Wilson loops and amplitudes in N=4 Super Yang-Mills

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\( \mathbf{N}=4 \) Super Yang-Mills

maximal supersymmetric theory (without gravity) conformally invariant, \( \beta \) fn. = 0

spin 1 gluon
4 spin 1/2 gluinos
6 spin 0 real scalars
N=4 Super Yang-Mills

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- only planar diagrams
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- \textbf{AdS/CFT duality} \quad \text{Maldacena 97}

- large-\( \lambda \) limit of 4dim CFT \( \leftrightarrow \) weakly-coupled string theory
  (aka weak-strong duality)
AdS/CFT duality, amplitudes & Wilson loops

planar scattering amplitude at strong coupling

\[ M \sim \exp \left( i \frac{\sqrt{\lambda}}{2\pi} (\text{Area}_{cl}) \right) \]

area of string world-sheet

(classical solution neglects \(O(1/\sqrt{\lambda})\) corrections)
AdS/CFT duality, amplitudes & Wilson loops

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amplitude has same form as ansatz for MHV amplitudes at weak coupling

\[ M_n = M_n^{(0)} \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + \text{Const}^{(l)} + E_n^{(l)}(\epsilon) \right) \right] \]
**AdS/CFT** duality, amplitudes & Wilson loops

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computation ```formally the same as ... the expectation value of a Wilson loop given by a sequence of light-like segments```
MHV amplitudes in planar $\textit{N}=4\ \text{SYM}$

At any order in the coupling, colour-ordered MHV amplitude in $\textit{N}=4\ \text{SYM}$ can be written as tree-level amplitude times helicity-free loop coefficient

$$M_{n}^{(L)} = M_{n}^{(0)} m_{n}^{(L)}$$
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at 1 loop

$$m_n^{(1)} = \sum_{pq} F_{2me}^{pq}(p, q, P, Q) \quad n \geq 6$$
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at 2 loops, iteration formula for the $n$-pt amplitude

\[ m^{(2)}(\epsilon) = \frac{1}{2} \left[ m^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) m^{(1)}(2\epsilon) + \text{Const}^{(2)} + R \]

Anastasiou Bern Dixon Kosower 03
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- at all loops, ansatz for a resummed exponent
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ansatz for MHV amplitudes in planar $N=4$ SYM

\[
M_n = M_n^{(0)} \left[ 1 + \sum_{L=1}^{\infty} a^L m_n^{(L)}(\epsilon) \right]
= M_n^{(0)} \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + \text{Const}^{(l)} + E_n^{(l)}(\epsilon) \right) \right]
\]

coupling \[ a = \frac{\lambda}{8\pi^2} (4\pi e^{-\gamma})^\epsilon \]
\[ \lambda = g^2 N \quad \text{`t Hooft parameter} \]

\[ f^{(l)}(\epsilon) = \frac{\hat{\gamma}^{(l)}}{4} + \epsilon \frac{l}{2} \hat{G}^{(l)} + \epsilon^2 f_2^{(l)} \]
\[ E_n^{(l)}(\epsilon) = O(\epsilon) \]

$\hat{\gamma}^{(l)}$ cusp anomalous dimension, known to all orders of $a$

$\hat{G}^{(l)}$ collinear anomalous dimension, known through $O(a^4)$

ansatz generalises the iteration formula for the 2-loop $n$-pt amplitude $m_n^{(2)}$

\[
m_n^{(2)}(\epsilon) = \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) + \text{Const}^{(2)} + O(\epsilon)
\]
Factorisation of a multi-leg amplitude in QCD

\[ \mathcal{M}_N(p_i/\mu, \epsilon) = \sum_L S_{NL}(\beta_i \cdot \beta_j, \epsilon) H_L \left( \frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2} \right) \prod_i J_i \left( \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \epsilon \right) \frac{J_i}{i^2} \left( \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \epsilon \right) \]

\[ p_i = \beta_i Q_0 / \sqrt{2} \]

value of \( Q_0 \) is immaterial in \( S, J \)

to avoid double counting of soft-collinear region (IR double poles),
\( J_i \) removes eikonal part from \( J_i \), which is already in \( S \)
\( J_i/J_i \) contains only single collinear poles
\( N = 4 \) \textit{SYM} in the planar limit

colour-wise, the planar limit is trivial: can absorb \( S \) into \( J_i \)

each slice is square root of Sudakov form factor

\[
\mathcal{M}_n = \prod_{i=1}^{n} \left[ \mathcal{M}^{[gg \rightarrow 1]} \left( \frac{S_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} h_n(p_i, \mu^2, \alpha_s, \epsilon)
\]

