

A tale of two Regge limits

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Amplitudes SLAC 22 June 2018

- In the last few years, a lot of progress has been made in understanding the analytic structure of multi-loop amplitudes
- we review what implications that progress has had on our understanding of:
 - the Regge limit of QCD
 - the Regge limit of $N=4$ Super Yang-Mills (SYM)

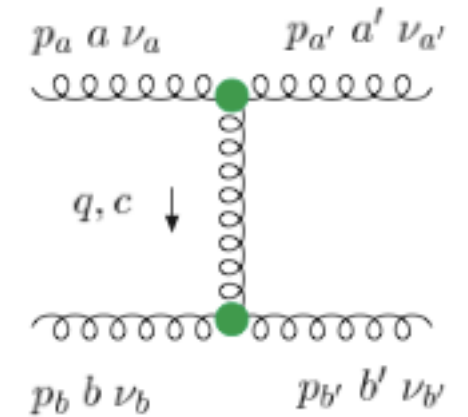
Regge limit of QCD

- In perturbative QCD, in the Regge limit $s \gg t$, any scattering process is dominated by gluon exchange in the t channel

- For a tree 4-gluon amplitude, we obtain

$$\mathcal{M}_{aa'bb'}^{gg \rightarrow gg}(s, t) = 2g_s^2 \left[(T^c)_{aa'} C_{\nu_a \nu_{a'}}(p_a, p_{a'}) \right] \frac{s}{t} \left[(T_c)_{bb'} C_{\nu_b \nu_{b'}}(p_b, p_{b'}) \right]$$

$C_{\nu_a \nu_{a'}}(p_a, p_{a'})$ are called *impact factors*



- we may break the amplitude into even/odd states under $s \leftrightarrow u$ crossing

$$\mathcal{M}^{(\pm)}(s, t) = \frac{\mathcal{M}(s, t) \pm \mathcal{M}(-s - t, t)}{2}$$

- we may decompose the amplitude into t -channel SU(3) representations. For gluon-gluon scattering, it is

$$\mathbf{8}_a \otimes \mathbf{8}_a = [\mathbf{1} \oplus \mathbf{8}_s \oplus \mathbf{27}] \oplus [\mathbf{8}_a \oplus \mathbf{10} \oplus \overline{\mathbf{10}}]$$

- at tree level, and at leading power in t/s , there is only $\mathbf{8}_a$ and only the odd amplitude under $s \leftrightarrow u$ crossing

$$\mathcal{M}_{ij \rightarrow ij}^{(0)}(s, t) = \mathcal{M}_{ij \rightarrow ij}^{(0,-)}(s, t) \quad \mathcal{M}_{ij \rightarrow ij}^{(0,+)}(s, t) = 0$$

LL accuracy

At leading logarithmic (LL) accuracy in s/t , there is still only $\mathbf{8}_a$ and loops corrections are obtained by the substitution $\frac{1}{t} \rightarrow \frac{1}{t} \left(\frac{s}{-t} \right)^{\alpha(t)}$

$\alpha(t)$ is the Regge gluon trajectory, with infrared coefficients

$$\alpha(t) = \frac{\alpha_s(-t, \epsilon)}{4\pi} \alpha^{(1)} + \left(\frac{\alpha_s(-t, \epsilon)}{4\pi} \right)^2 \alpha^{(2)} + \mathcal{O}(\alpha_s^3) \quad \alpha_s(-t, \epsilon) = \left(\frac{\mu^2}{-t} \right)^\epsilon \alpha_s(\mu^2)$$

$$\alpha^{(1)} = C_A \frac{\hat{\gamma}_K^{(1)}}{\epsilon} = C_A \frac{2}{\epsilon} \quad \alpha^{(2)} = \frac{b_0}{\epsilon^2} C_A + \frac{2\gamma_K^{(2)}}{\epsilon} C_A + \left(\frac{404}{27} - 2\zeta_3 \right) C_A^2 - \frac{56}{27} C_A n_f + \mathcal{O}(\epsilon)$$

the exponentiation through a Regge trajectory is called Reggeisation

in Mellin space, the amplitude displays a (Regge) pole

$$f_\ell^{(8_a)}(t) \propto \frac{\alpha(t)}{\ell - 1 - \alpha(t)}$$

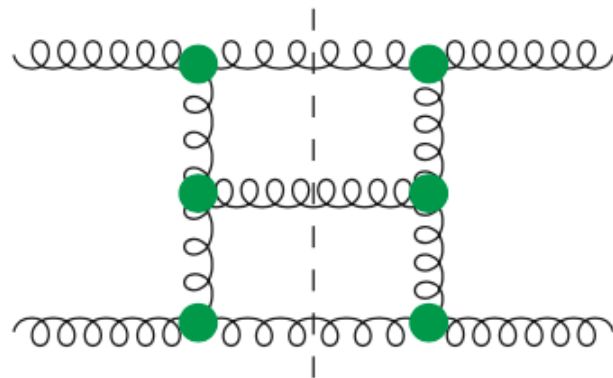
the Regge gluon trajectory is universal, i.e. process independent

Building blocks of **BFKL** at **LL** accuracy

The building blocks of the **BFKL** equation at **LL** accuracy are



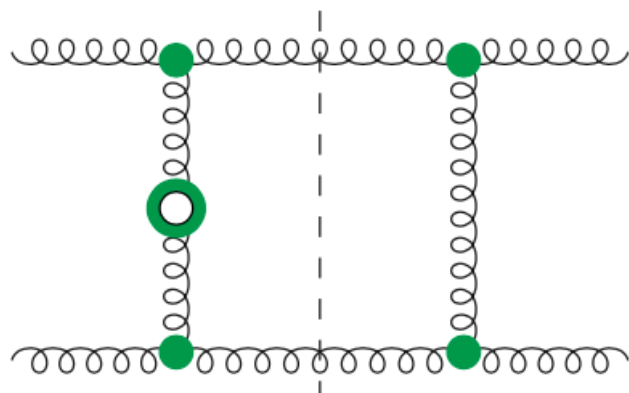
real: the emission of a gluon along the ladder



$$\begin{aligned} \mathcal{M}_{gg \rightarrow ggg}^{(0)}(s, t) &= 2s \left[g_s f^{ad_1 c_1} C_{\nu_a \nu_1}(p_a, p_1) \right] \\ &\times \frac{1}{t_1} \left[g_s f^{c_1 d_2 c_2} C_{\nu_2}(p_2) \right] \\ &\times \frac{1}{t_2} \left[g_s f^{bd_3 c_2} C_{\nu_b \nu_3}(p_b, p_3) \right] \end{aligned}$$



virtual: the one-loop Regge trajectory

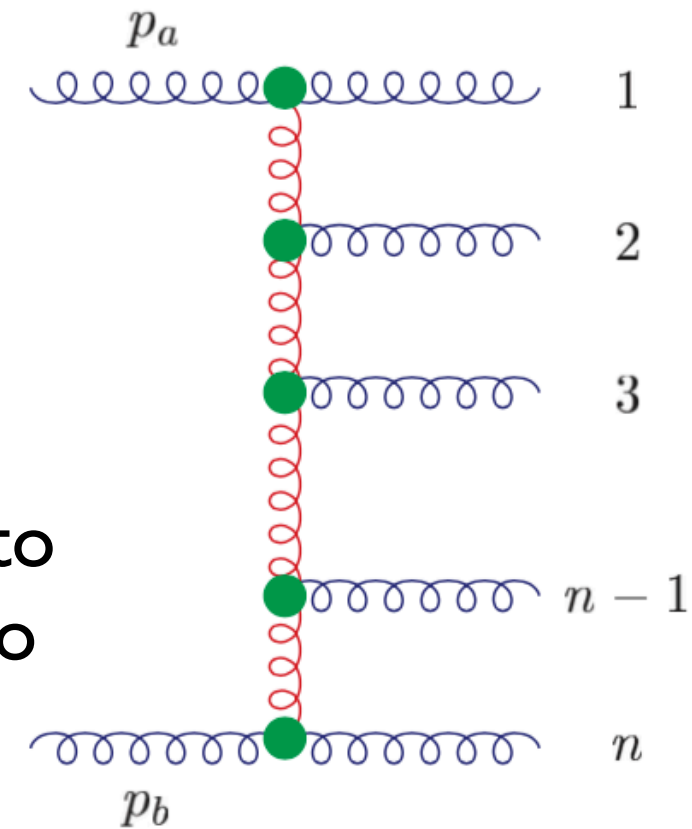


$$\mathcal{M}_{gg \rightarrow gg}^{(1)}(s, t) = \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{-t} \right)^\epsilon \frac{2C_A}{\epsilon} \ln \frac{s}{-t} \mathcal{M}_{gg \rightarrow gg}^{(0)}(s, t)$$

BFKL resummation

BFKL is a resummation of multiple gluon radiation out of the gluon exchanged in the t channel

the **LL** (Balitski Fadin Kuraev Lipatov 1976-77) and **Next-to-Leading Logarithmic** (Fadin-Lipatov 1998) contributions in $\log(s/|t|)$ of the radiative corrections to the gluon propagator in the t channel are resummed to all orders in α_s



the resummation yields an integral (**BFKL**) equation for the evolution of the gluon propagator in 2-dim transverse momentum space

the **BFKL** equation is obtained in the limit of strong rapidity ordering of the emitted gluons, with no ordering in transverse momentum - *multi-Regge kinematics* (**MRK**)

the solution is a Green's function of the momenta flowing in and out of the gluon ladder exchanged in the t channel

