The Infrared structure of gauge amplitudes in the high-energy limit

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17 May 2012

Why the infrared structure of gauge amplitudes ?

- Perturbation theory calculations of amplitudes beyond the leading order exhibit infrared divergences, which in physical processes must cancel between the virtual corrections and the real emissions

- While the finite part of an amplitude depends on the scattering process at hand, the infrared-divergent part is process independent (but for the parton species involved): it is *universal*, and reveals the infrared structure of the gauge theory

- Guesses have been made on the all-order structure of the infrared divergences (dipole formula). The high-energy limit is one more tool which allows us to constrain the all-order structure

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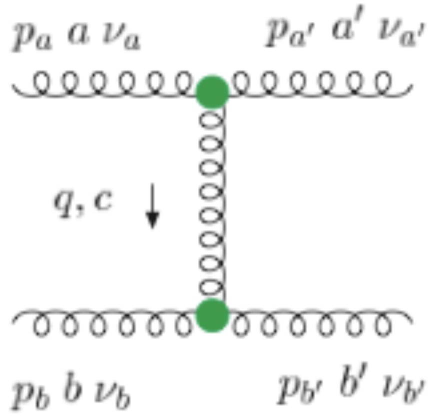

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) \;=\; 2\,g_s^2 \left[(T^c)_{aa'} C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right] \frac{s}{t} \left[(T_c)_{bb'} C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

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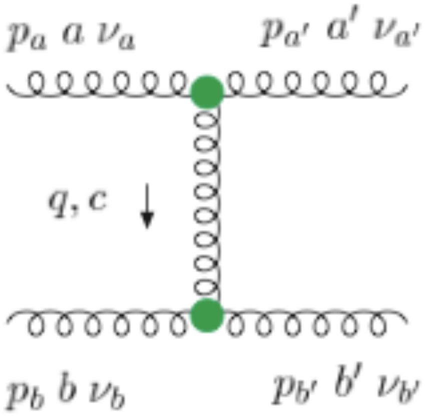

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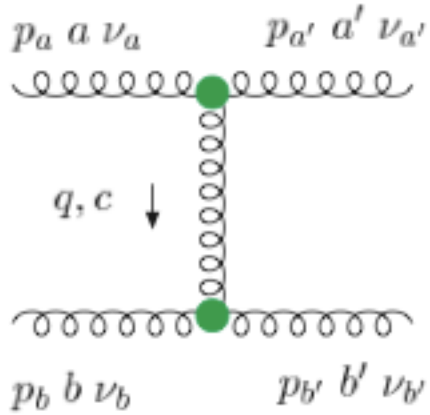

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- $\alpha(t)$ is the Regge gluon trajectory, with infrared coefficients

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right)$$

$$\alpha^{(1)} = C_A\,\frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A\,\frac{2}{\epsilon} \qquad \alpha^{(2)} = C_A\left[-\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)}\,\frac{2}{\epsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]$$

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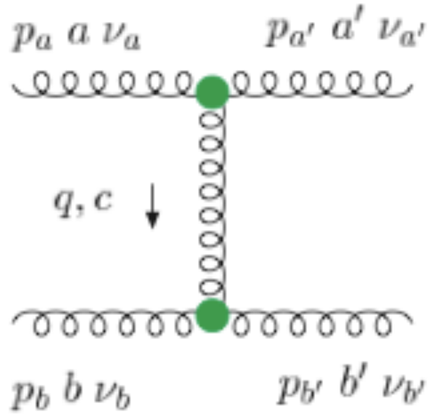

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- in the Regge limit, the amplitude is invariant under *s* ↔ *u* exchange.

  To NLL accuracy, the amplitude is given by

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\, g_s^2 \frac{s}{t} \left[(T^c)_{aa'} C_{\nu_a \nu_{a'}}(p_a,p_{a'})\right] \left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right] \left[(T_c)_{bb'} C_{\nu_b \nu_{b'}}(p_b,p_{b'})\right]$$

Resummation: Sudakov form factor

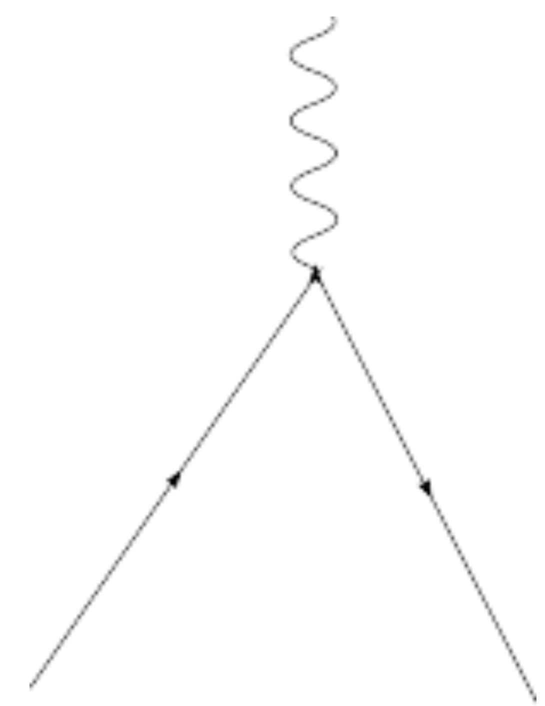
- Sudakov (quark) form factor as matrix element of EM current

$$\Gamma_\mu(p_1,p_2;\mu^2,\epsilon) \equiv < 0|J_\mu(0)|p_1,p_2 > = \bar{v}(p_2)\gamma_\mu u(p_1)\,\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)$$

obeys evolution equation

$$Q^2\frac{\partial}{\partial Q^2}\ln\left[\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] = \frac{1}{2}\left[K\left(\alpha_s(\mu^2),\epsilon\right) + G\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right]$$

K is a counterterm; G is finite as $\epsilon \to 0$



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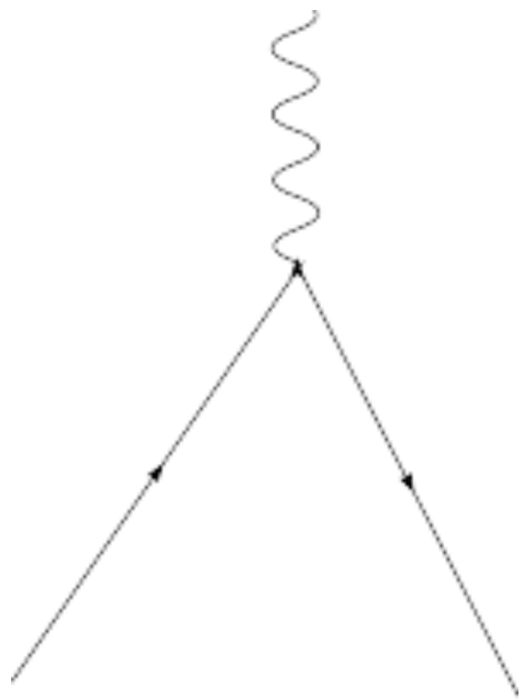
- RG invariance requires

$$\mu\frac{dG}{d\mu} = -\mu\frac{dK}{d\mu} = \gamma_K(\alpha_s(\mu^2))$$

Korchemsky Radyushkin 1987



γ_K is the cusp anomalous dimension

the solution is

$$\Gamma\left(Q^2,\epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[G\left(-1,\bar{\alpha}_s(\xi^2,\epsilon),\epsilon\right) - \frac{1}{2}\gamma_K\left(\bar{\alpha}_s(\xi^2,\epsilon)\right)\ln\left(\frac{-Q^2}{\xi^2}\right)\right]\right\}$$