\( \beta \) fn = 0 \Rightarrow \text{coupling runs only through dimension} \quad \bar{\alpha}_s(\mu^2)\mu^{2\epsilon} = \bar{\alpha}_s(\lambda^2)\lambda^{2\epsilon}

Sudakov form factor has simple solution

\[
\ln \left[ \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = -\frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^n \left( \frac{-Q^2}{\mu^2} \right)^{-n\epsilon} \left[ \frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right]
\]

\( \Rightarrow \text{IR structure of } N = 4 \) SUSY amplitudes

Magnea Sterman 90
Bern Dixon Smirnov 05
Brief history of the ansatz

the ansatz checked for the 3-loop 4-pt amplitude
2-loop 5-pt amplitude

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the ansatz fails on 2-loop 6-pt amplitude

Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
Alday Maldacena 07; Bartels Lipatov Sabio-Vera 08
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at 2 loops, the remainder function characterises the deviation from the ansatz

\[ R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 - f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) - Const^{(2)} \]

\[ R_6^{(2)} \text{ known numerically} \]

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\[ \text{analytically} \]

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Wilson loops

\[ W[C_n] = \text{Tr} \mathcal{P} \exp \left[ ig \int d\tau \dot{x}^\mu(\tau) A_\mu(x(\tau)) \right] \]

closed contour \( C_n \) made by light-like external momenta

\[ p_i = x_i - x_{i+1} \]

Alday Maldacena 07
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\[ \langle W[C_n] \rangle = 1 + \sum_{L=1}^{\infty} \alpha^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} \alpha^L w_n^{(L)} \]

through 2 loops

\[ w_n^{(1)} = W_n^{(1)} \quad w_n^{(2)} = W_n^{(2)} - \frac{1}{2} \left( W_n^{(1)} \right)^2 \]
Wilson loops

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through 2 loops \( w_n^{(1)} = W_n^{(1)} \quad w_n^{(2)} = W_n^{(2)} - \frac{1}{2} \left( W_n^{(1)} \right)^2 \)

relation between 1 loop amplitudes & Wilson loops

\[ w_n^{(1)} = \frac{\Gamma(1 - 2\epsilon)}{\Gamma^2(1 - \epsilon)} m_n^{(1)} = m_n^{(1)} - n \frac{\zeta_2}{2} + \mathcal{O}(\epsilon) \]

\( p_i = x_i - x_{i+1} \)

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Gatheral 83
Frenkel Taylor 84

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MHV amplitudes $\leftrightarrow$ Wilson loops

agreement between $n$-edged Wilson loop and $n$-point MHV amplitude at weak coupling (aka weak-weak duality)
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verified for $n$-edged 1-loop Wilson loop up to 6-edged 2-loop Wilson loop

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**MHV amplitudes ↔ Wilson loops**

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- $n$-edged 2-loop Wilson loops also computed (numerically)

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- verified for $n$-edged 1-loop Wilson loop up to 6-edged 2-loop Wilson loop

- $n$-edged 2-loop Wilson loops also computed (numerically)

- no amplitudes are known beyond the 6-point 2-loop amplitude

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Wilson loops & Ward identities

$N=4$ SYM is invariant under $SO(2,4)$ conformal transformations.
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**Wilson loops & Ward identities**

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- The solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + $R$. 

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Wilson loops & Ward identities

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- the solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + $R$
- for $n = 4, 5$, $R$ is a constant
- for $n \geq 6$, $R$ is an unknown function of conformally invariant cross ratios
Wilson loops & Ward identities

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for \( n = 4, 5, \) \( R \) is a constant

for \( n \geq 6, \) \( R \) is an unknown function of conformally invariant cross ratios

for \( n = 6, \) the conformally invariant cross ratios are

\[
\begin{align*}
    u_1 &= \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \\
    u_2 &= \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \\
    u_3 &= \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}
\end{align*}
\]

\( x_i \) are variables in a dual space s.t. \( p_i = x_i - x_{i+1} \)

thus \( x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2 \)
Wilson loops

Wilson loops fulfill a Ward identity for special conformal boosts; the solution is the BDS ansatz + $R$. 
Wilson loops

