Feynman Integrals, Polylogarithms and Symbols

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based on work with Claude Duhr & Volodya Smirnov

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- let us consider *l*-loop *n*-point Feynman integrals

example: 1-loop *n*-point scalar Feynman integral without internal or external masses

$$I_n^D\Big(\{\nu_i\};\{Q_i^2\}\Big) \propto \int \frac{d^D k}{i\pi^{D/2}} \prod_{i=1}^{n} \frac{1}{D_i^{\nu_i}}$$

$$D_1 = k^2 + i0, \qquad D_m = \left(k + \sum_{j=1}^{m-1} k_j\right)^2 + i0, \qquad m = 2, \ldots, n$$

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- suppose that we can compute analytically the Feynman integral,
  which is given as a combination of multiple polylogarithms

$$G(a, \vec{w}; z) = \int_0^z \frac{\mathrm{d}t}{t-a} G(\vec{w}; t)\,, \qquad G(a; z) = \ln\left(1 - \frac{z}{a}\right)$$

Goncharov 98

- suppose that the result is quite involved, which is the analytic outcome of the Feynman integral, but that we are interested in a ``simpler” analytic result, *e.g.* because either we want to study the underlying theory, or we just need a numeric result, but we hope to use simpler numeric routines of the functions involved

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- using those functional relations, it may become a frustating and unyielding exercise to simplify the result

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Duhr Smirnov VDD 09

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- it turned out to be unfeasible to ``simplify” the result through functional relations which related the multiple polylogarithms

N=4 Super Yang-Mills

- maximal supersymmetric theory (without gravity)
  conformally invariant, β fn. = 0

  - spin 1 gluon
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- AdS/CFT duality 		Maldacena 97
  - large-λ limit of 4dim CFT $\leftrightarrow$ weakly-coupled string theory
    (aka weak-strong duality)

MHV amplitudes in planar N=4 SYM

- at any order in the coupling, a colour-ordered MHV amplitude in N=4 SYM can be written as tree-level amplitude times helicity-free loop coefficient $M_n^{(L)} = M_n^{(0)} m_n^{(L)}$

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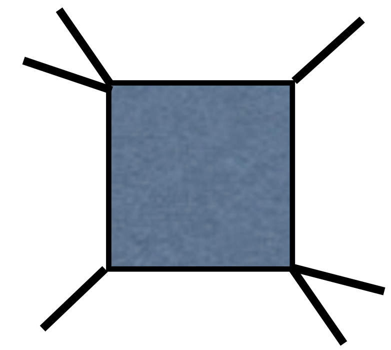

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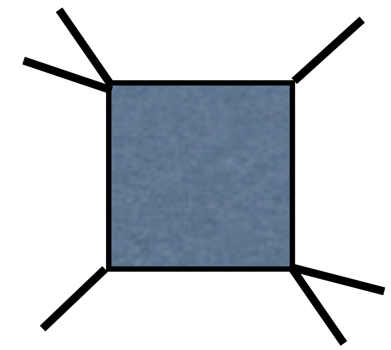

- at 2 loops, iteration formula for the *n*-pt amplitude

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + R$$

Anastasiou Bern Dixon Kosower 03

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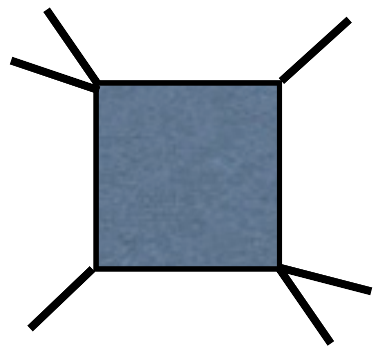

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Bern Dixon Smirnov 05

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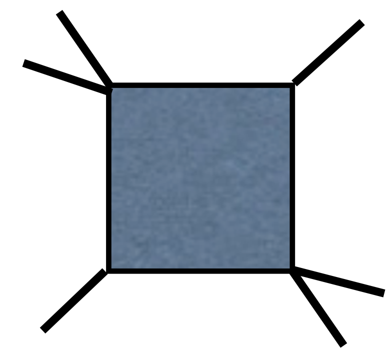

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$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - Const^{(2)}$$

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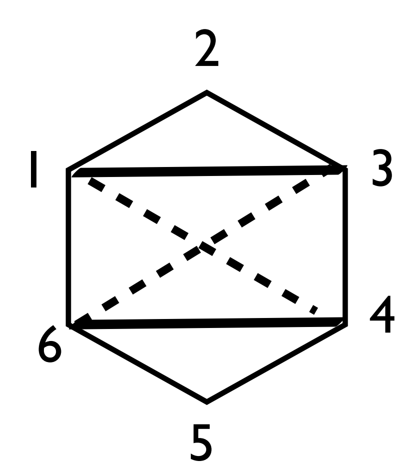
- for n = 6, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$



x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

thus $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$

2-loop 6-point remainder function $R_6^{(2)}$

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transcendental weights: $w(\ln x) = w(\pi) = 1$ $\quad w(\mathrm{Li}_2(x)) = w(\pi^2) = 2$

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↓ finite answer, but in intermediate steps many divergences
output is punishingly long: $O(10^3)$ multiple polylogarithms $G(u_1, u_2, u_3)$

- yet, our result was given in terms of polylogarithms

Goncharov Spradlin Vergu Volovich 10

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where

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$$L_4(x^+,x^-) = \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-)) + \frac{1}{8!!}\log(x^+x^-)^4$$

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↑ answer is short and simple
introduces *symbols* in TH physics

Symbols

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- take a fn. defined as an iterated integral of logs of rational functions R_i

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  is defined on the tensor product of the group of rational functions, modulo constants

