SYM amplitudes & pentagons in the high-energy limit

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Amplitudes09 Durham 1 April 2009

Bern-Dixon-Smirnov ansatz

an ansatz for MHV amplitudes in N=4 SYM

Bern Dixon Smirnov 05

$$\begin{aligned} m_n &= m_n^{(0)} \left[1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \right] \\ &= m_n^{(0)} \exp \left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right] \end{aligned}$$

coupling $a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^\epsilon$ $\lambda = g^2 N$ ‘t Hooft parameter

$$f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \, \frac{l}{2} \, \hat{G}^{(l)} + \epsilon^2 \, f_2^{(l)} \qquad E_n^{(l)}(\epsilon) = O(\epsilon)$$

$\hat{\gamma}_K^{(l)}$ cusp anomalous dimension, known to all orders of *a*

Korchemsky Radyuskin 86
Beisert Eden Staudacher 06

$\hat{G}^{(l)}$ collinear anomalous dimension, known through O(a^4)

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 07

Brief history of BDS ansatz

BDS ansatz checked for the 3-loop 4-pt amplitude — Bern Dixon Smirnov 05

2-loop 5-pt amplitude — Cachazo Spradlin Volovich 06; Bern Czakon Kosower Roiban Smirnov 06

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BDS ansatz shown to fail on 2-loop 6-pt amplitude

Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

Hints of break-up from strong-coupling expansion — Alday Maldacena 07

hexagon Wilson loop — Drummond Henn Korchemsky Sokatchev 07

multi-Regge limit (?) — Bartels Lipatov Sabio-Vera 08

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The BDS ansatz implies an iteration formula
for the 2-loop *n*-pt amplitude $m_n^{(2)}$ (rescaled by the tree amplitude)

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + \mathcal{O}(\epsilon)$$

Anastasiou Bern Dixon Kosower 03

The remainder function characterises the deviation from the ABDK/BDS iteration

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - Const^{(2)}$$

Why ?

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solid theory of the IR-divergent part

Mueller, Sen, Korchemsky, Radyuskin,
Collins, Sterman, Magnea, ...

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but not much theory of the remainder function,
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for $n = 4, 5$

Alday Maldacena
Drummond Henn Korchemsky Sokatchev

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What is the remainder function ?

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for $n = 4, 5$ Alday Maldacena
Drummond Henn Korchemsky Sokatchev

How ?

What is the remainder function ?

we are trying to move forward analytically Duhr Glover Smirnov VDD 09

MHV amplitudes ⇔ Wilson loops

agreement between *n*-edged Wilson loop and *n*-point MHV amplitude, verified for

Alday Maldacena 07

- *n*-edged 1-loop Wilson loop — Brandhuber Heslop Travaglini 07
- 6-edged 2-loop Wilson loop — Drummond Henn Korchemsky Sokatchev 07; Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

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7-edged & 8-edged 2-loop Wilson loops also computed (numerically)

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

if agreement holds up to 8-edged 2-loop Wilson loops,
then $R_7^{(2)}, R_8^{(2)}$ are known numerically

$R_n^{(2)}$ unknown analytically,
but functions of conformally-invariant cross-ratios

Drummond Henn Korchemsky Sokatchev 07

Colour decomposition of the tree *n*-point amplitude

$$\mathcal{M}_n^{(0)} = 2^{n/2}\, g^{n-2} \sum_{S_n/Z_n} \mathrm{tr}(T^{d_1} \cdots T^{d_n})\, m_n^{(0)}(1,\ldots,n)$$

$m_n^{(0)}(1,2,\ldots,n)$ colour-stripped amplitude

MHV amplitude $$m_n^{(0)}(1,2,\ldots,n) = \frac{\langle p_i p_j \rangle^4}{\langle p_1 p_2 \rangle \cdots \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

Regge factorisation of the 4-pt amplitude

colour-stripped 4-pt amplitude $g_1\, g_2 \to g_3\, g_4$ in the Regge limit $s \gg -t$

$$m_4(1,2,3,4) = s\left[g\, C(p_2,p_3,\tau)\right] \frac{1}{t} \left(\frac{-s}{\tau}\right)^{\alpha(t)} \left[g\, C(p_1,p_4,\tau)\right]$$

Glover VDD 08

$\alpha(t)$ Regge trajectory $C(p_2,p_3,\tau)$ coefficient function τ Regge-factorisation scale

$$\alpha(t) = \bar{g}^2 \bar{\alpha}^{(1)}(t) + \bar{g}^4 \bar{\alpha}^{(2)}(t) + \bar{g}^6 \bar{\alpha}^{(3)}(t) + O(\bar{g}^8) \qquad \bar{g}^2 = g^2 N c_\Gamma$$

$$C(p_i,p_j,\tau) = C^{(0)}(p_i,p_j)\left(1 + \bar{g}^2 \bar{C}^{(1)}(t,\tau) + \bar{g}^4 \bar{C}^{(2)}(t,\tau) + \bar{g}^6 \bar{C}^{(3)}(t,\tau) + \mathcal{O}(\bar{g}^8)\right)$$

$\bar{\alpha}^{(n)}(t)\,,\quad \bar{C}^{(n)}(t,\tau)$ are re-scaled loop coefficients

$$\bar{\alpha}^{(n)}(t) = \left(\frac{\mu^2}{-t}\right)^{n\epsilon} \alpha^{(n)}\,,\quad \bar{C}^{(n)}(t,\tau) = \left(\frac{\mu^2}{-t}\right)^{n\epsilon} C^{(n)}(t,\tau)$$

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Glover VDD 08

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Because the Regge limit is exponential in the Regge trajectory, one can use (the logarithm of) the BDS ansatz to obtain the Regge trajectory to all loops

Naculich Schnitzer 07
Drummond Korchemsky Sokatchev 07
Bartels Lipatov Sabio-Vera 08
Glover VDD 08

$$\alpha^{(l)}(\epsilon) = 2^{l-1} \alpha^{(1)}(l\epsilon) \left(\frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon\, \frac{l}{2}\, \hat{G}^{(l)}\right) + O(\epsilon) \qquad \alpha^{(1)}(\epsilon) = \frac{2}{\epsilon}$$

Caveat

In QCD the standard Regge factorisation is on the colour-dressed amplitude

$$M_4(1,2,3,4) = s\left[ig\, f^{abe}\, C(p_2,p_3,\tau)\right] \frac{1}{t} \left(\frac{-s}{\tau}\right)^{\alpha(t)} \left[ig\, f^{cde}\, C(p_1,p_4,\tau)\right]$$

Kuraev Fadin Lipatov 76
Fadin Lipatov 93

but it is known (of course also to Fadin & Lipatov) to be only approximate

new colour structures at one loop   C.R. Schmidt VDD 98

Regge factorisation of the 1-loop 4-pt amplitude

$$m_4^{(1)} = \bar{\alpha}^{(1)}(t)L + \ 2\bar{C}^{(1)}(t,\tau)$$

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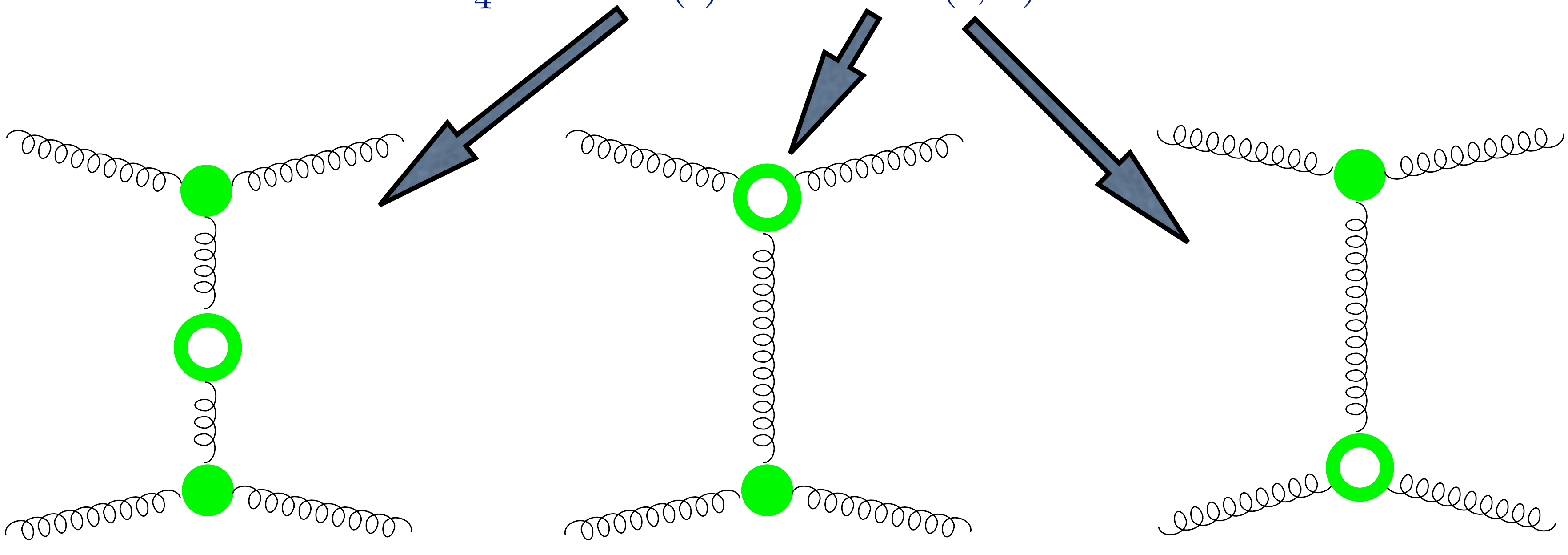