Wilson loops fulfill a Ward identity for special conformal boosts, the solution is the BDS ansatz + $R$

at 2 loops

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon) w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

with

$$f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3 \epsilon - 5\zeta_4 \epsilon^2$$

(to be compared with $f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2$ for the amplitudes)

$R_{4,WL} = R_{5,WL} = 0$
Wilson loops

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$R_{4, WL} = R_{5, WL} = 0$

$R_{n, WL}^{(2)}$ arbitrary function of conformally invariant cross ratios

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

with $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$

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Wilson loops

**Wilson** loops fulfill a **Ward** identity for special conformal boosts
the solution is the **BDS ansatz** + \( R \)

at 2 loops

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w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon) w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R^{(2)}_{n, WL} + \mathcal{O}(\epsilon)
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with \( f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3 \epsilon - 5\zeta_4 \epsilon^2 \)

(to be compared with \( f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2 \) for the amplitudes)

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\( R_{n, WL}^{(2)} \) arbitrary function of conformally invariant cross ratios

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u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2} \quad \text{with} \quad x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2
\]

duality **Wilson loop** ⇔ **MHV amplitude** is expressed by

\[
R_{n, WL}^{(2)} = R_n^{(2)}
\]
Collinear limits of Wilson loops

collinear limit $a \parallel b$

$R_6 \rightarrow 0$  $R_7 \rightarrow R_6$  $R_n \rightarrow R_{n-1}$
Collinear limits of Wilson loops

**collinear limit** $a||b$

\[ R_6 \to 0 \quad R_7 \to R_6 \quad R_n \to R_{n-1} \]

**triple collinear limit** $a||b||c$

\[ R_6 \to R_6 \quad R_7 \to R_6 \quad R_8 \to R_6 + R_6 \quad R_n \to R_{n-2} + R_6 \]
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**quadruple collinear limit** $a || b || c || d$

\[ R_7 \to R_7 \quad R_8 \to R_7 \quad R_9 \to R_6 + R_7 \quad R_n \to R_{n-3} + R_7 \]
Collinear limits of Wilson loops

collinear limit  \( a||b \)
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\]

\((k+1)\)-ple collinear limit  \( i_1||i_2||\cdots||i_{k+1} \)
\[
R_n \rightarrow R_{n-k} + R_{k+4}
\]

\((n-4)\)-ple collinear limit  \( i_1||i_2||\cdots||i_{n-4} \)
\[
R_{n-1} \rightarrow R_{n-1} \quad R_n \rightarrow R_{n-1}
\]

\((n-3)\)-ple collinear limit  \( i_1||i_2||\cdots||i_{n-3} \)
\[
R_n \rightarrow R_n
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Collinear limits of Wilson loops

Collinear limit $a||b$

$$R_6 \to 0 \quad R_7 \to R_6 \quad R_n \to R_{n-1}$$

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Quadruple collinear limit $a||b||c||d$

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$(k+1)$-ple collinear limit $i_1||i_2||\cdots||i_{k+1}$

$$R_n \to R_{n-k} + R_{k+4}$$

$(n-4)$-ple collinear limit $i_1||i_2||\cdots||i_{n-4}$

$$R_{n-1} \to R_{n-1} \quad R_n \to R_{n-1}$$

$(n-3)$-ple collinear limit $i_1||i_2||\cdots||i_{n-3}$

$$R_n \to R_n$$

thus $R_n$ is fixed by the $(n-3)$-ple collinear limit
Quasi-multi-Regge limit of hexagon Wilson loop

6-pt amplitude in the qmR limit of a pair along the ladder

\[ y_3 \gg y_4 \simeq y_5 \gg y_6; \quad \left| p_3 \perp \right| \simeq \left| p_4 \perp \right| \simeq \left| p_5 \perp \right| \simeq \left| p_6 \perp \right| \]

the conformally invariant cross ratios are

\[
\begin{align*}
    u_{36} &= \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}} \\
    u_{14} &= \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} = \frac{s_{23}s_{56}}{s_{234}s_{123}} \\
    u_{25} &= \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} = \frac{s_{34}s_{61}}{s_{234}s_{345}}
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the cross ratios are all \( O(1) \)

