INFN - Laboratori Nazionali di Frascati, May 1995
From: **The Second DaΦne Physics Handbook**[1]
Chapter 1: CP and CPT Violation
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Estimates of ϵ'/ϵ

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Abstract

We review the latest calculations of the next-to-leading $\Delta S=1$ effective Hamiltonian, relevant for $K\to\pi\pi$ transitions. Numerical results for the Wilson coefficients are given for different regularization schemes. Predictions of ϵ'/ϵ, obtained using different approaches to evaluate the relevant hadronic matrix elements, are compared. Given the present value of the top mass, $m_t=(174\pm 17)$ GeV, all the analyses, in spite of the large theoretical uncertainties, indicate that the value of ϵ'/ϵ is smaller than 1×10^{-3}.

1 Introduction

The understanding of mixing and CP-violation in hadronic systems is one of the crucial tests of the Standard Model. In the last few years considerable theoretical and experimental effort has been invested in this subject.

On the theoretical side, the complete next-to-leading expressions of the relevant effective $\Delta S=1$, $\Delta S=2$, $\Delta B=1$ and $\Delta B=2$ Hamiltonians have been computed [1]–[5], thus reducing the theoretical uncertainties[2]. Moreover, there is now increasing theoretical evidence that the value of the pseudoscalar B-meson decay constant is large, $f_B\sim 200$ MeV, and that the B–$\bar{B}$ parameter B_B is quite close to one. This strongly constrains the Cabibbo-Kobayashi-Maskawa parameters and it has remarkable consequences on CP-violation in

[1]Supported by the INFN, by the EC under the HCM contract number CHRX-CT920026 and by the authors home institutions

[2]Indeed only the top contribution to the $\Delta S=2$ Hamiltonian is fully known, at the next-to-leading order. To our knowledge, the other contributions have been only partially computed [6].

B decays, see refs. [7, 8]. Still, the evaluation of hadronic matrix elements is subject to large uncertainties, that are particularly severe for ϵ'/ϵ, where important cancellations of different contributions occur for large values of the top mass. Indeed, a significative reduction of the theoretical uncertainty on ϵ'/ϵ would require a substantial improvement in the calculation of the hadronic matrix elements, either from lattice simulations or from other non-perturbative techniques.

On the experimental side, more accurate measurements of the mixing angles are now available and the mass of the top quark, experimental evidence of which has recently been found by CDF [10], is constrained within tight limits [11]. Still, in spite of very accurate measurements, the experimental results for the CP violating parameter ϵ'/ϵ are far from conclusive [12, 13]. A better accuracy, at the level of 1×10^{-4}, should be achieved by the experiments of the next generation.

In the following, we briefly introduce the $\Delta S=1$ effective Hamiltonian and summarize the main results of the next-to-leading calculation of the relevant Wilson coefficients. An updated analysis of ϵ'/ϵ, along the lines followed in refs. [7]–[9], is presented. Particular emphasis is devoted to a realistic evaluation of the uncertainties. We also compare the results of refs. [7]–[9] to the next-to-leading order analysis of ref. [14]. In section 2, the basic formulae, which define the CP-violation parameters ϵ and ϵ', are presented; in section 3, the definition of the Cabibbo–Kobayashi–Maskawa matrix and the notation used in this work are introduced; in sections 4 and 5, we give several details about the $\Delta S=1$ effective Hamiltonian relevant for direct CP-violation. In particular, the Wilson coefficients in different regularization schemes are reported. In section 7, we give the formulae which has been used to obtain the theoretical predictions; in section 8, the theoretical predictions from the more recent analyses are given. Further details, including a theoretical discussion of the matching conditions, of the B-parameters and of the uncertainties coming from the choice of Λ_{QCD}, the renormalization scale, etc. can be found in ref. [9].

2 CP-violation in $K \to \pi\pi$ decays

In this section, we introduce the parameters ϵ and ϵ' that describe CP-violation in the neutral kaon-system. In the following, we assume CPT symmetry. A comprehensive discussion of the general case, including CPT-violation, can be found in ref. [15].

There are two possible sources of CP-violation in the decays of the neutral kaons into two pions. CP-violation can take place both in the kaon mixing matrix and at the decay vertices. Let us consider the mixing first. The most general CPT-conserving Hamiltonian of the K^0–$\bar{K}^0$ system at rest can be written as

$$H = M - \frac{i}{2}\Gamma = \begin{pmatrix} M_0 & M_{12} \\ M_{12}^* & M_0 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} \Gamma_0 & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_0 \end{pmatrix}, \tag{1}$$

where the bra (ket) can be represented as the two component vector $\langle K^0| \equiv (1,0)$, $\langle \bar{K}^0| \equiv (0,1)$, M is the "mass" matrix and Γ is the "width" matrix. Both M_0 and Γ_0 are real.

Notice that there is some freedom in the definition of the phases of the kaon field. In particular, one can make the change

$$|K^0\rangle \to e^{i\alpha}|K^0\rangle\,, \qquad |\bar{K}^0\rangle \to e^{-i\alpha}|\bar{K}^0\rangle\,. \tag{2}$$

Correspondingly, the off–diagonal matrix elements of any operator X, acting on the K^0–$\bar{K}^0$ system, undergo the changes

$$X_{12} \to e^{-2i\alpha}X_{12}\,, \qquad X_{21} \to e^{2i\alpha}X_{21}\,. \tag{3}$$

This arbitrariness enters in some popular definitions of the CP-violation parameters. For definiteness, we choose a particular phase convention, namely we require that the CP operator is given by

$$\mathrm{CP} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{4}$$

In this case,

$$|K_\pm\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle \pm |\bar{K}^0\rangle\right) \tag{5}$$

are the CP eigenstates. In the presence of CP-violation, $[\mathrm{CP}, H] \neq 0$ and the eigenstates of the Hamiltonian H are not CP eigenstates. We introduce the parameter $\bar{\epsilon}$ which defines the eigenstates of H as

$$|K_L\rangle = \frac{|K_-\rangle + \bar{\epsilon}|K_+\rangle}{\sqrt{1+|\bar{\epsilon}|^2}}\,, \qquad |K_S\rangle = \frac{|K_+\rangle + \bar{\epsilon}|K_-\rangle}{\sqrt{1+|\bar{\epsilon}|^2}}\,. \tag{6}$$

The corresponding (complex) eigenvalues are denoted as

$$\begin{aligned} \lambda_L &= m_L - i\Gamma_L/2 = \frac{1}{2}(H_{11}+H_{22}) - \sqrt{H_{12}H_{21}} \\ \lambda_S &= m_S - i\Gamma_S/2 = \frac{1}{2}(H_{11}+H_{22}) + \sqrt{H_{12}H_{21}}\,. \end{aligned} \tag{7}$$

In the phase convention (4), the parameter $\bar{\epsilon}$ controls the amount of CP-violation, namely the CP symmetric limit is recovered for $\bar{\epsilon} \to 0$. We can explicitly write $\bar{\epsilon}$ in terms of the matrix elements of H

$$\bar{\epsilon} = \frac{H_{21} - H_{12}}{\Delta\lambda - H_{12} - H_{21}}\,, \tag{8}$$

where $\Delta\lambda = \lambda_L - \lambda_S$.

Experimentally CP-violation is a small effect, i.e. $|\bar{\epsilon}| \ll 1$. For this reason, one can simplify eq. (8) to obtain

$$\bar{\epsilon} \simeq -\frac{i\mathrm{Im}M_{12} + \mathrm{Im}\Gamma_{12}/2}{\Delta\lambda}\,, \tag{9}$$

with

$$\begin{aligned}\Delta\lambda &= \Delta M - i\Delta\Gamma/2 \\ \Delta M &= -2\mathrm{Re}M_{12} \\ \Delta\Gamma &= -2\mathrm{Re}\Gamma_{12}\,.\end{aligned} \tag{10}$$

Moreover, since $\Delta M/\Delta\Gamma = -0.9565 \pm 0.0051 \approx -1$, eq. (9) becomes

$$\bar{\epsilon} \simeq \frac{1+i}{2}\frac{\mathrm{Im}M_{12}}{2\mathrm{Re}M_{12}} - \frac{1-i}{2}\frac{\mathrm{Im}\Gamma_{12}}{2\mathrm{Re}\Gamma_{12}}\,. \tag{11}$$

In view of the following discussion of the CP-violation parameters, let us introduce amplitudes of the weak decays of kaons into two pions states with definite isospin

$$A_I e^{i\delta_I} = \langle\pi\pi(I)|H_W|K^0\rangle\,, \tag{12}$$

where $I = 0, 2$ is the isospin of the final two-pion state and the δ_I's are the strong phases induced by final-state interaction. Watson's theorem ensures that

$$A_I^* e^{i\delta_I} = \langle\pi\pi(I)|H_W|\bar{K}^0\rangle\,. \tag{13}$$

Direct CP-violation, occurring at the decay vertices, appears as a difference between the amplitudes $\langle\pi\pi|H_W|K^0\rangle$ and $\langle\pi\pi|H_W|\bar{K}^0\rangle$. This corresponds to a phase difference between A_0 and A_2.