valid to all orders in ε

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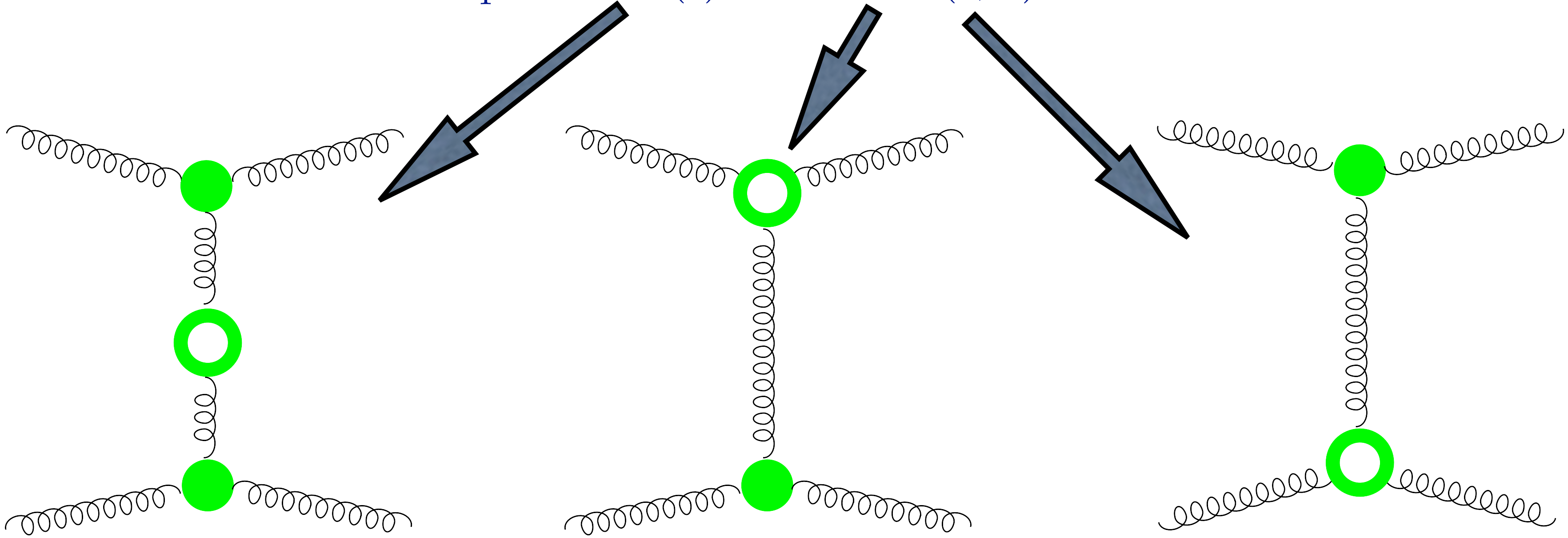


valid to all orders in ε

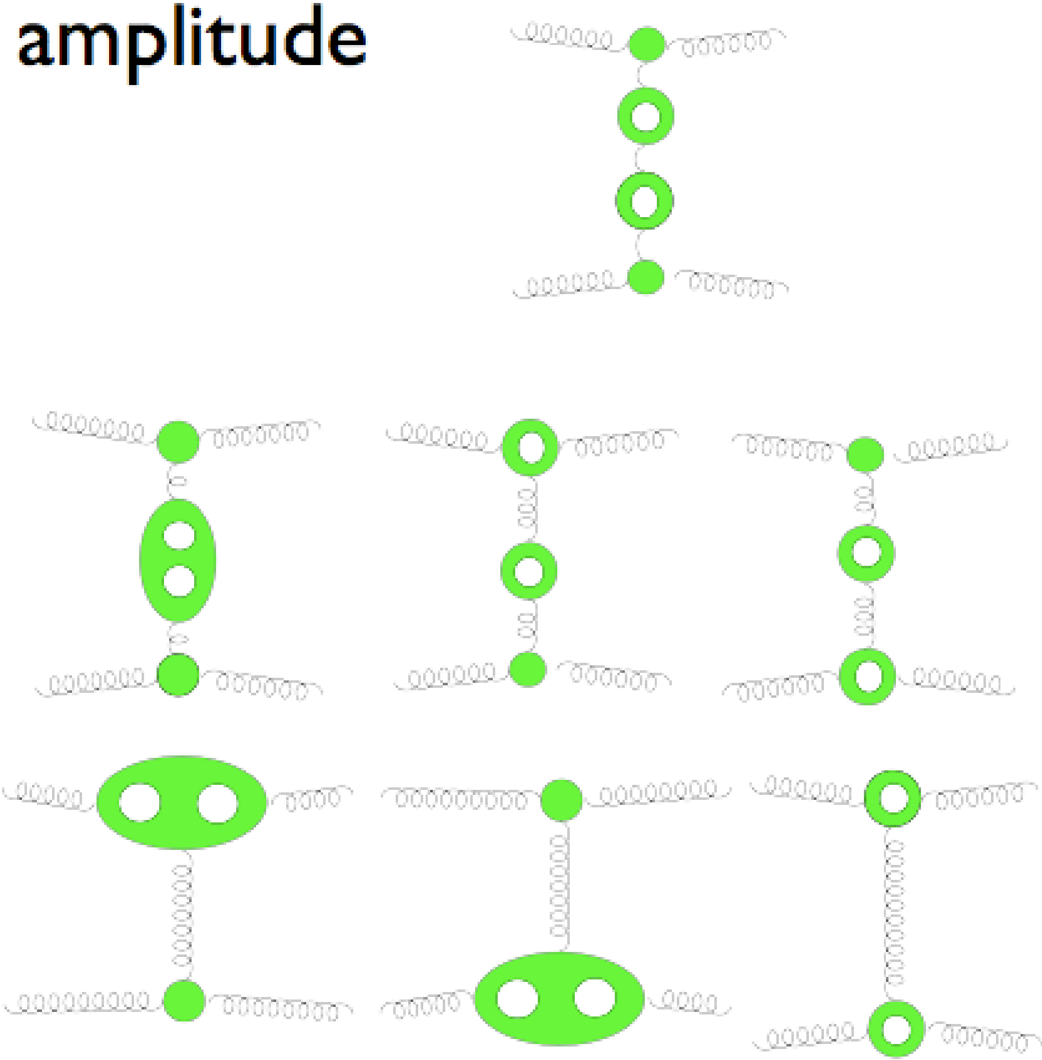
1-loop coefficient function

$$\begin{aligned} C^{(1)}(t,\tau) &= \frac{\psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1)}{\epsilon} - \frac{1}{\epsilon}\ln\frac{-t}{\tau} \\ &= \frac{1}{\epsilon^2}\left(-2 - \epsilon\,\ln\frac{-t}{\tau} + 3\sum_{n=1}^{\infty}\zeta_{2n}\,\epsilon^{2n} + \sum_{n=1}^{\infty}\zeta_{2n+1}\,\epsilon^{2n+1}\right) \end{aligned}$$

Factorisation of the 2-loop amplitude

$$
\begin{aligned}
m_4^{(2)} &= \frac{1}{2}\left(\bar{\alpha}^{(1)}(t)\right)^2 L^2 \\
&+ \left(\bar{\alpha}^{(2)}(t) + 2\,\bar{C}^{(1)}(t,\tau)\bar{\alpha}^{(1)}(t)\right) L \\
&+ 2\,\bar{C}^{(2)}(t,\tau) + \left(\bar{C}^{(1)}(t,\tau)\right)^2
\end{aligned}
$$

valid to all orders in ε



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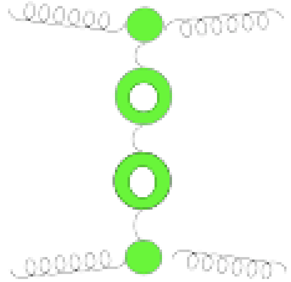

