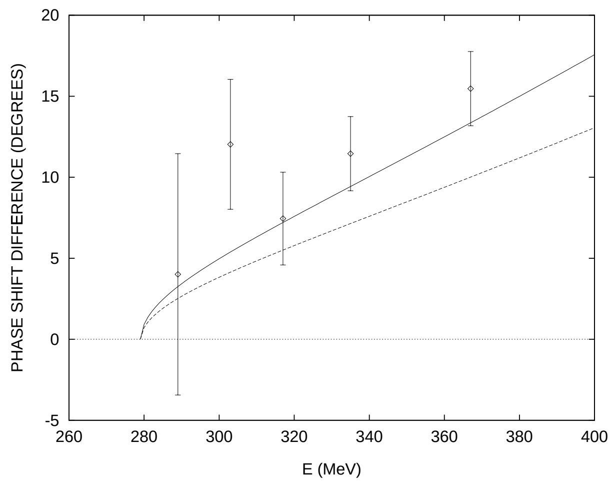

Figure 1: The phase shift difference $\Delta = \delta_0^0 - \delta_1^1$ from the chiral expansion. The data are from Ref. [4]. The solid line stands for the result at one-loop accuracy, $\Delta = \Delta^{(2)} + \Delta^{(4)}$, whereas the dashed line displays the leading order term $\Delta^{(2)}$, see Eq. (3) .

where

$$\begin{aligned} h(x) &= \rho^2(689x^3 - 4630x^2 + 11396x - 15240)x \\ & \quad 6\rho(50x^4 - 460x^3 + 1319x^2 - 1028x - 112)h_1(x) \\ & \quad +36(3x^2 - 36x + 106)h_1^2(x) \ , \\ h_1(x) &= \ln\left\{\frac{1-\rho}{1+\rho}\right\} \ , \ \rho = (1-4/x)^{1/2} \ , \ x = s/M_\pi^2 \ . \end{aligned} \tag{4}$$

The quantity $\Delta^{(2)}$ stems from the leading order term $(s - M_\pi^2)/F_\pi^2$ in Eq. (1). Numerical results are displayed in the figures. In Fig. 1, we show the data from Ref. [4], together with the full one-loop result $\Delta = \Delta^{(2)} + \Delta^{(4)}$ (solid line) and the leading order term $\Delta^{(2)}$ (dashed line). In Fig. 2, the various contributions to the next-to-leading order term $\Delta^{(4)}$ are resolved. Notice that the contribution from the low-energy constant $\bar{l}_3$ is very small.

For a discussion of the $\pi\pi$ amplitude in the framework of generalized chiral perturbation theory, see Ref. [13].

4 Improvements at DAΦNE

According to Baillargeon and Franzini [3], DAΦNE will allow one to determine the phase shift difference $\delta_0^0 - \delta_1^1$ with considerably higher precision than available now [4]. It will, therefore, be of considerable interest to confront the above predictions with these data. In particular, we note that a value of $a_0^0 = 0.26$ is not compatible with the chiral prediction $a_0^0 = 0.20$.

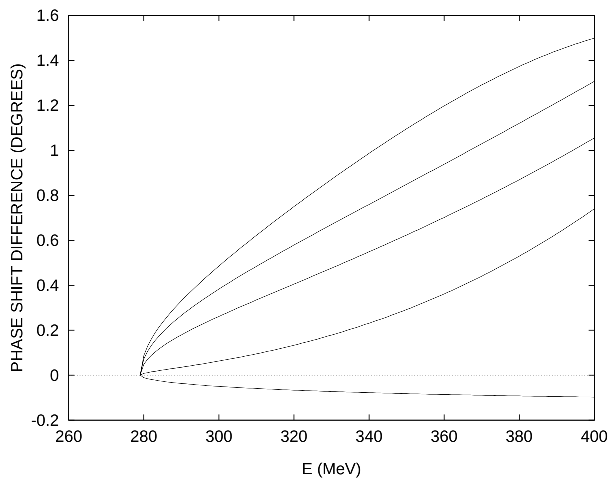

Figure 2: The various terms in $\Delta^{(4)}$ according to Eq. (3). From top to bottom, the solid lines display the contributions proportional to $h(x), \langle r^2\rangle^\pi_S, a^0_2, a^2_2$ and $\bar{l}_3$ in order. The sum of these terms generates the difference between the solid and the dashed line in figure 1.

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