One introduces the parameter ϵ' to account for direct CP-violation. A convenient definition is

$$\begin{aligned}\epsilon' &= \frac{\langle\pi\pi(0)|H_W|K_S\rangle\langle\pi\pi(2)|H_W|K_L\rangle - \langle\pi\pi(0)|H_W|K_L\rangle\langle\pi\pi(2)|H_W|K_S\rangle}{\sqrt{2}\langle\pi\pi(0)|H_W|K_S\rangle^2} \\ &\simeq i\frac{e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\mathrm{Im}(\frac{A_2}{A_0}) \simeq i\frac{e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\frac{\omega}{\mathrm{Re}A_0}\left(\omega^{-1}\mathrm{Im}A_2 - \mathrm{Im}A_0\right)\,,\end{aligned} \tag{14}$$

where $\omega = \mathrm{Re}A_2/\mathrm{Re}A_0$. Equation (14) is obtained in the approximation $\mathrm{Im}A_0 \ll \mathrm{Re}A_0$, $\mathrm{Im}A_2 \ll \mathrm{Re}A_2$ and also $\omega \ll 1$, as a consequence of the $\Delta I = 1/2$ enhancement in kaon decays; ϵ' is independent of the kaon phase convention. On the contrary, the parameter $\bar{\epsilon}$, defined in eq. (6), depends on the choice of the phase. Under a redefinition of the phases as in eq. (2), $\bar{\epsilon}$ changes as

$$\bar{\epsilon} \to \frac{-i\sin\alpha + \bar{\epsilon}\cos\alpha}{\cos\alpha - i\bar{\epsilon}\sin\alpha} \tag{15}$$

and the CP-symmetric limit does not correspond to $\bar{\epsilon} \to 0$[3].

Another parameter, which is independent of the phase convention and accounts for CP-violation in the mixing matrix, can be defined in terms of the $K \to \pi\pi$ transition amplitudes

$$\epsilon = \frac{\langle\pi\pi(0)|H_W|K_L\rangle}{\langle\pi\pi(0)|H_W|K_S\rangle} = \frac{i\sin\phi_0 + \bar{\epsilon}\cos\phi_0}{\cos\phi_0 + i\bar{\epsilon}\sin\phi_0} \simeq \bar{\epsilon} + i\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\,, \tag{16}$$

[3]In this case, the states $|K_\pm\rangle$ are not CP eigenstates.

where $A_0 = |A_0| e^{i\phi_0}$ and the last expression is obtained in the approximation $\bar{\epsilon}$, $\phi_0 \ll 1$. The two definitions, eqs. (6) and (16), coincide in the Wu-Yang phase convention, $\mathrm{Im}A_0 = 0$. One can check that ϕ_0 changes with the phase convention as $\phi_0 \to \phi_0 + \alpha$ and that ϵ is invariant.

From unitarity, one has

$$\Gamma_{12} = \sum_n 2\pi\delta(M_K - E_n)\langle K^0|H_W|n\rangle\langle n|H_W|\bar{K}^0\rangle \,. \tag{17}$$

Given the dominance of $K^0 \to \pi\pi\,(0)$ decay, one obtains the relation $\Gamma_{12} = (A_0^*)^2$. From eqs. (11) and (16), one has

$$\epsilon \simeq \frac{e^{i\pi/4}}{\sqrt{2}}\left(\frac{\mathrm{Im}M_{12}}{2\mathrm{Re}M_{12}} - \xi\right) \,, \tag{18}$$

where $\xi = -\mathrm{Im}A_0/\mathrm{Re}A_0$. In the Cabibbo–Kobayashi–Maskawa phase convention, the ξ contribution is small and can be safely neglected.

To make contact with the experiments, one defines the two amplitude ratios

$$\eta_{00} = \frac{\langle\pi^0\pi^0|H_W|K_L\rangle}{\langle\pi^0\pi^0|H_W|K_S\rangle} \,, \qquad \eta_{+-} = \frac{\langle\pi^+\pi^-|H_W|K_L\rangle}{\langle\pi^+\pi^-|H_W|K_S\rangle} \,. \tag{19}$$

Neglecting small terms, one has

$$\eta_{00} \simeq \epsilon - 2\epsilon' \,, \qquad \eta_{+-} \simeq \epsilon + \epsilon' \,, \tag{20}$$

namely

$$|\epsilon|^2 \simeq |\eta_{+-}|^2 \simeq |\eta_{00}|^2 \,,$$

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) \simeq \frac{1}{6}\left(1 - \frac{|\eta_{00}|^2}{|\eta_{+-}|^2}\right) \,. \tag{21}$$

Expressing η_{00} and η_{+-} in terms of the corresponding widths

$$\begin{aligned} |\eta_{00}|^2 &= \frac{\Gamma(K_L \to \pi^0\pi^0)}{\Gamma(K_S \to \pi^0\pi^0)} \\ |\eta_{+-}|^2 &= \frac{\Gamma(K_L \to \pi^+\pi^-)}{\Gamma(K_S \to \pi^+\pi^-)} \,, \end{aligned} \tag{22}$$

eq. (21) gives the CP-violation parameters in terms of measurable quantities. Notice that ϵ'/ϵ is approximately real, since experimentally $\delta_2 - \delta_0 \approx -\pi/4$.

3 The CKM matrix

In the Standard Model, the basic quark charged-current interactions are described by the Lagrangian

$$\mathcal{L}_{\mathrm{quark-W}} = \frac{g}{2\sqrt{2}} \bar{u}_i \gamma_\mu (1-\gamma_5) V_{ij} d_j W^\mu \,+\, \mathrm{h.c.}\,, \tag{23}$$

where u_i are the charged 2/3 quarks (u, c, t), d_j the charged $-1/3$ quarks (d, s, b) and g is the $SU(2)_L$ weak coupling constant ($G_F/\sqrt{2} = g^2/8M_W^2$, where G_F is the Fermi constant). V is the unitary CKM matrix [16]. A useful parametrization is [17, 18]

$$\begin{aligned} V &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &= \begin{pmatrix} C_\theta C_\sigma & S_\theta C_\sigma & S_\sigma e^{-i\delta} \\ -S_\theta C_\tau - C_\theta S_\sigma S_\tau e^{i\delta} & C_\theta C_\tau - S_\theta S_\tau S_\sigma e^{i\delta} & C_\sigma S_\tau \\ S_\theta S_\tau - C_\theta C_\tau S_\sigma e^{i\delta} & -C_\theta S_\tau - C_\tau S_\theta S_\sigma e^{i\delta} & C_\sigma C_\tau \end{pmatrix} . \end{aligned} \tag{24}$$

In eq. (24), θ, σ and τ are quark mixing angles (in particular, θ corresponds approximately to the Cabibbo angle); C_θ, S_θ, etc., mean $\cos\theta$, $\sin\theta$, etc.; δ is the CP-violating phase. Experimental determinations of $|V_{ud}|$, $|V_{cb}|$ and $|V_{ub}|$ from K and B decays show that there is a hierarchy in the mixing angles, so that the CKM matrix can be empirically expanded in powers of $\lambda = S_\theta \simeq 0.22$ [19]. Up to and including terms of order λ^3 (λ^5) for the real (imaginary) part, V is given by

$$V = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3\,(\rho - i\eta) \\ -\lambda\,(1 + A^2\lambda^4(\rho+i\eta)) & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3\left[1-\left(1-\frac{\lambda^2}{2}\right)(\rho+i\eta)\right] & -A\lambda^2\,(1+\lambda^2(\rho+i\eta)) & 1 \end{pmatrix}, \tag{25}$$

where $S_\tau = A\lambda^2$ and $S_\sigma e^{-i\delta} = A\lambda^3(\rho - i\eta)$. In this particular (quark) phase convention, the imaginary part of the matrix appears at order λ^3.

The unitarity of the CKM matrix implies

$$\sum_q V_{iq} V^*_{jq} = \delta_{ij}\,. \tag{26}$$

In particular, considering the condition

$$V^*_{ub} V_{ud} + V^*_{cb} V_{cd} + V^*_{tb} V_{td} = 0 \tag{27}$$

in the approximation $V_{ud} \simeq V_{tb} \simeq 1$, one obtains

$$\frac{V^*_{ub}}{A\lambda^3} + \frac{V_{td}}{A\lambda^3} - 1 = 0\,. \tag{28}$$

This relation identifies a triangle in the ρ–η plane (see fig. 1). The angles of this triangle, α, β and δ, are measures of CP-violation.

Recent phenomenological analyses of the CKM matrix elements can be found in refs. [9, 20, 21]. A brief discussion of these analyses together with the numerical results, can be found in section 8.

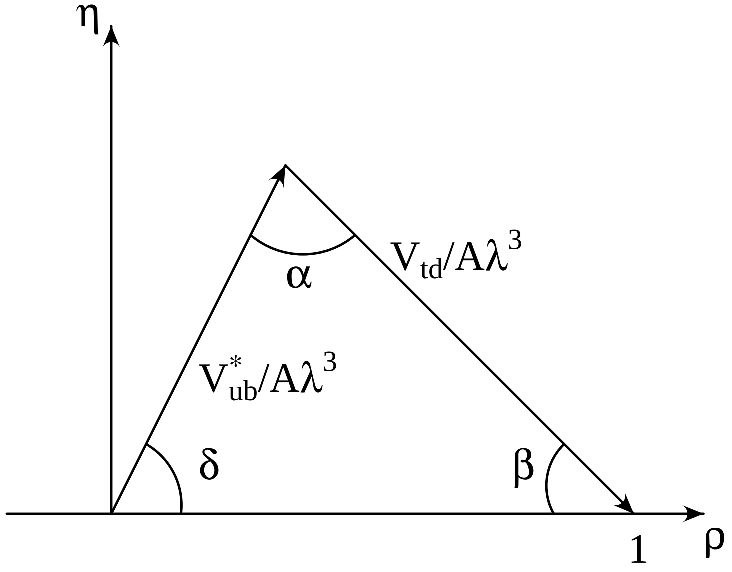

Figure 1: *The unitarity triangle in the ρ–η plane.*

4 The bare $\Delta S = 1$ effective Hamiltonian

Weak decays of light hadrons are more conveniently studied using the Wilson operator product expansion (OPE) [22]. With the OPE, it is possible to introduce an effective Hamiltonian, written in terms of renormalized local operators and of the corresponding Wilson coefficients [23]–[26]. Short-distance strong-interaction effects are contained in the coefficients and can be computed in perturbation theory, because of asymptotic freedom. Long-distance strong-interaction effects are included in the hadronic matrix elements of the local operators and must be evaluated with some non-perturbative technique (lattice, QCD sum rules, etc.). The convenience of the effective Hamiltonian approach is that all known non-perturbative methods are usually able to predict matrix elements of local operators only. In this section we introduce the bare $\Delta S = 1$ effective Hamiltonian, the renormalization of which will be discussed in the next section.