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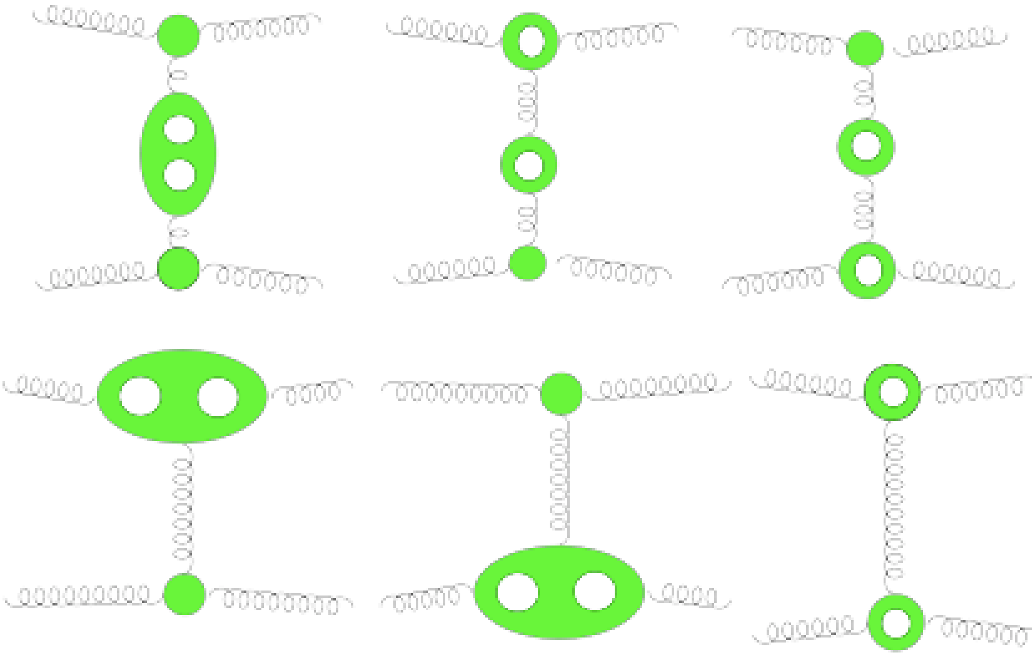
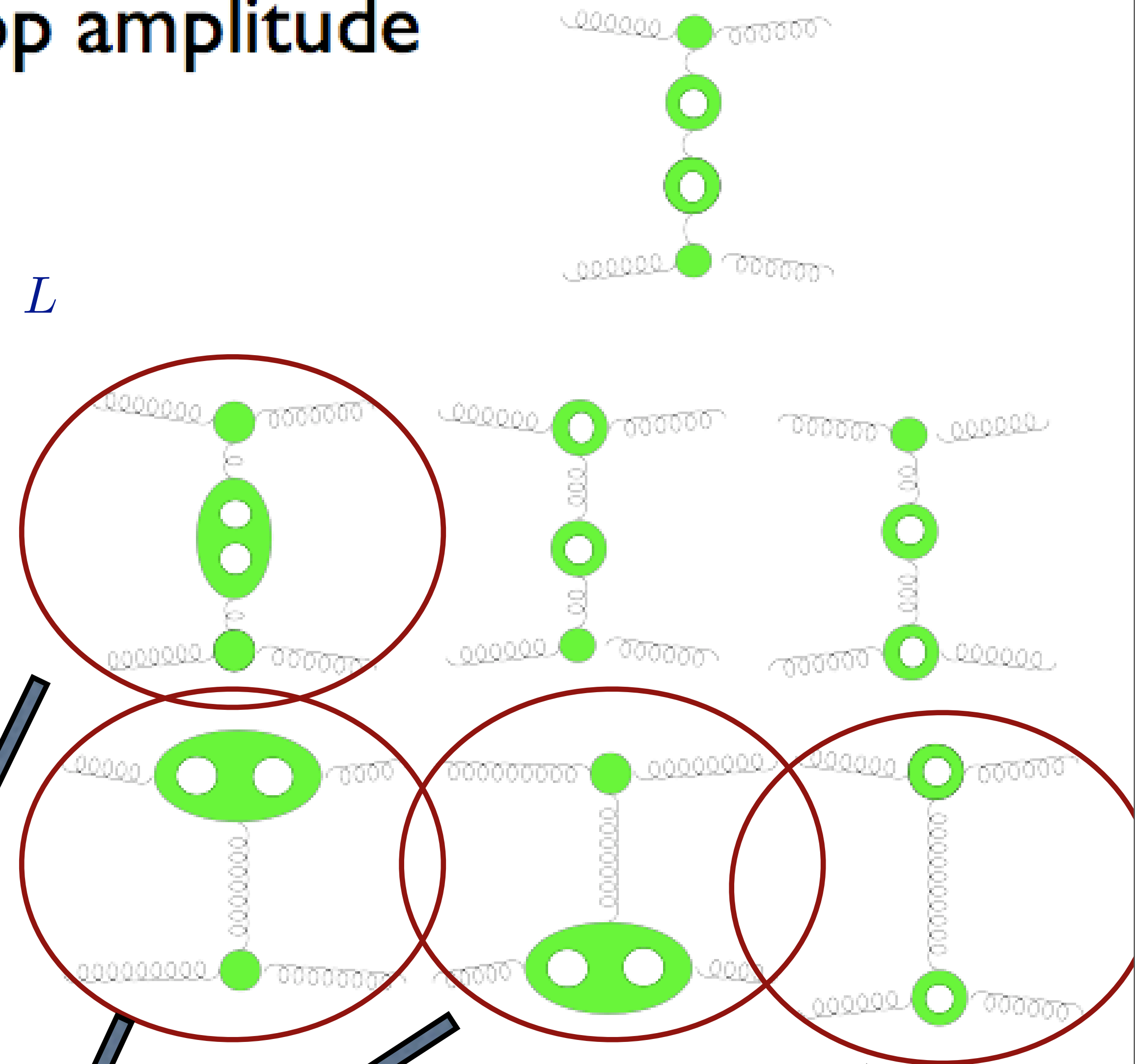


a more efficient way of writing it

$$
m_4^{(2)} = \frac{1}{2}\left(m_4^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t)L + 2\,\bar{C}^{(2)}(t,\tau) - \left(\bar{C}^{(1)}(t,\tau)\right)^2
$$

Factorisation of the 2-loop amplitude



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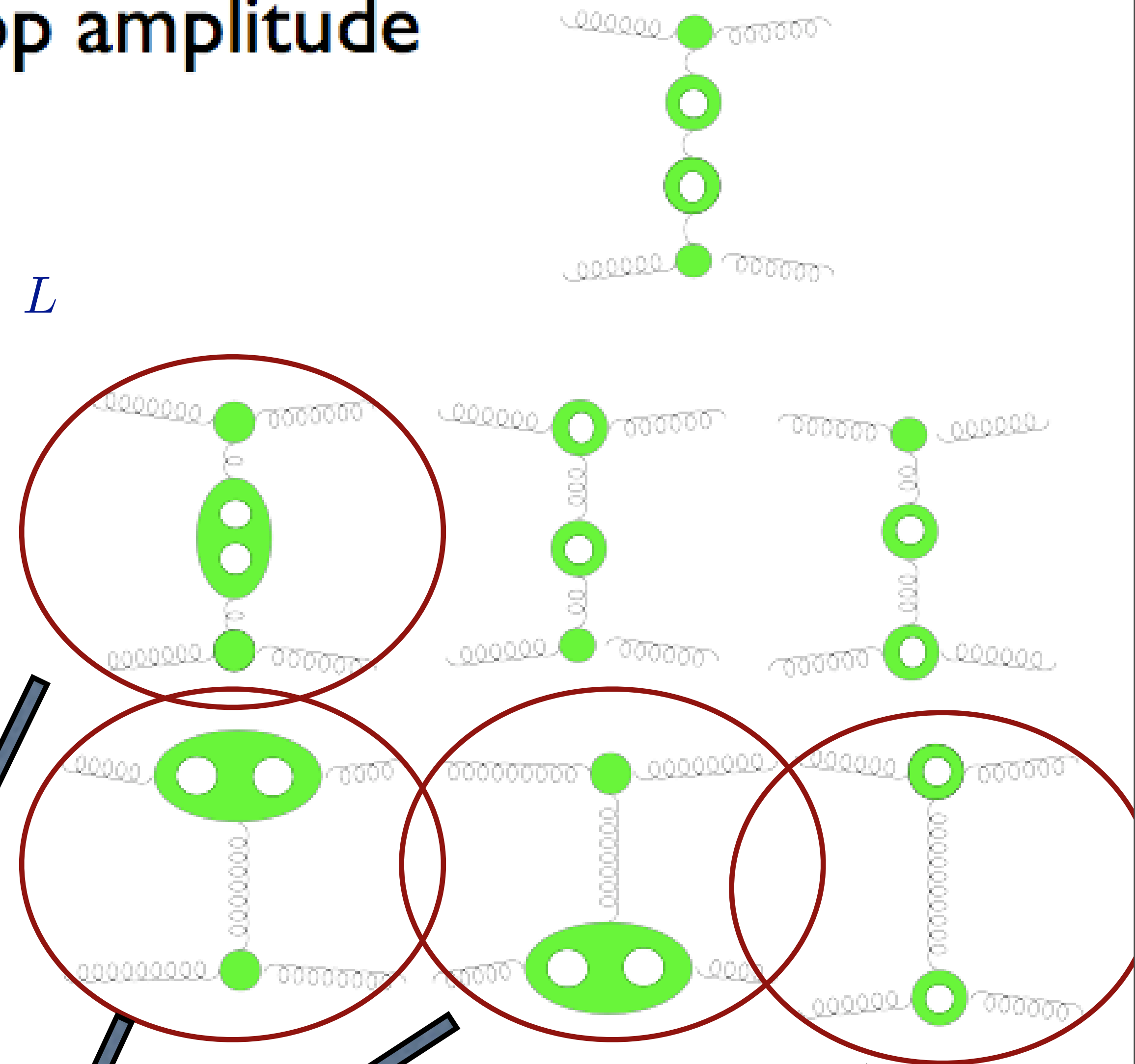

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where $m_4^{(1)}$ must be known at least through $\mathcal{O}(\epsilon^2)$

by direct calculation from
the 2-loop 4-pt amplitude $m_4^{(2)}$ to $O(\varepsilon^2)$ Bern Dixon Smirnov 05
we get 2-loop trajectory

$$\alpha^{(2)} = -\frac{2\zeta_2}{\epsilon} - 2\zeta_3 - 8\zeta_4\epsilon + (36\zeta_2\zeta_3 + 82\zeta_5)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

2-loop coefficient function

$$\begin{aligned} C^{(2)}(t,\tau) &= \frac{1}{2}\left[C^{(1)}(t,\tau)\right]^2 + \frac{\zeta_2}{\epsilon^2} + \left(\zeta_3 + \zeta_2 \ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &+ \left(\zeta_3 \ln\frac{-t}{\tau} - 19\zeta_4\right) + \left(4\zeta_4 \ln\frac{-t}{\tau} - 2\zeta_2\zeta_3 - 39\zeta_5\right)\epsilon \\ &- \left(48\zeta_3^2 + \frac{1773}{8}\zeta_6 + (18\zeta_2\zeta_3 + 41\zeta_5)\ln\frac{-t}{\tau}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \end{aligned}$$