BFKL theory

the **BFKL** equation describes the evolution of the gluon propagator in 2-dim transverse momentum space

$$\omega f_\omega(q_1, q_2) = \frac{1}{2} \delta^{(2)}(q_1 - q_2) + \int d^2k K(q_1, k) f_\omega(k, q_2)$$

the solution is given in terms of eigenfunctions $\Phi_{\nu n}$ and an eigenvalue $\omega_{\nu n}$

$$f_\omega(q_1, q_2) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \frac{1}{\omega - \omega_{\nu n}} \Phi_{\nu n}(q_1) \Phi_{\nu n}^*(q_2)$$

as a function of rapidity, the solution is

$$f(q_1, q_2, y) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \Phi_{\nu n}(q_1) \Phi_{\nu n}^*(q_2) e^{y \omega_{\nu n}}$$

we expand kernel K , eigenfunctions $\Phi_{\nu n}$ and eigenvalue $\omega_{\nu n}$ in powers of $\bar{\alpha}_\mu = \frac{N_C}{\pi} \alpha_S(\mu^2)$

$$K(q_1, q_2) = \bar{\alpha}_\mu \sum_{l=0}^{\infty} \bar{\alpha}_\mu^l K^{(l)}(q_1, q_2) \quad \omega_{\nu n} = \bar{\alpha}_\mu \sum_{l=0}^{\infty} \bar{\alpha}_\mu^l \omega_{\nu n}^{(l)} \quad \Phi_{\nu n}(q) = \sum_{l=0}^{\infty} \bar{\alpha}_\mu^l \Phi_{\nu n}^{(l)}(q)$$

At **LL** accuracy

$$\omega_{\nu n}^{(0)} = -2\gamma_E - \psi\left(\frac{|n|+1}{2} + i\nu\right) - \psi\left(\frac{|n|+1}{2} - i\nu\right) \quad \Phi_{\nu n}^{(0)}(q) = \frac{1}{2\pi} (q^2)^{-1/2+i\nu} e^{in\theta}$$

note that in **N=4 SYM** the eigenfunctions and the eigenvalue are the same

Regge-pole factorisation

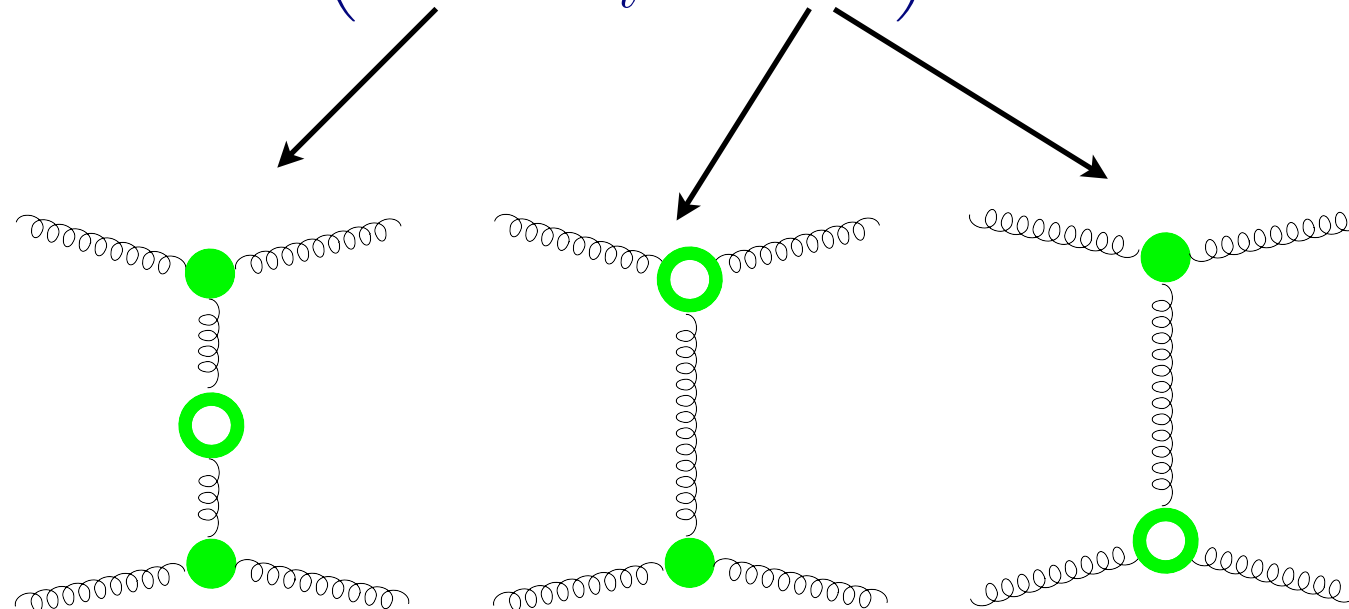
- gluon-gluon (odd) amplitude for $\mathbf{8}_a$

$$\mathcal{M}_{aa'bb'}^{gg \rightarrow gg}(s, t) = 2 g_s^2 \frac{s}{t} \left[(T^c)_{aa'} C_{\nu_a \nu_{a'}}(p_a, p_{a'}) \right] \left[\left(\frac{s}{-t} \right)^{\alpha(t)} + \left(\frac{-s}{-t} \right)^{\alpha(t)} \right] \left[(T_c)_{bb'} C_{\nu_b \nu_{b'}}(p_b, p_{b'}) \right]$$

strip colour off & expand at one loop

Fadin Lipatov 1993

$$\text{Re}[\mathcal{M}_{gg \rightarrow gg}^{(1,-)[8_a]}(s, t)] = \mathcal{M}_{gg \rightarrow gg}^{(0)}(s, t) \left(\alpha^{(1)}(t) \ln \frac{s}{-t} + 2C_{gg}^{(1)}(t) \right)$$

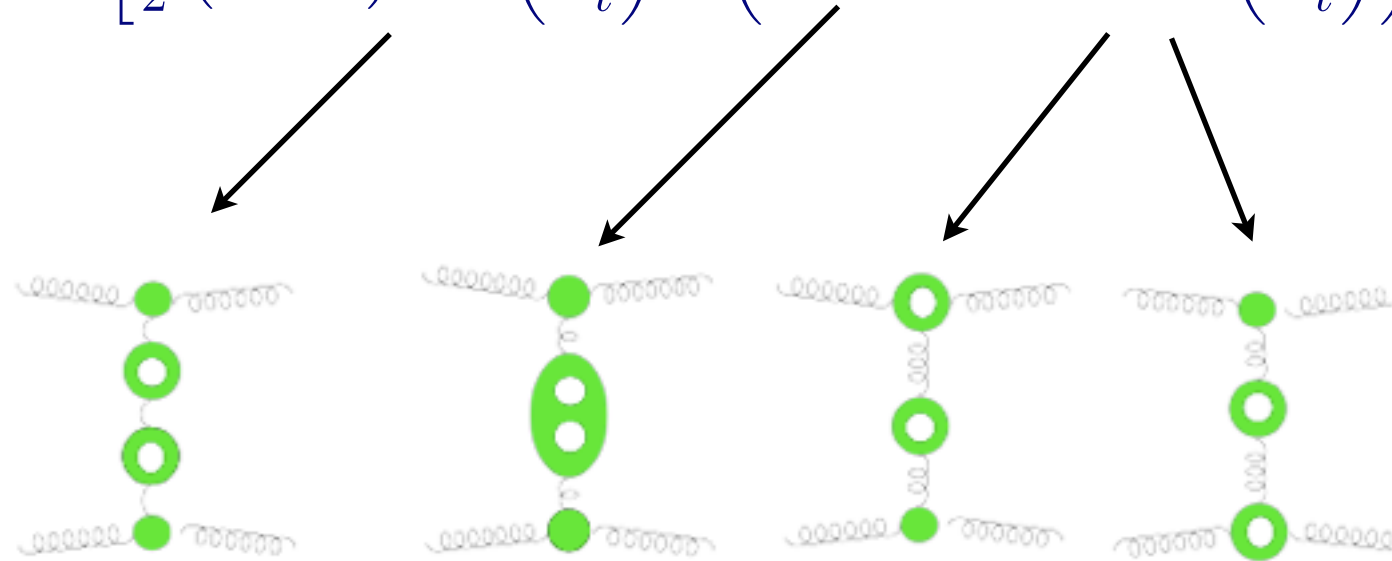


- the Regge gluon trajectory is universal, i.e. process independent
- the one-loop gluon impact factor $C_{gg}^{(1)}(t)$ is a polynomial in t, ϵ
- perform the Regge limit of the quark-quark amplitude
→ get one-loop quark impact factor $C_{qq}^{(1)}(t)$
- if factorisation holds, one can obtain the one-loop quark-gluon amplitude by assembling the Regge trajectory and the gluon and quark impact factors
the result should match the quark-gluon amplitude in the high-energy limit: it does

Regge-pole factorisation at **NLL** accuracy

- in the Regge limit, the two-loop expansion of the gluon-gluon (odd) amplitude for $\mathbf{8}_a$ is

$$\text{Re}[\mathcal{M}_{gg \rightarrow gg}^{(2,-)[\mathbf{8}_a]}(s, t)] = \left[\frac{1}{2} \left(\alpha^{(1)}(t) \right)^2 \ln^2 \left(\frac{s}{-t} \right) + \left(\alpha^{(2)}(t) + 2C_{gg}^{(1)}(t) \ln \left(\frac{s}{-t} \right) \right) \right] \mathcal{M}_{gg \rightarrow gg}^{(0)}(s, t)$$

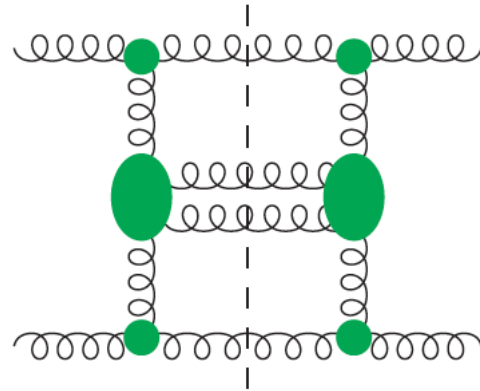


- for the real part of the amplitude, at **NLL** accuracy in s/t there is still only $\mathbf{8}_a$ which yields the 2-loop gluon trajectory
- gluon Reggeisation has been proven at **NLL** accuracy Fadin Fiore Kozlov Reznichenko 2006
- the two-loop Regge gluon trajectory is universal