cusp anomalous dimension

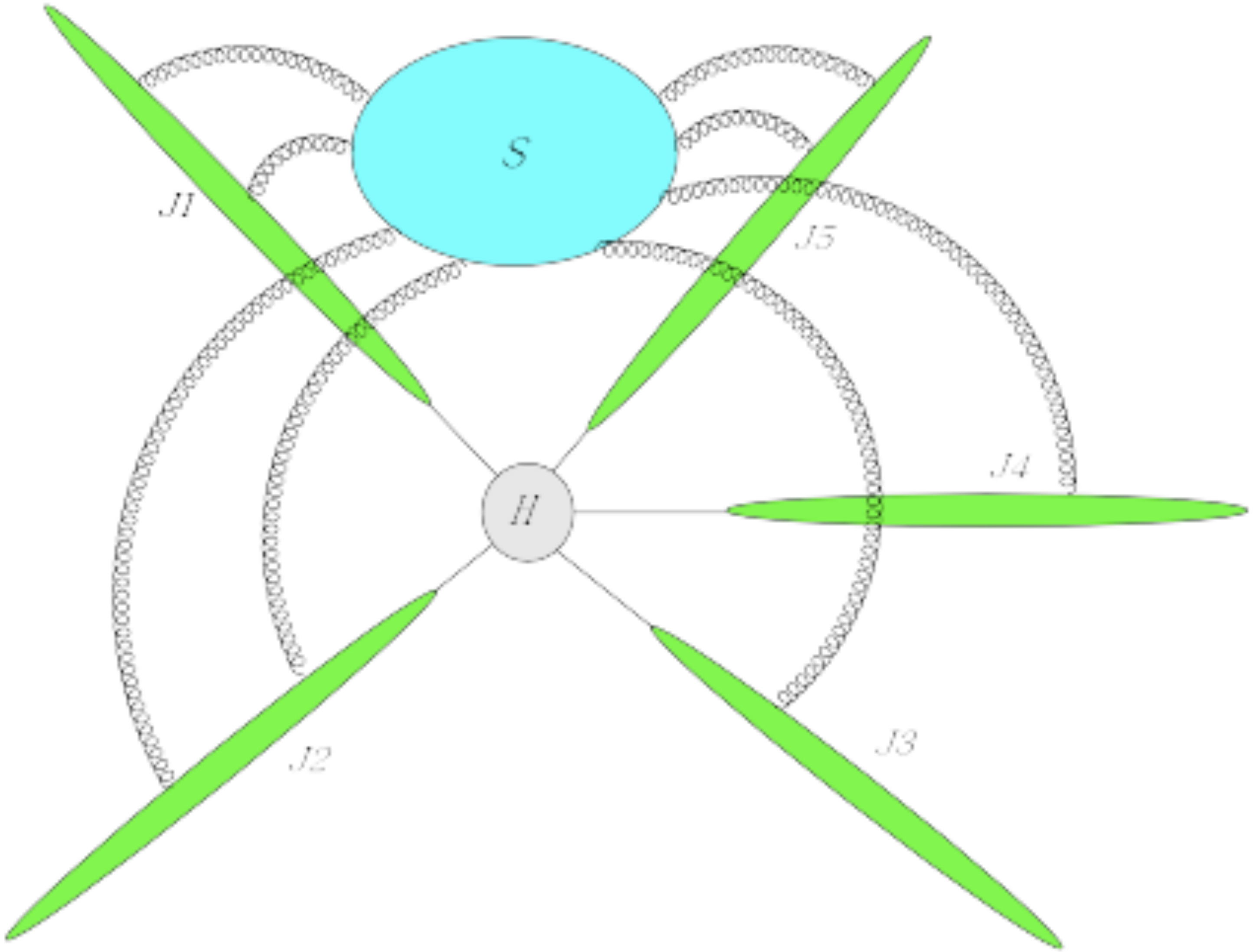
- loop expansion of the cusp anomalous dimension

$$\gamma_K^{(i)} = 2C_i \, \frac{\alpha_s(\mu^2)}{\pi} + K C_i \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^2 + \cdots$$

with

$$K = \left(\frac{67}{18} - \zeta_2 \right) C_A - \frac{10}{9} T_F N_f$$

Factorisation of a multi-leg amplitude



Mueller 1981
Sen 1983
Botts Sterman 1987
Kidonakis Oderda Sterman 1998
Catani 1998
Tejeda-Yeomans Sterman 2002
Kosower 2003
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$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \mathcal{S}_{NL}(\beta_i\cdot\beta_j,\epsilon)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right)\prod_i \frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

$p_i = \beta_i Q_0/\sqrt{2}$ value of Q_0 is immaterial in *S, J*

to avoid double counting of soft-collinear region (IR double poles),
$\mathcal{J}_i$ removes eikonal part from J_i, which is already in S
$J_i/\mathcal{J}_i$ contains only single collinear poles

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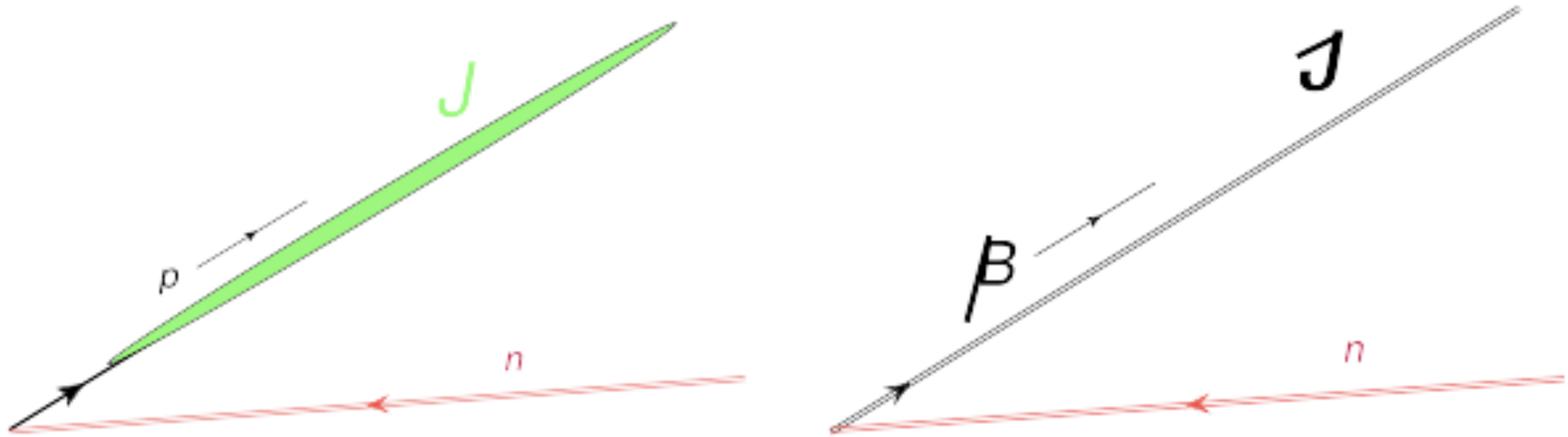
- Soft function is a matrix which mixes the colour representations and is driven by the anomalous dimension matrix Γ_S

Jet definition

- introduce auxiliary vector n_i ($n_i^2 \neq 0$) to separate collinear region
- define a jet using a Wilson line along n_i



partonic jet

$$\bar{u}(p)\, J\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \epsilon\right) = < p|\bar{\psi}(0)\Phi_n(0,-\infty)|0 >$$