Glover VDD 08

where $C^{(1)}(t,\tau,\epsilon)$ must be known at least through $\mathcal{O}(\epsilon^2)$

A similar factorisation holds also for QCD amplitudes.
In that case, the 2-loop 4-parton amplitude $m_4^{(2)}$
yields the 2-loop trajectory

Fadin Fiore 95
Glover VDD 01

$$\alpha^{(2)} = C_A \left[\beta_0 \frac{1}{\epsilon^2} + K \frac{2}{\epsilon} + C_A \left(\frac{404}{27} - 2\zeta_3 \right) - \frac{56}{27} N_F \right] + \mathcal{O}(\epsilon)$$

maximal trascendentality
Kotikov Lipatov 02

$$\beta_0 = \frac{11}{3} C_A - \frac{2}{3} N_F \qquad K = \left(\frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} N_F$$

maximal trascendentality:

$\zeta_n, \ \ln^n, \ \epsilon^{-n}$ have weight *n* in trascendentality

N=4 SYM amplitudes, and quantities derived from them,
are homogeneous polynomials of maximal trascendentality

BDS ansatz and Regge limit

the iteration formula for the 2-loop *n*-pt amplitude $m_n^{(2)}$

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + 4\,Const^{(2)} + \mathcal{O}(\epsilon)$$

Anastasiou Bern Dixon Kosower 03

valid for $n = 4, 5$

$$f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2 \qquad Const^{(2)} = -\frac{\zeta_2^2}{2}$$

(we use a different normalisation from BDS)

$$G(\epsilon) = \frac{e^{-\gamma\epsilon}\,\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)} = 1 + \mathcal{O}(\epsilon^2)$$

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from the iteration formula and Regge factorisation
we obtain iteration formulae for the Regge trajectory and the coefficient function

$$\alpha^{(2)}(\epsilon) = 2\,f^{(2)}(\epsilon)\,\alpha^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

$$C^{(2)}(t,\tau,\epsilon) = \frac{1}{2}\left[C^{(1)}(t,\tau,\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, C^{(1)}(t,\tau,2\epsilon) + 2\,Const^{(2)} + \mathcal{O}(\epsilon)$$

Glover VDD 08

where $C^{(1)}(t,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

the formulae for $n = 4$ implied by
the BDS ansatz and by Regge factorisation differ in that
BDS: valid for arbitrary kinematics, but to $O(\varepsilon^0)$
Regge: valid to all orders in ε, but only in the Regge kinematics.
They overlap and agree in the Regge kinematics to $O(\varepsilon^0)$

Regge factorisation at 3 loops

$$m_4^{(3)} = m_4^{(2)} m_4^{(1)} - \frac{1}{3}\left(m_4^{(1)}\right)^3$$
$$+ \bar{\alpha}^{(3)}(t) L + 2\,\bar{C}^{(3)}(t,\tau) - 2\,\bar{C}^{(2)}(t,\tau)\,\bar{C}^{(1)}(t,\tau) + \frac{2}{3}\left(\bar{C}^{(1)}(t,\tau)\right)^3$$

valid to all orders in ε

with 3-loop trajectory

$$\alpha^{(3)} = \frac{44\zeta_4}{3\epsilon} + \frac{40}{3}\zeta_2\zeta_3 + 16\zeta_5 + \mathcal{O}(\epsilon)$$

3-loop coefficient function

$$C^{(3)}(t,\tau) = C^{(2)}(t,\tau)\,C^{(1)}(t,\tau) - \frac{1}{3}\left[C^{(1)}(t,\tau)\right]^3$$
$$- \frac{44}{9}\frac{\zeta_4}{\epsilon^2} - \left(\frac{40}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5 + \frac{22}{3}\zeta_4 \ln\frac{-t}{\tau}\right)\frac{1}{\epsilon}$$
$$+ \frac{3982}{27}\zeta_6 - \frac{68}{9}\zeta_3^2 - \left(8\zeta_5 + \frac{20}{3}\zeta_2\zeta_3\right)\ln\frac{-t}{\tau} + \mathcal{O}(\epsilon)$$

Glover VDD 08

where $C^{(1)}(t,\tau,\epsilon)$ must be known at least through $\mathcal{O}(\epsilon^4)$

$C^{(2)}(t,\tau,\epsilon)$ $\mathcal{O}(\epsilon^2)$

BDS ansatz and 3-loop Regge factorisation

from BDS's iteration formula for the 3-loop 4-point amplitude and Regge factorisation, we get iteration formulae for the 3-loop Regge trajectory and coefficient function

$$\alpha^{(3)}(\epsilon) = 4\, f^{(3)}(\epsilon)\, \alpha^{(1)}(3\epsilon) + \mathcal{O}(\epsilon)$$

$$\begin{aligned} C^{(3)}(t,\tau,\epsilon) &= C^{(2)}(t,\tau,\epsilon)\, C^{(1)}(t,\tau,\epsilon) - \frac{1}{3}\left[C^{(1)}(t,\tau,\epsilon)\right]^3 \\ &+ \frac{4\, G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, C^{(1)}(t,\tau,3\epsilon) + 4\, Const^{(3)} + \mathcal{O}(\epsilon) \end{aligned}$$

with

$$f^{(3)}(\epsilon) = \frac{11}{2}\zeta_4 + (6\zeta_5 + 5\zeta_2\zeta_3)\epsilon + (c_1\zeta_6 + c_2\zeta_3^2)\epsilon^2$$

$$Const^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2$$

with c_1 and c_2 known constants (which drop out of the recursive formula above)

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$$\alpha^{(3)}(\epsilon) = 4\, f^{(3)}(\epsilon)\, \alpha^{(1)}(3\epsilon) + \mathcal{O}(\epsilon)$$

$$\begin{aligned} C^{(3)}(t,\tau,\epsilon) &= C^{(2)}(t,\tau,\epsilon)\, C^{(1)}(t,\tau,\epsilon) - \frac{1}{3}\left[C^{(1)}(t,\tau,\epsilon)\right]^3 \\ &+ \frac{4\, G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, C^{(1)}(t,\tau,3\epsilon) + 4\, Const^{(3)} + \mathcal{O}(\epsilon) \end{aligned}$$

with

$$f^{(3)}(\epsilon) = \frac{11}{2}\zeta_4 + (6\zeta_5 + 5\zeta_2\zeta_3)\epsilon + (c_1\zeta_6 + c_2\zeta_3^2)\epsilon^2$$

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with c_1 and c_2 known constants (which drop out of the recursive formula above)

To $\mathcal{O}(\epsilon^0)$, the BDS iteration formulae above are in agreement with the Regge formulae of the previous slide

Regge factorisation is valid also for amplitudes with 5 or more points in generalised Regge limits.

The strategy is to use the modular form of the amplitudes dictated by high-energy factorisation, to obtain information on n-point amplitudes in terms of building blocks derived from m-point amplitudes, with $m < n$

Regge factorisation of the 5-pt amplitude

5-pt amplitude $g_1 g_2 \to g_3 g_4 g_5$ in the multi-Regge limit $s \gg s_1, s_2 \gg -t_1, -t_2$

$$m_5 = s\,[g\,C(p_2, p_3, \tau)]\;\frac{1}{t_2}\;\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}\;[g\,V(q_2, q_1, \kappa, \tau)]\;\frac{1}{t_1}\;\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}\;[g\,C(p_1, p_5, \tau)]$$

gluon-production vertex

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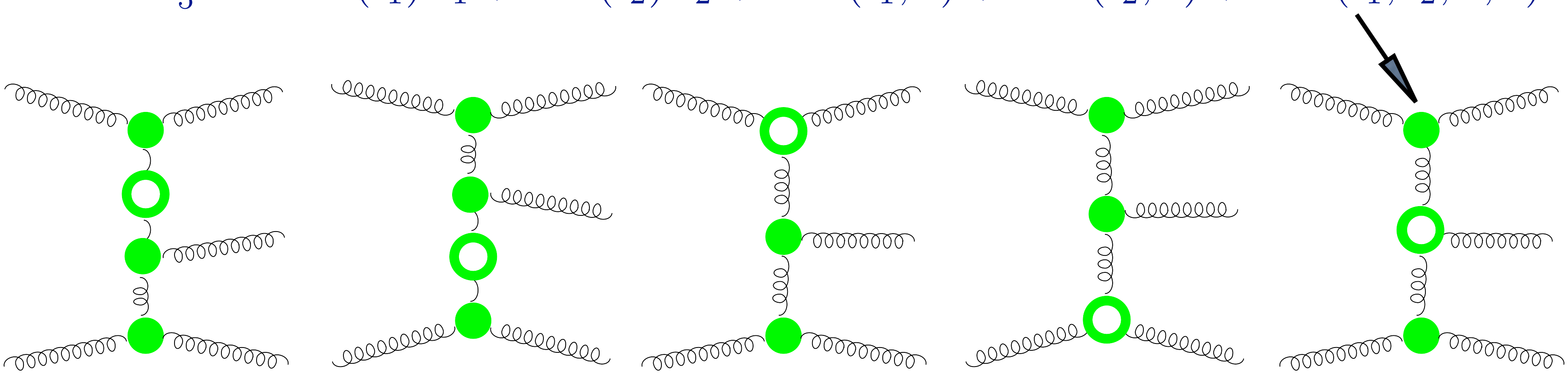
$$m_5 = s \left[g\, C(p_2, p_3, \tau)\right] \frac{1}{t_2} \left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)} \left[g\, V(q_2, q_1, \kappa, \tau)\right] \frac{1}{t_1} \left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)} \left[g\, C(p_1, p_5, \tau)\right]$$

gluon-production vertex

1 loop

$$m_5^{(1)} = \bar{\alpha}^{(1)}(t_1) L_1 + \bar{\alpha}^{(1)}(t_2) L_2 + \bar{C}^{(1)}(t_1, \tau) + \bar{C}^{(1)}(t_2, \tau) + \bar{V}^{(1)}(t_1, t_2, \kappa, \tau)$$



