The Infrared structure of QCD amplitudes in the high-energy limit

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Why the infrared structure of QCD amplitudes ?

- Perturbation theory calculations of amplitudes beyond the leading order exhibit infrared divergences, which in physical processes must cancel between the virtual corrections and the real emissions

- While the finite part of an amplitude depends on the scattering process at hand, the infrared-divergent part is process independent (but for the parton species involved): it is *universal*, and reveals the infrared structure of the gauge theory

- Guesses have been made on the all-order structure of the infrared divergences (dipole formula). The high-energy limit is one more tool which allows us to constrain the all-order structure

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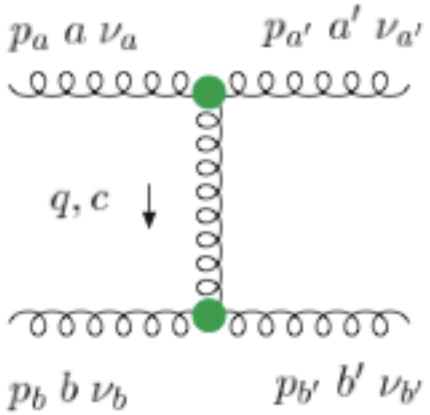

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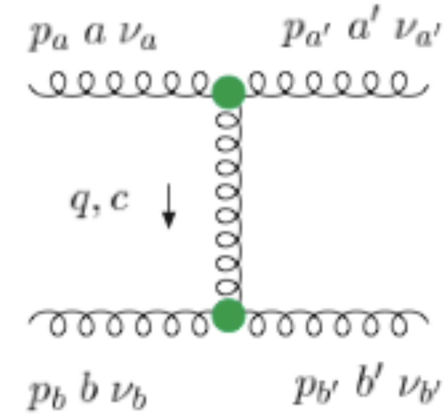

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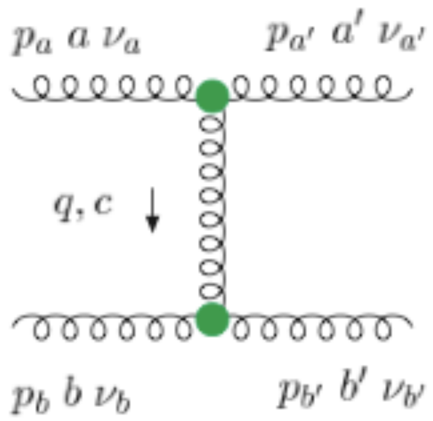

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- $\alpha(t)$ is the Regge gluon trajectory, with infrared coefficients

$$\alpha(t) \;=\; \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right)$$

$$\alpha^{(1)} \;=\; C_A\,\frac{\widehat{\gamma}_K^{(1)}}{\epsilon} \;=\; C_A\,\frac{2}{\epsilon} \qquad\qquad \alpha^{(2)} \;=\; C_A\left[-\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)}\,\frac{2}{\epsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]$$

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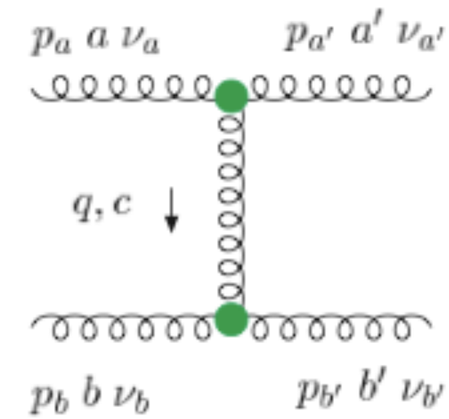

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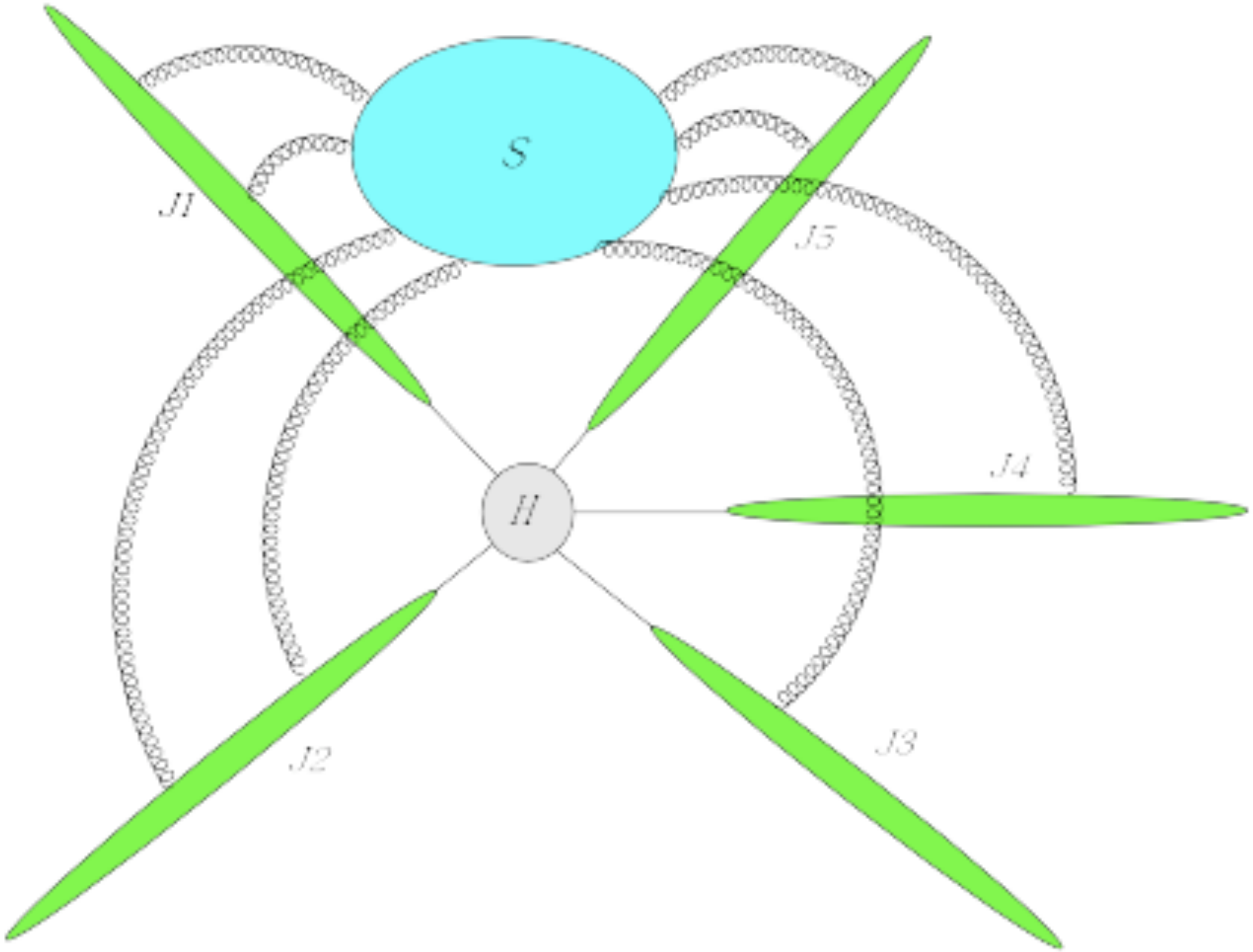
- in the Regge limit, the amplitude is invariant under *s* ↔ *u* exchange.

  To NLL accuracy, the amplitude is given by

  Fadin Lipatov 1993

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2 \frac{s}{t} \left[(T^c)_{aa'} C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right] \left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right] \left[(T_c)_{bb'} C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

Factorisation of a multi-leg amplitude



Mueller 1981
Sen 1983
Botts Sterman 1987
Kidonakis Oderda Sterman 1998
Catani 1998
Tejeda-Yeomans Sterman 2002
Kosower 2003
Aybat Dixon Sterman 2006
Becher Neubert 2009
Gardi Magnea 2009

$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \mathcal{S}_{NL}(\beta_i\cdot\beta_j,\epsilon)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2}, \frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right) \prod_i \frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

$p_i = \beta_i Q_0/\sqrt{2}$ value of Q_0 is immaterial in *S*, *J*

to avoid double counting of soft-collinear region (IR double poles),
$\mathcal{J}_i$ removes eikonal part from J_i, which is already in S
$J_i/\mathcal{J}_i$ contains only single collinear poles

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    representing the evolution of the external legs
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- Soft function is a matrix which mixes the colour representations
  and is driven by the anomalous dimension matrix Γ_S

Reduced soft anomalous dimension

- the evolution equation for the reduced soft anomalous dimension

$$\sum_{j\neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Gamma^{\bar{S}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\gamma_K^{(i)}(\alpha_s)$$

(simplest) solution: *dipole formula*

$$\Gamma^{\bar{S}}(\rho_{ij}, \alpha_s)\Big|_{\rm dip} = -\frac{1}{8}\hat{\gamma}_K(\alpha_s)\sum_{i\neq j}\ln(\rho_{ij})\, T_i \cdot T_j + \frac{1}{2}\,\hat{\delta}_{\bar{S}}(\alpha_s)\sum_{i=1}^{n} C_i$$

Becher Neubert 2009
Gardi Magnea 2009

with

$$\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s) \qquad \hat{\gamma}_K(\alpha_s) = 2\frac{\alpha_s(\mu^2)}{\pi} + K\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 + K^{(2)}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3 + \cdots$$

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- only 2-eikonal-line correlations
- generalises 2-loop solution
- colour matrix structure fixed at one loop
- cusp anomalous dimension plays role of IR coupling

Dipole formula for the amplitude

- combining the dipole-formula solution for the reduced soft function with the jet functions, one obtains a dipole formula for the amplitude

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) \mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

where all the collinear and soft singularities are in the dipole operator Z

$$Z\left(\frac{p_l}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left[\frac{\widehat{\gamma}_K\left(\alpha_s(\lambda^2)\right)}{4} \sum_{(i,j)} \ln\left(\frac{-s_{ij}}{\lambda^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j - \sum_{i=1}^{L} \gamma_{J_i}\left(\alpha_s(\lambda^2)\right)\right]\right\}$$

Possible corrections to the dipole formula

- the cusp anomalous dimension might violate Casimir scaling at 4 loops

$$\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s) + \tilde{\gamma}_K^{(i)}(\alpha_s)$$

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$$\Gamma^{\bar{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \Gamma^{\bar{\mathcal{S}}}(\rho_{ij}, \alpha_s)\Big|_{\rm dip} + \Delta(\rho_{ijkl}, \alpha_s)$$

$$\rho_{ijkl} = \frac{\rho_{ij}\rho_{kl}}{\rho_{ik}\rho_{jl}}$$

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Becher Neubert 2009
Dixon Gardi Magnea 2009