At second order in the weak coupling constant and at zero order in the strong coupling constant, the $\Delta S = 1$ effective Hamiltonian can be written in terms of a local product of two charged currents

$$\mathcal{H}_W^{\Delta S=1} = -\frac{G_F}{\sqrt{2}}\left[\lambda_u\left(\bar{s}_L\gamma_\mu u_L\right)\left(\bar{u}_L\gamma_\mu d_L\right) + \lambda_c\left(u \to c\right)\right] =$$

$$-\frac{G_F}{\sqrt{2}}\left[\lambda_u\left(\bar{s}_\alpha u_\alpha\right)_{(V-A)}\left(\bar{u}_\beta d_\beta\right)_{(V-A)} + \lambda_c\left(u \to c\right)\right], \qquad (29)$$

where $(\bar{s}_\alpha d_\alpha)_{(V-A)} = \bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha$; α and β are colour indices and the sum over repeated indices is understood. We have introduced the notation

$$\lambda_q = V_{qd} V_{qs}^* \tag{30}$$

for $q = u, c, t$. In terms of the λ_q, the unitarity condition of the CKM matrix can be written as

$$\lambda_u + \lambda_c + \lambda_t = 0 \,. \tag{31}$$

Equation (29) has been obtained from the original theory, by neglecting all masses and momenta with respect to M_W. In practice, the effective Hamiltonian is obtained by taking $1/(M_W^2 - q^2) \to 1/M_W^2$ in the T-product of the two charged currents and by putting the u, d and s masses to zero. In order to discuss CP-violation, it is convenient to write $\mathcal{H}_W^{\Delta S=1}$ as

$$\mathcal{H}_W^{\Delta S=1} = -\lambda_u \frac{G_F}{\sqrt{2}} \left[(1-\tau) \left(Q_2^u - Q_2^c \right) + \tau Q_2^u \right] , \tag{32}$$

where $\tau = -\lambda_t / \lambda_u$ contains the CP-violating phase and

$$Q_2^q = (\bar{s}_\alpha q_\alpha)_{(V-A)} \, (\bar{q}_\beta d_\beta)_{(V-A)} \,. \tag{33}$$

5 QCD corrections

Strong interactions play a crucial role in non-leptonic weak decays. The perturbative short-distance effects, included in the calculation of the Wilson coefficients, may be very important because of the presence of large logarithms $\sim \alpha_s^n \ln^m (M_W/\mu)$, where μ is a scale of the order of the mass of the decaying hadron. For an accurate estimate of the short-distance contributions, the large logarithms have to be resummed to all orders using renormalization group (RG) techniques.

The starting point is the T-product of the two weak currents expanded at short distances in terms of local operators. Taking into account the renormalization effects due to strong interactions, we write

$$\langle F | \mathcal{H}_W^{\Delta S=1} | I \rangle = \frac{g^2}{8} \int d^4x D^{\mu\nu} \left(x, M_W \right) \langle F | T \Big(J_\mu(x), J_\nu(0) \Big) | I \rangle =$$

$$-\frac{G_F}{\sqrt{2}} \sum_i C_i(\mu) \langle F | Q_i(\mu) | I \rangle + \dots , \tag{34}$$

where $\langle F|$ and $|I\rangle$ are the generic final and initial states; the $Q_i(\mu)$ form a complete basis of operators renormalized at the scale μ; the $C_i(\mu)$ are the corresponding Wilson coefficients and the dots represent terms which are suppressed with respect to the dominant ones as powers of $\Lambda_{\mathrm{QCD}}^2 / M_W^2$ (m_b^2/M_W^2 for B-decays). The effective Hamiltonian is independent of renormalization scale μ. On the lattice, the renormalization scale can be replaced by the inverse lattice spacing a^{-1} and the effective Hamiltonian can be expressed in terms of bare lattice operators [9]. The OPE in eq. (34) must be valid for all possible initial and final states. This implies that the effective Hamiltonian is defined from an operator relation

$$\mathcal{H}_W^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} \sum_i C_i(\mu) Q_i(\mu) = -\frac{G_F}{\sqrt{2}} \vec{Q}^T(\mu) \cdot \vec{C}(\mu) \,. \tag{35}$$

The important features of $\mathcal{H}_W^{\Delta S=1}$ are the following:

- the Wilson coefficients can be calculated using (RG-improved) perturbation theory, provided that one chooses a sufficiently large renormalization scale $\mu \simeq 2$–3 GeV $\gg \Lambda_{\mathrm{QCD}}$. In the leading logarithmic approximation (LLA), all terms of $O\left(\alpha_s(\mu)^n \log(M_W/\mu)^n\right)$ are taken into account;

- all non-perturbative effects are contained in the matrix elements of the local operators, the calculation of which requires a non-perturbative technique.

Since $\mathcal{H}_W^{\Delta S=1}$, eq. (35), is independent of μ, the coefficients $\vec{C}(\mu) = (C_1(\mu), C_2(\mu), \dots)$ must satisfy the RG equations

$$\mu^2 \frac{d}{d\mu^2} \vec{C}(\mu) = \frac{1}{2} \hat{\gamma}^T \vec{C}(\mu) \,, \tag{36}$$

which can be more conveniently written as

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \frac{1}{2} \hat{\gamma}^T(\alpha_s) \right) \vec{C}(\mu) = 0 \,, \tag{37}$$

where

$$\beta(\alpha_s) = \mu^2 \frac{d\alpha_s}{d\mu^2} \tag{38}$$

is the QCD β-function and

$$\hat{\gamma}(\alpha_s) = 2 \hat{Z}^{-1} \mu^2 \frac{d}{d\mu^2} \hat{Z} \tag{39}$$

is the anomalous-dimension matrix of the renormalized operators; $\hat{Z}$ is defined by the relation which connects the bare operators to the renormalized ones, $\vec{Q}(\mu) = \hat{Z}^{-1}(\mu, \alpha_s) \vec{Q}^B$.

The solution of the system of linear equations (37) is found by introducing a suitable evolution matrix $U(\mu, M_W)$ and by imposing an appropriate set of initial conditions, usually called matching conditions. The coefficients $\vec{C}(\mu)$ are given by[4]

$$\vec{C}(\mu) = \hat{U}(\mu, M_W) \vec{C}(M_W) \,, \tag{40}$$

[4]The problem of the thresholds due to the presence of heavy quarks with a mass $M_W \gg m_Q \gg \Lambda_{\mathrm{QCD}}$ will be discussed below.

with

$$\hat{U}(m_1, m_2) = T_{\alpha_s} \exp\left(\int_{\alpha_s(m_1)}^{\alpha_s(m_2)} \frac{d\alpha_s}{\beta(\alpha_s)} \hat{\gamma}^T(\alpha_s)\right) ; \tag{41}$$

T_{α_s} is the ordered product with increasing couplings from right to left. The matching conditions are found by imposing that, at $\mu = M_W$, the matrix elements of the original T-product of the currents coincide, up to terms suppressed as inverse powers of M_W, with the corresponding matrix elements of $\mathcal{H}_W^{\Delta S=1}$. To this end, we introduce the vector $\vec{T}$ defined by the relation

$$\langle\alpha|T(J^\dagger J)|\beta\rangle = -\frac{G_F}{\sqrt{2}}\langle\alpha|\vec{Q}^T|\beta\rangle_0 \cdot \vec{T} + \dots \tag{42}$$

where $\langle\alpha|\vec{Q}^T|\beta\rangle_0$ are the matrix elements of the operators at tree level. We also introduce the matrix $\hat{M}(\mu)$ such that

$$\langle\alpha|\mathcal{H}_W^{\Delta S=1}|\beta\rangle = -\frac{G_F}{\sqrt{2}}\langle\alpha|\vec{Q}^T(\mu)|\beta\rangle\vec{C}(\mu) =$$

$$-\frac{G_F}{\sqrt{2}}\langle\alpha|\vec{Q}^T|\beta\rangle_0 \hat{M}^T(\mu)\vec{C}(\mu) \, . \tag{43}$$

In terms of $\vec{T}$ and $\hat{M}$, the matching condition

$$\langle\alpha|T(J^\dagger J)|\beta\rangle = \langle\alpha|\mathcal{H}_W^{\Delta S=1}|\beta\rangle \tag{44}$$

fixes the value of the Wilson coefficients at the scale M_W

$$\vec{C}(M_W) = [\hat{M}^T(M_W)]^{-1}\vec{T} \, . \tag{45}$$

Notice that the matching could be imposed at any scale $\bar{\mu}$, such that large logarithms do not appear in the calculation of the Wilson coefficients at the scale $\bar{\mu}$, i.e. $\alpha_s \ln M_W/\bar{\mu} \ll 1$.

Equation (40) is correct if no threshold corresponding to a quark mass between μ and M_W is present. Indeed, as α_s, $\hat{\gamma}$ and $\beta(\alpha_s)$ all depend on the number of active flavours, it is necessary to change the evolution matrix $\hat{U}$ defined in eq. (41), when passing the threshold. The general case then corresponds to a sequence of effective theories with a decreasing number of "active" flavours. By "active" flavour, we mean a dynamical massless ($\mu \gg m_Q$) quark field. The theory with k "active" flavours is matched to the one with $k+1$ "active" flavours at the threshold. This procedure changes the solution for the Wilson coefficients. For instance, if one starts with five "active" flavours at the scale M_W and chooses $m_c \ll \mu \ll m_b$, the Wilson coefficients become

$$\vec{C}(\mu) = \hat{W}[\mu, M_W]\vec{C}(M_W) = \hat{U}_4(\mu, m_b)\hat{U}_5(m_b, M_W)\vec{C}(M_W) \, . \tag{46}$$

The inclusion of the charm threshold proceeds along the same lines.