\( \rightarrow R_6 \) does not change its functional dependence on the \( u \)'s
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the cross ratios are all \(O(1)\)

\[ R_6 \] does not change its functional dependence on the \(u\)'s

\( R_6 \) is invariant under the qmR limit of a pair along the ladder

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Quasi-multi-Regge limit of $n$-sided Wilson loop

7-pt amplitude in the qmR limit of a triple along the ladder

$$y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_7; \quad |p_3\perp| \simeq |p_4\perp| \simeq |p_5\perp| \simeq |p_6\perp| \simeq |p_7\perp|$$

7 cross ratios, which are all $O(1)$

$R_7$ is invariant under the qmR limit of a triple along the ladder
Quasi-multi-Regge limit of \( n \)-sided Wilson loop

7-pt amplitude in the qmR limit of a triple along the ladder

\[
y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_7; \quad |p_3\perp| \simeq |p_4\perp| \simeq |p_5\perp| \simeq |p_6\perp| \simeq |p_7\perp|
\]

7 cross ratios, which are all \( O(1) \)
\( R_7 \) is invariant under the qmR limit of a triple along the ladder

can be generalised to the \( n \)-pt amplitude in the qmR limit of a \((n-4)\)-ple along the ladder

\[
y_3 \gg y_4 \simeq \ldots \simeq y_{n-1} \gg y_n; \quad |p_3\perp| \simeq \ldots \simeq |p_n\perp|
\]
**Quasi-multi-Regge limit of Wilson loops**

$L$-loop Wilson loops are Regge exact

\[ w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon) w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n, WL}^{(L)}(u_{ij}) + O(\epsilon) \]
Quasi-multi-Regge limit of Wilson loops

$L$-loop Wilson loops are Regge exact

\[ w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon) w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n,WL}^{(L)}(u_{ij}) + O(\epsilon) \]

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\[ \ln(s_{ij}) + \text{Li}_2(1 - u_{ij}) \]

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Duhr Smirnov VDD 09

Monday, August 8, 2011
Quasi-multi-Regge limit of Wilson loops

$L$-loop Wilson loops are Regge exact

\[
\begin{align*}
    w_n^{(L)}(\epsilon) &= f_{WL}^{(L)}(\epsilon) w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n,WL}^{(L)}(u_{ij}) + O(\epsilon) \\
    w_n^{(1)} &= \frac{\Gamma(1 - 2\epsilon)}{\Gamma^2(1 - \epsilon)} m_n^{(1)} \\
    \ln(s_{ij}) &+ \text{Li}_2(1 - u_{ij})
\end{align*}
\]

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u's are invariant in the qmRk
**Quasi-multi-Regge limit of Wilson loops**

* L-loop Wilson loops are Regge exact

\[
w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon) w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n, WL}^{(L)}(u_{ij}) + \mathcal{O}(\epsilon)
\]

\[w_n^{(1)} = \frac{\Gamma(1 - 2\epsilon)}{\Gamma^2(1 - \epsilon)} m_n^{(1)}\]

\[\ln(s_{ij}) + \text{Li}_2(1 - u_{ij})\]

* u’s are invariant in the qmRk
* log’s are not power suppressed

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Monday, August 8, 2011
Quasi-multi-Regge limit of Wilson loops

$L$-loop Wilson loops are Regge exact

\[ w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon) w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n,WL}^{(L)}(u_{ij}) + O(\epsilon) \]

\[ w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} m_n^{(1)} \]

\[ \ln(s_{ij}) + \text{Li}_2(1-u_{ij}) \]

\[ \log's \text{ are not power suppressed} \]

we may compute the Wilson loop in qmRk

the result will be correct in general kinematics !!!
Diagrams of 2-loop Wilson loops

Each diagram yields an integral, similar to a Feynman-parameter integral

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
Wilson loops: analytic calc

1. Use Mellin-Barnes (MB) representation of the Feynman-parameter integrals: replace each denominator by a contour integral

\[
\frac{1}{(A + B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{\Gamma(-z) \Gamma(\lambda + z)}{\Gamma(z)} \frac{A^z}{B^{\lambda+z}}
\]

integral turns into a sum of residues

\[
\text{Res}_{z = -n} \Gamma(z) = \frac{(-1)^n}{n!}
\]
Wilson loops: analytic calc

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\]

integral turns into a sum of residues

\[
\text{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}
\]

2. Use Regge exactness in the qmR limit:
   retain only leading behaviour
   (i.e. leading residues) of the integral
Wilson loops: analytic calc

3. Use Regge exactness again: iterate the qmR limit $n$ times, by taking the $n$ cyclic permutations of the external legs.
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Wilson loops: analytic calc

3. Use Regge exactness again: iterate the qmR limit \( n \) times, by taking the \( n \) cyclic permutations of the external legs.