Dipole formula in the high-energy limit

- we introduce the colour operators

$$\mathbf{T}_s = \mathbf{T}_a + \mathbf{T}_b\,, \qquad \mathbf{T}_t = \mathbf{T}_a + \mathbf{T}_{a'}\,, \qquad \mathbf{T}_u = \mathbf{T}_a + \mathbf{T}_{b'}$$

$$\mathbf{T}_a + \mathbf{T}_b + \mathbf{T}_{a'} + \mathbf{T}_{b'} = 0$$

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i = \mathcal{C}_{\rm tot}$$

- in the limit *s » t*, the dipole operator *Z* becomes, to power accuracy in *s/t*

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \exp\left[-i\frac{\pi}{2}K\Big(\alpha_s(\mu^2), \epsilon\Big)\mathcal{C}_{\rm tot}\right]$$

Duhr Gardi Magnea White VDD 2011

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- colour and *s* dependence are in the operator $\tilde{Z}$

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which is determined by the cusp anomalous dimension, through

$$K\Big(\alpha_s(\mu^2), \epsilon\Big) = -\frac{1}{4}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\, \widehat{\gamma}_K\left(\alpha_s(\lambda^2, \epsilon)\right) \,,$$

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- in the limit $s \gg t$, the dipole operator Z becomes, to power accuracy in s/t

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \exp\left[-i\frac{\pi}{2}K\Big(\alpha_s(\mu^2), \epsilon\Big)\mathcal{C}_{\rm tot}\right]$$

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- the dipole operator fixes the Regge pole structure

- the operator Z_1 is colourless and s independent

$$Z_{\mathbf{1}}\left(\frac{t}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \exp\left\{\sum_{i=1}^{4} B_i\Big(\alpha_s(\mu^2),\epsilon\Big) + \frac{1}{2}\left[K\Big(\alpha_s(\mu^2),\epsilon\Big)\,\ln\left(\frac{-t}{\mu^2}\right) + D\Big(\alpha_s(\mu^2),\epsilon\Big)\right]\,\mathcal{C}_{\rm tot}\right\}$$

with

$$D\Big(\alpha_s(\mu^2),\epsilon\Big) = -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda^2,\epsilon)\right)\ln\left(\frac{\mu^2}{\lambda^2}\right)\,,$$

$$B_i\Big(\alpha_s(\mu^2),\epsilon\Big) \equiv -\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\gamma_{J_i}\left(\alpha_s(\lambda^2,\epsilon)\right)$$

one may think of the operator Z_1 as a jet operator,
which, as we will see, yields the divergent parts of the impact factor

Possible corrections to the dipole formula in the HEL

- the high-energy limit puts constraints on 4-line correlations which may appear at 3 loops
- we know that corrections to Δ like

$$\Delta^{(212)}(\rho_{ijkl}, \alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f^{ade} f^{cbe} L_{1234}^2 \left(L_{1423} L_{1342}^2 + L_{1423}^2 L_{1342} \right) + \mathrm{cycl} \right]$$

fulfill Bose symmetry, collinear limits and transcendentality bounds Dixon Gardi Magnea 2009

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- however, in the high-energy limit

$$\rho_{1234} = \frac{(-s_{12})(-s_{34})}{(-s_{13})(-s_{24})} = \left(\frac{s}{-t}\right)^2 \mathrm{e}^{-2\mathrm{i}\pi}\,; \qquad L_{1234} = 2(L - \mathrm{i}\pi) \qquad L = \ln\left(\frac{s}{t}\right)$$

$$\rho_{1342} = \frac{(-s_{13})(-s_{24})}{(-s_{14})(-s_{23})} = \left(\frac{-t}{s+t}\right)^2\,; \qquad L_{1342} \simeq -2L$$

$$\rho_{1423} = \frac{(-s_{14})(-s_{23})}{(-s_{12})(-s_{34})} = \left(\frac{s+t}{s}\right)^2 \mathrm{e}^{2\mathrm{i}\pi}\,; \qquad L_{1423} \simeq 2\mathrm{i}\pi$$

$$\Delta^{(212)}(\rho_{ijkl},\alpha_s)) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\, 32\,\mathrm{i}\,\pi\Big[\Big(-L^4 - \mathrm{i}\pi L^3 - \pi^2 L^2 - \mathrm{i}\pi^3 L\Big) f^{ade} f^{cbe}$$
$$+ \left(2\mathrm{i}\pi L^3 - 3\pi^2 L^2 - \mathrm{i}\pi^3 L\right) f^{cae} f^{dbe}\Big] + \mathcal{O}\left(|t/s|\right)$$

has super-leading logs, which are incompatible with the high-energy limit

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Dipole formula & leading logs

- to leading logarithmic accuracy in *s/t*,
  the dipole operator loses the imaginary part (*s*-channel)

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big) \ln\left(\frac{s}{-t}\right) \mathbf{T}_t^2\right\} Z_{\mathbf{1}}\, \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

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- in the Regge limit *s* » *t*, any scattering process is dominated by gluon exchange in the *t* channel
  in particular, in parton-parton scattering *t*-channel gluon exchange occurs at leading order,
  the other channel contributions being power suppressed
  - the *t*-channel exchange colour structure is an eigenstate of the operator

$$\mathbf{T}_t^2\, \mathcal{H}^{ff\to ff} \xrightarrow{|t/s|\to 0} C_t\, \mathcal{H}_t^{ff\to ff}$$

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$$\mathbf{T}_t^2\, \mathcal{H}^{ff\to ff} \xrightarrow{|t/s|\to 0} C_t\, \mathcal{H}_t^{ff\to ff}$$

- to leading logarithmic accuracy in *s/t*, the parton-parton scattering amplitude becomes

$$\mathcal{M}^{ff\to ff} = \left(\frac{s}{-t}\right)^{C_A\, K\left(\alpha_s(\mu^2), \epsilon\right)} Z_{\mathbf{1}}\, \mathcal{H}_t^{ff\to ff}$$

to leading order, the cusp anomalous dimension is

$$\widehat{\gamma}_K(\alpha_s) = 2\,\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \quad \Longrightarrow \quad K(\alpha_s, \epsilon) = \frac{1}{2\epsilon}\,\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)$$

so the singular part of the one-loop Regge gluon trajectory becomes

$$\alpha^{(1)} = C_A\,\frac{2}{\epsilon} + \mathcal{O}\left(\epsilon^0\right)$$

in agreement with the high-energy limit of parton-parton amplitudes

Dipole formula beyond the leading logs

- to power accuracy in *s/t* (thus to arbitrary logarithmic accuracy), the dipole operator *Z* can be rewritten as

$$\widetilde{Z}\left(\frac{s}{t},\alpha_s(\mu^2),\epsilon\right) = \left(\frac{s}{-t}\right)^{K(\alpha_s,\epsilon)\,\mathbf{T}_t^2} \exp\left\{\mathrm{i}\,\pi\,K(\alpha_s,\epsilon)\,\mathbf{T}_s^2\right\}$$
$$\times\,\exp\left\{-\mathrm{i}\,\frac{\pi}{2}\Big[K(\alpha_s,\epsilon)\Big]^2\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right\}$$
$$\times\,\exp\left\{\frac{1}{6}\Big[K(\alpha_s,\epsilon)\Big]^3\left(-2\pi^2\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]+\mathrm{i}\pi\,\ln^2\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]\right)\right\}$$
$$\times\,\exp\left\{\mathcal{O}\left(\Big[K(\alpha_s,\epsilon)\Big]^4\right)\right\}$$

- NLL accuracy:

$$K(\alpha_s,\epsilon) = \frac{\alpha_s}{4\pi}\,\frac{\widehat{\gamma}_K^{(1)}}{\epsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{2\widehat{\gamma}_K^{(2)}}{\epsilon} - \frac{b_0\,\widehat{\gamma}_K^{(1)}}{2\epsilon^2}\right) + \left(\frac{\alpha_s}{4\pi}\right)^3\left(\frac{16\widehat{\gamma}_K^{(3)}}{3\epsilon} - 4\,\frac{b_0\,\widehat{\gamma}_K^{(2)}+b_1\,\widehat{\gamma}_K^{(1)}}{3\epsilon^2} + \frac{b_0^2\,\widehat{\gamma}_K^{(1)}}{3\epsilon^3}\right) + \mathcal{O}(\alpha_s^4)$$

reproduces the singular part of the one- and two-loop Regge gluon trajectory, while the imaginary part does not Reggeise

Dipole formula beyond the leading logs

- to power accuracy in *s/t* (thus to arbitrary logarithmic accuracy), the dipole operator Z can be rewritten as

$$\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \left(\frac{s}{-t}\right)^{K(\alpha_s,\epsilon)\,\mathbf{T}_t^2} \exp\left\{\mathrm{i}\,\pi\,K(\alpha_s,\epsilon)\,\mathbf{T}_s^2\right\}$$
$$\times \exp\left\{-\mathrm{i}\,\frac{\pi}{2}\left[K(\alpha_s,\epsilon)\right]^2 \ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right\}$$
$$\times \exp\left\{\frac{1}{6}\left[K(\alpha_s,\epsilon)\right]^3\left(-2\pi^2 \ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right] + \mathrm{i}\pi\,\ln^2\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right]\right)\right\}$$
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- NNLL accuracy:

$\mathcal{O}(\alpha_s^2)$: $\rightarrow$ $-\frac{1}{2}\pi^2 K^2(\alpha_s,\epsilon)\,\left(\mathbf{T}_s^2\right)^2$ which is non-logarithmic and non-diagonal in the *t* channel

$\mathcal{O}(\alpha_s^3)$: $\rightarrow$ $-\frac{\pi^2}{3} K^3(\alpha_s,\epsilon)\,\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right]$ breaks down the Regge-pole picture