6 The operators of the $\Delta S = 1$ effective Hamiltonian

So far, we have presented the exact solutions of the renormalization group equations for the Wilson coefficients. In practice, it is only possible to calculate the relevant functions in perturbation theory. For illustrative purposes, we consider the calculation of the $\Delta S=1$ effective Hamiltonian at the leading order in QCD. The bare Hamiltonian is given in eq. (29). In the presence of QCD interactions, other operators appear in the Wilson expansion. A complete basis is given by the following operators

$$\begin{aligned}
Q_1 &= (\bar{s}_\alpha d_\alpha)_{(V-A)}(\bar{u}_\beta u_\beta)_{(V-A)} \\
Q_2 &= (\bar{s}_\alpha d_\beta)_{(V-A)}(\bar{u}_\beta u_\alpha)_{(V-A)} \\
Q_3 &= (\bar{s}_\alpha d_\alpha)_{(V-A)}\sum_q(\bar{q}_\beta q_\beta)_{(V-A)} \\
Q_4 &= (\bar{s}_\alpha d_\beta)_{(V-A)}\sum_q(\bar{q}_\beta q_\alpha)_{(V-A)} \\
Q_5 &= (\bar{s}_\alpha d_\alpha)_{(V-A)}\sum_q(\bar{q}_\beta q_\beta)_{(V+A)} \\
Q_6 &= (\bar{s}_\alpha d_\beta)_{(V-A)}\sum_q(\bar{q}_\beta q_\alpha)_{(V+A)} \\
Q_1^c &= (\bar{s}_\alpha d_\alpha)_{(V-A)}(\bar{c}_\beta c_\beta)_{(V-A)} \\
Q_2^c &= (\bar{s}_\alpha d_\beta)_{(V-A)}(\bar{c}_\beta c_\alpha)_{(V-A)}\,.
\end{aligned} \tag{47}$$

The q index runs over the "active" flavours. The above operators are generated by gluon exchanges in the Feynman diagrams of fig. 2. In particular, Q_1 is generated by current–current diagrams and Q_3–Q_6 are generated by penguin diagrams. The choice of the operator basis in not unique, and different possibilities have been considered in the literature [27]. If the electromagnetic correction, are also taken into account, the operator basis enlarges to include the following operators

$$\begin{aligned}
Q_7 &= \frac{3}{2}(\bar{s}_\alpha d_\alpha)_{(V-A)}\sum_q e_q(\bar{q}_\beta q_\beta)_{(V+A)} \\
Q_8 &= \frac{3}{2}(\bar{s}_\alpha d_\beta)_{(V-A)}\sum_q e_q(\bar{q}_\beta q_\alpha)_{(V+A)} \\
Q_9 &= \frac{3}{2}(\bar{s}_\alpha d_\alpha)_{(V-A)}\sum_q e_q(\bar{q}_\beta q_\beta)_{(V-A)} \\
Q_{10} &= \frac{3}{2}(\bar{s}_\alpha d_\beta)_{(V-A)}\sum_q e_q(\bar{q}_\beta q_\alpha)_{(V-A)}\,.
\end{aligned} \tag{48}$$

Below the bottom threshold, the following relation holds

$$Q_{10} - Q_9 - Q_4 + Q_3 = 0\,, \tag{49}$$

so that there are nine independent operators. The basis is further reduced below the charm threshold by using the relations

$$Q_4 - Q_3 - Q_2 + Q_1 = 0$$

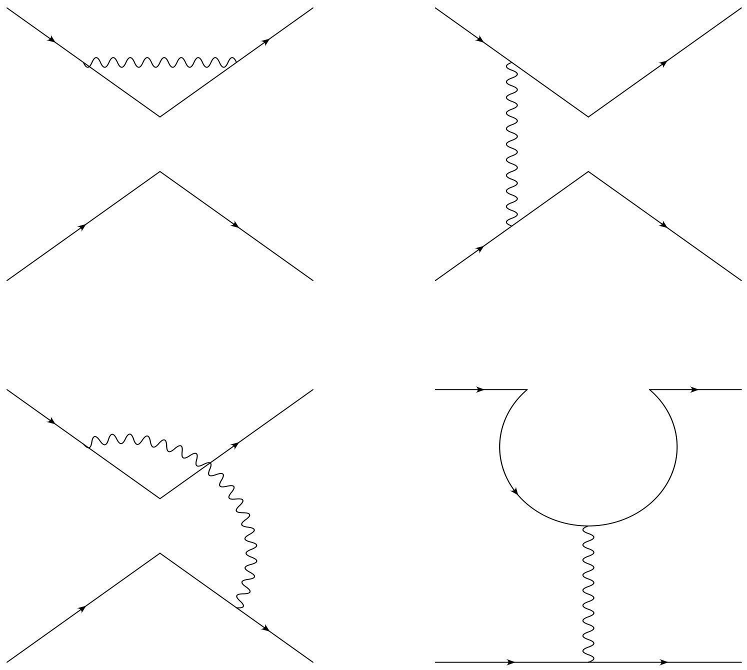

Figure 2: *One-loop corrections to the $\Delta S=1$ effective Hamiltonian.*

$$Q_9 - \frac{3}{2}Q_1 + \frac{1}{2}Q_3 \;=\; 0\,. \tag{50}$$

All the operators considered above are dimension-six operators. In principle, two dimension-five operators

$$\begin{aligned} Q_{11} &= \frac{Q_d e}{16\pi^2} m_s \bar{s}_\alpha \sigma^{\mu\nu}_{(V-A)} d_\alpha F_{\mu\nu} \\ Q_{12} &= \frac{g}{16\pi^2} m_s \bar{s}_\alpha \sigma^{\mu\nu}_{(V-A)} t^A_{\alpha\beta} d_\beta G^A_{\mu\nu} \end{aligned} \tag{51}$$

should also be included in the operator basis. The matrix elements of Q_{11} and Q_{12}, however, enter only at $O(p^4)$ in chiral perturbation theory. Since the phenomenological analysis presented in the following is only valid up to terms of $O(p^2)$, we do not need to include the contribution of the dimension-five operators in the calculation of ϵ'/ϵ. The effect of these operators on ϵ'/ϵ has recently been analysed in ref. [29]. Other operators of lower dimensionality (e.g. two-fermion operators) are also potentially present. However, it can be shown that their effect can be reabsorbed in a suitable redefinition of the fermion fields and by diagonalizing the quark mass matrix at first order in G_F [23]–[26].

In summary, the $\Delta S=1$ effective Hamiltonian, renormalized at a scale $\mu \gg m_c$, can be written as

$$\mathcal{H}^{\Delta S=1}_{eff} \;=\; -\lambda_u \frac{G_F}{\sqrt{2}}\Big\{(1-\tau)\Big[C_1(\mu)\,(Q_1(\mu) - Q^c_1(\mu)) + C_2(\mu)\,(Q_2(\mu) - Q^c_2(\mu))\Big]$$

$$+\tau\vec{Q}(\mu)^T\vec{C}(\mu)\Big\}\,, \tag{52}$$

where, in order to find the Wilson coefficients to a given order in α_s, we have to calculate eqs. (41), (45) in perturbation theory.

The explicit expressions of $\hat{\gamma}_s(\alpha_s)$ and $\beta(\alpha_s)$, in the LLA

$$\hat{\gamma}(\alpha_s)=\frac{\alpha_s}{4\pi}\hat{\gamma}_s^{(0)}\,,\qquad \beta(\alpha_s)=-\frac{\alpha_s^2}{4\pi}\beta_0\,, \tag{53}$$

can be found for example in ref. [5]. In eq. (41), using $\hat{\gamma}_s^{(0)}$ and β_0, one obtains

$$U(\mu,M_W)=\left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right)^{\hat{\gamma}_s^{(0)T}/2\beta_0}\,. \tag{54}$$

At this order, the matching conditions are trivial: $\hat{M}$, eq. (43), is the identity matrix; $\vec{T}$, eq. (42), has all vanishing components with the only exception of $T_2=1$. Thus the Wilson coefficients at the leading order for $m_c\ll\mu\ll m_b$ are given by

$$\vec{C}(\mu)=\left(\frac{\alpha_s^{n_f=4}(m_b)}{\alpha_s^{n_f=4}(\mu)}\right)^{(\hat{\gamma}_s^{(0)T}/2\beta_0)_{n_f=4}}\left(\frac{\alpha_s^{n_f=5}(M_W)}{\alpha_s^{n_f=5}(m_b)}\right)^{(\hat{\gamma}_s^{(0)T}/2\beta_0)_{n_f=5}}\vec{C}(M_W)\,, \tag{55}$$

with $C_2(M_W)=1$ and all the other Wilson coefficients at the scale M_W vanish.