4. Sum remaining towers of residues

\[
\sum_{n=1}^{\infty} \frac{u^n}{n} = - \ln(1 - u)
\]

\[
\sum_{n=1}^{\infty} \frac{u^n}{n^k} = \text{Li}_k(u)
\]
Wilson loops: analytic calc

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\[
\sum_{n=1}^{\infty} \frac{u^n}{n^k} = \text{Li}_k(u)
\]

in general, get nested harmonic sums \( \rightarrow \) Goncharov polylogarithms

\[
\sum_{n_1=1}^{\infty} \frac{u_1^{n_1}}{n_1^{m_1}} \sum_{n_2=1}^{n_1 - 1} \ldots \sum_{n_k=1}^{n_{k-1} - 1} \frac{u_k^{n_k}}{n_k^{m_k}} = (-1)^k \ G \left( \frac{0, \ldots, 0, \frac{1}{u_1}, \ldots, 0, \ldots, 0, \frac{1}{u_1 \ldots u_k}; 1}{m_1 - 1, \ldots, m_k - 1} \right)
\]
Analytic 2-loop 6-edged Wilson loop

- compute 2-loop 6-edged Wilson loop

- in MB representation of the integrals in general kinematics, get up to 8-fold integrals
Analytic 2-loop 6-edged Wilson loop

compute 2-loop 6-edged Wilson loop

in MB representation of the integrals in general kinematics, get up to 8-fold integrals

after procedure in qmR limit, at most 3-fold integrals
in fact, only one 3-fold integral, which comes from \( f_H(p_1, p_3, p_5; p_4, p_6, p_2) \)

\[
\int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_1 z_2 + z_2 z_3 + z_3 z_1) u_1^{z_1} u_2^{z_2} u_3^{z_3} 
\]

\[
\times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1 + z_2) \Gamma(z_2 + z_3) \Gamma(z_3 + z_1)
\]

the result is in terms of Goncharov polylogarithms

\[
G(a, \vec{w}; z) = \int_0^z \frac{dt}{t - a} G(\vec{w}; t), \quad G(a; z) = \ln \left( 1 - \frac{z}{a} \right)
\]
Analytic 2-loop 6-edged Wilson loop

compute 2-loop 6-edged Wilson loop

in MB representation of the integrals in general kinematics, get up to 8-fold integrals

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\int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_1 z_2 + z_2 z_3 + z_3 z_1) u_1^{\bar{z}_1} u_2^{\bar{z}_2} u_3^{\bar{z}_3}
\]

\[\times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1 + z_2) \Gamma(z_2 + z_3) \Gamma(z_3 + z_1)\]

the result is in terms of Goncharov polylogarithms

\[G(a, \vec{w}; z) = \int_0^z \frac{dt}{t - a} G(\vec{w}; t), \quad G(a; z) = \ln \left(1 - \frac{z}{a}\right)\]

the remainder function \( R_6^{(2)} \) is given in terms of \( O(10^3) \) Goncharov polylogarithms \( G(u_1, u_2, u_3) \)

Duhr Smirnov VDD 09
2-loop 6-edged remainder function $R_6^{(2)}$

the remainder function $R_6^{(2)}$ is explicitly dependent on the cross ratios $u_1, u_2, u_3$
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it is symmetric in all its arguments
(in general it’s symmetric under cyclic permutations and reflections)
2-loop 6-edged remainder function $R_{6}^{(2)}$

- the remainder function $R_{6}^{(2)}$ is explicitly dependent on the cross ratios $u_1, u_2, u_3$
- it is symmetric in all its arguments (in general it’s symmetric under cyclic permutations and reflections)
- it is of uniform, and irreducible, transcendental weight 4

Transcendental weights: $w(\ln x) = w(\pi) = 1 \quad w(\text{Li}_2(x)) = w(\pi^2) = 2$
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transcendental weights: $w(ln x) = w(\pi) = 1 \quad w(Li_2(x)) = w(\pi^2) = 2$

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it is in agreement with the numeric calculation by

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
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  Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
- Straightforward computation
  qmR kinematics make it technically feasible
2-loop 6-edged remainder function $R_{6}^{(2)}$

- The remainder function $R_{6}^{(2)}$ is explicitly dependent on the cross ratios $u_1, u_2, u_3$.
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- It is in agreement with the numeric calculation by Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09.