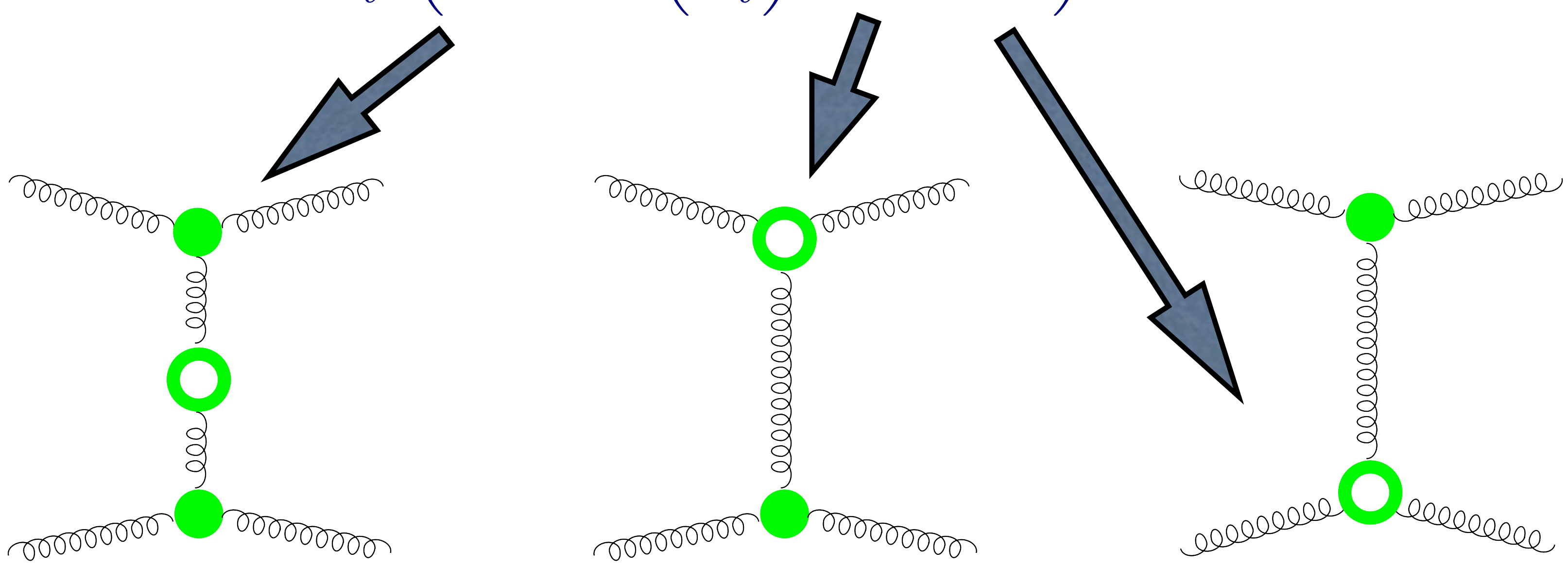
Amplitudes in the high-energy limit

- Regge limit of the gluon-gluon amplitude

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^c)_{aa'}C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[(T_c)_{bb'}C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

strip colour off & expand at one loop

$$m^{(1)}_{gg\to gg} = 2\,g_s^2\,\frac{s}{t}\left(\alpha^{(1)}(t)\ln\left(\frac{s}{-t}\right)+2\,C^{(1)}_{gg}(t)\right)$$



the Regge gluon trajectory is universal;
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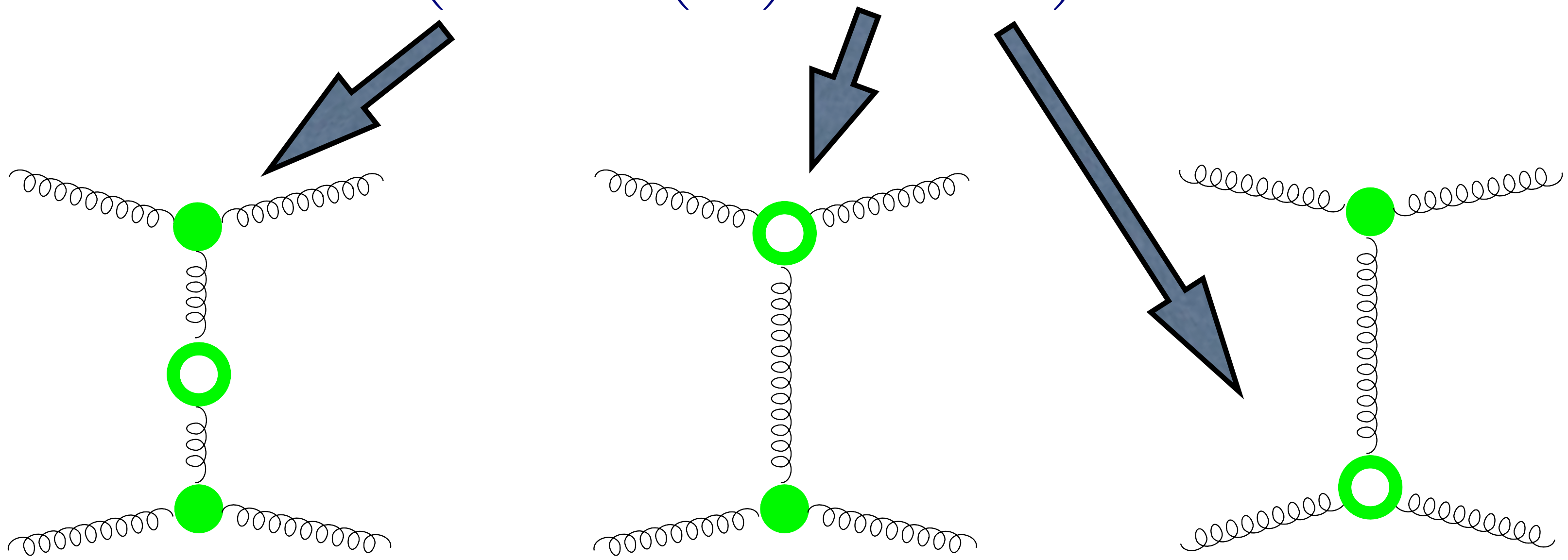

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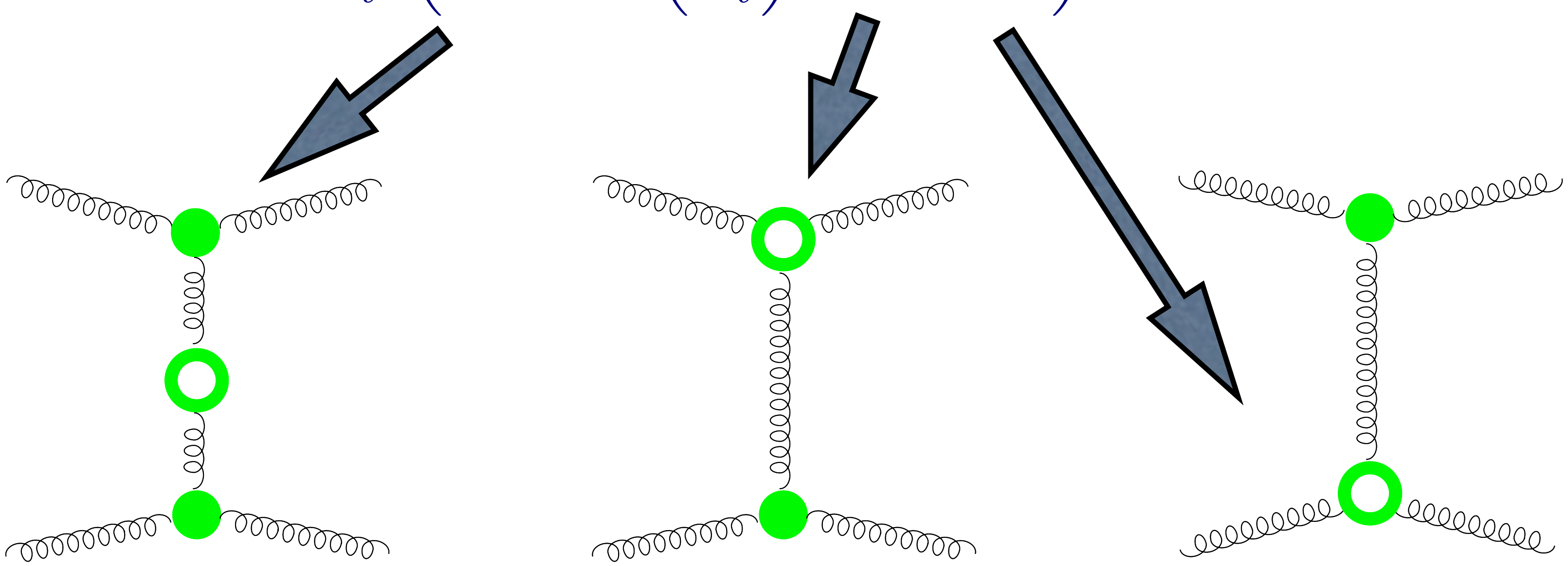

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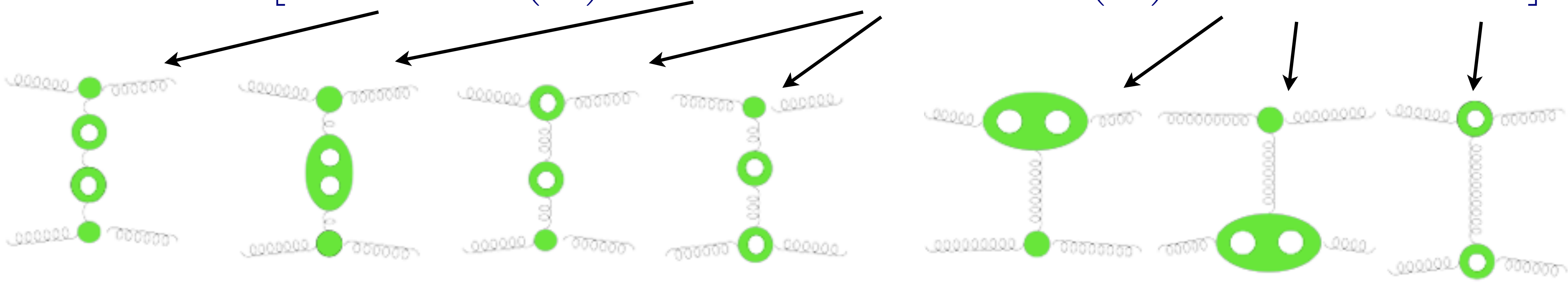
  the result should match the quark-gluon amplitude in the high-energy limit: it does

High-energy limit at 2 loops

- in the Regge limit, the 2-loop expansion of the gluon-gluon amplitude is

$$m^{(2)}_{gg\to gg} = 2\, g_s^2\, \frac{s}{t} \left[\frac{1}{2} \left(\alpha^{(1)}(t) \right)^2 \ln^2 \left(\frac{s}{-t} \right) + \left(\alpha^{(2)}(t) + 2\, C^{(1)}_{gg}(t)\, \alpha^{(1)}(t) \right) \ln \left(\frac{s}{-t} \right) + 2\, C^{(2)}_{gg}(t) + \left(C^{(1)}_{gg}(t) \right)^2 \right]$$



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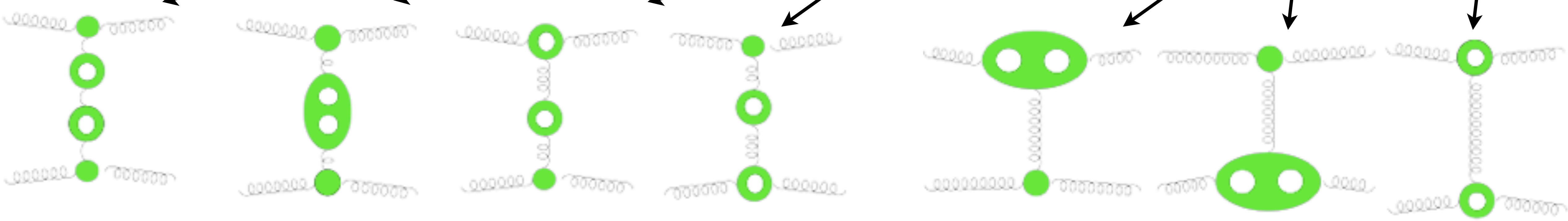