Building blocks of **BFKL** at **NLL** accuracy

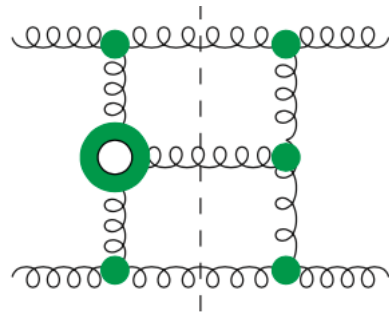
The building blocks of the **BFKL** equation at **NLL** accuracy are

RR: the emission of two gluons, or a qq pair, along the ladder



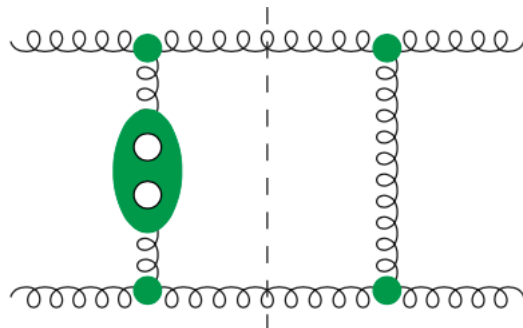
Fadin Lipatov 1989
VDD 1996; Fadin Lipatov 1996

RV: the one-loop correction to the emission of a gluon along the ladder



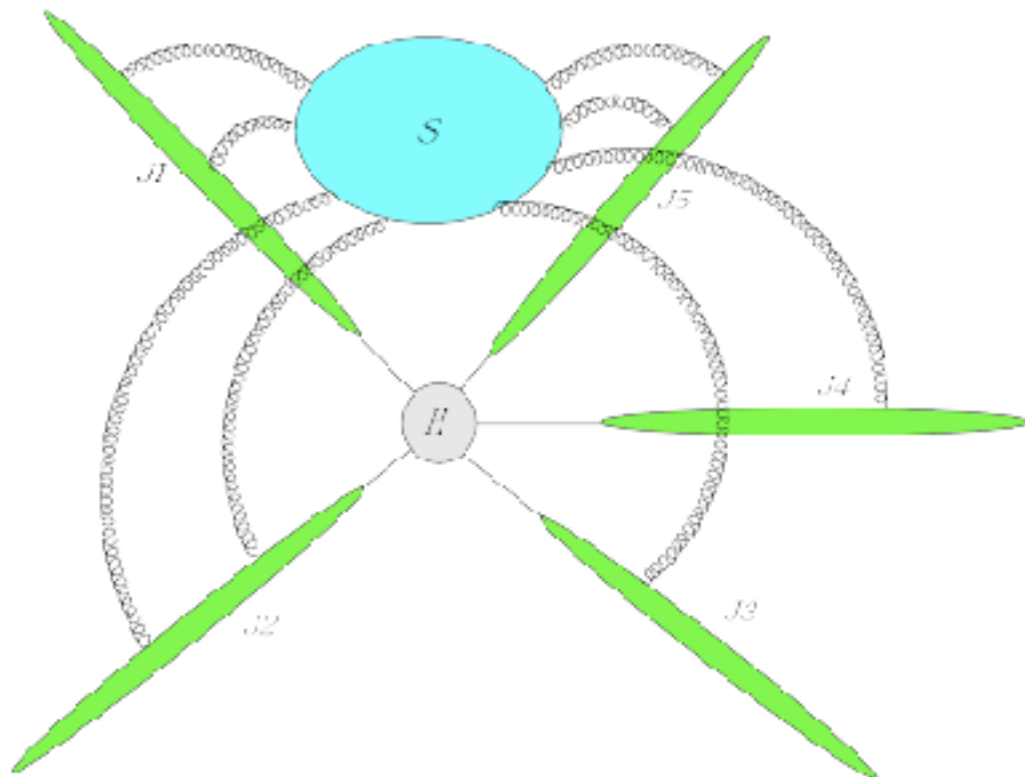
Fadin Lipatov 1993
Fadin Fiore Quartarolo 1994
Fadin Fiore Kotsky 1996
VDD Schmidt 1998

VV: the two-loop Regge trajectory



Fadin Fiore Quartarolo 1995
Fadin Fiore Kotsky 1995, 1996
VDD Glover 2001

Infrared factorisation



$$\mathcal{M}_n(\{p_i\}, \alpha_s) = Z_n(\{p_i\}, \alpha_s, \mu) \mathcal{H}_n(\{p_i\}, \alpha_s, \mu)$$

Z_n is solution to the RGE equation

$$Z_n = \text{P exp} \left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma_n(\{p_i\}, \lambda, \alpha_s(\lambda^2)) \right\}$$

Γ_n is the soft anomalous dimension

$$\Gamma_n(\{p_i\}, \lambda, \alpha_s) = \Gamma_n^{\text{dip}}(\{p_i\}, \lambda, \alpha_s) + \Delta_n(\{\rho_{ijkl}\}, \alpha_s)$$

dipole form

$$\Gamma_n^{\text{dip}}(\{p_i\}, \lambda, \alpha_s) = -\frac{1}{2} \hat{\gamma}_K(\alpha_s) \sum_{i < j} \log \left(\frac{-s_{ij}}{\lambda^2} \right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{i=1}^n \gamma_{\mathbf{J}_i}(\alpha_s)$$

$$\rho_{ijkl} = \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

Becher Neubert; Gardi Magnea 2009

At 2 loops, $\Delta^{(2)} = 0$, Γ_2 : Catani 1998; Aybat Dixon Sterman 2006

At 3 loops,

$$\Delta_4^{(3)}(\rho_{1234}, \rho_{1432}, \alpha_s) = 16 \mathbf{T}_1^{a_1} \mathbf{T}_2^{a_2} \mathbf{T}_3^{a_3} \mathbf{T}_4^{a_4} \left\{ f^{a_1 a_2 b} f^{a_3 a_4 b} \left[F \left(1 - \frac{1}{z} \right) - F \left(\frac{1}{z} \right) \right] \right. \\ \left. + f^{a_1 a_3 b} f^{a_4 a_2 b} [F(z) - F(1 - z)] + f^{a_1 a_4 b} f^{a_2 a_3 b} \left[F \left(\frac{1}{1 - z} \right) - F \left(\frac{z}{z - 1} \right) \right] \right\}$$

$$\rho_{1234} = z\bar{z} \quad \rho_{1432} = (1 - z)(1 - \bar{z})$$

$$F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2[\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)] + 6\zeta_4\mathcal{L}_1(z)$$

is given in terms of SVHPLs

Almelid Duhr Gardi 2015

Infrared factorisation in the Regge limit

we introduce the colour operators

$$\mathbf{T}_s = \mathbf{T}_a + \mathbf{T}_b,$$

$$\mathbf{T}_t = \mathbf{T}_a + \mathbf{T}_{a'},$$

$$\mathbf{T}_u = \mathbf{T}_a + \mathbf{T}_{b'}$$

$$\mathbf{T}_a + \mathbf{T}_b + \mathbf{T}_{a'} + \mathbf{T}_{b'} = 0$$

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^4 C_i = \mathcal{C}_{\text{tot}}$$

in the limit $s \gg t$, the dipole operator Z becomes

$$Z(\{p_i\}, \alpha_s(\mu^2), \mu) = \tilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \mu\right) Z_i(t, \alpha_s(\mu^2), \mu) Z_j(t, \alpha_s(\mu^2), \mu)$$

VDD Duhr Gardi Magnea White 2011
VDD Falcioni Magnea Vernazza 2014
Caron-Huot Gardi Vernazza 2017

Z_i are scalar factors which define the impact factors in terms of cusp and collinear anomalous dimensions

$$Z_n = \text{P exp} \left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left[\frac{\gamma_K(\alpha_s(\lambda^2))}{4} C_i \ln \frac{-t}{\lambda^2} \right] + \gamma_i(\alpha_s(\lambda^2)) \right\}$$

colour and $\ln(s/t)$ dependence are in the operator \tilde{Z}

$$\tilde{Z} = \exp \left\{ K(\alpha_s(\mu^2)) \left[\left(\ln \left(\frac{s}{-t} \right) - i\frac{\pi}{2} \right) \mathbf{T}_t^2 + i\pi \mathbf{T}_{s-u}^2 \right] + Q_{\Delta}^{(3)} \right\}$$

which is determined by the cusp anomalous dimension and by Q , through

$$K(\alpha_s(\mu^2), \epsilon) = -\frac{1}{4} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \hat{\gamma}_K(\alpha_s(\lambda^2), \epsilon),$$

$$Q_{\Delta}^{(3)} = -\frac{\Delta^{(3)}}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left(\frac{\alpha_s(\lambda^2)}{\pi} \right)^3$$

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comparing infrared and Regge factorizations

- the pole terms of the Regge trajectory are fixed by the operator K and thus by the cusp anomalous dimension Korchenskaya Korchemsky 1994
- the pole terms of the (one-loop) impact factor are fixed by the cusp and collinear anomalous dimensions VDD Falcioni Magnea Vernazza 2014
- in infrared factorisation, gluon Reggeisation at **LL** and **NLL** accuracy is due to the operator \tilde{Z} being diagonal in the ***t***-channel colour basis VDD Duhr Gardi Magnea White 2011

a mysterious relation ...

- in infrared factorisation, we have a precise knowledge of how the infrared poles in ϵ occur in the impact factors and in the Regge trajectory. Their finite parts, though, are treated as free parameters

- the Regge limit is an expansion in $\ln(s/t)$ and is valid to all orders of ϵ

- the one-loop gluon impact factor $C_{gg}^{(1)}(\epsilon)$ is known, in CDR/HV, to all orders of ϵ

$$C_{gg}^{(1)}(\epsilon) = -\frac{\gamma_K^{(1)}}{\epsilon^2} C_A + \frac{4\gamma_g^{(1)}}{\epsilon} + \frac{b_0}{2\epsilon} + \left(3\zeta_2 - \frac{67}{18}\right) C_A + \frac{5}{9}n_f \\ + \left[\left(\zeta_3 - \frac{202}{27}\right) C_A + \frac{28}{27}n_f\right] \epsilon + \mathcal{O}(\epsilon^2)$$

Fadin Fiore 1992
Fadin Lipatov 1993
Bern VDD Schmidt 1998

- the two-loop Regge trajectory is

$$\alpha^{(2)}(\epsilon) = \frac{b_0}{\epsilon^2} C_A + \frac{2\gamma_K^{(2)}}{\epsilon} C_A + \left(\frac{404}{27} - 2\zeta_3\right) C_A^2 - \frac{56}{27} C_A n_f + \mathcal{O}(\epsilon)$$

- the $\mathcal{O}(\epsilon)$ term of the one-loop gluon impact factor predicts the $\mathcal{O}(\epsilon^0)$ term of the two-loop Regge trajectory

VDD 2017

What is the $O(\epsilon^0)$ term of the two-loop Regge trajectory?