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$$\mathrm{Li}_1(z) = -\ln(1-z) \qquad \mathrm{Li}_k(z) = \int_0^z \mathrm{d}\ln t \; \mathrm{Li}_{k-1}(t) \qquad \mathrm{Sym}[\mathrm{Li}_k(z)] = -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{k\text{-1 times}}$$

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- in general, if $Disc(f\,g) = Disc(f)\,g + f\,Disc(g)$

  and $\mathrm{Sym}[f] = \otimes_{i=1}^{n} R_i \qquad \mathrm{Sym}[g] = \otimes_{i=n+1}^{m} R_i$

  then $\mathrm{Sym}[fg] = \sum_\sigma \otimes_{i=1}^{n} R_{\sigma(i)}$

  where σ denotes the set of all shuffles of $n+(m-n)$ elements

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  become algebraic identities satisfied by its symbol

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- through some symbol-processing procedure,
  find a simpler form of the integral in terms of multiple polylogarithms

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let us prove the identity $$\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(x) - \ln x \ln(1-x) + \frac{\pi^2}{6}$$

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at $x = 1$ $\quad 0 = -\frac{\pi^2}{6} - 0 + c\,\pi^2 \Longrightarrow c = \frac{1}{6}$

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$$\mathrm{Li}_{m_1,\ldots,m_k}(u_1,\ldots,u_k) = \sum_{n_k=1}^{\infty} \frac{u_k^{n_k}}{n_k^{m_k}} \sum_{n_{k-1}=1}^{n_k-1} \cdots \sum_{n_1=1}^{n_2-1} \frac{u_1^{n_1}}{n_1^{m_1}}$$

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- special values as polylogarithms and Nielsen logarithms

$$G(\vec{0}_n;x) = \frac{1}{n!}\ln^n x \qquad G(\vec{a}_n;x) = \frac{1}{n!}\ln^n\left(1-\frac{x}{a}\right)$$

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- when the root equals 1, multiple polylogarithms become harmonic polylogarithms (HPLs)

$$\mathrm{Li}_n(x) = H(\vec{0}_{n-1},1;x) \qquad S_{n,m}(x) = H(\vec{0}_n,\vec{1}_m;x)$$

HPLs are defined through iterated integrals

$$H(a,\vec{w};z) = \int_0^z \mathrm{d}t\, f(a;t)\, H(\vec{w};t) \qquad f(-1;t) = \frac{1}{1+t}, \quad f(0;t) = \frac{1}{t}, \quad f(1;t) = \frac{1}{1-t}$$

with $\{a,\vec{w}\} \in \{-1,0,1\}$

... on to symbols

$$\mathrm{Sym}[\ln x] = x \qquad \mathrm{Sym}\left[\frac{1}{n!}\ln^n x\right] = \underbrace{x \otimes \cdots \otimes x}_{n \text{ times}} \equiv x^{\otimes n}$$

$$\mathrm{Sym}[\mathrm{Li}_n(x)] = -(1-x) \otimes x^{\otimes(n-1)}$$

$$\mathrm{Sym}[S_{n,m}(x)] = (-1)^m (1-x)^{\otimes m} \otimes x^{\otimes n}$$

$$\mathrm{Sym}[H(a_1,\ldots,a_n;x)] = (-1)^k (a_n - x) \otimes \cdots \otimes (a_1 - x) \qquad \{a_i\} \in \{0,1\}$$

k is the number of *a*'s equal to 1

- using symbols, one can reduce the HPLs to a minimal set — Buehler Duhr 11

weight 1: $B_1^{(1)}(x) = \ln x\,, \qquad B_1^{(2)}(x) = \ln(1-x)\,, \qquad B_1^{(3)}(x) = \ln(1+x)$

weight 2: $B_2^{(1)}(x) = \mathrm{Li}_2(x)\,, \qquad B_2^{(2)}(x) = \mathrm{Li}_2(-x)\,, \qquad B_2^{(3)}(x) = \mathrm{Li}_2\left(\frac{1-x}{2}\right)$

weight 3: polylogarithms of type Li_3 of various arguments

weight 4: polylogarithms of type Li_4 of various arguments, plus a few polylogarithms of type $\mathrm{Li}_{2,2}$, like $\mathrm{Li}_{2,2}(-1, x)$ etc. Alternatively, the polylogarithms of type $\mathrm{Li}_{2,2}$ can be replaced by the HPLs: $H(0,1,0,-1; x)$ and $H(0,1,1,-1; x)$

if needed numerically, any combination of HPLs up to weight 4 can be evaluated in terms of a minimal set of numerical routines

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- These features generalise to multiple polylogarithms — Duhr Gangl Rhodes (in progress)

weight 1: one needs functions of type $\ln x$
weight 2: $\mathrm{Li}_2(x)$
weight 3: $\mathrm{Li}_3(x)$
weight 4: $\mathrm{Li}_4(x)$, $\mathrm{Li}_{2,2}(x,y)$
weight 5: $\mathrm{Li}_5(x)$, $\mathrm{Li}_{2,3}(x,y)$
weight 6: $\mathrm{Li}_6(x)$, $\mathrm{Li}_{2,4}(x,y)$, $\mathrm{Li}_{3,3}(x,y)$, $\mathrm{Li}_{2,2,2}(x,y)$

2-loop 6-edged Wilson loop

integral

↓

function

↓

symbol

↓

simpler function

2-loop 6-edged Wilson loop

integral

function

symbol

simpler function

ideally

integral

differential equations

Drummond Henn Trnka 10

discontinuities

Caron-Huot 11

symbol

algorithm

Duhr Gangl Rhodes (in progress)

(simpler) function