Wilson line

$$\Phi_n(\lambda_2,\lambda_1) = P\exp\left[ig\int_{\lambda_1}^{\lambda_2} d\lambda n\cdot A(\lambda n)\right]$$

eikonal jet

$$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2}, \epsilon\right) = < 0|\bar{\Phi}_\beta(\infty,0)\Phi_n(0,-\infty)|0 >$$

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- double poles and kinematic dependence of single poles are controlled by cusp γ_K,
  like in the quark form factor

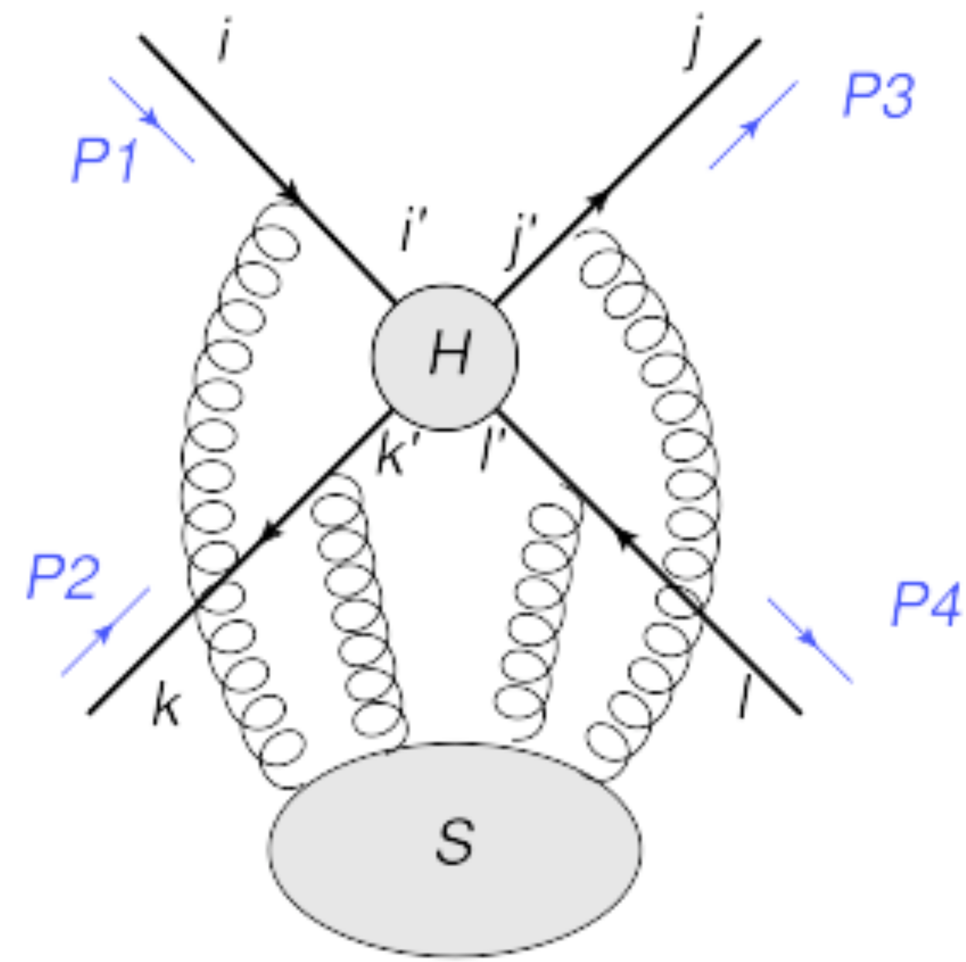
$$\mathcal{J}\left(\frac{2(\beta \cdot n)^2}{n^2}, \epsilon\right) = \exp\left\{\frac{1}{4}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left[\delta_{\mathcal{J}_i}\left(\alpha_s(\lambda^2,\epsilon)\right) - \frac{1}{2}\gamma_K\left(\bar{\alpha}_s(\lambda^2,\epsilon)\right)\ln\left(\frac{2(\beta \cdot n)^2\mu^2}{n^2\lambda^2}\right)\right]\right\}$$

δ_j is a constant

Soft function S

- soft function is a matrix which mixes the colour representations

$$(c_N)_{ijkl}\mathcal{S}_{NL}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)$$
$$=\sum_{i'j'k'l'} <0|\Phi^{k,k'}_{-\beta_2}(0,\infty)\Phi^{i,i'}_{\beta_1}(\infty,0)\Phi^{j,j'}_{\beta_3}(0,\infty)\Phi^{l,l'}_{-\beta_4}(\infty,0)|0> \ (c_L)_{i'j'k'l'}$$



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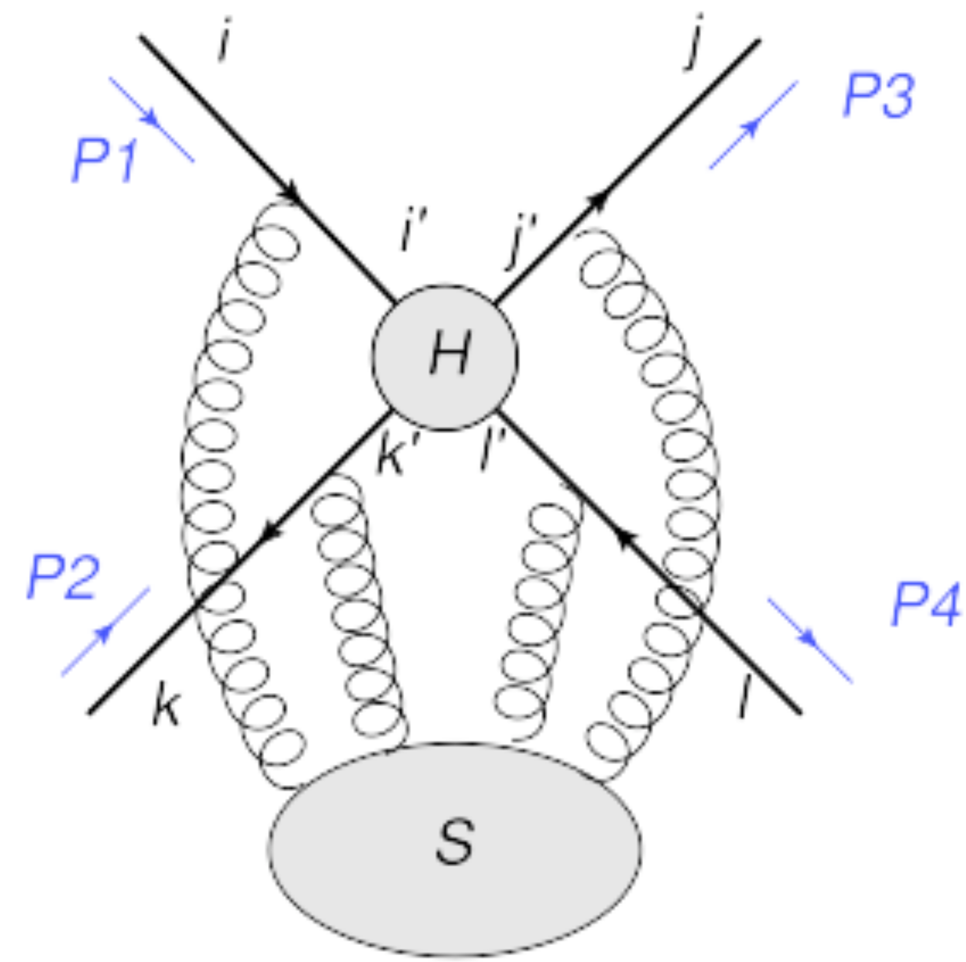
- matrix evolution equation

$$\mu\frac{d}{d\mu}\mathcal{S}_{JL}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)$$
$$=-\sum_N\left[\Gamma_{\mathcal{S}}\right]_{JN}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)\mathcal{S}_{NL}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)$$