In planar $N=4$ SYM, the BDS ansatz fixes the $O(\epsilon^0)$ term of the Regge trajectory to be the gluon collinear anomalous dimension

$$\alpha_{N=4}^{(l)}(\epsilon) = 2^{l-1} \alpha^{(1)}(l\epsilon) \left(f_0^{(l)} + \epsilon f_1^{(l)} \right) + \mathcal{O}(\epsilon)$$

Drummond Korchemsky Sokatchev 2007
Naculich Schnitzer 2007
VDD Glover 2008

In QCD, the eikonal anomalous dimension, the collinear anomalous dimension on a polygonal Wilson loop, the threshold soft anomalous dimension in SCET, the D term in threshold resummation, all include a block which up to the ζ_3 term coincide with the $O(\epsilon^0)$ term of the Regge trajectory

the “eikonal” anomalous dimension of Erdogan Sterman 2011 agrees with the $O(\epsilon^0)$ term of the Regge trajectory

Why does the $O(\epsilon)$ term of the one-loop gluon impact factor know the $O(\epsilon^0)$ term of the two-loop Regge trajectory?

it hints at more structure in infrared factorisation than we currently know (perhaps related to this being a two-hard-scale problem)

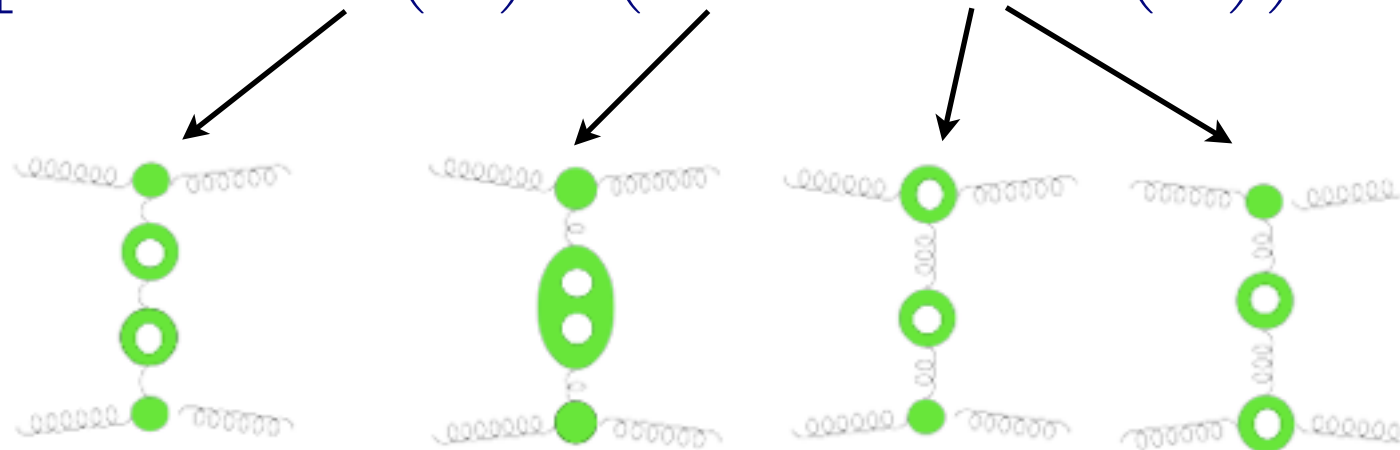
Regge-pole factorisation breaks at NNLO

- at LL accuracy for the amplitude, and at NLL accuracy for the real part of the amplitude, Regge-pole factorisation is based on the t -channel exchange of $\mathbf{8}_a$ only as one Reggeised gluon
- one can see in 3 ways that this is not correct at NNLO:
 - if pole factorisation holds, one can obtain the two-loop quark-gluon amplitude by assembling the two-loop Regge trajectory and gluon and quark impact factors. The result should match the quark-gluon amplitude in the high-energy limit. It doesn't by an N_c -subleading π^2/ϵ^2 factor VDD Glover 2001
 - in infrared factorisation at NNLL accuracy, the operator \tilde{Z} is non-diagonal in the t -channel colour basis VDD Duhr Gardi Magnea White 2011
VDD Falcioni Magnea Vernazza 2014
 - at NNLO, the picture based on one Reggeised-gluon exchange breaks down. Using the Balitsky-JIMWLK rapidity evolution equation, or a direct computation, one can see that a N_c -subleading 3-Reggeised-gluons exchange occurs at NNLO and NNLL accuracy Caron-Huot Gardi Vernazza 2017
Fadin Lipatov 2017
- It is still possible, though, to define a 2-loop impact factor, based on one Reggeised-gluon exchange VDD Falcioni Magnea Vernazza 2014
Caron-Huot Gardi Vernazza 2017

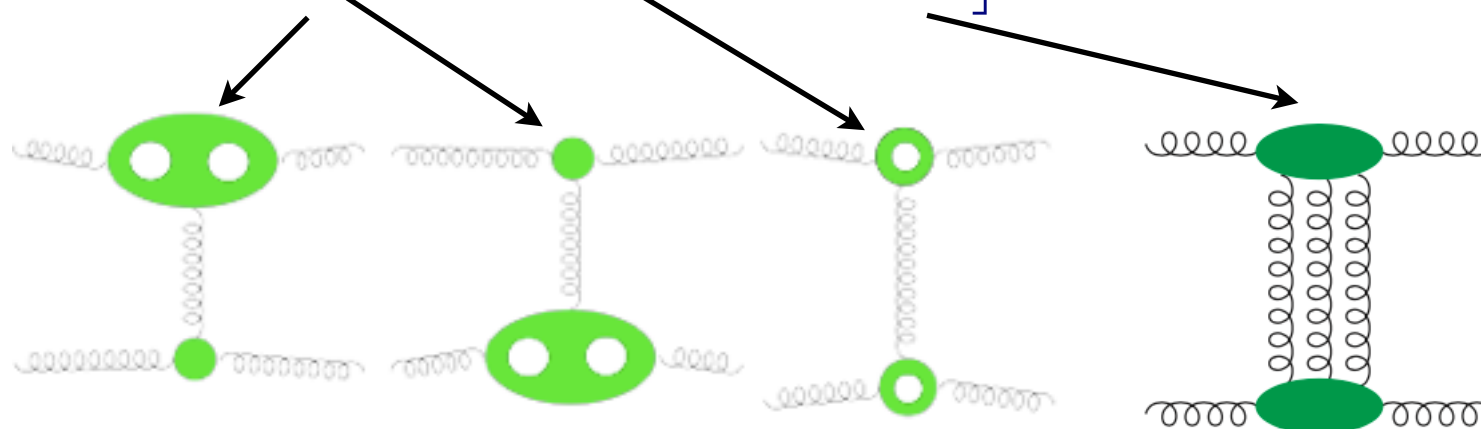
Regge factorisation at 2 loops

in the Regge limit, the two-loop expansion of the gluon-gluon (odd) amplitude for $\mathbf{8}_a$ is


$$\text{Re}[\mathcal{M}_{gg \rightarrow gg}^{(2,-)[\mathbf{8}_a]}(s, t)] = \left[\frac{1}{2} \left(\alpha^{(1)}(t) \right)^2 \ln^2 \left(\frac{s}{-t} \right) + \left(\alpha^{(2)}(t) + 2 C_{gg}^{(1)}(t) \ln \left(\frac{s}{-t} \right) \right) \right.$$




$$+ 2 C_{gg}^{(2)}(t) + \left(C_{gg}^{(1)}(t) \right)^2 + R_{gg}^{(2)}(t) \Big] \mathcal{M}_{gg \rightarrow gg}^{(0)}(s, t)$$



Regge factorisation at NNLL accuracy


 $\mathcal{M}^{(2,0,-)}$: $\mathbf{8}_a$, Regge pole, one Reggeised gluon
 $\mathbf{8}_a$, Regge cut, three Reggeised gluons (N_c -subleading)


 $\mathcal{M}^{(3,1,-)}$: $\mathbf{8}_a$, Regge pole, one Reggeised gluon
 $\mathbf{8}_a$, Regge cut, three Reggeised gluons
 $10 \oplus \overline{10}$, Regge cut, three Reggeised gluons Caron-Huot Gardi Vernazza 2017

the N_c -subleading pole-factorisation violation ($\mathbf{8}_a$, Regge cut, three Reggeised gluons) predicted for $\mathcal{M}^{(3,1,-)}$ in VDD Falcioni Magnea Vernazza 2014

confirmed by the 3-loop 4-pt amplitude computation in full $N=4$ SYM Henn Mistlberger 2016


 one must also consider the imaginary parts at NLL accuracy,
 since their squares would be relevant to resummations at NNLL accuracy

$\mathcal{M}^{(1,0,+)}$: $\mathbf{8}_s$, Regge pole, one Reggeised gluon
 $\mathbf{1}$ and $\mathbf{27}$, Regge cut, two Reggeised gluons

$\mathcal{M}^{(2,1,+)}$: $\mathbf{1}$ and $\mathbf{27}$, Regge cut, two Reggeised gluons

Caron-Huot Gardi Reichel Vernazza 2017


 finally, we may ignore $Q_{\Delta}^{(3)}$ since it contributes to the imaginary parts at NNLL accuracy,
 and to the real parts at N^3 LL accuracy