In the next-to-leading logarithmic approximation (NLLA), one proceeds along the general scheme described above. In this case, all quantities entering in the matching procedure have to be computed at order α_s (α_e for the electromagnetic case). The β-function and the anomalous dimension matrix have to be computed at second order in the coupling constants. Thus, for example, the anomalous dimension matrix in the NLLA has the form

$$\hat{\gamma}=\frac{\alpha_s}{4\pi}\hat{\gamma}_s^{(0)}+\frac{\alpha_e}{4\pi}\hat{\gamma}_e^{(0)}+(\frac{\alpha_s}{4\pi})^2\hat{\gamma}_s^{(1)}+\frac{\alpha_s}{4\pi}\frac{\alpha_e}{4\pi}\hat{\gamma}_e^{(1)}\,, \tag{56}$$

where $O(\alpha_e^2)$ corrections have been neglected. We will not give here any details of the NLLA calculations. They can be found in refs. [1]–[5]. At the next-to-leading order, it is necessary to solve numerically eq. (37). Table 1 contains the coefficients, calculated at the leading (LO) and at the next-to-leading (NLO) order, using the 't Hooft–Veltman (HV) and the naïve dimensional (NDR) regularization schemes, for different values of the renormalization scale μ. The errors in the table take into account the variation of the values of the coefficients due to $\Lambda^{(4)}_{QCD}=(330\pm100)$ MeV and $m_t=(174\pm17)$ GeV. Notice that the next-to-leading Wilson coefficients and operators both depend on the regularization scheme, while the effective Hamiltonian is scheme-independent up to terms $O(\alpha_s^2)$. Actually the dependence of the effective Hamiltonian on the regularization scheme, due to the unknown next-to-next-to-leading terms, can be estimated and contributes to the uncertainties in the prediction of ϵ'/ϵ, see ref. [9].

The coefficients in table 1 have been computed independently by the Munich group [4, 14]. The definition of the renormalized operators in the HV scheme used here differ from those defined in ref. [14]. This is due to the different way of taking into account the two-loop anomalous dimension of the weak current, which does not vanish in the HV calculation. One can relate the HV coefficients of table 1 ($\vec{C}$) and those of ref. [14] ($\vec{C}'$). The relation is

$$\vec{C}(\mu) = \Big(1 - \frac{\alpha_s(\mu)}{4\pi}\frac{\gamma_J}{\beta_0}\hat{1}\Big)\vec{C}'(\mu)\,, \tag{57}$$

where

$$\gamma_J = 4\frac{N_c^2-1}{2N_c}\beta_0\,. \tag{58}$$

Once these differences in the definition of the renormalized operators and the reduction of the operator basis, eq. (49), are properly taken into account, the numerical results presented here agree with those of ref. [14].

7 Relevant formulae

In order to estimate ϵ'/ϵ, we have to constrain the CP-violating phase δ in the CKM matrix, by using the available experimental information. To this end, we consider the CP-violating term in the K^0–$\bar{K}^0$ mixing amplitude and the CP-conserving term for B^0–$\bar{B}^0$ mixing. In the following, we present all the formulae used in our analysis, namely the expressions of ϵ, x_d and ϵ'/ϵ from the $\Delta S=2$, $\Delta B=2$, $\Delta S=1$ effective Hamiltonian, respectively.

The effective Hamiltonian governing the $\Delta S=2$ amplitude is given by

$$\mathcal{H}_{eff}^{\Delta S=2} = \frac{G_F^2}{16\pi^2}M_W^2(\bar{s}\gamma_L^\mu d)^2\left\{\lambda_c^2F(x_c)+\lambda_t^2F(x_t)+2\lambda_c\lambda_tF(x_c,x_t)\right\}\,, \tag{59}$$

where $x_q = m_q^2/M_W^2$ and the functions $F(x_i)$ and $F(x_i,x_j)$ are the so-called *Inami–Lim* functions [28], including QCD corrections [2]; $F(x_t)$ is known at the next-to-leading order and has been included in our calculation. From eqs. (18) and (59), one can derive the CP-violation parameter

$$\begin{aligned} |\epsilon|_{\xi=0} \;=\; & C_\epsilon B_K A^2\lambda^6\sigma\sin\delta\,\{F(x_c,x_t)+ \\ & F(x_t)[A^2\lambda^4(1-\sigma\cos\delta)]-F(x_c)\Big\}\,, \end{aligned} \tag{60}$$

where

$$C_\epsilon = \frac{G_F^2 f_K^2 M_K M_W^2}{6\sqrt{2}\pi^2\Delta M}\,. \tag{61}$$

In eq. (60), $\rho = \sigma\cos\delta$, $\eta = \sigma\sin\delta$ and λ, A, ρ and η are the parameters of the CKM matrix in the Wolfenstein parametrization [19]. B_K is the renormalization group invariant B-factor, the definition of which at the leading order is

$$\langle\bar{K}|\,(\bar{s}\gamma_L^\mu d)^2\,|K\rangle = \frac{8}{3}f_K^2M_K^2\alpha_s(\mu)^{6/25}B_K\,. \tag{62}$$

	LO	NLO HV	NLO NDR
		$\mu = 1.5$ GeV	
C_1	$(-4.22 \pm 0.65 \pm 0.00) \times 10^{-1}$	$(-3.91 \pm 0.51 \pm 0.00) \times 10^{-1}$	$(-3.80 \pm 0.55 \pm 0.00) \times 10^{-1}$
C_2	$(11.62 \pm 0.38 \pm 0.00) \times 10^{-1}$	$(106.13 \pm 0.82 \pm 0.00) \times 10^{-2}$	$(11.95 \pm 0.35 \pm 0.00) \times 10^{-1}$
C_3	$(1.99 \pm 0.35 \pm 0.00) \times 10^{-2}$	$(2.17 \pm 0.41 \pm 0.00) \times 10^{-2}$	$(2.60 \pm 0.52 \pm 0.00) \times 10^{-2}$
C_4	$(-4.16 \pm 0.56 \pm 0.02) \times 10^{-2}$	$(-4.51 \pm 0.60 \pm 0.01) \times 10^{-2}$	$(-0.63 \pm 0.11 \pm 0.00) \times 10^{-1}$
C_5	$(1.19 \pm 0.12 \pm 0.00) \times 10^{-2}$	$(1.37 \pm 0.15 \pm 0.00) \times 10^{-2}$	$(10.52 \pm 0.61 \pm 0.01) \times 10^{-3}$
C_6	$(-0.66 \pm 0.13 \pm 0.00) \times 10^{-1}$	$(-0.63 \pm 0.11 \pm 0.00) \times 10^{-1}$	$(-0.93 \pm 0.21 \pm 0.00) \times 10^{-1}$
C_7	$(0.16 \pm 0.04 \pm 0.19) \times 10^{-3}$	$(-0.04 \pm 0.00 \pm 0.17) \times 10^{-3}$	$(0.02 \pm 0.06 \pm 0.20) \times 10^{-3}$
C_8	$(0.63 \pm 0.14 \pm 0.16) \times 10^{-3}$	$(0.97 \pm 0.19 \pm 0.15) \times 10^{-3}$	$(1.06 \pm 0.26 \pm 0.19) \times 10^{-3}$
C_9	$(-6.77 \pm 0.27 \pm 0.71) \times 10^{-3}$	$(-6.32 \pm 0.37 \pm 0.64) \times 10^{-3}$	$(-7.24 \pm 0.19 \pm 0.73) \times 10^{-3}$
		$\mu = 2$ GeV	
C_1	$(-3.47 \pm 0.44 \pm 0.00) \times 10^{-1}$	$(-3.29 \pm 0.37 \pm 0.00) \times 10^{-1}$	$(-3.13 \pm 0.39 \pm 0.00) \times 10^{-1}$
C_2	$(11.16 \pm 0.23 \pm 0.00) \times 10^{-1}$	$(104.13 \pm 0.54 \pm 0.00) \times 10^{-2}$	$(11.54 \pm 0.23 \pm 0.00) \times 10^{-1}$
C_3	$(1.59 \pm 0.23 \pm 0.00) \times 10^{-2}$	$(1.73 \pm 0.26 \pm 0.00) \times 10^{-2}$	$(2.07 \pm 0.33 \pm 0.00) \times 10^{-2}$
C_4	$(-3.50 \pm 0.40 \pm 0.01) \times 10^{-2}$	$(-3.82 \pm 0.44 \pm 0.01) \times 10^{-2}$	$(-5.19 \pm 0.71 \pm 0.01) \times 10^{-2}$
C_5	$(10.40 \pm 0.94 \pm 0.04) \times 10^{-3}$	$(1.20 \pm 0.11 \pm 0.00) \times 10^{-2}$	$(10.54 \pm 0.16 \pm 0.02) \times 10^{-3}$
C_6	$(-5.23 \pm 0.80 \pm 0.03) \times 10^{-2}$	$(-5.08 \pm 0.72 \pm 0.03) \times 10^{-2}$	$(-0.72 \pm 0.13 \pm 0.00) \times 10^{-1}$
C_7	$(0.18 \pm 0.02 \pm 0.19) \times 10^{-3}$	$(0.01 \pm 0.00 \pm 0.18) \times 10^{-3}$	$(0.01 \pm 0.04 \pm 0.20) \times 10^{-3}$
C_8	$(0.50 \pm 0.08 \pm 0.12) \times 10^{-3}$	$(0.77 \pm 0.12 \pm 0.12) \times 10^{-3}$	$(0.81 \pm 0.16 \pm 0.14) \times 10^{-3}$
C_9	$(-7.03 \pm 0.20 \pm 0.74) \times 10^{-3}$	$(-6.71 \pm 0.27 \pm 0.68) \times 10^{-3}$	$(-7.49 \pm 0.15 \pm 0.75) \times 10^{-3}$
		$\mu = 3$ GeV	
C_1	$(-2.68 \pm 0.28 \pm 0.00) \times 10^{-1}$	$(-2.59 \pm 0.25 \pm 0.00) \times 10^{-1}$	$(-2.41 \pm 0.25 \pm 0.00) \times 10^{-1}$
C_2	$(10.71 \pm 0.12 \pm 0.00) \times 10^{-1}$	$(101.88 \pm 0.24 \pm 0.00) \times 10^{-2}$	$(11.12 \pm 0.14 \pm 0.00) \times 10^{-1}$
C_3	$(1.20 \pm 0.14 \pm 0.01) \times 10^{-2}$	$(1.29 \pm 0.16 \pm 0.00) \times 10^{-2}$	$(1.56 \pm 0.19 \pm 0.01) \times 10^{-2}$
C_4	$(-2.78 \pm 0.26 \pm 0.01) \times 10^{-2}$	$(-3.06 \pm 0.29 \pm 0.01) \times 10^{-2}$	$(-4.10 \pm 0.46 \pm 0.00) \times 10^{-2}$
C_5	$(8.60 \pm 0.67 \pm 0.04) \times 10^{-3}$	$(9.99 \pm 0.84 \pm 0.04) \times 10^{-3}$	$(9.75 \pm 0.46 \pm 0.03) \times 10^{-3}$
C_6	$(-3.89 \pm 0.47 \pm 0.03) \times 10^{-2}$	$(-3.84 \pm 0.44 \pm 0.03) \times 10^{-2}$	$(-5.33 \pm 0.73 \pm 0.03) \times 10^{-2}$
C_7	$(0.22 \pm 0.01 \pm 0.20) \times 10^{-3}$	$(0.08 \pm 0.00 \pm 0.19) \times 10^{-3}$	$(0.04 \pm 0.02 \pm 0.20) \times 10^{-3}$
C_8	$(3.73 \pm 0.50 \pm 0.92) \times 10^{-4}$	$(5.84 \pm 0.74 \pm 0.91) \times 10^{-4}$	$(0.59 \pm 0.09 \pm 0.10) \times 10^{-3}$
C_9	$(-7.29 \pm 0.14 \pm 0.77) \times 10^{-3}$	$(-7.04 \pm 0.18 \pm 0.73) \times 10^{-3}$	$(-7.66 \pm 0.11 \pm 0.78) \times 10^{-3}$
		$\mu = 4.5$ GeV	
C_1	$(-2.07 \pm 0.19 \pm 0.00) \times 10^{-1}$	$(-2.03 \pm 0.18 \pm 0.00) \times 10^{-1}$	$(-1.84 \pm 0.17 \pm 0.00) \times 10^{-1}$
C_2	$(103.85 \pm 0.67 \pm 0.00) \times 10^{-2}$	$(1001.15 \pm 0.43 \pm 0.00) \times 10^{-3}$	$(108.20 \pm 0.87 \pm 0.00) \times 10^{-2}$
C_3	$(1.13 \pm 0.11 \pm 0.03) \times 10^{-2}$	$(1.19 \pm 0.12 \pm 0.03) \times 10^{-2}$	$(1.40 \pm 0.14 \pm 0.03) \times 10^{-2}$
C_4	$(-2.42 \pm 0.20 \pm 0.02) \times 10^{-2}$	$(-2.67 \pm 0.23 \pm 0.02) \times 10^{-2}$	$(-3.51 \pm 0.34 \pm 0.02) \times 10^{-2}$
C_5	$(7.06 \pm 0.49 \pm 0.04) \times 10^{-3}$	$(8.36 \pm 0.66 \pm 0.04) \times 10^{-3}$	$(8.75 \pm 0.52 \pm 0.04) \times 10^{-3}$
C_6	$(-2.94 \pm 0.29 \pm 0.02) \times 10^{-2}$	$(-2.97 \pm 0.29 \pm 0.02) \times 10^{-2}$	$(-4.09 \pm 0.46 \pm 0.02) \times 10^{-2}$
C_7	$(0.29 \pm 0.01 \pm 0.20) \times 10^{-3}$	$(0.17 \pm 0.00 \pm 0.19) \times 10^{-3}$	$(0.09 \pm 0.01 \pm 0.20) \times 10^{-3}$
C_8	$(2.88 \pm 0.32 \pm 0.69) \times 10^{-4}$	$(4.53 \pm 0.49 \pm 0.70) \times 10^{-4}$	$(4.39 \pm 0.52 \pm 0.77) \times 10^{-4}$
C_9	$(-0.96 \pm 0.01 \pm 0.10) \times 10^{-2}$	$(-0.93 \pm 0.00 \pm 0.10) \times 10^{-2}$	$(-0.97 \pm 0.01 \pm 0.10) \times 10^{-2}$
C_{10}	$(2.12 \pm 0.19 \pm 0.24) \times 10^{-3}$	$(1.95 \pm 0.17 \pm 0.24) \times 10^{-3}$	$(1.86 \pm 0.16 \pm 0.23) \times 10^{-3}$