Γ_S soft anomalous dimension,
singular due to the UV and collinear poles



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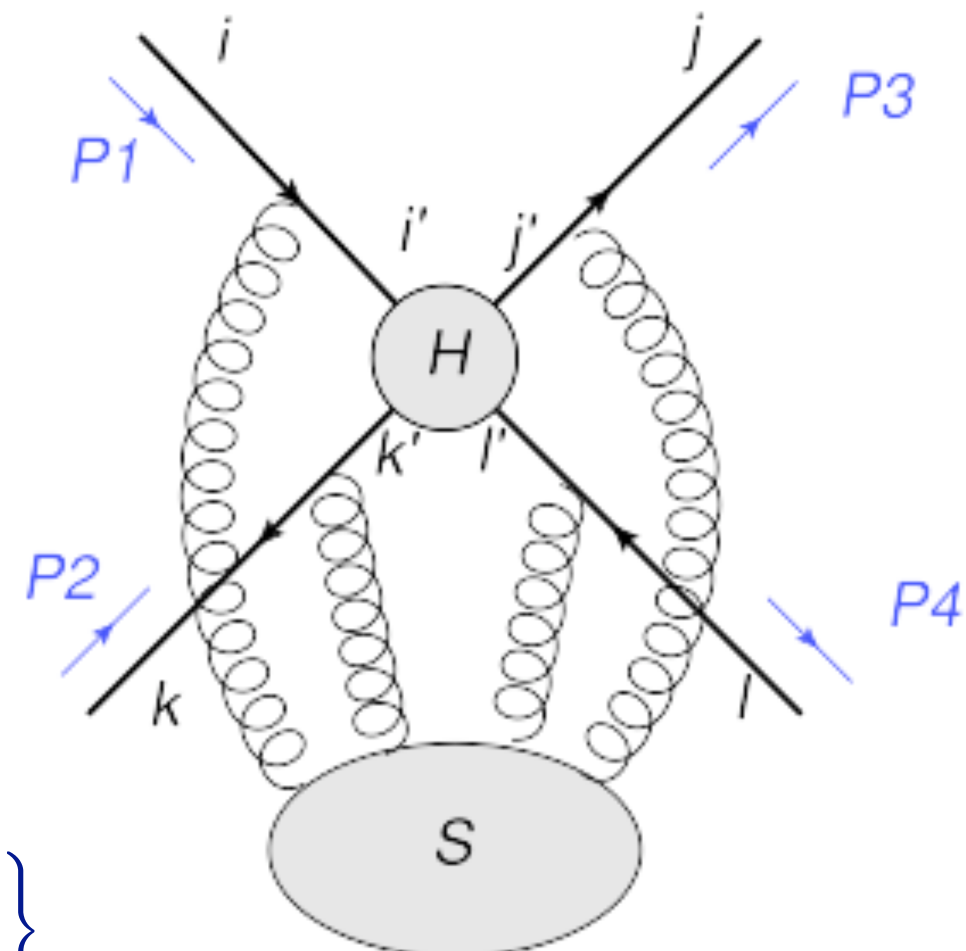


- in DimReg the solution is

$$\mathcal{S}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)=P\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma_{\mathcal{S}}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)\right\}$$

$$\Gamma_{\mathcal{S}}=\sum_{n=1}^{\infty}\Gamma_{\mathcal{S}}^{(n)}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n$$

Soft anomalous dimension

- for an amplitude with an arbitrary # of legs

$$\Gamma_{\mathcal{S}}^{(2)} = \frac{K}{2}\Gamma_{\mathcal{S}}^{(1)}$$

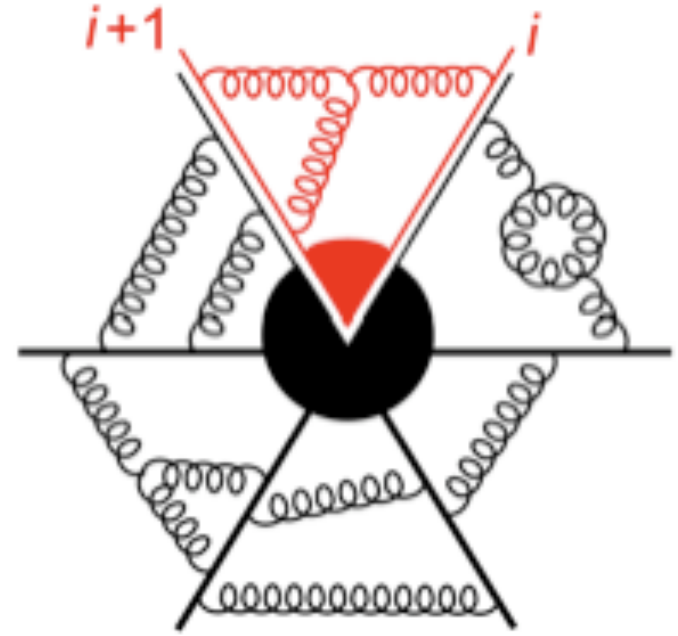
Aybat Dixon Sterman 2006

K is 2-loop coefficient of cusp anomalous dimension

Γ_S has cusp singularities like γ_J

$N = 4$ SUSY in the planar limit

- colour-wise, the planar limit is trivial:can absorb S into J_i



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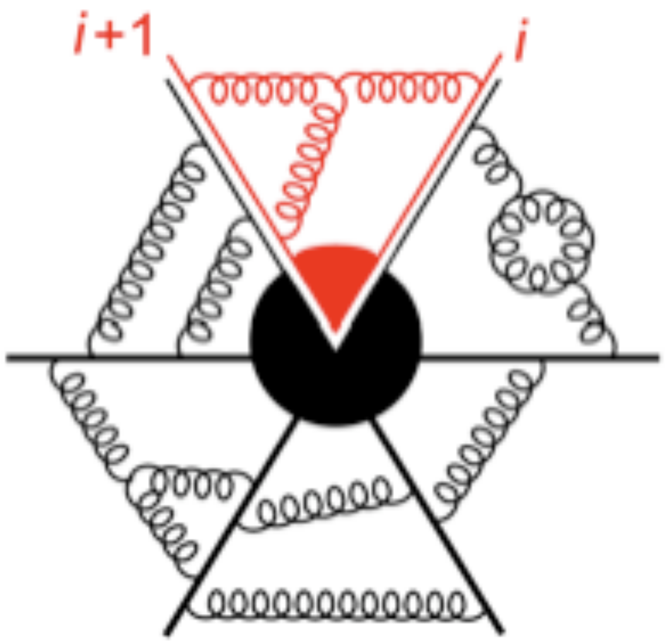

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} h_n(\{p_i\}, \mu^2, \alpha_s, \epsilon)$$