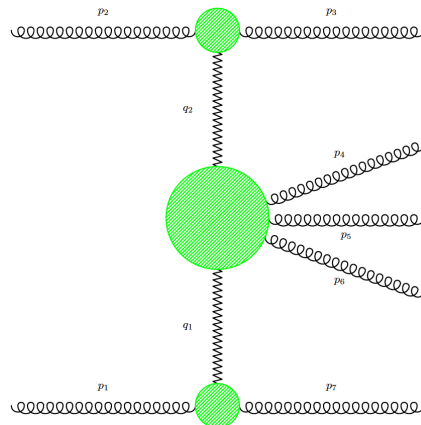
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Building blocks of **BFKL** at **NNLL** accuracy

The building blocks of a would-be **BFKL** ladder at **NNLL** accuracy



RRR: the emission of three partons along the ladder



VDD Frizzo Maltoni 1999



VVV: the three-loop Regge trajectory

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still unknown

RRV: the one-loop correction to the emission of two gluons, or a qq pair, along the ladder

RVV: the two-loop correction to the emission of a gluon along the ladder

Planar $N=4$ Super Yang Mills

- In the last years, a huge progress has been made in understanding the analytic structure of the S -matrix of planar $N=4$ SYM
- Besides the ordinary conformal symmetry, in the planar limit the S -matrix exhibits a dual conformal symmetry
Drummond Henn Smirnov Sokatchev 2006
- Accordingly, the analytic structure of the scattering amplitudes is highly constraint
- 4- and 5-point amplitudes are fixed to all loops by the symmetries in terms of the one-loop amplitudes and the cusp anomalous dimension
Anastasiou Bern Dixon Kosower 2003, Bern Dixon Smirnov 2005
Drummond Henn Korchemsky Sokatchev 2007
- Beyond 5 points, the finite part of the amplitudes is given in terms of a remainder function R . The symmetries only fix the variables of R (some conformally invariant cross ratios) but not the analytic dependence of R on them

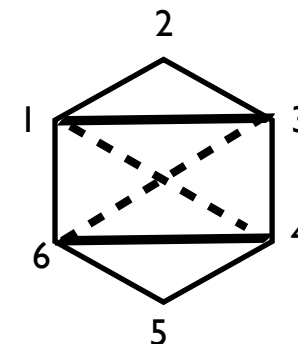


for $n = 6$, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

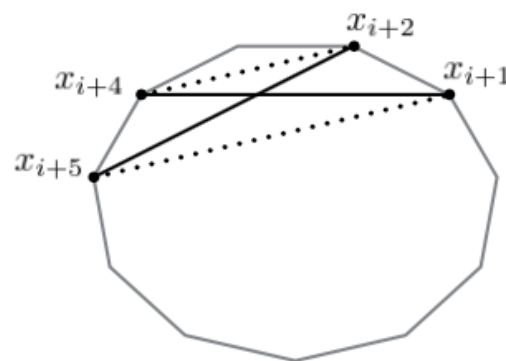
x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

thus $x_{k,k+r}^2 = (p_k + \dots + p_{k+r-1})^2$

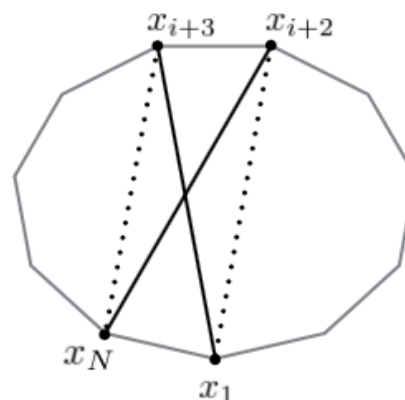


for n points, dual conformal invariance implies dependence on $3n-15$ independent cross ratios

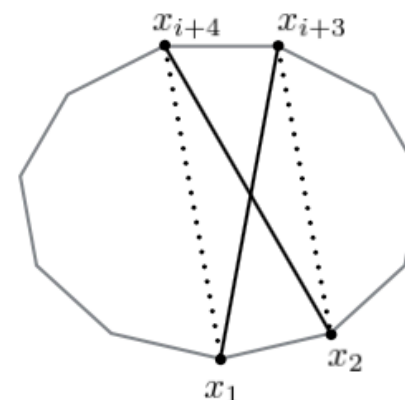
$$u_{1i} = \frac{x_{i+1,i+5}^2 x_{i+2,i+4}^2}{x_{i+1,i+4}^2 x_{i+2,i+5}^2}, \quad u_{2i} = \frac{x_{N,i+3}^2 x_{1,i+2}^2}{x_{N,i+2}^2 x_{1,i+3}^2}, \quad u_{3i} = \frac{x_{1,i+4}^2 x_{2,i+3}^2}{x_{1,i+3}^2 x_{2,i+4}^2}$$



u_{1i}



u_{2i}



u_{3i}

Multi-Regge kinematics in planar $N=4$ SYM

Amplitudes in multi-Regge kinematics (MRK) at LL accuracy factorise in terms of building blocks, which are expressed through Regge poles and can be determined through the 4-pt and 5-pt amplitudes

In planar $N=4$ SYM, the symmetries (BDS ansatz) fix the 4-pt and 5-pt amplitudes to all orders. Thus, it comes as no surprise that (in the Euclidean region) the remainder functions R vanish at all points

Brower Nastase Schnitzer Tan; Bartels Lipatov Sabio-Vera; VDD Duhr Glover 2008

If, before taking the multi-Regge limit, we analytically continue to regions of the Minkowski space where some Mandelstam invariants may pick up a phase, the amplitude may develop cuts, due to 2-Reggeon exchange.

The discontinuity of the amplitude is described by a dispersion relation for the adjoint, which is similar to the singlet BFKL equation in QCD

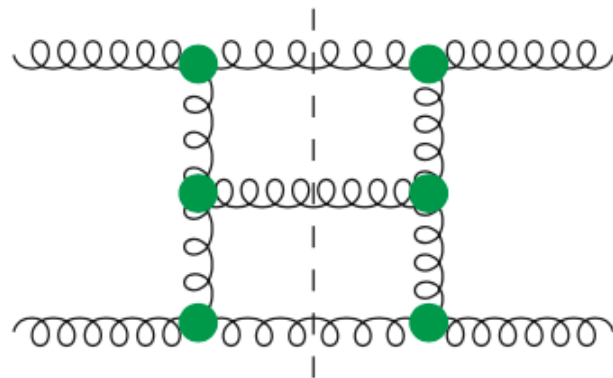
Bartels Lipatov Sabio-Vera 2008

Building blocks of BFKL at LL accuracy

The building blocks of the BFKL equation at LL accuracy are



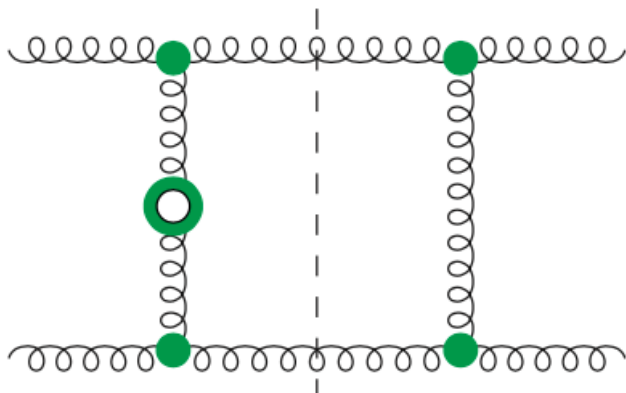
real: the emission of a gluon along the ladder



$$\begin{aligned} \mathcal{M}_{gg \rightarrow ggg}^{(0)}(s, t) &= 2s \left[g_s f^{ad_1 c_1} C_{\nu_a \nu_1}(p_a, p_1) \right] \\ &\times \frac{1}{t_1} \left[g_s f^{c_1 d_2 c_2} C_{\nu_2}(p_2) \right] \\ &\times \frac{1}{t_2} \left[g_s f^{bd_3 c_2} C_{\nu_b \nu_3}(p_b, p_3) \right] \end{aligned}$$



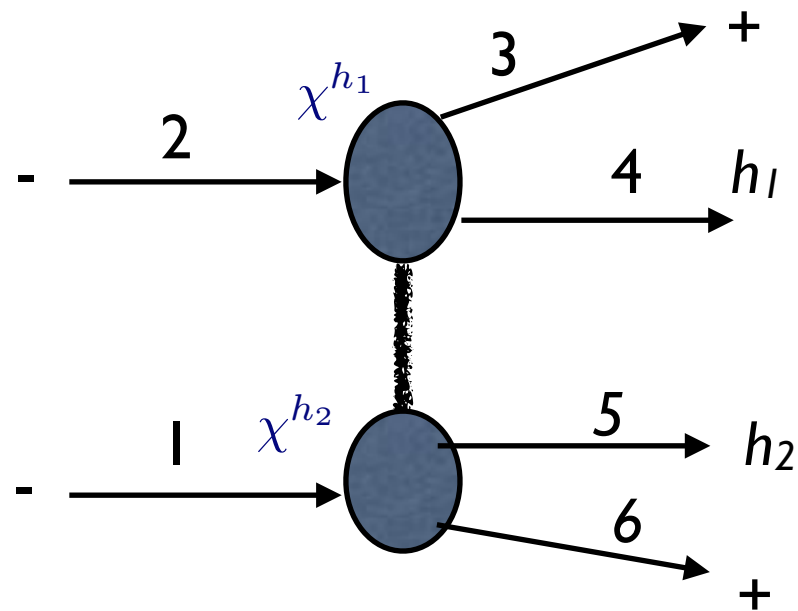
virtual: the one-loop Regge trajectory



$$\mathcal{M}_{gg \rightarrow gg}^{(1)}(s, t) = \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{-t} \right)^\epsilon \frac{2C_A}{\epsilon} \ln \frac{s}{-t} \mathcal{M}_{gg \rightarrow gg}^{(0)}(s, t)$$