Table 1: *Wilson coefficients of the $\Delta S = 1$ effective Hamiltonian at $\mu = 1.5, 2, 3, 4.5$ GeV. For $\mu < m_b$, the relation (49) has been used to reduce the operator basis. We take $\Lambda^{(4)}_{QCD} = (330 \pm 100)$ MeV and $m_t = (174 \pm 17)$ GeV. The values of the coefficients shown here correspond to the central values of these parameters. The first error is due to the uncertainty on Λ_{QCD}, the second is due to m_t.*

The $\Delta B=2$ effective Hamiltonian is given by

$$\mathcal{H}_{eff}^{\Delta B=2} = \frac{G_F^2}{16\pi^2} M_W^2 \lambda_t^2 (\bar{b}\gamma_L^\mu d)^2 F(x_t)\,. \tag{63}$$

Here $\lambda_t = V_{td}V_{tb}^*$. From eq. (63), one finds the B^0–$\bar{B}^0$ mixing parameter

$$\begin{aligned} x_d &= \frac{\Delta M_B}{\Gamma} = C_B \frac{\tau_B f_B^2}{M_B} B_B A^2 \lambda^6 \Big(1+\sigma^2 - 2\sigma\cos\delta\Big) F(x_t), \\ C_B &= \frac{G_F^2 M_W^2 M_B^2}{6\pi^2}\,, \end{aligned} \tag{64}$$

where B_B is the B-parameter relevant for $B-\bar{B}$ mixing, the definition of which is analogous to the B_K one.

We can write ϵ' as

$$\epsilon' = i\frac{e^{i(\delta_2-\delta_0)}}{\sqrt{2}} \frac{\omega}{\mathrm{Re}A_0} \Big[\omega^{-1}(\mathrm{Im}A_2)' - (1-\Omega_{IB})\,\mathrm{Im}A_0\Big]\,. \tag{65}$$

With respect to eq. (14), we have here explicitly written the isospin-breaking contribution Ω_{IB}, see for example ref. [30],

$$(\mathrm{Im}A_2)' = (\mathrm{Im}A_2) - \Omega_{IB}(\omega\,\mathrm{Im}A_0)\,. \tag{66}$$

To compute $\mathrm{Im}A_0$ and $(\mathrm{Im}A_2)'$, we need the hadronic matrix elements of the operators Q_i between a kaon and two pions. Usually they are given in terms of the so-called B-parameters:

$$\begin{aligned} \langle\pi\pi(I=0)|Q_i(\mu)|K\rangle &= B_i^{1/2}(\mu)\langle\pi\pi(I=0)|Q_i|K\rangle_{VIA} \\ \langle\pi\pi(I=2)|Q_i(\mu)|K\rangle &= B_i^{3/2}(\mu)\langle\pi\pi(I=2)|Q_i|K\rangle_{VIA}\,, \end{aligned} \tag{67}$$

where the subscripts VIA means that the matrix elements are computed in the vacuum insertion approximation. The relevant VIA matrix elements can be expressed in terms of three quantities

$$X = f_\pi\left(M_K^2 - M_\pi^2\right), \tag{68}$$

$$Y = f_\pi\left(\frac{M_K^2}{m_s(\mu)+m_d(\mu)}\right)^2 \sim 12\,X\left(\frac{0.15\,\mathrm{GeV}}{m_s(\mu)}\right)^2, \tag{69}$$

$$Z = 4\left(\frac{f_K}{f_\pi}-1\right)Y\,. \tag{70}$$

From $\mathcal{H}_{eff}^{\Delta S=1}$, the expressions of $(\mathrm{Im}A_2)'$ and $\mathrm{Im}A_0$ in terms of Wilson coefficients and of the B-parameters are obtained

$$\begin{aligned} \mathrm{Im}A_0 = &-\frac{G_F}{\sqrt{2}}\mathrm{Im}\Big(V_{ts}^*V_{td}\Big)\Big\{-\Big(C_6B_6+\frac{1}{3}C_5B_5\Big)Z + (C_4B_4+ \\ &\frac{1}{3}C_3B_3\Big)X + C_7B_7^{1/2}\left(\frac{2Y}{3}+\frac{Z}{6}+\frac{X}{2}\right) + C_8B_8^{1/2}(2Y+ \\ &\frac{Z}{2}+\frac{X}{6}\Big) - C_9B_9^{1/2}\frac{X}{3} + \left(\frac{C_1B_1^c}{3}+C_2B_2^c\right)X\Big\}\,, \end{aligned} \tag{71}$$

$$
\begin{aligned}
(\mathrm{Im}A_2)' \;=\; & -G_F \mathrm{Im}\Big(V_{ts}^* V_{td}\Big)\Big\{C_7 B_7^{3/2}\left(\frac{Y}{3}-\frac{X}{2}\right)+ \\
& C_8 B_8^{3/2}\left(Y-\frac{X}{6}\right)+C_9 B_9^{3/2}\frac{2X}{3}\Big\}\,. \qquad (72)
\end{aligned}
$$

Notice that the matrix elements of the electromagnetic left–right operators $Q_{7,8}$, which belong to the $(8_L, 8_R)$ representation of $SU(3)_L \otimes SU(3)_R$, contain Y and do not vanish in the chiral limit.

The evaluation of the B-factors requires a non-perturbative technique. The Wilson coefficients and the hadronic matrix elements both depend on the regularization scheme. In order to cancel this dependence (up to $O(\alpha_s^2)$), it is necessary to control the matching between the B-parameters and the coefficients at the next-to-leading order. Notice that many non-perturbative methods (e.g. $1/N$ expansion) do not fulfil this requirement.