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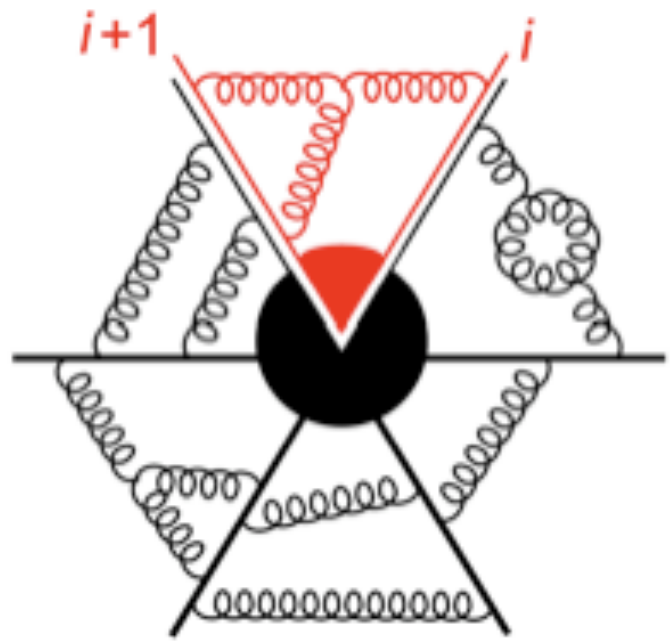

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- β fn = 0 $\Rightarrow$ coupling runs only through dimension $\qquad \bar{\alpha}_s(\mu^2)\mu^{2\epsilon} = \bar{\alpha}_s(\lambda^2)\lambda^{2\epsilon}$

  the Sudakov form factor has a simple solution

$$\ln \left[\Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = -\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^n \left(\frac{-Q^2}{\mu^2} \right)^{-n\epsilon} \left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right]$$

  $\Rightarrow$ IR structure of $\mathcal{N} = 4$ SUSY amplitudes

Reduced soft function

- $$\bar{\mathcal{S}}_{JL}\left(\rho_{ij},\epsilon\right) = \frac{\mathcal{S}_{JL}\left(\beta_i\cdot\beta_j,\epsilon\right)}{\prod_{i=1}^{n}\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

Dixon Magnea Sterman 2008

the reduced soft function is made such that the double poles cancel.
It does not have cusp singularities ⇒ must respect rescaling $\beta_i \to \kappa_i\, \beta_i$

⇒ $\bar{\mathcal{S}}$ depends only on $$\rho_{ij} = \frac{(\beta_i\cdot\beta_j)^2}{\frac{2(\beta_i\cdot n_i)^2}{n_i^2}\frac{2(\beta_j\cdot n_j)^2}{n_j^2}}$$

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- the factorisation becomes

$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \bar{\mathcal{S}}_{NL}(\rho_{ij},\epsilon)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right)\prod_i J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)$$

$\bar{\mathcal{S}}$ has only single poles due to large-angle soft emissions

Reduced soft anomalous dimension

- the evolution equation for the reduced soft anomalous dimension

$$\sum_{j\neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Gamma^{\bar{S}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\gamma_K^{(i)}(\alpha_s)$$

(simplest) solution: *dipole formula*

$$\Gamma^{\bar{S}}(\rho_{ij}, \alpha_s)\Big|_{\rm dip} = -\frac{1}{8}\hat{\gamma}_K(\alpha_s)\sum_{i\neq j}\ln(\rho_{ij})\, T_i \cdot T_j + \frac{1}{2}\,\hat{\delta}_{\bar{S}}(\alpha_s)\sum_{i=1}^{n} C_i$$

Becher Neubert 2009
Gardi Magnea 2009

with

$$\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s) \qquad \hat{\gamma}_K(\alpha_s) = 2\frac{\alpha_s(\mu^2)}{\pi} + K\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 + K^{(2)}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3 + \cdots$$

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- only 2-eikonal-line correlations

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Becher Neubert 2009
Gardi Magnea 2009

with

$$\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s) \qquad \hat{\gamma}_K(\alpha_s) = 2 \frac{\alpha_s(\mu^2)}{\pi} + K \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^2 + K^{(2)} \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^3 + \cdots$$

- only 2-eikonal-line correlations
- generalises 2-loop solution

Reduced soft anomalous dimension

- the evolution equation for the reduced soft anomalous dimension

$$\sum_{j\neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Gamma^{\bar{S}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\gamma_K^{(i)}(\alpha_s)$$

(simplest) solution: *dipole formula*

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- only 2-eikonal-line correlations
- generalises 2-loop solution
- colour matrix structure fixed at one loop
- cusp anomalous dimension plays role of IR coupling

Dipole formula for the amplitude

- combining the dipole-formula solution for the reduced soft function with the jet functions, one obtains a dipole formula for the amplitude

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) \mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

where all the collinear and soft singularities are in the dipole operator Z

$$Z\left(\frac{p_l}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left[\frac{\widehat{\gamma}_K\left(\alpha_s(\lambda^2)\right)}{4} \sum_{(i,j)} \ln\left(\frac{-s_{ij}}{\lambda^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j - \sum_{i=1}^{L} \gamma_{J_i}\left(\alpha_s(\lambda^2)\right)\right]\right\}$$

Possible corrections to the dipole formula

- the cusp anomalous dimension might violate Casimir scaling at 4 loops

$$\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s) + \tilde{\gamma}_K^{(i)}(\alpha_s)$$

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$$\Gamma^{\bar{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \Gamma^{\bar{\mathcal{S}}}(\rho_{ij}, \alpha_s)\Big|_{\rm dip} + \Delta(\rho_{ijkl}, \alpha_s)$$

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- Δ is constrained by Bose symmetry, collinear limits and transcendentality bounds

Becher Neubert 2009
Dixon Gardi Magnea 2009

Dipole formula in the high-energy limit

- we introduce the colour operators

$$\mathbf{T}_s = \mathbf{T}_a + \mathbf{T}_b\,, \qquad \mathbf{T}_a + \mathbf{T}_b + \mathbf{T}_{a'} + \mathbf{T}_{b'} = 0$$

$$\mathbf{T}_t = \mathbf{T}_a + \mathbf{T}_{a'}\,,$$

$$\mathbf{T}_u = \mathbf{T}_a + \mathbf{T}_{b'} \qquad \mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i$$

- in the limit *s* » *t*, the dipole operator ***Z*** becomes, to power accuracy in *s/t*

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right)\, Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$

Duhr Gardi Magnea White VDD 2011

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which is determined by the cusp anomalous dimension, through

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- the dipole operator fixes the Regge pole structure, and beyond

- for completeness, the operator Z_1 is

$$Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{\sum_{i=1}^{4} B_i\Big(\alpha_s(\mu^2), \epsilon\Big) + \frac{1}{2}\left[K\Big(\alpha_s(\mu^2), \epsilon\Big)\left(\ln\left(\frac{-t}{\mu^2}\right) - \mathrm{i}\pi\right) + D\Big(\alpha_s(\mu^2), \epsilon\Big)\right]\sum_{i=1}^{4} C_i\right\}$$

with

$$D\Big(\alpha_s(\mu^2), \epsilon\Big) = -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda^2, \epsilon)\right)\ln\left(\frac{\mu^2}{\lambda^2}\right),$$

$$B_i\Big(\alpha_s(\mu^2), \epsilon\Big) \equiv -\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\gamma_{J_i}\left(\alpha_s(\lambda^2, \epsilon)\right)$$

Dipole formula & leading logs

- to leading logarithmic accuracy in *s/t*,
  the dipole operator loses the imaginary part (*s*-channel)

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big) \ln\left(\frac{s}{-t}\right) \mathbf{T}_t^2\right\} Z_{\mathbf{1}}\, \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