Regge factorisation of the 5-pt amplitude

5-pt amplitude $g_1 g_2 \to g_3 g_4 g_5$ in the multi-Regge limit $s \gg s_1, s_2 \gg -t_1, -t_2$

$$m_5 = s\left[g\, C(p_2,p_3,\tau)\right] \frac{1}{t_2} \left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)} \left[g\, V(q_2,q_1,\kappa,\tau)\right] \frac{1}{t_1} \left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)} \left[g\, C(p_1,p_5,\tau)\right]$$

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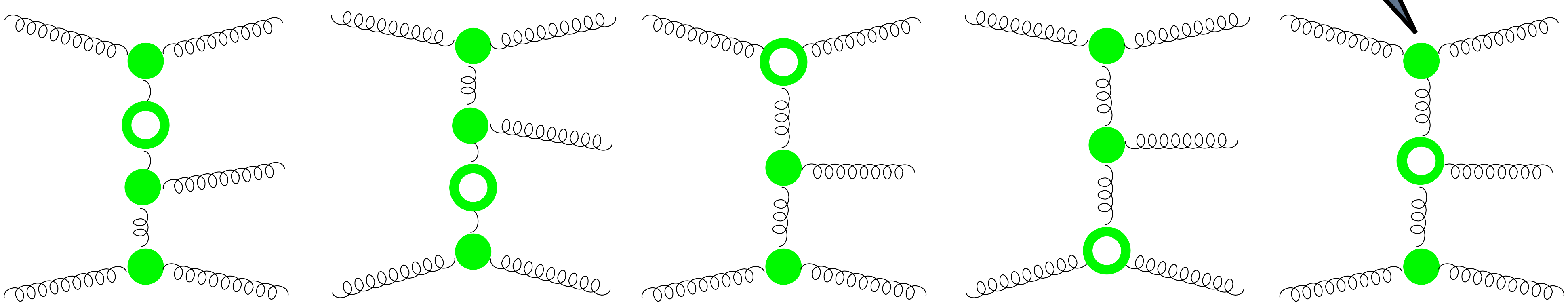

2 loops

$$\begin{aligned} m_5^{(2)} &= \frac{1}{2}\left(m_5^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t_1)L_1 + \bar{\alpha}^{(2)}(t_2)L_2 \\ &+ \bar{C}^{(2)}(t_1,\tau) + \bar{V}^{(2)}(t_1,t_2,\kappa,\tau) + \bar{C}^{(2)}(t_2,\tau) \\ &- \frac{1}{2}\left(\bar{C}^{(1)}(t_1,\tau)\right)^2 - \frac{1}{2}\left(\bar{V}^{(1)}(t_1,t_2,\kappa,\tau)\right)^2 - \frac{1}{2}\left(\bar{C}^{(1)}(t_2,\tau)\right)^2 \end{aligned}$$

where $m_5^{(1)}$ must be known at least through $\mathcal{O}(\epsilon^2)$

BDS ansatz and Regge limit for the 5-pt amplitude

Using the BDS and Regge 2-loop iteration formula for the 5-pt amplitude $m_5^{(2)}$ and the iteration formulae for the trajectory and the coefficient functions, one obtains a 2-loop iteration formula for the gluon-production vertex

$$V^{(2)}(t_1, t_2, \kappa, \tau, \epsilon) = \frac{1}{2}\left[V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, V^{(1)}(t_1, t_2, \kappa, \tau, 2\epsilon) + \mathcal{O}(\epsilon)$$

Duhr Glover VDD 08

where $V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

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Duhr Glover VDD 08

where $V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

Similarly, at 3 loops

$$\begin{aligned} V^{(3)}(t_1,t_2,\kappa,\tau,\epsilon) &= V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon)V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon) - \frac{1}{3}\left[V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)\right]^3 \\ &+ \frac{4\,G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, V^{(1)}(t_1,t_2,\kappa,\tau,3\epsilon) + \mathcal{O}(\epsilon) \end{aligned}$$

where $V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^4)$

$V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon)$ $\mathcal{O}(\epsilon^2)$

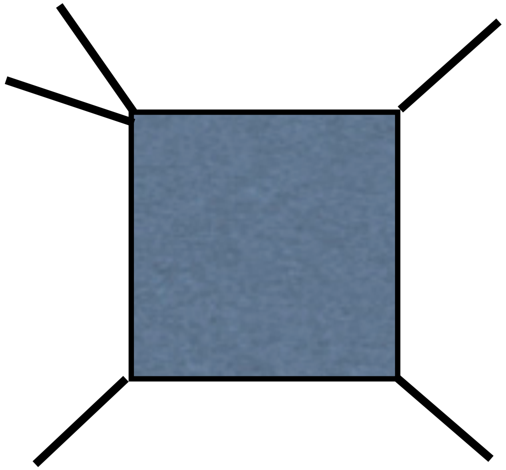
1-loop 5-pt amplitude

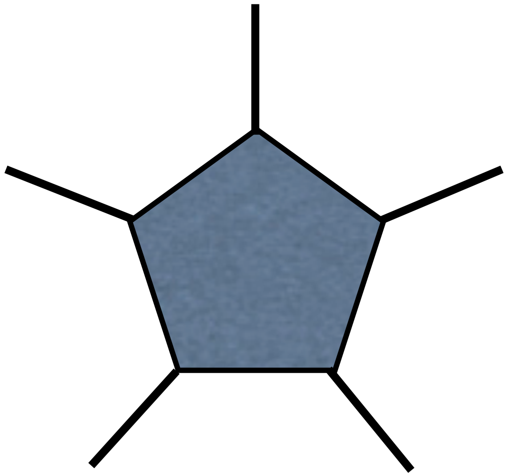
Bern Dixon Dunbar Kosower 97

$$m_5^{(1)} = -\frac{1}{4}\sum_{\text{cyclic}} s_{12}\, s_{23}\, I_4^{1m}(1,2,3,45,\epsilon) - \frac{\epsilon}{2}\,\epsilon_{1234}\, I_5^{6-2\epsilon}(\epsilon)$$

parity-even and $O(\varepsilon^{-2})$ parity-odd and $O(\varepsilon)$

$$\epsilon_{1234} = \mathrm{tr}[\gamma_5 \not{k}_1 \not{k}_2 \not{k}_3 \not{k}_4]$$





one-mass boxes known to all orders in ε

(6-2ε)-dim pentagon IR finite, but irreducible, and unknown analytically

1-loop 5-pt amplitude computed through $O(\varepsilon^2)$ numerically

Cachazo Spradlin Volovich 06
Bern Czakon Kosower Roiban Smirnov 06

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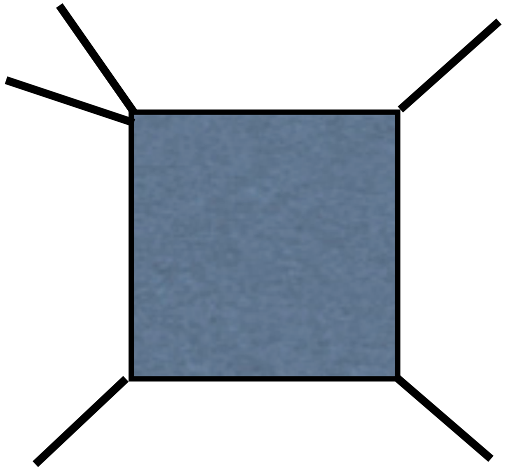

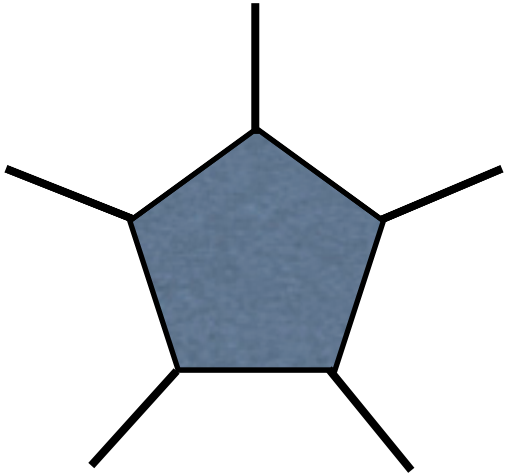

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in multi-Regge kinematics, we have computed analytically Duhr Glover Smirnov VDD 09
the 1-loop 5-pt amplitude to all orders in ε, expanded through $O(\varepsilon^2)$

Negative-dimension (NDIM) method

1-loop *n*-pt (massless) integral in D=d-2ε dimensions

$$I_n^D\Big(\{\nu_i\};\{Q_i^2\}\Big) = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \prod_{i=1}^{n} \frac{1}{D_i^{\nu_i}}$$

$$D_1 = k^2 + i0$$

$$D_i = \left(k + \sum_{j=1}^{i-1} k_i\right)^2 + i0$$

$$Q_i^2 = s_{i,i+1}$$

Schwinger parametrization

$$\frac{1}{D_i^{\nu_i}} = \frac{(-1)^{\nu_1}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \, e^{\alpha_i \, D_i}$$