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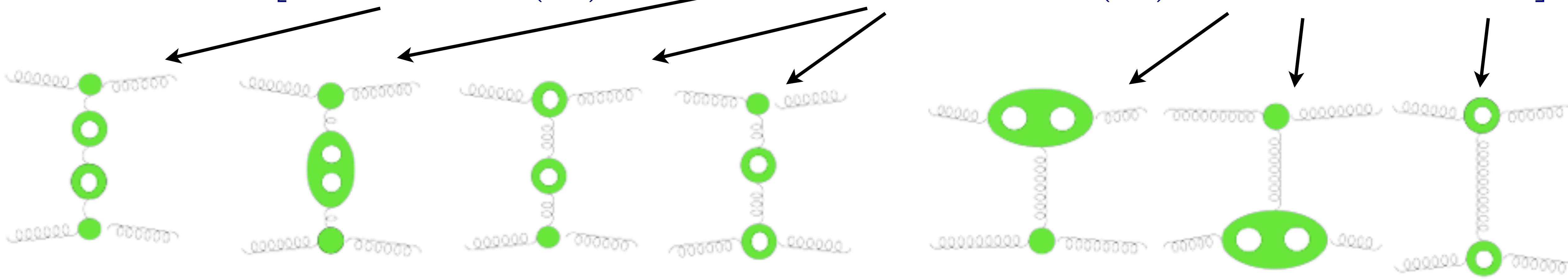

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  the result should match the quark-gluon amplitude in the high-energy limit:
  it doesn't! by a π^2/ε^2 factor

Glover VDD 2001

- to account for a factorization breaking remainder, we write the amplitude in the high-energy limit as

$$\mathcal{M}^{[8]}_{\rm rs}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s\right) = 2\pi\alpha_s\, H^{(0),[8]}_{\rm rs}$$
$$\times \left\{C_{\rm r}\left(\frac{t}{\mu^2},\alpha_s\right)\left[A_+\left(\frac{s}{t},\alpha_s\right) + \kappa_{\rm rs}\, A_-\left(\frac{s}{t},\alpha_s\right)\right]C_{\rm s}\left(\frac{t}{\mu^2},\alpha_s\right) + \mathcal{R}^{[8]}_{\rm rs}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s\right) + \mathcal{O}\left(\frac{t}{s}\right)\right\}$$

with $A_\pm\left(\frac{s}{t},\alpha_s\right) = \left(\frac{-s}{-t}\right)^{\alpha(t)} \pm \left(\frac{s}{-t}\right)^{\alpha(t)}$ for the $s \leftrightarrow u$ exchange symmetry

$\kappa_{gg} = \kappa_{qg} = 0 \qquad \kappa_{qq} = \frac{4-N_c^2}{N_c^2}$ for qq scattering

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we include a possible factorization breaking remainder at 2 loops and beyond

Expanded in α_S and *ln(s/t)*, they are

$$\mathcal{M}^{[j]}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s\right) = 4\pi\alpha_s\sum_{n=0}^{\infty}\sum_{i=0}^{n}\left(\frac{\alpha_s}{\pi}\right)^n \ln^i\left(\frac{s}{-t}\right) M^{(n),i,[j]}\left(\frac{t}{\mu^2}\right)$$

$$\mathcal{R}^{[8]}_{\rm rs}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s\right) = \sum_{n=2}^{\infty}\sum_{k=0}^{n-2}\left(\frac{\alpha_s}{\pi}\right)^n \ln^k\left(\frac{s}{-t}\right) R^{(n),k,[8]}_{\rm rs}\left(\frac{t}{\mu^2}\right)$$

- to compare with the amplitude in the high-energy limit, it is better to define the infrared operator as

$$\widetilde{Z}_S\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) \exp\left[-i\frac{\pi}{2}K\Big(\alpha_s(\mu^2), \epsilon\Big)\mathcal{C}_{\rm tot}\right]$$

$$= \exp\left\{K(\alpha_s)\left[\left(\log\left(\frac{s}{-t}\right) - {\rm i}\,\frac{\pi}{2}(1+\kappa_{\rm rs})\right)\mathbf{T}_t^2 + {\rm i}\,\frac{\pi}{2}\,\left(\mathbf{T}_s^2 - \mathbf{T}_u^2 + \kappa_{\rm rs}\mathbf{T}_t^2\right)\right]\right\}$$

The terms proportional to T_t^2 reproduce the structure of the high-energy amplitude. The breaking of high-energy factorization arises from the last term, which is a color- and process-dependent phase. This is in accordance with the expectation that the failure of high-energy factorization should come from the mixing of different color amplitudes

Falcioni Magnea Vernazza VDD 2013, and in progress

Comparing infrared and high-energy factorizations

- 1 loop

$M^{(1),1}$ → Regge trajectory $\alpha^{(1)} = C_A \frac{\widehat{\gamma}_K^{(1)}}{\epsilon}$ is universal

$M^{(1),0}$ → impact factor $C_{\rm r}^{(1)} = \frac{1}{2} Z_{1,{\rm rr}}^{(1)} + \frac{1}{2} \widehat{H}_{\rm rr}^{(1),0,[8]}$ $\quad r = q, g$

with constraints $Z_{1,\mathbf{R},qg}^{(1)} = \frac{1}{2}\left[Z_{1,\mathbf{R},qq}^{(1)} + Z_{1,\mathbf{R},gg}^{(1)}\right]$

$$\mathrm{Re}\left(\widehat{H}_{qg}^{(1),0,[8]}\right) = \frac{1}{2}\left[\mathrm{Re}\left(\widehat{H}_{gg}^{(1),0,[8]}\right) + \mathrm{Re}\left(\widehat{H}_{qq}^{(1),0,[8]}\right)\right]$$

confirming that the quark-gluon amplitude in the high-energy limit can be obtained by assembling the Regge trajectory with the gluon and quark impact factors

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confirming that the quark-gluon amplitude in the high-energy limit can be obtained by assembling the Regge trajectory with the gluon and quark impact factors

- 2 loops

$M^{(2),1}$ ➔ Regge trajectory $\alpha^{(2)} = C_A \frac{2\widehat{\gamma}_K^{(2)}}{\epsilon} + \mathrm{Re}\left[\widehat{H}_{\rm rs}^{(2),1,[8]}\right]$

with constraint $\mathrm{Re}\left[\widehat{H}_{gg}^{(2),1,[8]}\right] = \mathrm{Re}\left[\widehat{H}_{qg}^{(2),1,[8]}\right] = \mathrm{Re}\left[\widehat{H}_{qq}^{(2),1,[8]}\right]$

which ensures that the trajectory is universal

- 2 loops

$M^{(2),0}$ we may collect all factorization-breaking terms in a remainder

$$R^{(2),0,[j]} = -\frac{\pi^2}{4}\left(\frac{\widehat{\gamma}_K^{(1)}}{\epsilon}\right)^2 \frac{1}{H^{(0),[8]}}\left[\left((\mathbf{O}_{s-u})^2 - \left(1 - \kappa_{\rm rs}^2\left(\mathbf{T}_t^2\right)^2\right)\right)H^{(0)}\right]^{[j]}$$

with $\mathbf{O}_{s-u} = 2\mathbf{T}_s^2 - \mathcal{C}_{\rm tot} + (1+\kappa_{\rm rs})\mathbf{T}_t^2$

specifically $R_{qq}^{(2),0,[8]} = \frac{\pi^2}{4\epsilon^2}\left(1 - \frac{3}{N_c^2}\right)$, $R_{gg}^{(2),0,[8]} = -\frac{3\pi^2}{2\epsilon^2}$, $R_{qg}^{(2),0,[8]} = -\frac{\pi^2}{4\epsilon^2}$

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→ impact factor

$$\begin{aligned}
C_{\rm r}^{(2)} = &-\frac{1}{8}\left(Z_{1,\mathbf{R},{\rm rr}}^{(1)}\right)^2 + \frac{1}{2}Z_{1,\mathbf{R},{\rm rr}}^{(2)} + \frac{1}{4}Z_{1,\mathbf{R},{\rm rr}}^{(1)}\,{\rm Re}\left[\widehat{H}_{\rm rr}^{(1),0,[8]}\right] \\
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&-\frac{1}{8}\left({\rm Re}\left[\widehat{H}_{\rm rr}^{(1),0,[8]}\right]\right)^2 + \frac{\pi^2}{8}(1+\kappa_{\rm rr})\left({\rm Re}\left[\widehat{H}_{\rm rr}^{(1),1,[8]}\right]\right)^2 + \frac{1}{2}{\rm Re}\left[\widehat{H}_{\rm rr}^{(2),0,[8]}\right]
\end{aligned}$$

- 2 loops

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$$R^{(2),0,[j]} = -\frac{\pi^2}{4}\left(\frac{\widehat{\gamma}_K^{(1)}}{\epsilon}\right)^2 \frac{1}{H^{(0),[8]}}\left[\left(\left(\mathbf{O}_{s-u}\right)^2 - \left(1-\kappa_{\rm rs}^2\left(\mathbf{T}_t^2\right)^2\right)\right)H^{(0)}\right]^{[j]}$$

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real, color singlet, from jet function

- 2 loops

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real, color singlet, from jet function

potentially factorization breaking, but

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real, color singlet, from jet function

potentially factorization breaking, but

$$\left[\mathbf{O}_{s-u}\,\mathrm{Im}\left[\widehat{H}_{qg}^{(1),0}\right]\right]^{[8]} = \frac{1}{2}\left[\mathbf{O}_{s-u}\,\mathrm{Im}\left[\widehat{H}_{qq}^{(1),0}\right] + \mathbf{O}_{s-u}\,\mathrm{Im}\left[\widehat{H}_{gg}^{(1),0}\right]\right]^{[8]}$$

impact factor so defined preserves factorization

- mismatch between true amplitude and NLL expansion may be quantified by

$$
\begin{aligned}
\Delta_{(2),0,[8]} &= \frac{M_{qg}^{(2),0}}{H_{qg}^{(0),[8]}} - \left[C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{4}(1+\kappa)\left(\alpha^{(1)}\right)^2\right] \\
&= \frac{1}{2}\left[R_{qg}^{(2),0,[8]} - \frac{1}{2}\left(R_{qq}^{(2),0,[8]} + R_{gg}^{(2),0,[8]}\right)\right]
\end{aligned}
$$

- mismatch between true amplitude and NLL expansion may be quantified by

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$$= \frac{1}{2} \left[R_{qg}^{(2),0,[8]} - \frac{1}{2} \left(R_{qq}^{(2),0,[8]} + R_{gg}^{(2),0,[8]} \right) \right]$$

which agrees with the mismatch found by Glover and me in 2001

$$\Delta_{(2),0,[8]} = \frac{\pi^2 (K^{(1)})^2}{2} \left[\frac{3}{2} \left(\frac{N_c^2+1}{N_c^2} \right) \right] = \frac{\pi^2}{\epsilon^2} \frac{3}{16} \left(\frac{N_c^2+1}{N_c^2} \right)$$