Two different approaches to the matrix element evaluation have been used in recent next-to-leading ϵ'/ϵ analyses:

- In our previous analysis [8, 9], the numerical values of the B-parameters have been taken from lattice calculations [31]. Suitable renormalization factors are introduced to take into account the difference between the HV, NDR and lattice regularization schemes. For those B-parameters not yet computed on the lattice[5], we have made educated guesses, which are discussed in detail in ref. [9].

- A phenomenological approach has been implemented in ref. [14], where the B-parameters are constrained by using the experimental information from CP-conserving processes, by assuming $SU(3)$ flavour symmetry and deducing some constraints relating hadronic matrix elements at the charm threshold. Unfortunately, there is no way to determine the most important B-factors necessary to estimate ϵ'/ϵ, namely B_6 and B_8, which remain essentially unconstrained in this approach.

8 Results

In this section, the main results of our analysis are summarized. These results have been obtained by varying the experimental quantities, e.g. the value of the top mass m_t, τ_B, etc., and the theoretical parameters, e.g. the B-parameters, the strange quark mass m_s, etc., according to their errors. Values and errors of the input quantities used in the following are reported in tables 2–4. We assume a Gaussian distribution for the experimental quantities and a flat distribution (with a width of 2σ) for the theoretical ones. The only exception is m_s, taken from quenched lattice QCD calculations, for which we have assumed a Gaussian distribution, according to the results of ref. [32].

The theoretical predictions ($\cos\delta$, ϵ'/ϵ, etc.) depend on several fluctuating parameters. We have obtained their distributions numerically, from which we have calculated the central values and the errors reported below.

[5] Indeed B-parameters,which give the main contribution to the value of ϵ'/ϵ, namely B_6, $B_8^{(3/2)}$ and $B_9^{(3/2)}$, have already been computed on the lattice, see table 4.

Parameters	Values
m_t	(174 ± 17) GeV
$m_s(2\ \mathrm{GeV})$	(128 ± 18) MeV
$\Lambda_{QCD}^{n_f=5}$	(230 ± 80) MeV
$V_{cb} = A\lambda^2$	0.040 ± 0.006
$\vert V_{ub}/V_{cb}\vert = \lambda\sigma$	0.080 ± 0.015
τ_B	$(1.49 \pm 0.12) \times 10^{-12}$ sec
x_d	0.685 ± 0.076
$(f_B B_B^{1/2})_{th}$	(200 ± 40) MeV
Ω_{IB}	0.25 ± 0.10

Table 2: *Values of the fluctuating parameters used in the numerical analysis.*

Using the values given in the tables and the formulae given in the previous sections, we have obtained the following results:

a) The distribution for $\cos\delta$, obtained by comparing the experimental value of ϵ with its theoretical prediction, is given in fig. 3. As already noticed in refs. [7, 8] and [20, 21], large values of f_B and m_t favour $\cos\delta > 0$, given the current measurement of x_d. When the condition 160 MeV $\leq f_B B_B^{1/2} \leq$ 240 MeV is imposed (f_B-cut), most of the negative solutions disappear, giving the dashed histogram of fig. 3, from which we estimate

$$\cos\delta = 0.47 \pm 0.32 \ . \tag{73}$$

b) A contour plot in the ρ–η plane is given in fig. 4. It shows the current limits on the unitarity triangle defined in fig. 1.

c) In fig. 5, several pieces of information on ϵ'/ϵ are provided. Lego plots of the distribution of the generated events in the ϵ'/ϵ–$\cos\delta$ plane are shown, without and with the f_B-cut. In the same figure, the corresponding contour plots are displayed. One notices a very mild dependence of ϵ'/ϵ on $\cos\delta$. As a consequence, one obtains approximately the same prediction in the two cases (see also fig. 3). In the HV scheme the results are

$$\epsilon'/\epsilon = (2.3 \pm 2.1) \times 10^{-4} \ \ \mathrm{no-cut} \tag{74}$$

and

$$\epsilon'/\epsilon = (2.8 \pm 2.4) \times 10^{-4} \ \ f_B - \mathrm{cut}\,, \tag{75}$$

whereas in the NDR scheme we obtain

$$\epsilon'/\epsilon = (2.8 \pm 2.2) \times 10^{-4} \ \ \mathrm{no-cut} \tag{76}$$

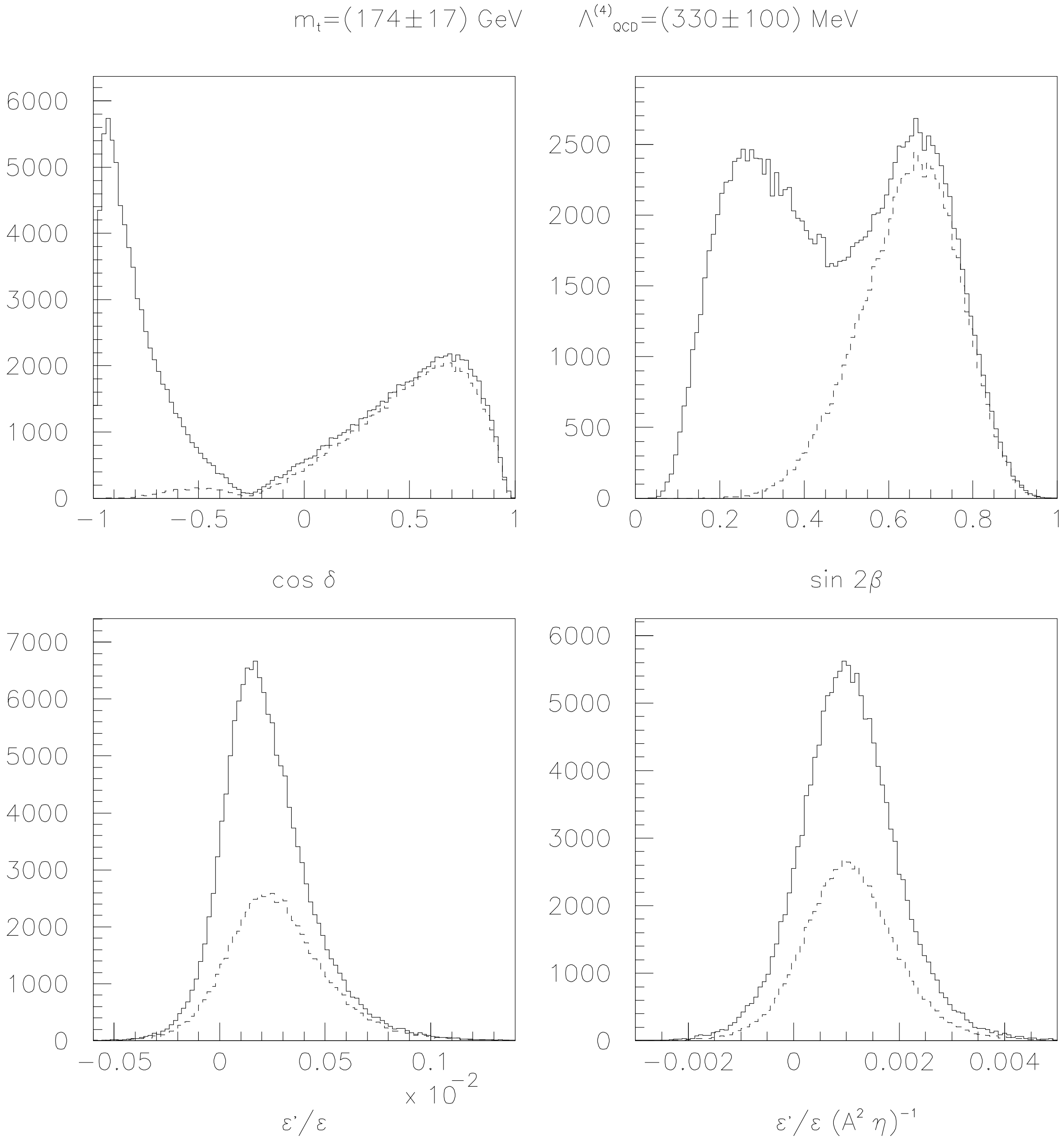

Figure 3: *Distributions of values for* $\cos\delta$, $\sin 2\beta$, ϵ'/ϵ *and* $\epsilon'/\epsilon(A^2\eta)^{-1}$, *for* $m_t = (174{\pm}17)$ *GeV, using the values of the parameters given in tabs. 2–4. The solid histograms are obtained without using the information coming from* B_d*–*$\bar{B}_d$ *mixing. The dashed ones use the* x_d *information, assuming that* $160\ MeV \leq f_B B_B^{1/2} \leq 240\ MeV$.