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Halliday Ricotta 87
Suzuki A.G.M. Schmidt 97
Anastasiou Glover Oleari 99

Gaussian integral is an analytic function of *D*

$$\int \frac{\mathrm{d}^D k}{i\pi^{D/2}}\, e^{\alpha k^2} = \frac{1}{\alpha^{D/2}} = \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} \int \frac{\mathrm{d}^D k}{i\pi^{D/2}}\, \left(k^2\right)^m$$

for *D* < 0, define $\int \frac{\mathrm{d}^D k}{i\pi^{D/2}}\, \left(k^2\right)^m = m!\, \delta_{m+\frac{D}{2},0}$

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- expand the exponential before and after the loop integration
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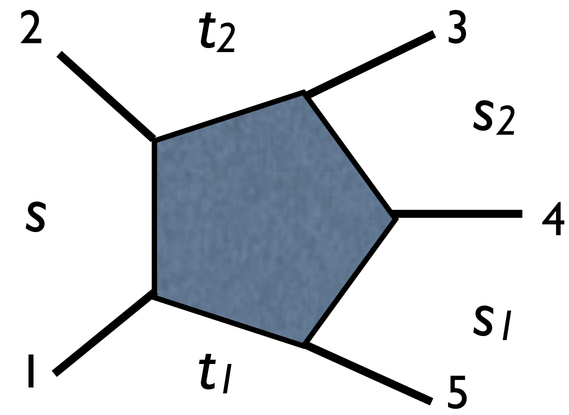
⟹ get the loop integral as a polynomial in multiple infinite sums

Pentagon integral in NDIM

$s \equiv s_{12}, \quad t_2 \equiv s_{23}, \quad s_2 \equiv s_{34}, \quad s_1 \equiv s_{45}, \quad t_1 \equiv s_{51}$



- the (6-2ε)-dim pentagon integral is written in terms of quadruple sums, functions of

$$\frac{s_2}{s}, \quad \frac{s_1 s_2}{s t_2}, \quad \frac{s_1 t_1}{s t_2}, \quad \frac{t_1}{t_2}$$

we don't know (yet) how to handle those quadruple sums

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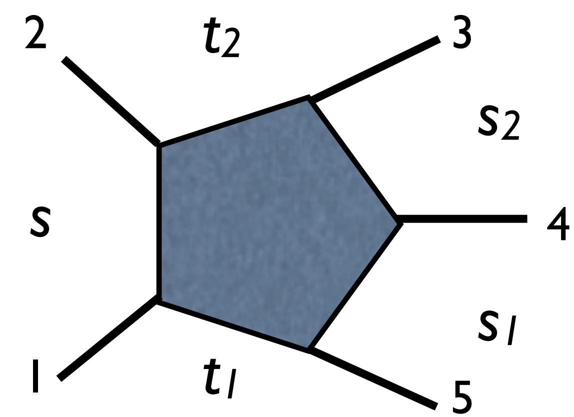

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- multi-Regge kinematics (in Euclidean region)

$$-s \gg -s_1, -s_2 \gg -t_1, -t_2$$

$$s_1 \to \lambda s_1, \quad s_2 \to \lambda s_2, \quad t_1 \to \lambda^2 t_1, \quad t_2 \to \lambda^2 t_2, \qquad \lambda \ll 1$$

$$\frac{s_2}{s}, \frac{s_1 t_1}{s t_2} = O(\lambda) \qquad \frac{s_1 s_2}{s t_2}, \frac{t_1}{t_2} = O(1)$$

to all orders in ε, the pentagon integral is reduced to double sums, functions of $\frac{s_1 s_2}{s t_2}, \frac{t_1}{t_2}$

double sums

Appell function $$F_4(a,b,c,d;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}}{(c)_m\,(d)_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}$$

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Kampé de Fériet (KdF) function

$$F^{p,q}_{p',q'}\left(\begin{array}{c|cc|} \alpha_i & \beta_j & \gamma_j \\ \alpha'_k & \beta'_\ell & \gamma'_\ell \end{array} x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_i (\alpha_i)_{m+n}\,\prod_j (\beta_j)_m\,(\gamma_j)_n}{\prod_k (\alpha'_k)_{m+n}\,\prod_\ell (\beta'_\ell)_m\,(\gamma'_\ell)_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!},$$

$$1 \le i \le p, \quad 1 \le j \le q, \quad 1 \le k \le p', \quad 1 \le \ell \le q'$$

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particular case $$F^{2,0}_{0,1}\left(\begin{array}{cc|cc|}a & b & - & - \\ - & - & c & d\end{array}\,x,y\right) = F_4(a,b,c,d;x,y)$$

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examples

$$F^{0,3}_{2,0}\left(\begin{array}{cc|cccccc|c} - & - & 1 & 1 & 1 & 1 & 1-\epsilon & 1\pm\epsilon & \\ 2 & 2\pm\epsilon & - & - & - & - & - & - & -\frac{s\,t_1}{s_1 s_2},\frac{t_1}{t_2} \end{array}\right)$$

$$\frac{\partial}{\partial\delta}F^{2,1}_{0,2}\left(\begin{array}{cc|cccc|c} 1+\delta & 1-\epsilon+\delta & - & - & - & 1 & \\ - & - & 1\pm\epsilon & 1\mp\epsilon+\delta & - & 1+\delta & -\frac{s\,t_1}{s_1 s_2},-\frac{s\,t_2}{s_1 s_2} \end{array}\right)_{|\delta=0}$$

$$\frac{\partial}{\partial\delta}F^{2,1}_{0,2}\left(\begin{array}{cc|cccc|c} 1+\delta & 1+\delta\pm\epsilon & 1 & - & - & - & \\ - & - & 1+\delta & 1\pm\epsilon & 1+\epsilon+\delta & - & -\frac{s_1 s_2}{s\,t_2},\frac{t_1}{t_2} \end{array}\right)_{|\delta=0}$$

after quite a bit of work (mostly Claude's), we were able to expand the KdF functions into *M* functions

$$I_5^{6-2\epsilon} = c_0 + c_1\epsilon + O(\epsilon^2)$$

c_0, c_1 polynomials (of uniform transcendentality) of the *M* functions

$$\mathcal{M}(\vec{\imath}, \vec{\jmath}, \vec{k}; x_1, x_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \binom{n_1+n_2}{n_1}^2 S_{\vec{\imath}}(n_1)\, S_{\vec{\jmath}}(n_2)\, S_{\vec{k}}(n_1+n_2)\, x_1^{n_1}\, x_2^{n_2}$$

nested harmonic sums

$$S_i(n) = \sum_{k=1}^{n} \frac{1}{k^i}$$

$$S_{i\vec{\jmath}}(n) = \sum_{k=1}^{n} \frac{S_{\vec{\jmath}}(k)}{k^i}$$

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in some cases, *i.e.* whenever the parent KdF function can be reduced to an Appell F_4 function, the *M* functions reduce to logs

Pentagon integral through Mellin-Barnes

a 4-fold integral in general kinematics

Bern Czakon Kosower Roiban Smirnov 06

$$I_5^{6-2\epsilon}(Q_i^2) =$$

$$\frac{-1}{\Gamma(1-2\epsilon)} \int_{-i\infty}^{+i\infty} \prod_{i=1}^{4} \frac{\mathrm{d}z_i}{2\pi i} \Gamma(-z_i) \, (-s)^{-2-\epsilon} \left(\frac{s_1}{s}\right)^{z_4} \left(\frac{s_2}{s}\right)^{z_1} \left(\frac{t_1}{s}\right)^{z_2} \left(\frac{t_2}{s}\right)^{z_3}$$

$$\times \Gamma(z_1 + z_2 + 1) \, \Gamma(-\epsilon - z_1 - z_2 - z_3 - 1) \, \Gamma(z_2 + z_3 + 1)$$

$$\times \Gamma(-\epsilon - z_2 - z_3 - z_4 - 1) \, \Gamma(z_3 + z_4 + 1) \, \Gamma(\epsilon + z_1 + z_2 + z_3 + z_4 + 2)$$

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$$\times \Gamma(z_1+z_2+1)\,\Gamma(-\epsilon-z_1-z_2-z_3-1)\,\Gamma(z_2+z_3+1)$$
$$\times \Gamma(-\epsilon-z_2-z_3-z_4-1)\,\Gamma(z_3+z_4+1)\,\Gamma(\epsilon+z_1+z_2+z_3+z_4+2)$$