Discontinuity of the amplitude in MRK

6-pt amplitude



continue to a Minkowski region

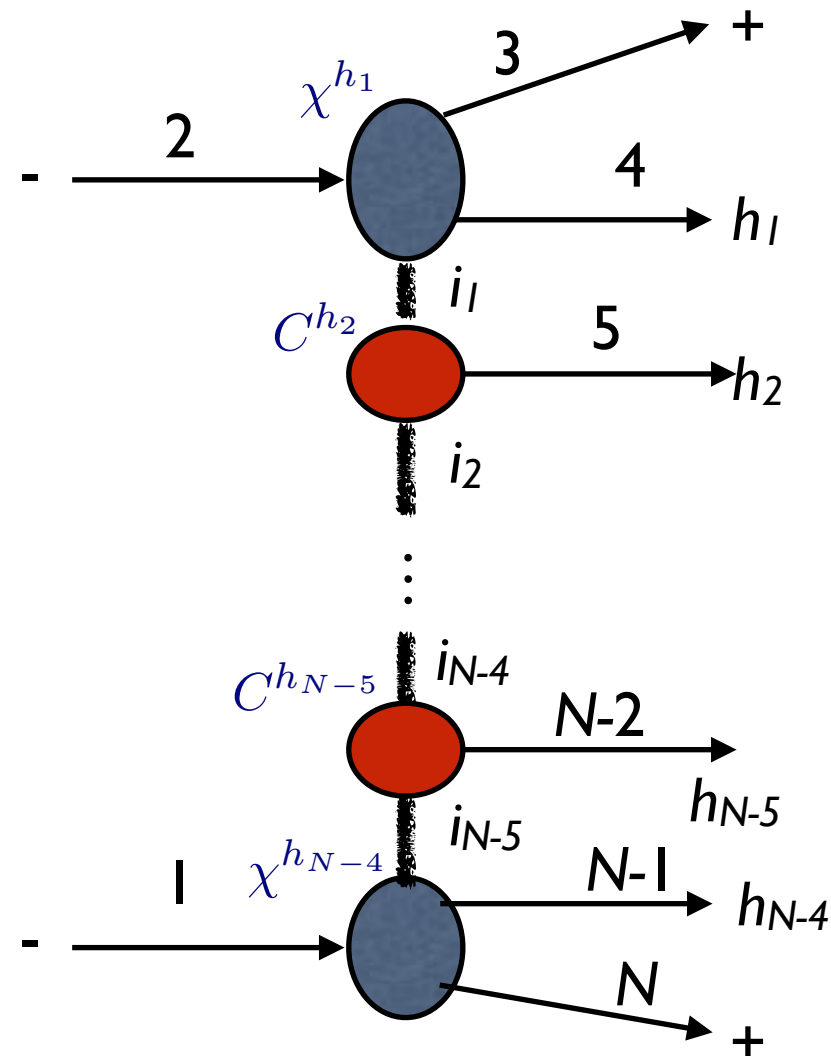
$$s_{34}, s_{56} < 0 \quad s, s_{45} > 0$$

one cross ratio picks up a phase

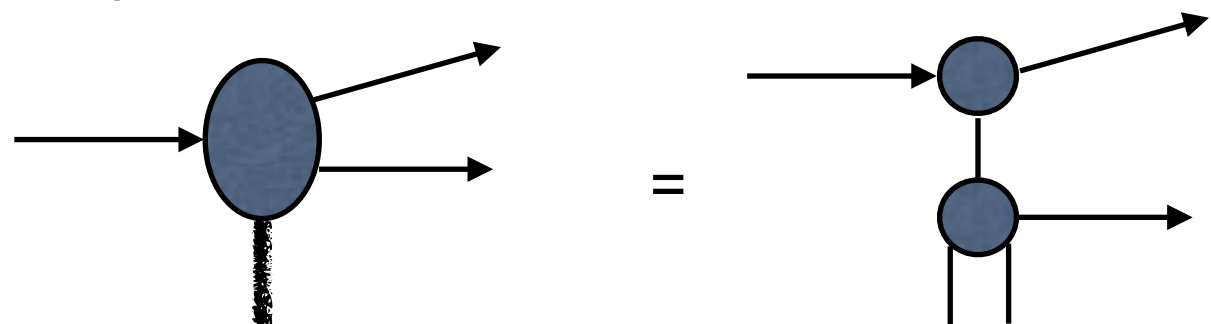
$$u_1 = \frac{s_{12}s_{45}}{s_{34}s_{56}} \rightarrow |u_1| e^{-2\pi i}$$

compute $\text{Disc}(\mathcal{M})|_{s_{45}}$

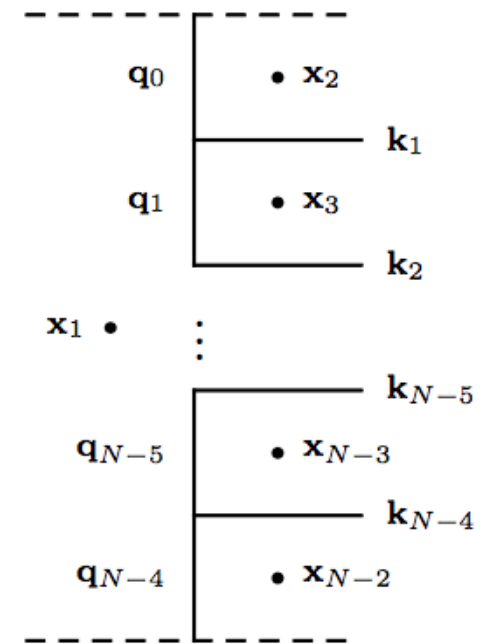
n -pt amplitude



impact factor




Moduli space of Riemann spheres



- in **MRK**, there is no ordering in transverse momentum, i.e. only the $n-2$ transverse momenta are non-trivial
- dual conformal invariance in transverse momentum space implies dependence on $n-5$ cross ratios of the transverse momenta

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})} = -\frac{\mathbf{q}_{i+1} \mathbf{k}_i}{\mathbf{q}_{i-1} \mathbf{k}_{i+1}} \quad i = 1, \dots, n-5$$

- $\mathcal{M}_{0,p}$ = space of configurations of p points on the Riemann sphere
Because we can fix 3 points at 0, 1, ∞ , its dimension is $\dim(\mathcal{M}_{0,p}) = p-3$

 see Y. Geyer's talk
- $\mathcal{M}_{0,n-2}$ is the space of the n -pt amplitudes in **MRK**, with $\dim(\mathcal{M}_{0,n-2}) = n-5$
Its coordinates can be chosen to be the z_i 's,
i.e. the cross ratios of the transverse momenta

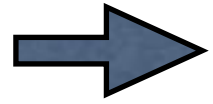
VDD Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek 2016
- on $\mathcal{M}_{0,n-2}$, the singularities are associated to degenerate configurations when two points merge $x_i \rightarrow x_{i+1}$
i.e. when momentum p_i becomes soft $p_i \rightarrow 0$

Iterated integrals on $\mathcal{M}_{0,n-2}$



iterated integrals on $\mathcal{M}_{0,p}$ can be written as multiple polylogarithms (MPL)

Brown 2006



amplitudes in **MRK** can be written in terms of MPLs



unitarity implies that for massless amplitudes

$$\Delta(M) = \ln(s_{ij}) \otimes \dots$$



dual conformal invariance requires that the first entry be a cross ratio
in particular, for amplitudes in **MRK** $\Delta(M) = \ln |\mathbf{x}_i - \mathbf{x}_j|^2 \otimes \dots$



except for the soft limit $p_i \rightarrow 0$, in **MRK** the transverse momenta never vanish

$|\mathbf{x}_i - \mathbf{x}_j|^2 \neq 0 \implies$ single-valued functions

thus, n -point amplitudes in **MRK** of planar **$N=4$ SYM** can be written
in terms of single-valued iterated integrals on $\mathcal{M}_{0,n-2}$

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for $n=6$, iterated integrals on $\mathcal{M}_{0,4}$ are harmonic polylogarithms
so, 6-point amplitudes in **MRK** of can be written in terms of
single-valued harmonic polylogarithms (SVHPL)

Dixon Duhr Pennington 2012

Unitarity on massless amplitudes

analytic structure of amplitudes is constrained by unitarity $\text{Disc}(M) = iMM^\dagger$

massless amplitudes may have branch points when Mandelstam invariants vanish $s_{ij} \rightarrow 0$ or become infinite $s_{ij} \rightarrow \infty$

discontinuity acts in the first entry of the coproduct $\Delta \text{Disc} = (\text{Disc} \otimes \text{id})\Delta$

then the coproduct of an amplitude is related to unitarity,

Duhr 2012

and for massless amplitudes $\Delta(M) = \ln(s_{ij}) \otimes \dots$



see A. McLeod's talk

MRK at LL accuracy

In MRK, 6-pt MHV and NMHV amplitudes are known at any number of loops

Lipatov Prygarin 2010-2011

Dixon Duhr Pennington 2012

Lipatov Prygarin Schnitzer 2012

knowing the space of functions of the n -point amplitudes in MRK,
(i.e. that is made of single-valued iterated integrals on $\mathcal{M}_{0,n-2}$)
allowed us to compute all MHV amplitudes at ℓ loops in LL accuracy
in terms of amplitudes with up to $(\ell+4)$ points, in practice up to 5 loops,
and all non-MHV amplitudes in LL accuracy up to 8 points and 4 loops

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for MHV amplitudes in MRK at LL accuracy at:

- at 2 loop, the n -pt remainder function $R_n^{(2)}$ can be written as a sum
of 2-loop 6-pt remainder functions $R_6^{(2)}$

Prygarin Spradlin Vergu Volovich 2011

- ...

Bartels Kormilitzin Lipatov Prygarin 2011

- ...