Straightforward computation
qmR kinematics make it technically feasible

Finite answer, but in intermediate steps many divergences
Output is punishingly long

Monday, August 8, 2011
our result has been simplified and given in terms of polylogarithms

\[
R_{6, WL}^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right)
\]

\[
- \frac{1}{8} \left( \sum_{i=1}^{3} \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{J^4}{24} + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}
\]
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\[
\begin{align*}
R^{(2)}_{6,WL}(u_1, u_2, u_3) &= \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) \\
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\end{align*}
\]

where

\[
x_i^\pm = u_i x^\pm \\
x^\pm = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2 u_1 u_2 u_3}
\]

\[
\Delta = (u_1 + u_2 + u_3 - 1)^2 - 4 u_1 u_2 u_3
\]

\[
L_4(x^+, x^-) = \sum_{m=0}^{3} \frac{(-1)^m}{(2m)!!} \log(x^+ x^-)^m (\ell_{4-m}(x^+) + \ell_{4-m}(x^-)) + \frac{1}{8!!} \log(x^+ x^-)^4
\]

\[
\ell_n(x) = \frac{1}{2} (\text{Li}_n(x) - (-1)^n \text{Li}_n(1/x))
\]

\[
J = \sum_{i=1}^{3} (\ell_1(x_i^+) - \ell_1(x_i^-))
\]
our result has been simplified and given in terms of polylogarithms

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R_{6, WL}^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right)
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not a new, independent, computation
just a manipulation of our result
our result has been simplified and given in terms of polylogarithms

\[ R_{6,WL}^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) \]

where

\[ x_i^\pm = u_i x^\pm \quad x^\pm = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2u_1u_2u_3} \]

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\[ J = \sum_{i=1}^{3} (\ell_1(x_i^+) - \ell_1(x_i^-)) \]

not a new, independent, computation
just a manipulation of our result
answer is short and simple
introduces symbols in TH physics
Symbols

Fn. $F$ of $\deg(F) = n$ : fn. with log cuts, s.t. $Disc = 2\pi i \times f$, with $\deg(f) = n-1$
Symbols

Fn. \( F \) of \( \deg(F) = n \): fn. with log cuts, s.t. \( Disc = 2\pi i \times f \), with \( \deg(f) = n-1 \)

\[ \deg(const) = 0 \implies \deg(\pi) = 0 \]

\( \ln x \): cut along \([-\infty, 0]\) with \( Disc = 2\pi i \implies \deg(\ln x) = 1 \)

\( \text{Li}_2(x) \): cut along \([1, \infty]\) with \( Disc = -2\pi i \ln x \implies \deg(\text{Li}_2(x)) = 2 \)
Symbols

Fn. $F$ of $\text{deg}(F) = n$ : fn. with log cuts, s.t. $\text{Disc} = 2\pi i \times f$, with $\text{deg}(f) = n-1$

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take a fn. defined as an iterated integral

$R_i$ rational functions

$T_k = \int_a^b d\ln R_1 \circ \cdots \circ d\ln R_k$

the symbol is

$\text{Sym}[T_k] = R_1 \otimes \cdots \otimes R_k$

defined on the tensor product of the group of rational functions, modulo constants

$\cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots$
Symbols

Fn. $F$ of $\text{deg}(F) = n$ : fn. with log cuts, s.t. $\text{Disc} = 2\pi i \times f$, with $\text{deg}(f) = n-1$

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\[ \cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots \]

\[ \text{Sym}[\ln x] = x \quad \text{Sym}[\text{Li}_2(x)] = -(x - 1) \otimes x \]
Symbols

Fn. $F$ of $\text{deg}(F) = n$ : fn. with log cuts, s.t. $\text{Disc} = 2\pi i \times f$, with $\text{deg}(f) = n - 1$

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take a fn. defined as an iterated integral $R_i$ rational functions

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the symbol is $\text{Sym}[T_k] = R_1 \otimes \cdots \otimes R_k$

defined on the tensor product of the group of rational functions, modulo constants

\[ \cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots \]