- in multi-Regge kinematics, it reduces to a 2-fold integral example

$$I_5^{6-2\epsilon}(Q_i^2) = \frac{-(-s)^{-\epsilon}(-s_1)^{\epsilon}(-s_2)^{\epsilon}}{s_1 s_2 \Gamma(1-2\epsilon)}$$
$$\times \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}z_1}{2\pi i} \frac{\mathrm{d}z_2}{2\pi i} \left(\frac{s\,t_1}{s_1 s_2}\right)^{z_1} \left(\frac{s\,t_2}{s_1 s_2}\right)^{z_2} \Gamma(-\epsilon-z_1)\,\Gamma(-z_1)\,\Gamma(-\epsilon-z_2)$$
$$\times \Gamma(-\epsilon-z_1-z_2)\,\Gamma(-z_2)\,\Gamma(z_1+z_2+1)\,\Gamma(\epsilon+z_1+z_2+1)^2$$

with $\sqrt{\frac{s\,t_1}{s_1 s_2}} + \sqrt{\frac{s\,t_2}{s_1 s_2}} < 1$

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$$\times\Gamma(z_1+z_2+1)\,\Gamma(-\epsilon-z_1-z_2-z_3-1)\,\Gamma(z_2+z_3+1)$$
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- in multi-Regge kinematics, it reduces to a 2-fold integral example

$$I_5^{6-2\epsilon}(Q_i^2) = \frac{-(-s)^{-\epsilon}(-s_1)^{\epsilon}(-s_2)^{\epsilon}}{s_1 s_2\,\Gamma(1-2\epsilon)}$$
$$\times \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i} \left(\frac{s\,t_1}{s_1 s_2}\right)^{z_1} \left(\frac{s\,t_2}{s_1 s_2}\right)^{z_2} \Gamma(-\epsilon-z_1)\,\Gamma(-z_1)\,\Gamma(-\epsilon-z_2)$$
$$\times\Gamma(-\epsilon-z_1-z_2)\,\Gamma(-z_2)\,\Gamma(z_1+z_2+1)\,\Gamma(\epsilon+z_1+z_2+1)^2$$

with $\sqrt{\frac{s\,t_1}{s_1 s_2}} + \sqrt{\frac{s\,t_2}{s_1 s_2}} < 1$

- which, after taking residues, agrees with the NDIM result

Gluon-production vertex

1-loop gluon-production vertex, needed at least through $O(\varepsilon^2)$

$$V_e^{(1)}(t_1,t_2,\tau,\kappa) = m_{5e}^{(1)}(1,2,3,4,5) - \bar{\alpha}^{(1)}(t_1)L_1 - \bar{\alpha}^{(1)}(t_2)L_2 - \bar{C}^{(1)}(t_1,\tau) - \bar{C}^{(1)}(t_2,\tau)$$
$$V_o^{(1)}(t_1,t_2,\tau,\kappa) = m_{5o}^{(1)}(1,2,3,4,5)$$

notice that coefficient functions and Regge trajectory are parity-even,
so the odd part of the amplitude equals the odd part of the gluon-production vertex

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Through the BDS ansatz, the 2-loop gluon-production vertex is

$$
\begin{aligned}
V_e^{(2)}(\epsilon) &= \frac{1}{2}\left[V_e^{(1)}(\epsilon)\right]^2 + \frac{2G^2(\epsilon)}{G(2\epsilon)}\, f^{(2)}(\epsilon)\, V_e^{(1)}(2\epsilon) + \mathcal{O}(\epsilon) \\
V_o^{(2)}(\epsilon) &= V_e^{(1)}(\epsilon) V_o^{(1)}(\epsilon) + \mathcal{O}(\epsilon)
\end{aligned}
$$

Regge factorisation of the 6-pt amplitude

6-pt amplitude $g_1 g_2 \to g_3 g_4 g_5 g_6$

in the multi-Regge limit $y_3 \gg y_4 \gg y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq | p_{5\perp}| \simeq | p_{6\perp}|$

$$s \gg s_1, s_2, s_3 \gg -t_1, -t_2, -t_3$$

$$m_6 = s\,[g\,C(p_2, p_3, \tau)]\,\frac{1}{t_3}\left(\frac{-s_3}{\tau}\right)^{\alpha(t_3)}[g\,V(q_2, q_3, \kappa_2, \tau)]$$
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The *l*-loop 6-pt amplitude can then be assembled using the *l*-loop trajectories, gluon-production vertices and coefficient functions, which can be determined through the *l*-loop 4-pt and 5-pt amplitudes

Thus, also the *l*-loop BDS iterative formula for *n* = 6 will be fulfilled

⇒ the multi-Regge limit is not able to detect the BDS-ansatz violation for *n* = 6

Remainder function

the remainder function of the 6-pt amplitude depends on
3 conformally-invariant cross-ratios

Drummond Henn Korchemsky Sokatchev 07

$$R_6^{(2)} = R_6^{(2)}(u_1.u_2, u_3)$$

$$u_1 = \frac{s_{12}\, s_{45}}{s_{345}\, s_{456}}, \qquad u_2 = \frac{s_{23}\, s_{56}}{s_{234}\, s_{456}}, \qquad u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}}$$

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in the multi-Regge kinematics

$$u_1 = 1 + \mathcal{O}\left(\frac{t}{s}\right)\,, \qquad u_2 = \mathcal{O}\left(\frac{t}{s}\right)\,, \qquad u_3 = \mathcal{O}\left(\frac{t}{s}\right)$$

like in the collinear limit

1-loop 6-pt amplitude

- computed through $O(\varepsilon^2)$ numerically

  even Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
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through $O(\varepsilon^0)$, it is given in terms of 1*m* and 2*me* boxes
at $O(\varepsilon)$ a hexagon occurs in the even part

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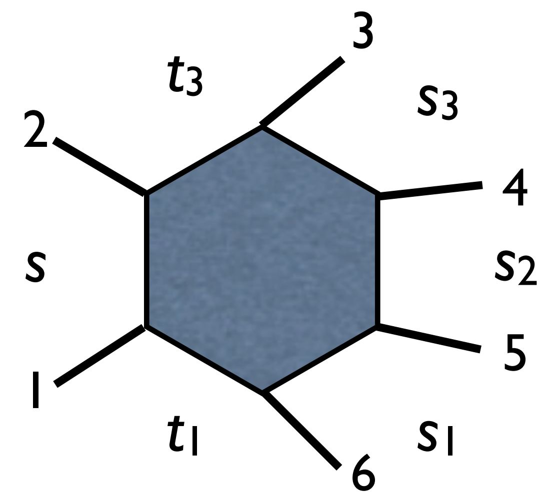

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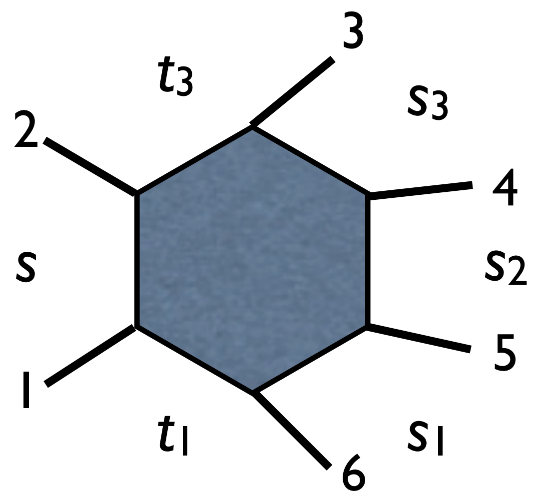
- multi-Regge kinematics (in Euclidean region)

$$-s \gg -s_1, -s_2, -s_3 \gg -t_1, -t_2, -t_3$$

$$s_1 \to \lambda^2 s_1, \quad s_2 \to \lambda^2 s_2, \quad s_3 \to \lambda^2 s_3, \quad t_1 \to \lambda^3 t_1, \quad t_3 \to \lambda^3 t_3, \qquad \lambda \ll 1$$



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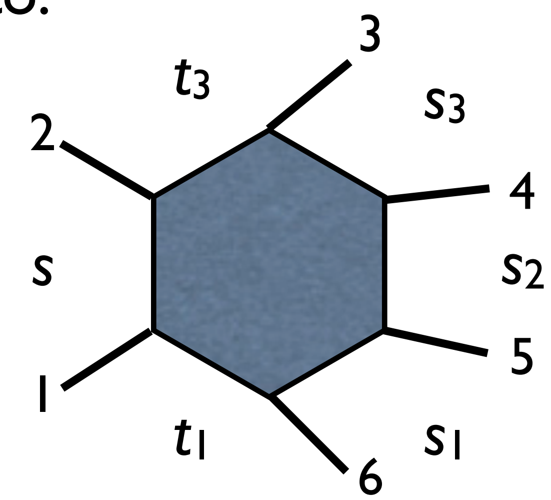
- to all orders in ε, the hexagon integral is reduced to:
  triple sums in NDIM,
  3-fold integrals through Mellin-Barnes



neglecting the (so far unknown) hexagon,
the 1-loop 6-pt amplitude is given through 1*m* and *2me* boxes

to all orders in ε, the *2me* boxes are given in terms of $_2F_1$ functions

Brandhuber Spence Travaglini 05

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Brandhuber Spence Travaglini 05

in the multi-Regge kinematics (in Euclidean region),

$$s_{345} \to \lambda s_{345}, \quad s_{456} \to \lambda s_{456}, \qquad 1 - \frac{s_{345} s_{456}}{s\, s_2} = \mathcal{O}(\lambda)$$

$$|p_{4\perp} + p_{5\perp}|^2 = s_{45} \left(1 - \frac{s\, s_2}{s_{345} s_{456}}\right) = \mathcal{O}(\lambda^3)$$

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the leading contribution from the 2*me* boxes yields

$$\frac{1}{\epsilon^2} \left(\frac{-s_2}{\mu^2} \right)^{-\epsilon} \left[1 - {}_2F_1\left(1, \epsilon, 1+\epsilon; 1 - \Phi\right) \right], \qquad \Phi = \frac{s_{345} s_{456}}{s\, s_2}$$