Bargheer Papathanasiou Schomerus 2015

- at 5 loops, the n -pt remainder function $R_n^{(5)}$ can be written as a
sum of 5-loop 6-, 7-, 8- and 9-pt amplitudes

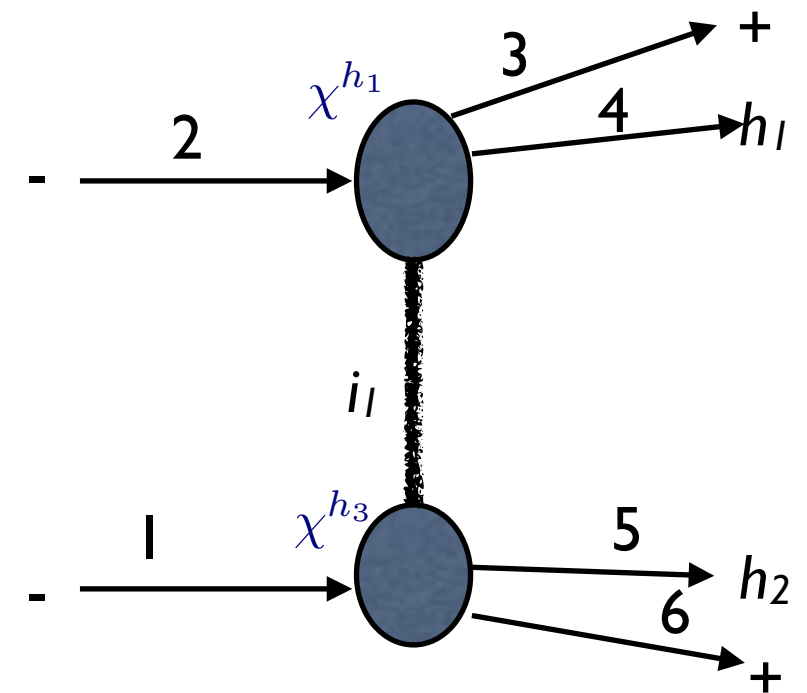
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MRK factorisation works also for non-MHV amplitudes,
however at each loop the number of building blocks is infinite

Beyond the LL accuracy

The building blocks of 6-pt amplitudes:
impact factors and 2-Reggeon exchange,
have been determined at finite coupling

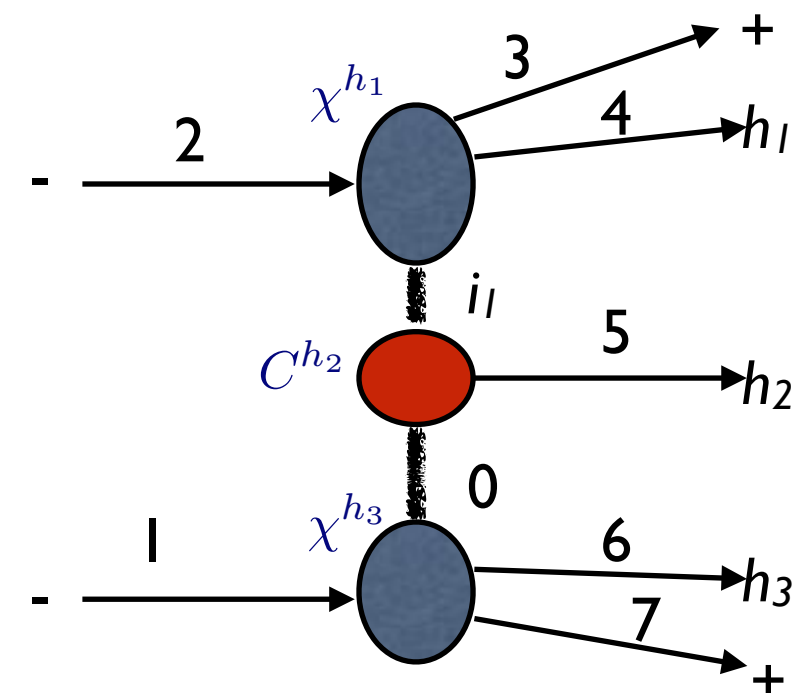
Basso Caron-Huot Sever 2014



Beyond 6 points, the only additional building block
is the central-emission vertex.

That has been determined at NLO, which allows for
computing the 7-pt amplitudes at NLL accuracy

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BFKL eigenvalue at LL accuracy in QCD



The singlet LL BFKL ladder in QCD, and thus the dijet cross section in the high-energy limit, can also be expressed in terms of SVHPLs, i.e. in terms of single-valued iterated integrals on $\mathcal{M}_{0,4}$

VDD Dixon Duhr Pennington 2013



Mueller & Navelet evaluated analytically the inclusive dijet cross section up to 5 loops. We evaluated it analytically up to 13 loops



Also, we could evaluate analytically the dijet cross section differential in the jet transverse energies or the azimuthal angle between the jets (up to 6 loops)

BFKL eigenvalue at NLL accuracy in QCD

At NLL accuracy in QCD and in N=4 SYM, the eigenvalue is

$$\omega_{\nu n}^{(1)} = \frac{1}{4} \delta_{\nu n}^{(1)} + \frac{1}{4} \delta_{\nu n}^{(2)} + \frac{1}{4} \delta_{\nu n}^{(3)} + \gamma_K^{(2)} \chi_{\nu n} - \frac{1}{8} \beta_0 \chi_{\nu n}^2 + \frac{3}{2} \zeta_3$$

Fadin Lipatov 1998
Kotikov Lipatov 2000, 2002

with one-loop beta function and two-loop cusp anomalous dimension

$$\beta_0 = \frac{11}{3} - \frac{2N_f}{3N_c} \quad \gamma_K^{(2)} = \frac{1}{4} \left(\frac{64}{9} - \frac{10N_f}{9N_c} \right) - \frac{\zeta_2}{2}$$

and with

$$\delta_{\nu n}^{(1)} = \partial_\nu^2 \chi_{\nu n} \quad \chi_{\nu n} = \omega_{\nu n}^{(0)}$$

$$\delta_{\nu n}^{(2)} = -2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma)$$

$$\delta_{\nu n}^{(3)} = - \frac{\Gamma(\frac{1}{2} + i\nu) \Gamma(\frac{1}{2} - i\nu)}{2i\nu} \left[\psi \left(\frac{1}{2} + i\nu \right) - \psi \left(\frac{1}{2} - i\nu \right) \right] \\ \times \left[\delta_{n0} \left(3 + \left(1 + \frac{N_f}{N_c^3} \right) \frac{2 + 3\gamma(1 - \gamma)}{(3 - 2\gamma)(1 + 2\gamma)} \right) - \delta_{|n|2} \left(\left(1 + \frac{N_f}{N_c^3} \right) \frac{\gamma(1 - \gamma)}{2(3 - 2\gamma)(1 + 2\gamma)} \right) \right]$$

$\Phi(n, \gamma)$ is a sum over linear combinations of ψ functions
and γ is a shorthand $\gamma = l/2 + i\nu$

In blue we labeled the terms which occur only in QCD,
in red the ones which occur in QCD and in N=4 SYM

Fourier-Mellin transform



At **NLL** accuracy, the **BFKL** ladder is

$$f^{NLL}(q_1, q_2, \eta_{s_0}) = \frac{1}{2\pi \sqrt{q_1^2 q_2^2}} \sum_{k=1}^{\infty} \frac{\eta_{s_0}^k}{k!} f_{k+1}^{NLL}(z) \quad \eta_{s_0} = \bar{\alpha}_S(s_0) y$$

with coefficients given by the Fourier-Mellin transform

$$f_k^{NLL}(z) = \mathcal{F} \left[\omega_{\nu n}^{(1)} \chi_{\nu n}^{k-2} \right] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}} \right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} |z|^{2i\nu} \omega_{\nu n}^{(1)} \chi_{\nu n}^{k-2} \quad \chi_{\nu n} = \omega_{\nu n}^{(0)}$$

using the explicit form of the eigenvalue

$$\omega_{\nu n}^{(1)} = \frac{1}{4} \delta_{\nu n}^{(1)} + \frac{1}{4} \delta_{\nu n}^{(2)} + \frac{1}{4} \delta_{\nu n}^{(3)} + \gamma_K^{(2)} \chi_{\nu n} - \frac{1}{8} \beta_0 \chi_{\nu n}^2 + \frac{3}{2} \zeta_3$$

the coefficients can be written as

$$f_k^{NLL}(z) = \frac{1}{4} C_k^{(1)}(z) + \frac{1}{4} C_k^{(2)}(z) + \frac{1}{4} C_k^{(3)}(z) + \gamma_K^{(2)} f_{k-1}^{LL}(z) - \frac{1}{8} \beta_0 f_k^{LL}(z) + \frac{3}{2} \zeta_3 f_{k-2}^{LL}(z)$$

with $C_k^{(i)}(z) = \mathcal{F} \left[\delta_{\nu n}^{(i)} \chi_{\nu n}^{k-2} \right]$

the weight of f_k^{NLL} is

$$\text{weight}(f_k^{NLL}) = \quad k \quad \quad k \quad \quad 0 \leq w \leq k \quad k-2 \leq w \leq k \quad k-1 \quad \quad k$$

SV functions



$C_k^{(1)}(z)$ are SVHPLs of uniform weight k with singularities at $z=0$ and $z=1$

$C_k^{(3)}(z)$ are MPLs of type $G(a_1, \dots, a_n; |z|)$ with $a_k \in \{-i, 0, i\}$

they are SV functions of z because they have no branch cut on the positive real axis, and have weight $0 \leq w \leq k$

For $C_k^{(2)}(z)$ one needs Schnetz' generalised SVMPLs with singularities at

$$z = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}$$

Schnetz 2016

then one can show that $C_k^{(2)}(z)$ are Schnetz' generalised SVMPLs

$\mathcal{G}(a_1, \dots, a_n; z)$ with singularities at $a_i \in \{-1, 0, 1, -1/\bar{z}\}$

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In moment space, the maximal weight of the **BFKL** eigenvalue and of the anomalous dimensions of the leading twist operators which control the Bjorken scaling violations in **QCD** is the same as the corresponding quantities in **N=4 SYM** (Principle of Maximal Transcendentality)

Kotikov Lipatov 2000, 2002

Kotikov Lipatov Velizhanin 2003



see B. Penante's talk

Interestingly, in transverse momentum space at **NLL** accuracy, the maximal weight of the **BFKL** ladder in **QCD** is *not* the same as the one of the ladder in **N=4 SYM**