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3 loops

$M^{(3),1}$ we expect that the single-log terms, which define the 3-loop Regge trajectory, are plagued by non-factoring contributions
we may define the 3-loop remainder suitably

$$R^{(3),1} = \frac{\pi^2}{4}\left(K^{(1)}\right)^3 \left[-\frac{4}{3}\mathbf{O}_{s,t,s} + \mathbf{O}_{s-u}\mathbf{O}_{t,s} - \frac{1}{2}\mathbf{O}_t\mathbf{O}_{s-u}^2 + \frac{1}{2}\mathbf{O}_t^3(1-\kappa_{\mathrm{rs}}^2)\right]\frac{2H^{(0)}}{H^{(0),[8]}}$$
$$+\left(K^{(1)}\right)^2\left[\pi\mathbf{O}_{t,s}\frac{\mathrm{Im}(H^{(1),0})}{H^{(0),[8]}} - \frac{\pi^2}{4}\mathbf{O}_{s-u}^2\frac{\mathrm{Re}(H^{(1),1})}{H^{(0),[8]}} + \frac{\pi^2}{4}\mathbf{O}_t^2(1-\kappa_{\mathrm{rs}}^2)\frac{\mathrm{Re}(H^{(1),1})}{H^{(0),[8]}}\right]$$
$$-\pi K^{(1)}\mathbf{O}_{s-u}\frac{\mathrm{Im}(H^{(2),1})}{H^{(0),[8]}} + \mathcal{O}(\epsilon^0)$$

with
$$\mathbf{O}_t = \mathbf{T}_t^2\,,$$
$$\mathbf{O}_{t,s} = \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\,,$$
$$\mathbf{O}_{s,t,s} = \left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right]$$

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$$\mathbf{O}_{s,t,s} = \left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right]$$

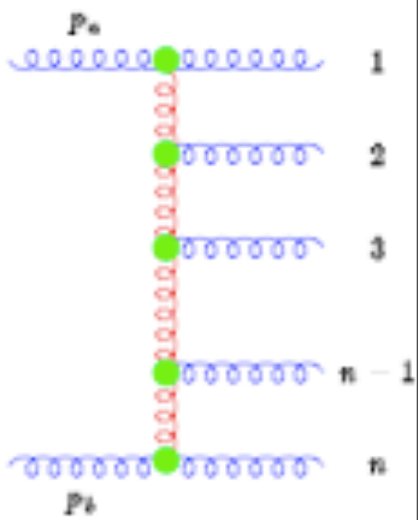
such that the trajectory be $\alpha^{(3)} = C_A\frac{16\widehat{\gamma}_K^{(3)}}{3\epsilon} + \mathcal{O}(\epsilon^0)$

Falcioni Magnea Vernazza VDD, in progress

BFKL resummation

- In perturbative QCD, in the Regge limit *s » t*,
  any scattering process is dominated by gluon exchange in the *t* channel

- BFKL is a resummation of multiple gluon radiation out of the gluon exchanged in the *t* channel, at LL and NLL accuracy in *ln(s/t)*



- the LL terms are obtained in the approximation of a strong rapidity ordering and no k_T ordering of the emitted gluons

- the resummation yields a 2-dim integral equation in k_T for the evolution of the gluon propagator exchanged in the *t* channel

- the solution is a Green's function of the momenta flowing in and out of the gluon ladder exchanged in the *t* channel

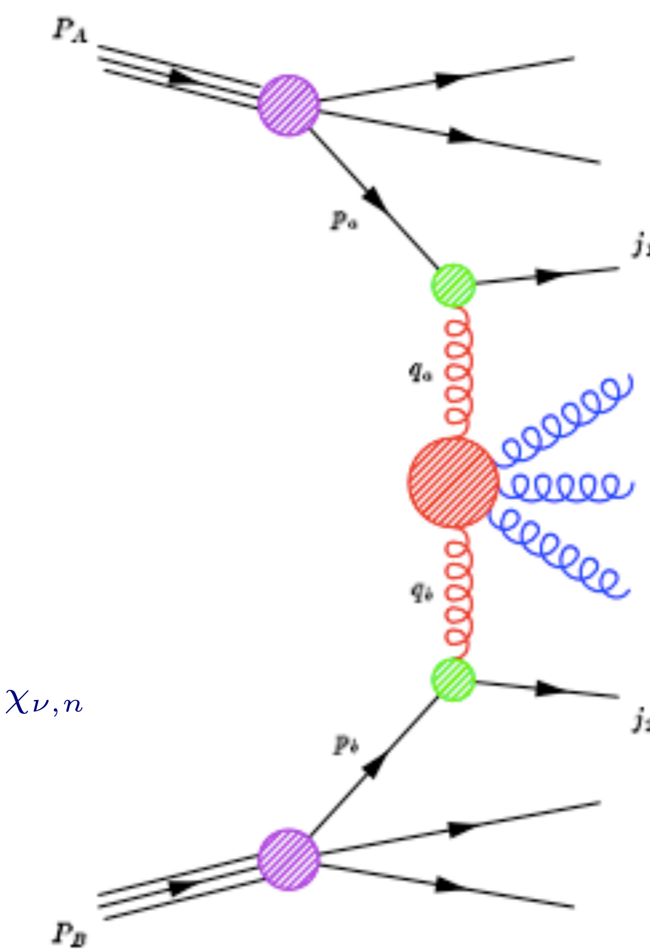
Balitsky Fadin Kuraev Lipatov 1977-78
Fadin Lipatov 1998

Mueller-Navelet jets

- the cross section for dijet production at large rapidity intervals $\Delta y = y_1 - y_2 = \ln\left(\frac{\hat{s}}{-t}\right) \gg 1$

  with $\hat{s} = x_a x_b S\,, \quad t = -\sqrt{p_{1\perp}^2 p_{2\perp}^2}$

$$\frac{d\hat{\sigma}_{gg}}{dp_{1\perp}^2 dp_{2\perp}^2 d\phi_{jj}} = \frac{\pi}{2}\left[\frac{C_A \alpha_s}{p_{1\perp}^2}\right] f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y)\left[\frac{C_A \alpha_s}{p_{2\perp}^2}\right]$$



can be described through the BFKL Green's function

$$f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y) = \frac{1}{(2\pi)^2\sqrt{q_{1\perp}^2\, q_{2\perp}^2}} \sum_{n=-\infty}^{+\infty} e^{in\phi} \int_{-\infty}^{+\infty} d\nu \left(\frac{q_{1\perp}^2}{q_{2\perp}^2}\right)^{i\nu} e^{\eta\,\chi_{\nu,n}}$$

with $\eta \equiv \frac{C_A\,\alpha_s}{\pi}\Delta y$ and ϕ the angle between q_1^2 and q_2^2

and the LL BFKL eigenvalue

$$\chi_{\nu,n} = -2\gamma_E - \psi\left(\frac{1}{2} + \frac{|n|}{2} + i\nu\right) - \psi\left(\frac{1}{2} + \frac{|n|}{2} - i\nu\right)$$

Mueller-Navelet dijet cross section

- azimuthal angle distribution ($\phi_{jj} = \phi - \pi$)

$$\frac{d\hat{\sigma}_{gg}}{d\phi_{jj}} = \frac{\pi(C_A\alpha_s)^2}{2E_\perp^2}\left[\delta(\phi_{jj}-\pi) + \sum_{k=1}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{e^{in\phi}}{2\pi}f_{n,k}\right)\eta^k\right]$$

with $$f_{n,k} = \frac{1}{2\pi}\,\frac{1}{k!}\int_{-\infty}^{\infty} d\nu\,\frac{\chi_{\nu,n}^k}{\nu^2+\frac{1}{4}}$$

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- azimuthal angle distribution ($\phi_{jj} = \phi - \pi$)

$$\frac{d\hat{\sigma}_{gg}}{d\phi_{jj}} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \left[\delta(\phi_{jj} - \pi) + \sum_{k=1}^{\infty} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in\phi}}{2\pi} f_{n,k} \right) \eta^k \right]$$

with $\quad f_{n,k} = \frac{1}{2\pi} \frac{1}{k!} \int_{-\infty}^{\infty} d\nu \, \frac{\chi_{\nu,n}^k}{\nu^2 + \frac{1}{4}}$

- the dijet cross section is $\quad \hat{\sigma}_{gg} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \sum_{k=0}^{\infty} f_{0,k} \, \eta^k$ Mueller Navelet 1987

with

$$\begin{aligned} f_{0,0} &= 1\,, \\ f_{0,1} &= 0\,, \\ f_{0,2} &= 2\zeta_2\,, \\ f_{0,3} &= -3\zeta_3\,, \\ f_{0,4} &= \frac{53}{6}\,\zeta_4\,, \\ f_{0,5} &= -\frac{1}{12}\,(115\zeta_5 + 48\zeta_2\zeta_3) \end{aligned}$$

BFKL Green's function and single-valued functions

- use complex transverse momentum $p_{k\perp} \to \tilde{p}_k = p_k^x + i p_k^y$

  and a complex variable $w = \dfrac{\tilde{p}_1}{\tilde{p}_2}$ $\qquad \left(\dfrac{w}{w^*}\right)^{1/2} = e^{-i\phi_{jj}} = -e^{i\phi}$

  the Green's function can be expanded into a power series in η

$$f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y) = \frac{1}{2}\delta^{(2)}(\vec{q}_{1\perp} - \vec{q}_{2\perp}) + \frac{1}{2\pi\sqrt{q_{1\perp}^2\, q_{2\perp}^2}} \sum_{k=1}^{\infty} \eta^k\, f_k(w, w^*)$$

  where the coefficient functions f_k are given by the inverse Fourier-Mellin transform

$$f_k(w, w^*) = \frac{1}{k!} \sum_{n=-\infty}^{+\infty} (-1)^n \left(\frac{w}{w^*}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\, |w|^{2i\nu}\, \chi_{\nu,n}^k$$

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  - f_k should be real-analytic functions of w

    they should have a unique, well-defined value for every ratio of the magnitudes of the two jet transverse momenta and angle between them

Harmonic polylogarithms

- classical polylogarithms $\mathrm{Li}_m(z) = \int_0^z dz' \frac{\mathrm{Li}_{m-1}(z')}{z'}$

- harmonic polylogarithms (HPLs)

$$H(a,\vec{w};z) = \int_0^z \mathrm{d}t\, f(a;t)\, H(\vec{w};t) \qquad f(-1;t) = \frac{1}{1+t}\,, \quad f(0;t) = \frac{1}{t}\,, \quad f(1;t) = \frac{1}{1-t}$$

with $\{a,\vec{w}\} \in \{-1,0,1\}$ Remiddi Vermaseren 1999

- HPLs obey the differential equations

$$\frac{d}{dz} H_{0\,\omega}(z) = \frac{H_\omega(z)}{z}\,, \qquad \frac{d}{dz} H_{1\,\omega}(z) = \frac{H_\omega(z)}{1-z}$$

subject to the constraints

$$H(z) = 1, \qquad H_{\vec{0}_n}(z) = \frac{1}{n!} \ln^n z, \qquad \lim_{z\to 0} H_{\omega \neq \vec{0}_n}(z) = 0$$