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- in the Regge limit *s* » *t*, any scattering process is dominated by gluon exchange in the *t* channel
  in particular, in parton-parton scattering *t*-channel gluon exchange occurs at leading order,
  the other channel contributions being power suppressed
  - the *t*-channel exchange colour structure is an eigenstate of the operator

$$\mathbf{T}_t^2\, \mathcal{H}^{ff\to ff} \xrightarrow{|t/s|\to 0} C_t\, \mathcal{H}_t^{ff\to ff}$$

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- to leading logarithmic accuracy in *s/t*, the parton-parton scattering amplitude becomes

$$\mathcal{M}^{ff\to ff} = \left(\frac{s}{-t}\right)^{C_A\, K\left(\alpha_s(\mu^2),\epsilon\right)} Z_{\mathbf{1}}\, \mathcal{H}_t^{ff\to ff}$$

to leading order, the cusp anomalous dimension is

$$\widehat{\gamma}_K(\alpha_s) = 2\,\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \quad \Longrightarrow \quad K(\alpha_s, \epsilon) = \frac{1}{2\epsilon}\,\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)$$

so the singular part of the one-loop Regge gluon trajectory becomes

$$\alpha^{(1)} = C_A\,\frac{2}{\epsilon} + \mathcal{O}\left(\epsilon^0\right)$$ in agreement with the high-energy limit of parton-parton amplitudes

Dipole formula beyond the leading logs

- to power accuracy in *s/t* (thus to arbitrary logarithmic accuracy), the dipole operator *Z* can be rewritten as

$$
\begin{aligned}
\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \left(\frac{s}{-t}\right)^{K(\alpha_s,\epsilon)\,\mathbf{T}_t^2} & \exp\left\{\mathrm{i}\,\pi\, K\left(\alpha_s, \epsilon\right) \mathbf{T}_s^2\right\} \\
\times\ \exp & \left\{-\mathrm{i}\,\frac{\pi}{2}\left[K\left(\alpha_s, \epsilon\right)\right]^2 \ln\left(\frac{s}{-t}\right)\ \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right\} \\
\times\ \exp & \left\{\frac{1}{6}\left[K\left(\alpha_s, \epsilon\right)\right]^3 \left(-2\pi^2 \ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right] + \mathrm{i}\pi\ \ln^2\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right]\right)\right\} \\
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$$\times\ \exp\left\{\mathcal{O}\left(\left[K(\alpha_s,\epsilon)\right]^4\right)\right\}$$

- NLL accuracy:

$$K(\alpha_s,\epsilon) = \frac{\alpha_s}{\pi}\,\frac{1}{2\epsilon} + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\widehat{\gamma}_K^{(2)}}{8\epsilon} - \frac{b_0}{16\epsilon^2}\right) + \mathcal{O}(\alpha_s^3)$$

reproduces the singular part of the one- and two-loop Regge gluon trajectory, while the imaginary part does not Reggeise

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- NNLL accuracy:

$\mathcal{O}(\alpha_s^2):\ \longrightarrow\ -\frac{1}{2}\pi^2 K^2(\alpha_s,\epsilon)\ \left(\mathbf{T}_s^2\right)^2$ which is non-logarithmic and non-diagonal in the *t* channel

$\mathcal{O}(\alpha_s^3):\ \longrightarrow\ -\frac{\pi^2}{3}K^3(\alpha_s,\epsilon)\ \ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]$ breaks down the Regge-pole picture

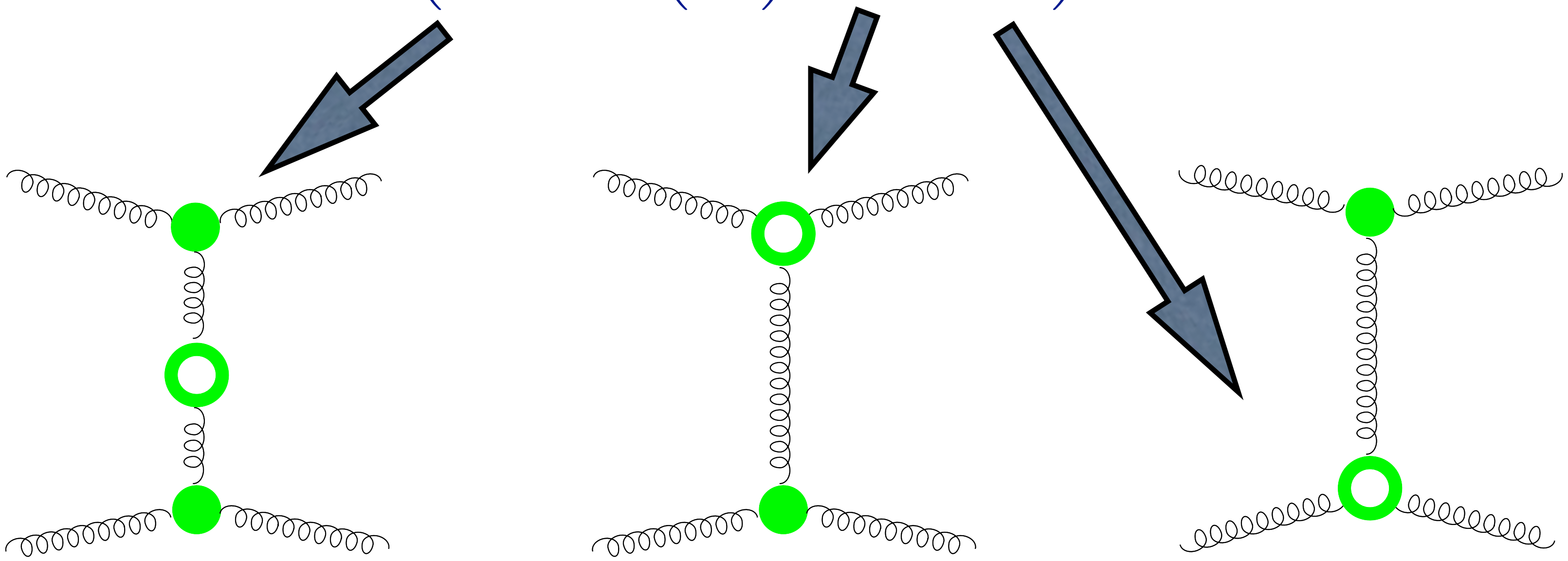
Amplitudes in the high-energy limit

- Regge limit of the gluon-gluon amplitude

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^c)_{aa'}C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[(T_c)_{bb'}C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

strip colour off & expand at one loop

$$m^{(1)}_{gg\to gg} = 2\,g_s^2\,\frac{s}{t}\left(\alpha^{(1)}(t)\ln\left(\frac{s}{-t}\right)+2\,C^{(1)}_{gg}(t)\right)$$