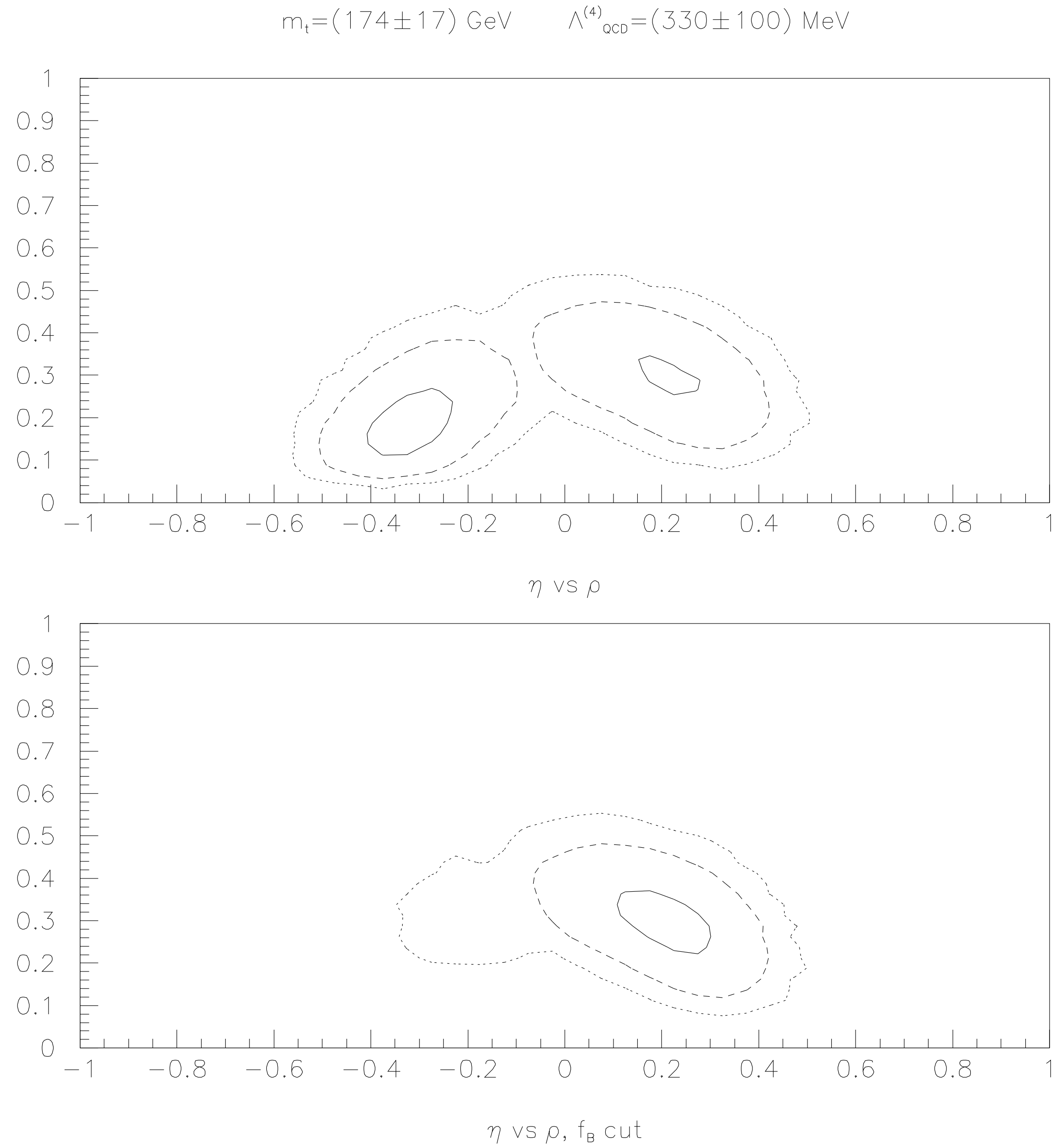

Figure 4: *Contour plots in the ρ–η plane. The solid, dashed and dotted contours contain 5%, 68% and 95% of the generated events respectively. The contours are given by excluding or including the f_B-cut. Similar results can be found in refs. [20, 21].*

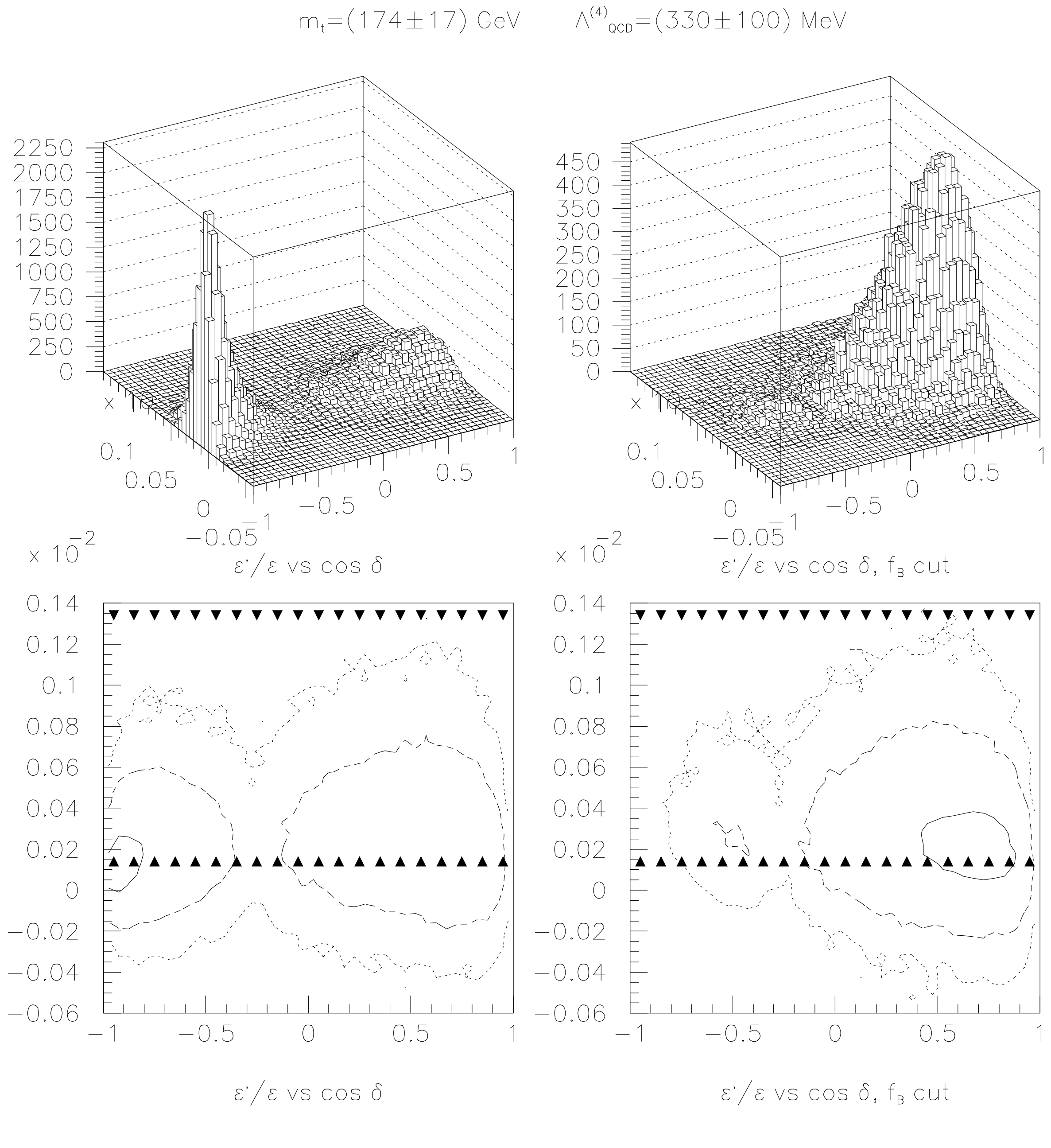

Figure 5: *Distributions of the events in the plane ϵ'/ϵ–$\cos\delta$ without and with the f_B-cut. The corresponding contour plots are displayed below the Lego plots.*

Constants	Values
G_F	$1.16634 \times 10^{-5}\mathrm{GeV}^{-2}$
m_c	1.5 GeV
m_b	4.5 GeV
M_W	80.6 GeV
M_π	140 MeV
M_K	490 MeV
M_B	5.278 GeV
ΔM_K	3.521×10^{-12} MeV
f_π	132 MeV
f_K	160 MeV
$\lambda = \sin\theta_c$	0.221
ϵ_{exp}	2.268×10^{-3}
$\mathrm{Re}A_0$	2.7×10^{-7} GeV
ω	0.045
μ	2 GeV

Table 3: *Constants used in the numerical analysis.*

and

$$\epsilon'/\epsilon = (3.4 \pm 2.5) \times 10^{-4} \quad f_B - \mathrm{cut}\,. \tag{77}$$

By averaging the results given in eqs. (75) and (77), we obtain our best estimate

$$\epsilon'/\epsilon = (3.1 \pm 2.5 \pm 0.3) \times 10^{-4} \quad f_B - \mathrm{cut}\,, \tag{78}$$

where the third error comes from the difference of the central values in the two schemes and gives an estimate of the uncertainty due to higher-order corrections.

A similar result has been obtained in ref. [14], using a different approach to the

B_K^{rgi}	$B_9^{(3/2)}$	B_{1-2}^c	$B_{3,4}$	$B_{5,6}$	$B_{7-8-9}^{(1/2)}$	$B_{7-8}^{(3/2)}$
0.75 ± 0.15	0.62 ± 0.10	$0 - 0.15^{(*)}$	$1 - 6^{(*)}$	1.0 ± 0.2	$1^{(*)}$	1.0 ± 0.2

Table 4: *Values of the B-parameters, for operators renormalized at the scale $\mu = 2$ GeV. The only exception is B_K^{rgi}, which is the renormalization group invariant B-parameter. $B_9^{3/2}$ has been taken equal to B_K, at any renormalization scale. The value reported in the table is $B_9^{3/2}(\mu = 2$ GeV). Entries with a $^{(*)}$ are educated guesses, the others are taken from lattice QCD calculations.*

hadronic-matrix-element evaluation. They quote

$$\frac{\epsilon'}{\epsilon} = (6.7 \pm 2.6) \times 10^{-4} \tag{79}$$

for $m_t = 130$ GeV. For this value of the top mass, the cancellation between penguin and electropenguin contributions is less effective, thus their ϵ'/ϵ prediction is significantly larger than ours. Actually the two predictions agree, once the difference in the top mass is taken into account[6]. It is reassuring that theoretical predictions, obtained by using quite different approaches to matrix elements evaluation, are in good agreement.

On the basis of the latest analyses, it seems very difficult that ϵ'/ϵ is larger than 10×10^{-4}. Theoretically, this may happen by taking the matrix elements of the dominant operators, Q_6 and Q_8, much more different than it is usually assumed. One possibility, discussed in ref. [14], is to take $B_6 \sim 2$ and $B_8 \sim 1$, instead of the usual values $B_6 \sim B_8 \sim 1$. To our knowledge, no coherent theoretical approach can accommodate such large values of B_6.

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