- HPLs form a shuffle algebra

$$H_{\omega_1}(z)\, H_{\omega_2}(z) = \sum_\omega H_\omega(z)$$ with ω the shuffle of ω_1 and ω_2

- HPLs are multi-valued functions on the complex plane

Single-valued harmonic polylogarithms

- define a function $\mathscr{L}$ that is real-analytic and single-valued on $\mathbb{C}/\{0,1\}$ and that has the same properties as the HPLs

it obeys the differential equations

$$\frac{\partial}{\partial z}\mathcal{L}_{0\,\omega}(z) = \frac{\mathcal{L}_\omega(z)}{z} \qquad \frac{\partial}{\partial z}\mathcal{L}_{1\,\omega}(z) = \frac{\mathcal{L}_\omega(z)}{1-z}$$

subject to the constraints

$$\mathcal{L}_e(z) = 1\,, \qquad \mathcal{L}_{\vec{0}_n}(z) = \frac{1}{n!}\ln^n|z|^2 \qquad \lim_{z\to 0}\mathcal{L}_{\omega\neq\vec{0}_n}(z) = 0$$

the SVHPLs $\mathscr{L}_\omega(z)$ also form a shuffle algebra

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- SVHPLs can be explicitly expressed as combinations of HPLs such that all the branch cuts cancel

Brown 2004

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Brown 2004

- examples

$$\mathcal{L}_0(z) = H_0(z) + H_0(\bar{z}) = \ln|z|^2$$

$$\mathcal{L}_1(z) = H_1(z) + H_1(\bar{z}) = -\ln|1+z|^2$$

$$\mathcal{L}_{0,1}(z) = \frac{1}{4}\left[-2H_{1,0} + 2\bar{H}_{1,0} + 2H_0\bar{H}_1 - 2\bar{H}_0H_1 + 2H_{0.1} - 2\bar{H}_{0,1}\right]$$

$$= \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln|z|^2\left(\ln(1-z) - \ln(1-\bar{z})\right)$$

a generating functional of SVHPLs

- to all orders in η the BFKL Green's function can be written in terms of a generating functional of SVHPLs Dixon Duhr Pennington VDD 2013

writing the coefficient function f_k as

$$f_k(w,w^*) = \frac{|w|}{|1+w|^2}\, F_k(w,w^*)$$

we obtain that the first few functions F_k are

$$\begin{aligned}
F_1(w,w^*) &= 1\,,\\
F_2(w,w^*) &= -\mathcal{L}_1 - \frac{1}{2}\mathcal{L}_0\,,\\
F_3(w,w^*) &= \mathcal{L}_{1,1} + \frac{1}{2}(\mathcal{L}_{0,1} + \mathcal{L}_{1,0}) + \frac{1}{6}\mathcal{L}_{0,0}\,,\\
F_4(w,w^*) &= -\mathcal{L}_{1,1,1} - \frac{1}{2}(\mathcal{L}_{0,1,1} + \mathcal{L}_{1,0,1} + \mathcal{L}_{1,1,0}) - \frac{1}{4}\mathcal{L}_{0,1,0}\\
&\quad - \frac{1}{6}(\mathcal{L}_{0,0,1} + \mathcal{L}_{1,0,0}) - \frac{1}{24}\mathcal{L}_{0,0,0} + \frac{1}{3}\zeta_3
\end{aligned}$$

Azimuthal angle distribution

- this allows us to write the azimuthal angle distribution as

$$\frac{d\hat{\sigma}_{gg}}{d\phi_{jj}} = \frac{\pi(C_A\alpha_s)^2}{2E_\perp^2}\left[\delta(\phi_{jj}-\pi) + \sum_{k=1}^{\infty}\frac{a_k(\phi_{jj})}{\pi}\,\eta^k\right]$$

where the contribution of the *k*th loop is

$$a_k(\phi_{jj}) = \int_0^\infty \frac{d|w|}{|w|} f_k(w,w^*) = \frac{\mathrm{Im}\,A_k(\phi_{jj})}{\sin\phi_{jj}}$$

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with

$$\begin{aligned}
A_1(\phi_{jj}) &= -\frac{1}{2}H_0\,,\\
A_2(\phi_{jj}) &= H_{1,0}\,,\\
A_3(\phi_{jj}) &= \frac{2}{3}H_{0,0,0} - 2H_{1,1,0} + \frac{5}{3}\zeta_2 H_0 - i\pi\,\zeta_2\,,\\
A_4(\phi_{jj}) &= -\frac{4}{3}H_{0,0,1,0} - H_{0,1,0,0} - \frac{4}{3}H_{1,0,0,0} + 4H_{1,1,1,0} - \zeta_2\Big(2H_{0,1} + \frac{10}{3}H_{1,0}\Big) + \frac{4}{3}\zeta_3\,H_0 + i\pi\Big(2\zeta_2 H_1 - 2\zeta_3\Big)\,,\\
A_5(\phi_{jj}) &= -\frac{46}{15}H_{0,0,0,0,0} + \frac{8}{3}H_{0,0,1,1,0} + 2H_{0,1,0,1,0} + 2H_{0,1,1,0,0} + \frac{8}{3}H_{1,0,0,1,0} + 2H_{1,0,1,0,0}\\
&\quad + \frac{8}{3}H_{1,1,0,0,0} - 8H_{1,1,1,1,0} - \zeta_2\Big(\frac{33}{5}H_{0,0,0} - 4H_{0,1,1} - 4H_{1,0,1} - \frac{20}{3}H_{1,1,0}\Big)\\
&\quad - \zeta_3\Big(2H_{0,1} + \frac{8}{3}H_{1,0}\Big) + \frac{217}{15}\zeta_4 H_0 + i\pi\Big[\zeta_2\Big(\frac{10}{3}H_{0,0} - 4H_{1,1}\Big) + 4\zeta_3 H_1 - \frac{10}{3}\zeta_4\Big]
\end{aligned}$$

where $\quad H_{i,j,\ldots} \equiv H_{i,j,\ldots}(e^{-2i\phi_{jj}})$

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Transverse momentum distribution

- $$\frac{d\hat{\sigma}_{gg}}{dp_{1\perp}^2\, dp_{2\perp}^2} = \frac{\pi (C_A \alpha_s)^2}{2 p_{1\perp}^2 p_{2\perp}^2} \left[\delta(p_{1\perp}^2 - p_{2\perp}^2) + \frac{1}{2\pi \sqrt{p_{1\perp}^2\, p_{2\perp}^2}}\, b(\rho;\eta) \right]$$

  where $\rho = |w|$ $\qquad b(\rho;\eta) = \frac{2\,\pi\,\rho}{1-\rho^2} \sum_{k=1}^{\infty} B_k(\rho)\,\eta^k$

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with

$$\begin{aligned}
B_1(\rho) &= 1\,,\\
B_2(\rho) &= -\frac{1}{2}\,H_0 - 2H_1\,,\\
B_3(\rho) &= \frac{1}{6}H_{0,0} + 2H_{0,1} + H_{1,0} + 4H_{1,1}\,,\\
B_4(\rho) &= -\frac{1}{24}H_{0,0,0} - \frac{4}{3}H_{0,0,1} - H_{0,1,0} - 4H_{0,1,1} - \frac{1}{3}H_{1,0,0} - 4H_{1,0,1} - 2H_{1,1,0} - 8H_{1,1,1} + \frac{1}{3}\,\zeta_3\,,\\
B_5(\rho) &= \frac{1}{120}H_{0,0,0,0} + \frac{2}{3}H_{0,0,0,1} + \frac{2}{3}H_{0,0,1,0} + \frac{8}{3}H_{0,0,1,1} + \frac{1}{3}H_{0,1,0,0} + 4H_{0,1,0,1}\\
&\quad + 2H_{0,1,1,0} + 8H_{0,1,1,1} + \frac{1}{12}H_{1,0,0,0} + \frac{8}{3}H_{1,0,0,1} + 2H_{1,0,1,0} + 8H_{1,0,1,1}\\
&\quad + \frac{2}{3}H_{1,1,0,0} + 8H_{1,1,0,1} + 4H_{1,1,1,0} + 16H_{1,1,1,1} + \zeta_3\left(-\frac{1}{12}H_0 - \frac{2}{3}H_1\right),
\end{aligned}$$

where $H_{i,j,\ldots} \equiv H_{i,j,\ldots}(\rho^2)$

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Mueller-Navelet dijet cross section reloaded

- the dijet cross section $\hat{\sigma}_{gg} = \frac{\pi(C_A\alpha_s)^2}{2E_\perp^2} \sum_{k=0}^{\infty} f_{0,k}\,\eta^k$

the first 5 loops were computed by Mueller-Navelet. Here are a few more

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the first 5 loops were computed by Mueller-Navelet. Here are a few more