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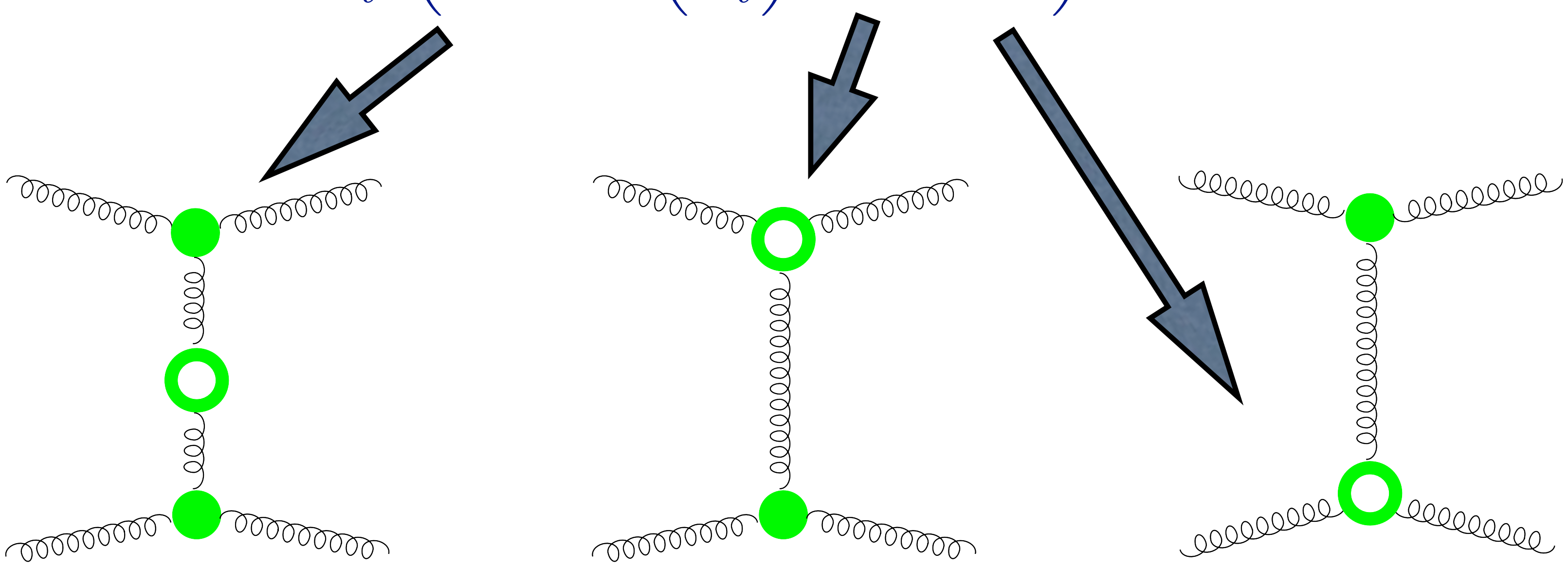

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- perform the Regge limit of the quark-quark amplitude
  → get one-loop quark impact factor

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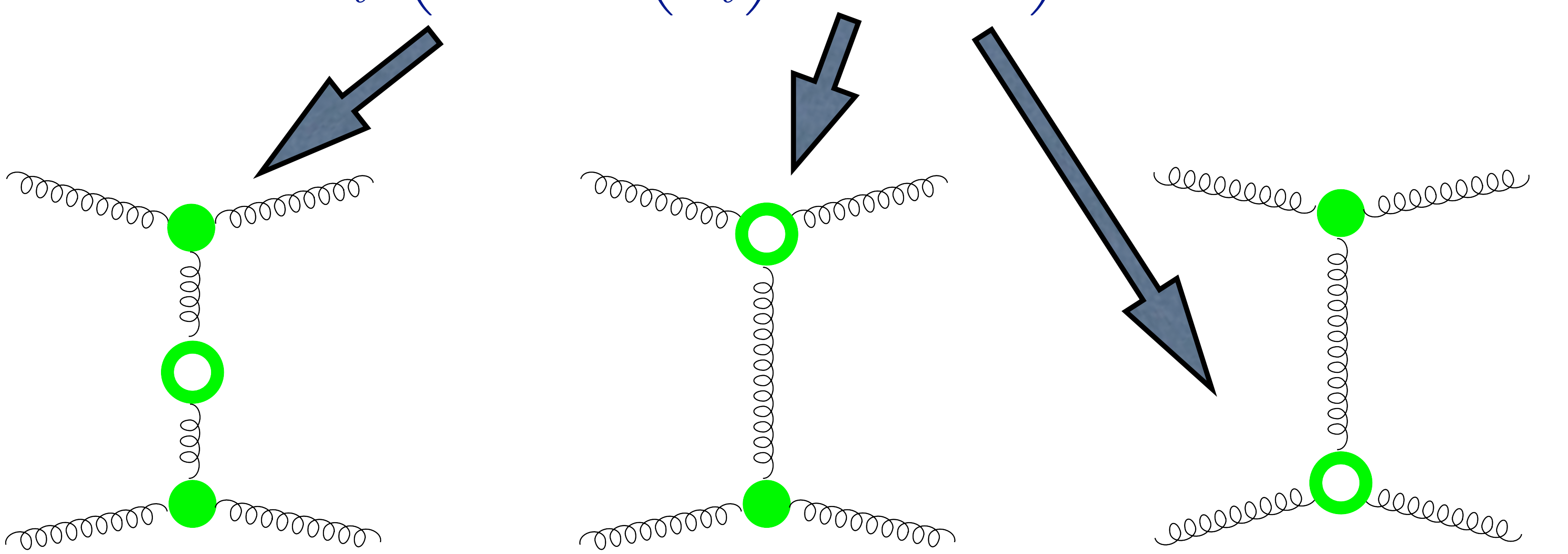

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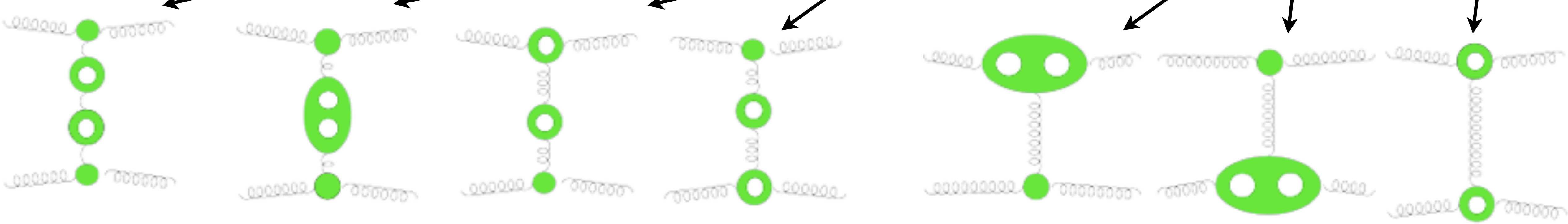
- if factorisation holds, one can obtain the one-loop quark-gluon amplitude
  by assembling the Regge trajectory and the gluon and quark impact factors
  the result should match the quark-gluon amplitude in the high-energy limit: it does

High-energy limit at 2 loops

- in the Regge limit, the 2-loop expansion of the gluon-gluon amplitude is

$$m^{(2)}_{gg\to gg} = 2\,g_s^2\,\frac{s}{t}\left[\frac{1}{2}\left(\alpha^{(1)}(t)\right)^2\ln^2\left(\frac{s}{-t}\right) + \left(\alpha^{(2)}(t) + 2\,C^{(1)}_{gg}(t)\,\alpha^{(1)}(t)\right)\ln\left(\frac{s}{-t}\right) + 2\,C^{(2)}_{gg}(t) + \left(C^{(1)}_{gg}(t)\right)^2\right]$$