$\text{Sym}[\ln x] = x$ $\quad$ $\text{Sym}[\text{Li}_2(x)] = -(x - 1) \otimes x$

take $f, g$ with $\text{deg}(f) = \text{deg}(g) = n$ and $\text{Sym}[f] = \text{Sym}[g]$

then $f - g = h$ with $\text{deg}(h) = n - 1$
Symbols

Fn. \( F \) of \( \text{deg}(F) = n \): fn. with log cuts, s.t. \( \text{Disc} = 2\pi i \times f \), with \( \text{deg}(f) = n-1 \)

\[
\text{deg(const)} = 0 \Rightarrow \text{deg}(\pi) = 0
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take a fn. defined as an iterated integral

\[
T_k = \int_a^b d\ln R_1 \circ \cdots \circ d\ln R_k
\]

defined on the tensor product of the group of rational functions, modulo constants

\[
\cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots
\]

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\text{Sym}[\ln x] = x \quad \text{Sym}[\text{Li}_2(x)] = -(x - 1) \otimes x
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take \( f, g \) with \( \text{deg}(f) = \text{deg}(g) = n \) and \( \text{Sym}[f] = \text{Sym}[g] \)

then \( f-g = h \) with \( \text{deg}(h) = n -1 \)

a symbol determines a polynomial of uniform degree up to a constant
2-loop 6-edged Wilson loop

integral

function

symbol

symbol
(simpler) function

ideally

integral

differential equations
Drummond Henn Trnka 10

discontinuities
Caron-Huot II

algorithm
Duhr Gangl Rhodes (in progress)
Symbols

- find suitable variables such that arguments of polylogarithms become rational functions
  (in $N=4$ SYM it helps to use twistors, but it’s not mandatory)

- through some symbol-processing procedure (e.g. DGR algorithm)
  find simpler form of the integral in terms of polylogarithms
Amplitudes in \textit{twistor} space

\textbf{twistors} live in the fundamental irrep of $\text{SO}(2,4)$

any point in \textbf{dual} space corresponds to a line in \textit{twistor} space

$x_a \leftrightarrow (Z_a, Z_{a+1})$
Amplitudes in **twistor** space

- **twistors** live in the fundamental irrep of $SO(2,4)$

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null separations in **dual** space correspond to intersections in **twistor** space
Amplitudes in **twistor** space

- **twistors** live in the fundamental irrep of $SO(2,4)$
- any point in **dual** space corresponds to a line in **twistor** space
  \[ x_a \leftrightarrow (Z_a, Z_{a+1}) \]

null separations in **dual** space correspond to intersections in **twistor** space

2-loop $n$-pt MHV amplitudes can be written as sum of pentaboxes in **twistor** space

\[
m^{(2)}_n = \frac{1}{2} \sum_{i<j<k<l<i} \]

Arkani-Hamed Bourjaily Cachazo Trnka 10
Recent results on symbols/integrals

- symbol of $n$-edged 2-loop Wilson loops (in principle one could get the $n$-edged 2-loop Wilson loop, but symbol is very complicated)

- symbols (and explicit expressions) of the 6-dimensional massless, 1-mass & 3-mass 1-loop 6-pt integrals

- explicit expression of massless 1-loop 6-pt integral is reminiscent of 6-edged 2-loop Wilson loop, but it has weight 3

$$I_6(u_1, u_2, u_3) = \frac{1}{\sqrt{\Delta}} \left[ -2 \sum_{i=1}^{3} L_3(x_i^+, x_i^-) 
\right.
\left. + \frac{1}{3} \left( \sum_{i=1}^{3} \ell_1(x_i^+) - \ell_1(x_i^-) \right)^3 + \frac{\pi^2}{3} \chi \sum_{i=1}^{3} (\ell_1(x_i^+) - \ell_1(x_i^-)) \right]$$

$$L_3(x^+, x^-) = \sum_{k=0}^{2} \frac{(-1)^k}{(2k)!!} \ln^k(x^+ x^-) \left( \ell_{3-k}(x^+) - \ell_{3-k}(x^-) \right)$$
In October 7-11, we shall have a School of Analytic Computing in Atrani, Italy

lectures on amplitudes & Wilson loops by
Fernando Alday
Simon Caron-Huot
Claude Duhr
Johannes Henn
Henrik Johansson
Vladimir Smirnov