Dixon Duhr Pennington VDD 2013

$$
\begin{aligned}
f_{0,6} &= \frac{13}{4}\zeta_3^2 + \frac{3737}{120}\zeta_6\,,\\
f_{0,7} &= -\frac{87}{5}\zeta_3\,\zeta_4 - \frac{116}{9}\zeta_2\,\zeta_5 - \frac{3983}{144}\zeta_7\,,\\
f_{0,8} &= -\frac{37}{75}\zeta_{5,3} + \frac{64}{15}\zeta_2\,\zeta_3^2 + \frac{369}{20}\zeta_5\,\zeta_3 + \frac{50606057}{453600}\zeta_8\,,\\
f_{0,9} &= -\frac{139}{60}\zeta_3^3 - \frac{15517}{252}\zeta_6\,\zeta_3 - \frac{3533}{63}\zeta_4\,\zeta_5 - \frac{557}{15}\zeta_2\,\zeta_7 - \frac{5215361}{60480}\zeta_9\,,\\
f_{0,10} &= -\frac{2488}{4725}\zeta_{5,3}\,\zeta_2 - \frac{94721}{211680}\zeta_{7,3} + \frac{1948}{105}\zeta_4\,\zeta_3^2 + \frac{2608}{105}\zeta_2\,\zeta_5\,\zeta_3 + \frac{12099}{224}\zeta_7\,\zeta_3 + \frac{1335931}{47040}\zeta_5^2 + \frac{25669936301}{63504000}\zeta_{10}\\
f_{0,11} &= \frac{62}{315}\zeta_{5,3}\,\zeta_3 + \frac{83}{120}\zeta_{5,3,3} - \frac{2872}{945}\zeta_2\zeta_3^3 - \frac{13211}{672}\zeta_5\,\zeta_3^2 - \frac{661411}{3024}\zeta_8\,\zeta_3\\
&\quad - \frac{242776937}{725760}\zeta_{11} - \frac{605321}{3024}\zeta_5\,\zeta_6 - \frac{2583643}{16200}\zeta_4\,\zeta_7 - \frac{28702763}{340200}\zeta_2\,\zeta_9\,,\\
f_{0,12} &= \frac{74711}{162000}\zeta_{5,3}\,\zeta_4 - \frac{13793}{7560}\zeta_{6,4,1,1} + \frac{3965011}{793800}\zeta_{7,3}\,\zeta_2 - \frac{33356851}{4082400}\zeta_{9,3}\\
&\quad + \frac{252163}{181440}\zeta_3^4 + \frac{620477}{10080}\zeta_6\,\zeta_3^2 + \frac{8101339}{75600}\zeta_4\,\zeta_5\,\zeta_3 + \frac{342869}{3780}\zeta_2\,\zeta_7\,\zeta_3\\
&\quad + \frac{101571047}{680400}\zeta_9\,\zeta_3 + \frac{71425871}{1587600}\zeta_2\,\zeta_5^2 + \frac{904497401571619}{620606448000}\zeta_{12} + \frac{484414571}{2721600}\zeta_5\,\zeta_7\,,\\
f_{0,13} &= \frac{4513}{1890}\zeta_{5,3}\,\zeta_5 + \frac{27248}{23625}\zeta_{5,3,3}\,\zeta_2 - \frac{97003}{235200}\zeta_{5,5,3} + \frac{13411}{75600}\zeta_{7,3}\,\zeta_3\\
&\quad + \frac{7997743}{12700800}\zeta_{7,3,3} - \frac{187318}{14175}\zeta_4\,\zeta_3^3 - \frac{125056}{4725}\zeta_2\,\zeta_5\,\zeta_3^2 - \frac{17411413}{302400}\zeta_7\,\zeta_3^2\\
&\quad - \frac{5724191}{100800}\zeta_5^2\,\zeta_3 - \frac{1874972477}{2376000}\zeta_{10}\,\zeta_3 - \frac{2418071698069}{2235340800}\zeta_{13}\\
&\quad - \frac{2379684877}{6048000}\zeta_{11}\,\zeta_2 - \frac{297666465053}{523908000}\zeta_6\,\zeta_7 - \frac{1770762319}{2494800}\zeta_5\,\zeta_8 - \frac{229717224973}{628689600}\zeta_4\,\zeta_9
\end{aligned}
$$

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- at LL accuracy, the solution of the BFKL resummation can be expressed in terms of real-analytic functions, more specifically in terms of single-valued harmonic polylogarithms
- we are able to compute differential distributions through 6 loops, and the Mueller-Navelet cross section through 13 loops

Back-up slides

Resummation: Sudakov form factor

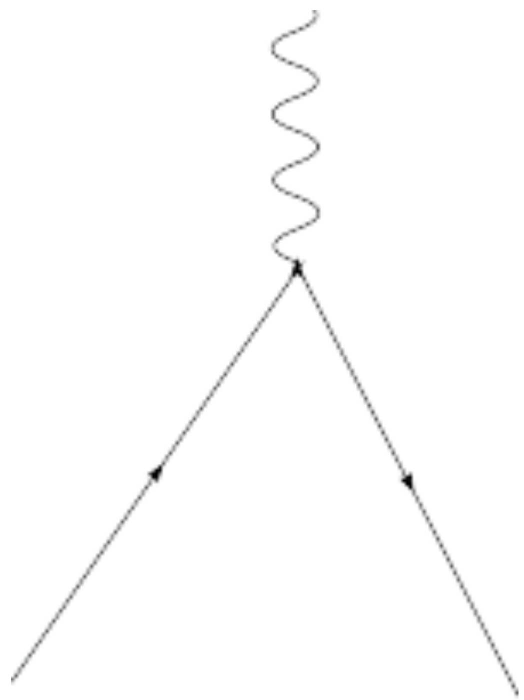
- Sudakov (quark) form factor as matrix element of EM current

$$\Gamma_\mu(p_1,p_2;\mu^2,\epsilon) \equiv < 0|J_\mu(0)|p_1,p_2 > = \bar{v}(p_2)\gamma_\mu u(p_1)\,\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)$$

obeys evolution equation

$$Q^2\frac{\partial}{\partial Q^2}\ln\left[\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] = \frac{1}{2}\left[K\left(\alpha_s(\mu^2),\epsilon\right) + G\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right]$$

K is a counterterm; G is finite as $\varepsilon \to 0$



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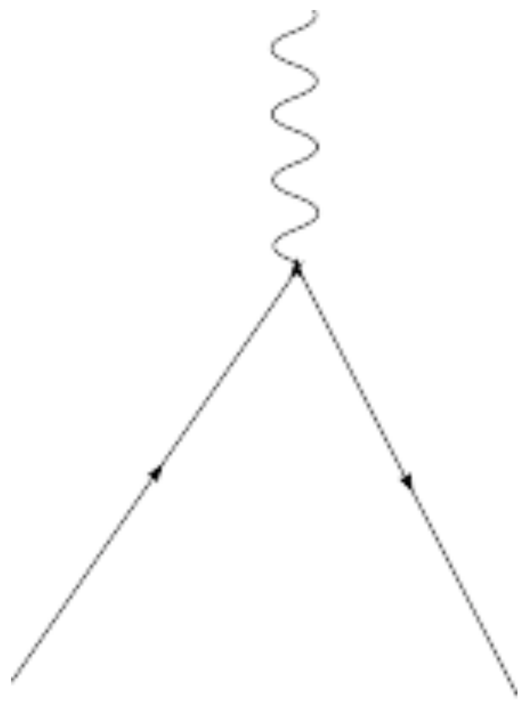
- RG invariance requires

$$\mu\frac{dG}{d\mu} = -\mu\frac{dK}{d\mu} = \gamma_K(\alpha_s(\mu^2))$$

Korchemsky Radyushkin 1987



γ_K is the cusp anomalous dimension

the solution is

$$\Gamma\left(Q^2,\epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[G\left(-1,\bar{\alpha}_s(\xi^2,\epsilon),\epsilon\right) - \frac{1}{2}\gamma_K\left(\bar{\alpha}_s(\xi^2,\epsilon)\right)\ln\left(\frac{-Q^2}{\xi^2}\right)\right]\right\}$$

cusp anomalous dimension

- loop expansion of the cusp anomalous dimension

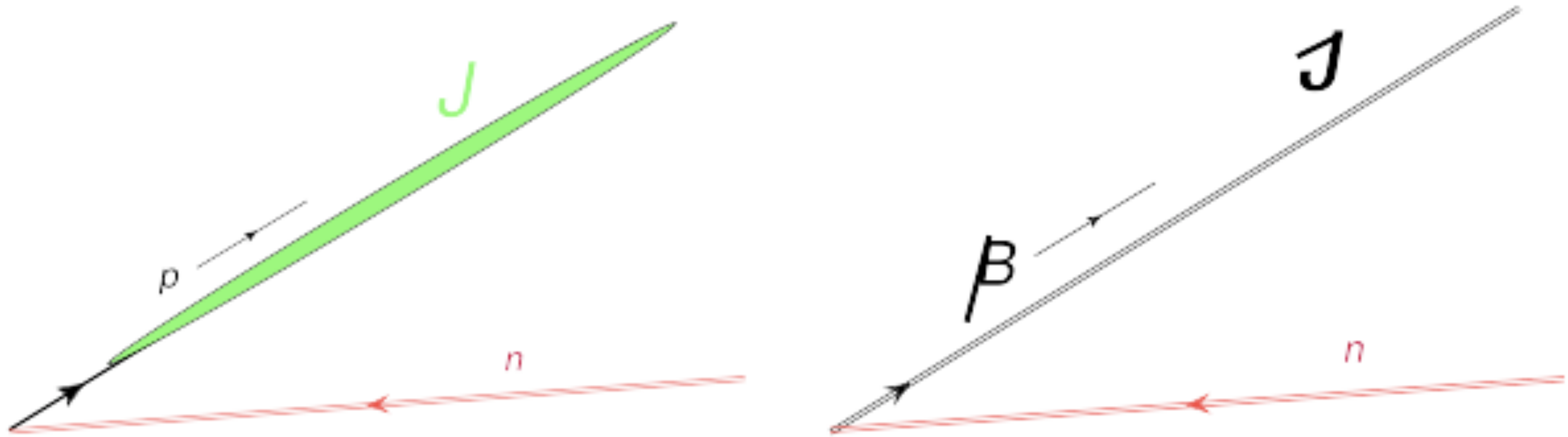
$$\gamma_K^{(i)} = 2C_i \frac{\alpha_s(\mu^2)}{\pi} + KC_i \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 + \cdots$$

with

$$K = \left(\frac{67}{18} - \zeta_2\right) C_A - \frac{10}{9} T_F N_f$$

Jet definition

- introduce auxiliary vector n_i ($n_i^2 \neq 0$) to separate collinear region
- define a jet using a Wilson line along n_i



partonic jet

$$\bar{u}(p)\, J\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \epsilon\right) = < p|\bar{\psi}(0)\Phi_n(0,-\infty)|0 >$$

Wilson line

$$\Phi_n(\lambda_2,\lambda_1) = P\exp\left[ig\int_{\lambda_1}^{\lambda_2} d\lambda n\cdot A(\lambda n)\right]$$

eikonal jet

$$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2}, \epsilon\right) = < 0|\bar{\Phi}_\beta(\infty,0)\Phi_n(0,-\infty)|0 >$$

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- eikonal jets are Wilson line correlators, introduced to avoid double counting

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- double poles and kinematic dependence of single poles are controlled by cusp γ_K,
  like in the quark form factor