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BFKL ladder in a generic $SU(N_c)$ gauge theory



one can consider the **BFKL** eigenvalue at **NLL** accuracy in a $SU(N_c)$ gauge theory with scalar or fermionic matter in arbitrary representations

$$\omega_{\nu n}^{(1)} = \frac{1}{4}\delta_{\nu n}^{(1)} + \frac{1}{4}\delta_{\nu n}^{(2)} + \frac{1}{4}\delta_{\nu n}^{(3)}(\tilde{N}_f, \tilde{N}_s) + \frac{3}{2}\zeta_3 + \gamma^{(2)}(\tilde{n}_f, \tilde{n}_s) \chi_{\nu n} - \frac{1}{8}\beta_0(\tilde{n}_f, \tilde{n}_s) \chi_{\nu n}^2$$

Kotikov Lipatov 2000

with $\beta_0(\tilde{n}_f, \tilde{n}_s) = \frac{11}{3} - \frac{2\tilde{n}_f}{3N_c} - \frac{\tilde{n}_s}{6N_c}$ $\gamma^{(2)}(\tilde{n}_f, \tilde{n}_s) = \frac{1}{4} \left(\frac{64}{9} - \frac{10\tilde{n}_f}{9N_c} - \frac{4\tilde{n}_s}{9N_c} \right) - \frac{\zeta_2}{2}$

$$\tilde{n}_f = \sum_R n_f^R T_R \quad \tilde{n}_s = \sum_R n_s^R T_R \quad \text{Tr}(T_R^a T_R^b) = T_R \delta^{ab} \quad T_F = \frac{1}{2}$$

$\tilde{n}_s(\tilde{n}_f) =$ number of scalars (Weyl fermions) in the representation R

$$\delta_{\nu n}^{(3)}(\tilde{N}_f, \tilde{N}_s) = \delta_{\nu n}^{(3,1)}(\tilde{N}_f, \tilde{N}_s) + \delta_{\nu n}^{(3,2)}(\tilde{N}_f, \tilde{N}_s)$$

with $\tilde{N}_x = \frac{1}{2} \sum_R n_x^R T_R (2C_R - N_c), \quad x = f, s$



Necessary and sufficient conditions for a $SU(N_c)$ gauge theory to have a **BFKL** ladder of maximal weight are:

- the one-loop beta function must vanish
- the two-loop cusp AD must be proportional to ζ_2
- $\delta_{\nu n}^{(3,2)}$ must vanish $\rightarrow 2\tilde{N}_f = N_c^2 + \tilde{N}_s$



There is no theory whose **BFKL** ladder has uniform maximal weight which agrees with the maximal weight terms of **QCD**

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Matter in the fundamental and in the adjoint



We solve the conditions above for matter in the fundamental F and in the adjoint A representations. We obtain:

$$2 n_f^F = n_s^F \qquad 2 n_f^A = 2 + n_s^A$$

which describes the spectrum of a gauge theory with N supersymmetries and $n^F = n_f^F$ chiral multiplets in F and $n^A = n_f^A - N$ chiral multiplets in A



There are four solutions to those conditions

\mathcal{N}	4	2	1	1
n_A	0	0	0	2
n_F	0	$4N_c$	$6N_c$	$2N_c$

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- the first is $N=4$ SYM
- the second is $N=2$ superconformal QCD with $N_f = 2N_c$ hypermultiplets
- the third is $N=1$ superconf. QCD



because the one-loop beta function is fixed by matter loops in gluon self-energies, we are only sensitive to the matter content of a theory, and not to its details (like scalar potential or Yukawa couplings)



Lipatov large N_c picture

- In **MRK**, amplitudes of **QCD** in the large N_c limit and amplitudes of planar **$N=4$ SYM** are described by similar (**BFKL**-like) Hamiltonians, corresponding to the t -channel exchange of n Reggeons

Lipatov 1993 - 2009

$$H = h + h^* \quad h = \sum_{i=1}^n h_{i,i+1} \quad h_{12} = \ln(p_1 p_2) + \frac{1}{p_1} \ln(\rho_{12}) p_1 + \frac{1}{p_2} \ln(\rho_{12}) p_2 - 2\psi(1)$$

$$\rho_{12} = \rho_1 - \rho_2 \quad \rho_k = x_k + i y_k \quad p_k = i \frac{\partial}{\partial \rho_k}$$

- those Hamiltonians coincide with the Hamiltonian of an integrable Heisenberg spin chain

Lipatov 1994

Faddeev Korchemsky 1995

- the Hamiltonians differ only by the boundary conditions, which one chooses for the t -channel exchange of an adjoint (\rightarrow open spin chain) in planar **$N=4$ SYM**, or of a singlet (\rightarrow closed spin chain) in large N_c **QCD**

$$\text{singlet} \quad h_{n,1} \rightarrow \ln \frac{p_1 p_n}{q^2} \quad \text{adjoint}$$

- the simplest case is the t -channel exchange of two Reggeons (\rightarrow two links on the spin chain), which corresponds to the BFKL equation in **QCD** and to the 6-pt amplitude in planar **$N=4$ SYM**

Double discontinuity of the amplitude in MRK

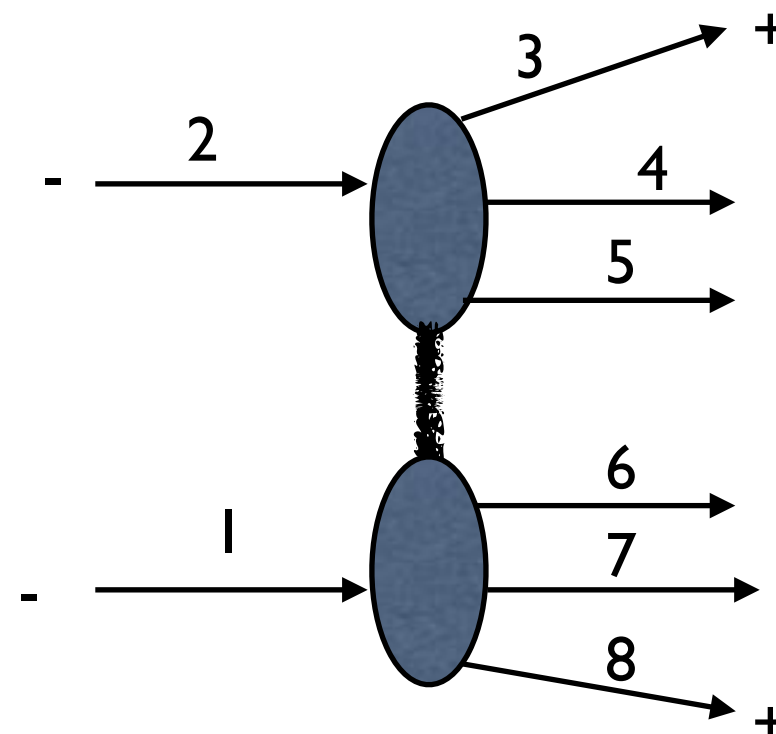
in planar $N=4$ SYM, 3-Reggeon exchange starts occurring with the 8-pt amplitude. We need take the double discontinuity

8-pt amplitude

continue to a Minkowski region

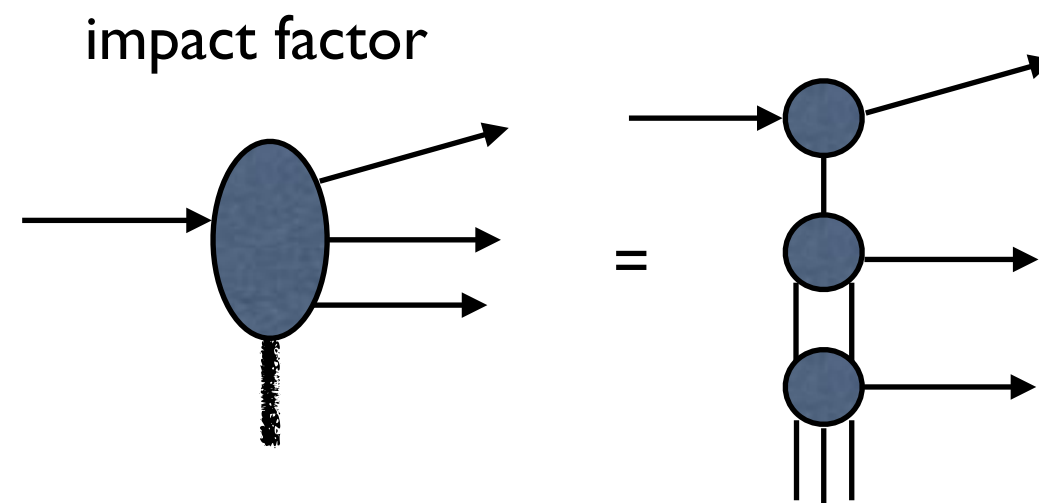
$$s, s_{4567}, s_{56} > 0$$

$$s_{34}, s_{45}, s_{67}, s_{78} < 0$$



we examined the double discontinuity of two-loop amplitudes, and found that it is determined to any number of points by building blocks which appear through 9 points.

Caron-Huot VDD Duhr Dulat Penante, in preparation



This is consistent with the picture above: the building blocks of double Disc are impact factors and 3-Reggeon exchange. Beyond 8 points, the only additional building block is the central-emission vertex, occurring at 9 points

Conclusions

- In **QCD**, amplitudes in the Regge limit features one-Reggeon exchange through **NLL** accuracy (for the real part, and 2-Reggeon exchange for the imaginary part)
3-Reggeon exchange appears in N_c -subleading pieces at **NNLL** accuracy
Although we are far from having a **BFKL** ladder, we understand the **NNLL** context in which it would arise
- In analogy to planar **$N=4$ SYM**, the functions which characterise the **BFKL** ladder in **QCD** are single-valued functions, specifically (generalised) SVMPLs
- In planar **$N=4$ SYM**, 2-Reggeon exchange is understood, even at finite coupling (where we just miss the central-emission vertex). At weak coupling, we know amplitudes at **LL** and **NLL** accuracy, in terms of SVMPLs
- We have just begun exploring 3-Reggeon exchange

Happy birthday!



BJ



Lance