- the two-loop Regge gluon trajectory is universal;
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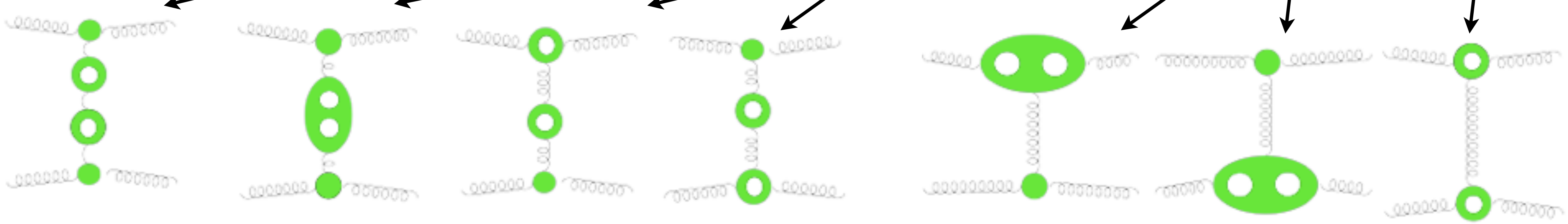

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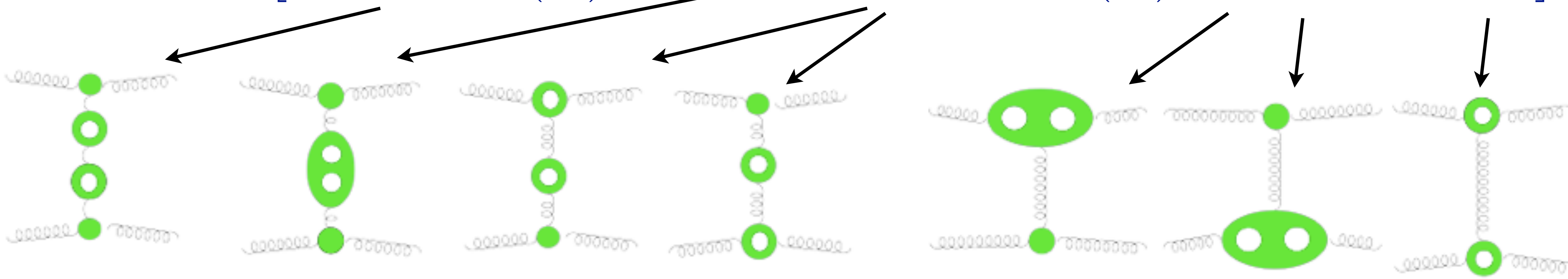
$$m^{(2)}_{gg\to gg} = 2\,g_s^2\,\frac{s}{t}\left[\frac{1}{2}\left(\alpha^{(1)}(t)\right)^2 \ln^2\left(\frac{s}{-t}\right) + \left(\alpha^{(2)}(t) + 2\,C^{(1)}_{gg}(t)\,\alpha^{(1)}(t)\right)\ln\left(\frac{s}{-t}\right) + 2\,C^{(2)}_{gg}(t) + \left(C^{(1)}_{gg}(t)\right)^2\right]$$



- the two-loop Regge gluon trajectory is universal;
  the two-loop gluon impact factor is a polynomial in t, ε, starting at $1/\varepsilon^4$

- perform the Regge limit of the quark-quark amplitude
  $\rightarrow$ get two-loop quark impact factor

- if factorisation holds, one can obtain the two-loop quark-gluon amplitude by assembling the Regge trajectory and the gluon and quark impact factors
  the result should match the quark-gluon amplitude in the high-energy limit:
  it doesn't! by a π^2/ε^2 factor

Glover VDD 2001

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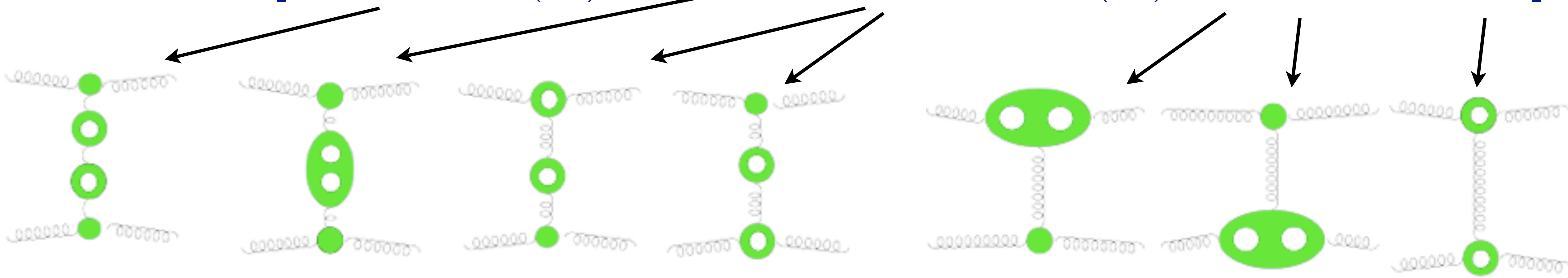

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- is it related to $\quad -\frac{1}{2}\pi^2\,K^2(\alpha_s,\epsilon)\,\left(\mathbf{T}_s^2\right)^2 \quad$?

Breaking down of the Regge-pole picture

- the analytic structure of an amplitude sports cuts and poles

- in the Regge limit, cuts should occur in 3-loop non-planar double-cross diagrams

Mandelstam 1965

- is it related to $-\frac{\pi^2}{3} K^3(\alpha_s, \epsilon) \ln\left(\frac{s}{-t}\right) \left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right]$?

Possible corrections to the dipole formula

- the high-energy limit puts constraints on 4-line correlations which may appear at 3 loops
- we know that corrections to Δ like

$$\Delta^{(212)}(\rho_{ijkl},\alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[\, f^{ade} f^{cbe}\, L_{1234}^2 \left(L_{1423}\, L_{1342}^2 + L_{1423}^2\, L_{1342}\right) + \mathrm{cycl}\Big]$$

fulfill Bose symmetry, collinear limits and transcendentality bounds Dixon Gardi Magnea 2009

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- however, in the high-energy limit

$$\rho_{1234} = \frac{(-s_{12})(-s_{34})}{(-s_{13})(-s_{24})} = \left(\frac{s}{-t}\right)^2 \mathrm{e}^{-2\mathrm{i}\pi}\,; \qquad L_{1234} = 2(L - \mathrm{i}\pi) \qquad L = \ln\left(\frac{s}{t}\right)$$

$$\rho_{1342} = \frac{(-s_{13})(-s_{24})}{(-s_{14})(-s_{23})} = \left(\frac{-t}{s+t}\right)^2 ; \qquad L_{1342} \simeq -2L$$

$$\rho_{1423} = \frac{(-s_{14})(-s_{23})}{(-s_{12})(-s_{34})} = \left(\frac{s+t}{s}\right)^2 \mathrm{e}^{2\mathrm{i}\pi}\,; \qquad L_{1423} \simeq 2\mathrm{i}\pi$$

$$\Delta^{(212)}(\rho_{ijkl},\alpha_s)) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \, 32\,\mathrm{i}\,\pi \Big[\left(- L^4 - \mathrm{i}\pi L^3 - \pi^2 L^2 - \mathrm{i}\pi^3 L \right) f^{ade} f^{cbe} + \left(2\mathrm{i}\pi L^3 - 3\pi^2 L^2 - \mathrm{i}\pi^3 L \right) f^{cae} f^{dbe} \Big] + \mathcal{O}\left(|t/s|\right)$$

has super-leading logs, which are incompatible with the high-energy limit

Duhr Gardi Magnea White VDD 2011

Conclusions

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- thus one can show that the Regge-pole picture breaks down at next-to-next-to-leading-logarithmic accuracy
- conversely, the high-energy limit allows us to put constraints on the possible corrections to the dipole formula