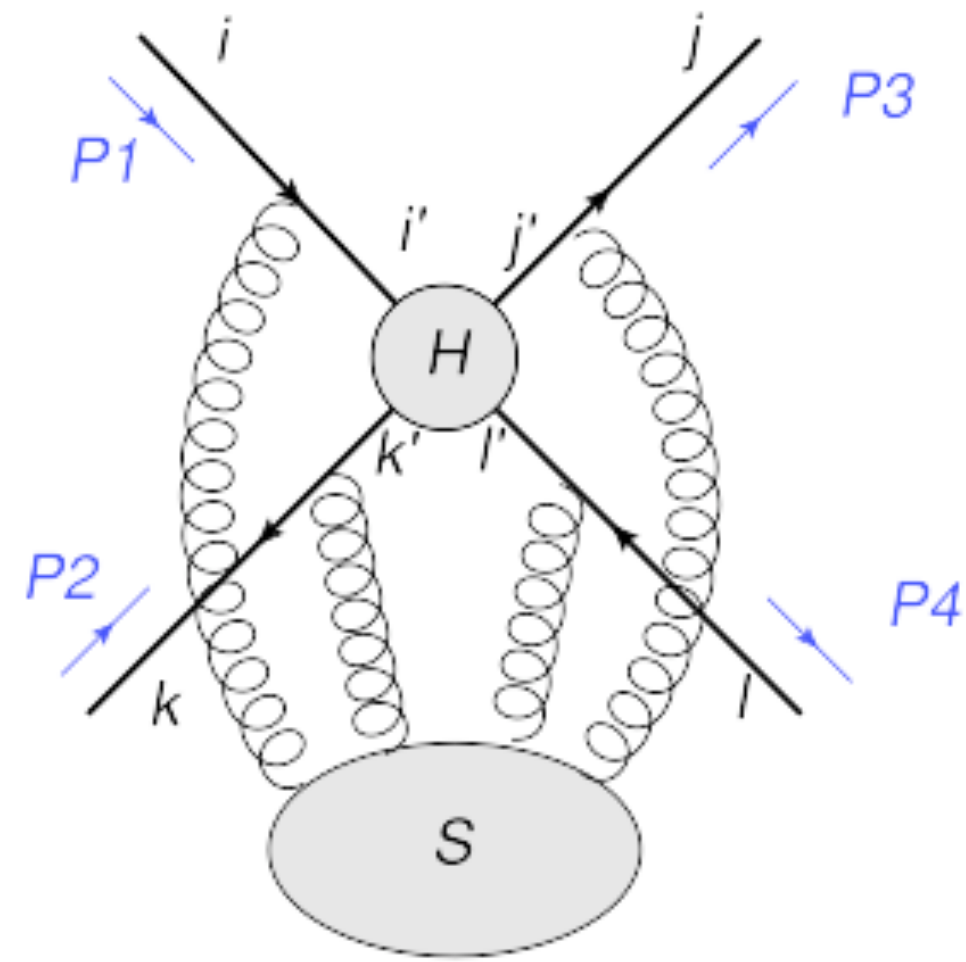
$$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2},\epsilon\right) = \exp\left\{\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[\delta_{\mathcal{J}_i}\left(\alpha_s(\lambda^2,\epsilon)\right) - \frac{1}{2}\gamma_K\left(\bar{\alpha}_s(\lambda^2,\epsilon)\right)\ln\left(\frac{2(\beta\cdot n)^2\mu^2}{n^2\lambda^2}\right)\right]\right\}$$

δ_j is a constant

Soft function S

- soft function is a matrix which mixes the colour representations

$$(c_N)_{ijkl}\mathcal{S}_{NL}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)$$
$$=\sum_{i'j'k'l'} <0|\Phi^{k,k'}_{-\beta_2}(0,\infty)\Phi^{i,i'}_{\beta_1}(\infty,0)\Phi^{j,j'}_{\beta_3}(0,\infty)\Phi^{l,l'}_{-\beta_4}(\infty,0)|0>\ (c_L)_{i'j'k'l'}$$



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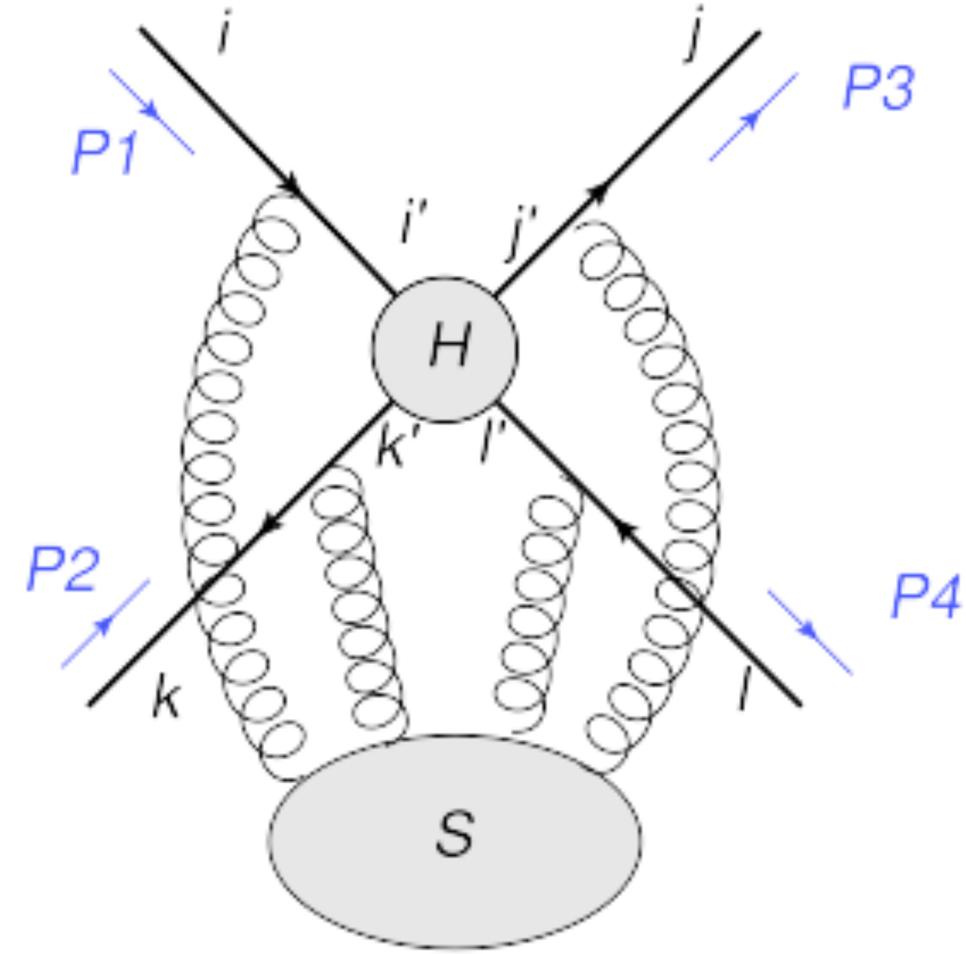
- matrix evolution equation

$$\mu\frac{d}{d\mu}\mathcal{S}_{JL}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)$$
$$=-\sum_N\left[\Gamma_{\mathcal{S}}\right]_{JN}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)\mathcal{S}_{NL}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)$$

Γ_S soft anomalous dimension,
singular due to the UV and collinear poles



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$$
\begin{aligned}
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&=\sum_{i'j'k'l'} <0|\Phi^{k,k'}_{-\beta_2}(0,\infty)\Phi^{i,i'}_{\beta_1}(\infty,0)\Phi^{j,j'}_{\beta_3}(0,\infty)\Phi^{l,l'}_{-\beta_4}(\infty,0)|0>\ (c_L)_{i'j'k'l'}
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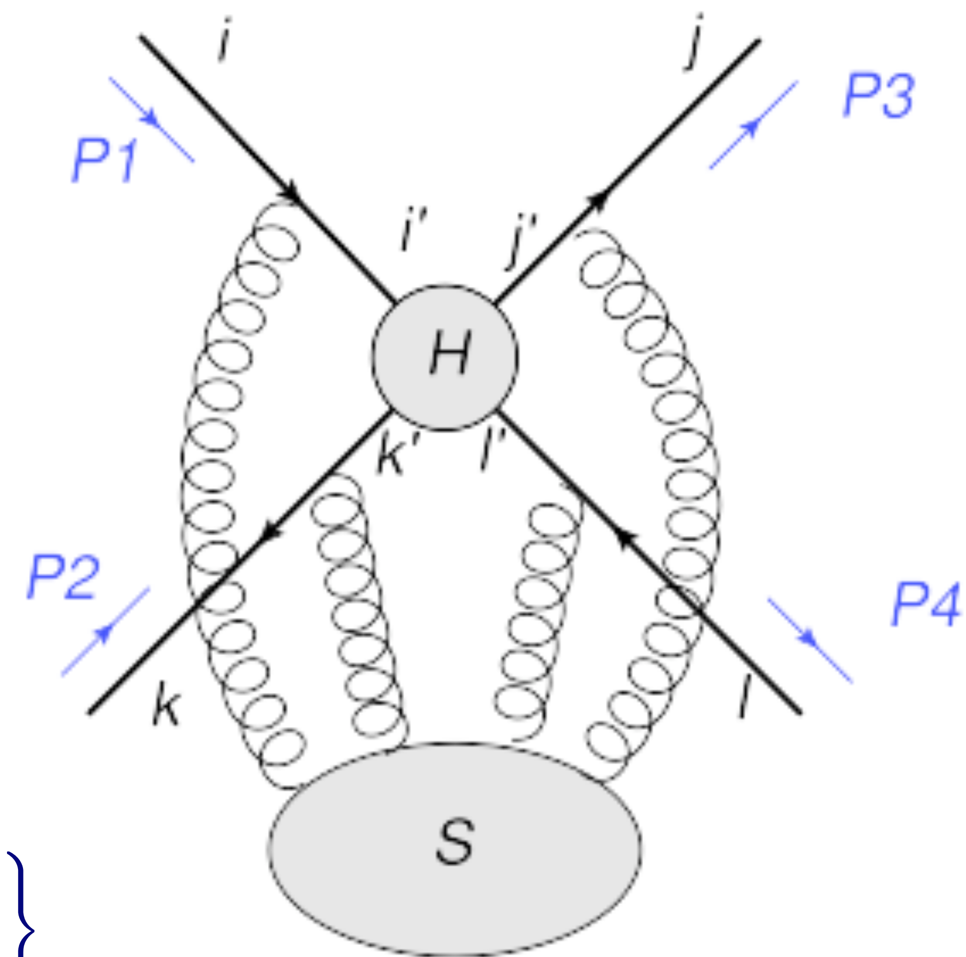

- in DimReg the solution is

$$
\mathcal{S}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)=P\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma_{\mathcal{S}}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)\right\}
$$

$$
\Gamma_{\mathcal{S}}=\sum_{n=1}^{\infty}\Gamma_{\mathcal{S}}^{(n)}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n
$$

Reduced soft function

- $$\bar{\mathcal{S}}_{JL}\left(\rho_{ij},\epsilon\right)=\frac{\mathcal{S}_{JL}\left(\beta_i\cdot\beta_j,\epsilon\right)}{\prod_{i=1}^{n}\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

Dixon Magnea Sterman 2008

the reduced soft function is made such that the double poles cancel.
It does not have cusp singularities ⇒ must respect rescaling $\beta_i \to \kappa_i\, \beta_i$

⇒ $\bar{\mathcal{S}}$ depends only on $$\rho_{ij}=\frac{(\beta_i\cdot\beta_j)^2}{\frac{2(\beta_i\cdot n_i)^2}{n_i^2}\,\frac{2(\beta_j\cdot n_j)^2}{n_j^2}}$$

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- the factorisation becomes

$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \bar{\mathcal{S}}_{NL}(\rho_{ij},\epsilon)\, H_L\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}\right) \prod_i J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \epsilon\right)$$

$\bar{\mathcal{S}}$ has only single poles due to large-angle soft emissions