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$$\lim_{\Phi \to 1} {}_2F_1 \left(1, \epsilon, 1 + \epsilon; 1 - \Phi \right) = 1$$

in the Euclidean region, the 2*me*-box contribution vanishes
it also vanishes where the *s*-type invariants are > 0, the *t*-type invariants are < 0

in the region where $s > 0, \quad s_2 > 0, \quad s_{345} < 0, \quad s_{456} < 0$

the analytic continuation of s, s_2 is $(-s_2) \to e^{-i\pi} s_2, \quad (-s) \to e^{-i\pi} s$

then $\Phi \to e^{-2i\pi} \Phi$

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$$ {}_2F_1(1, \epsilon, 1+\epsilon; 1 - e^{2\pi i}\Phi) = 1 - \sum_{n=1}^{\infty} (-\epsilon)^n \, \mathrm{Li}_n(1 - e^{2\pi i}\Phi) $$

the analytic properties of the polylogs are

$$ \mathrm{Im}\ \mathrm{Li}_n(1 - e^{2\pi i}\Phi) = -2\pi \frac{\ln^{n-1}(1-\Phi)}{(n-1)!} $$

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Bartels Lipatov Sabio-Vera 08

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the two limits do not commute

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we claim that the multi-Regge limit Φ → 1 should be taken first

Duhr Glover VDD 08

Regge factorisation of the *n*-pt amplitude

$$m_n(1,2,\ldots,n) = s\,[g\,C(p_2,p_3)]\,\frac{1}{t_{n-3}}\left(\frac{-s_{n-3}}{\tau}\right)^{\alpha(t_{n-3})}[g\,V(q_{n-3},q_{n-4},\kappa_{n-4})]$$

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n-pt amplitude in the multi-Regge limit

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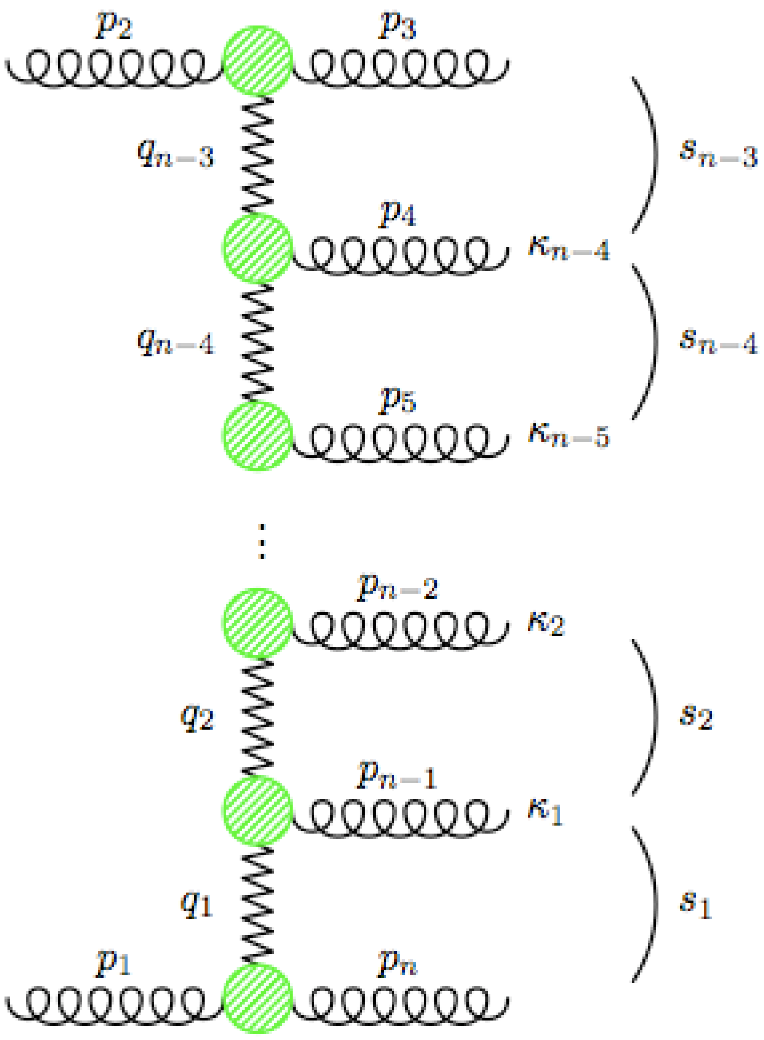

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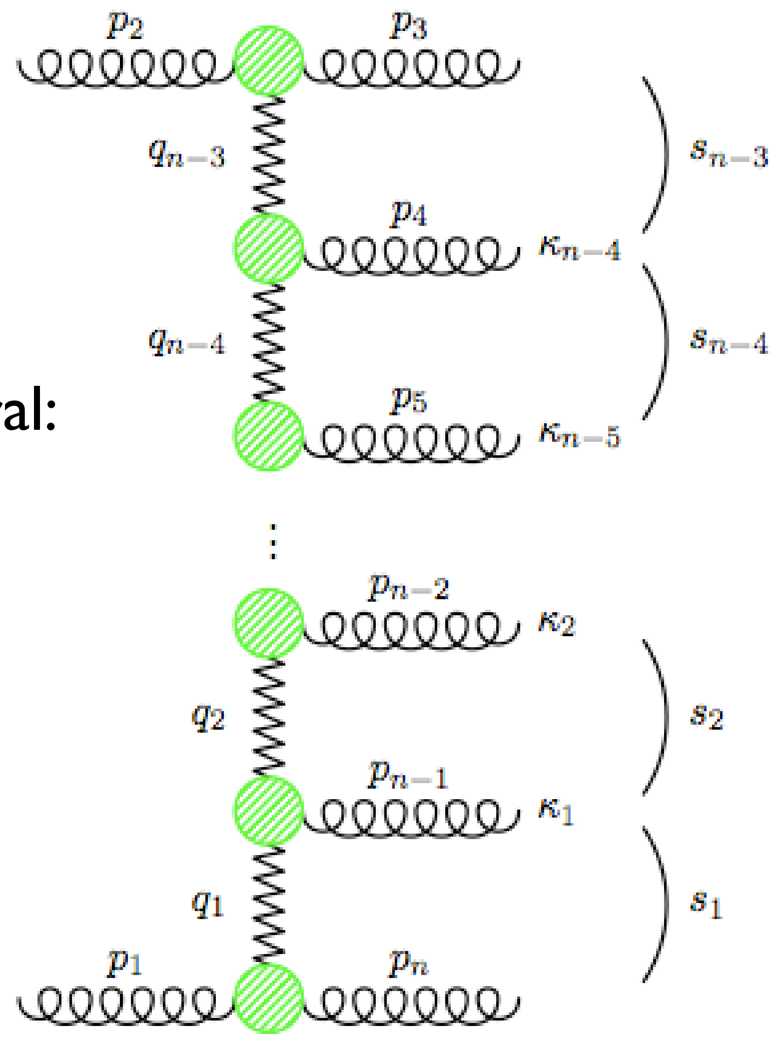


What we said for *n* = 6 can be repeated in general: the *l*-loop *n*-pt amplitude can be assembled using the *l*-loop trajectories, vertices and coefficient functions, determined through the *l*-loop 4-pt and 5-pt amplitudes

⇒ no violation of the BDS ansatz can be found in the multi-Regge limit

To have a chance to detect the violation of the BDS ansatz for the 2-loop 6-pt amplitude, that we see in arbitrary kinematics, we must relax the strong-ordering constraints of the multi-Regge kinematics

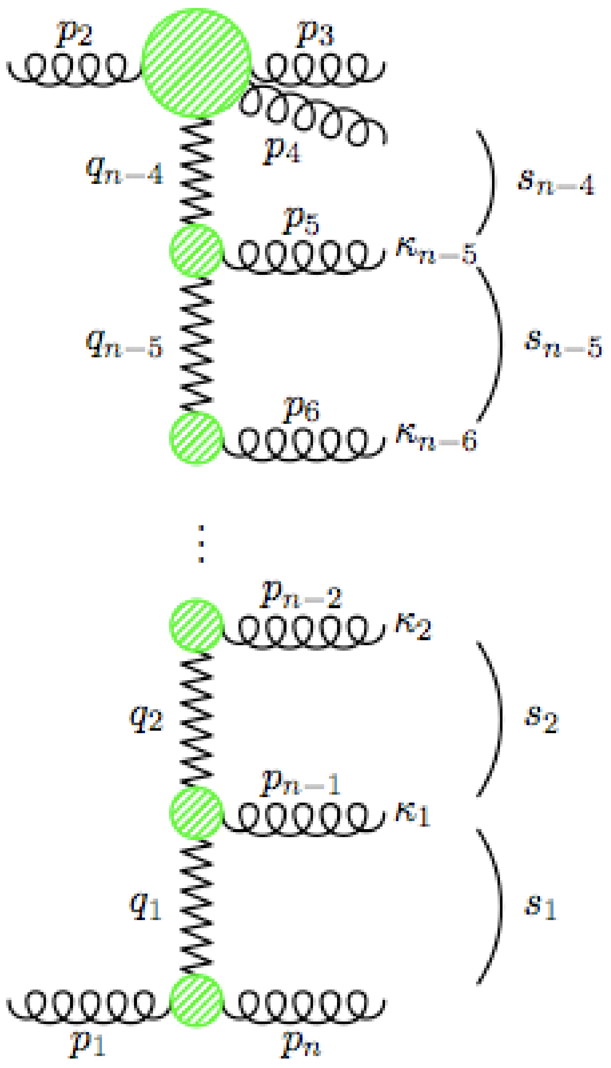
n-pt amplitude in quasi-multi-Regge kinematics

$$m_n(1,2,\ldots,n) = s\,[g^2\,A(p_2,p_3,p_4)]\;\frac{1}{t_{n-4}}\left(\frac{-s_{n-4}}{\tau}\right)^{\alpha(t_{n-4})}[g\,V(q_{n-4},q_{n-5},\kappa_{n-5})]$$

$$\cdots\times\;\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa_1)]\;\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_n)]$$

quasi-multi-Regge kinematics

$$y_3 \simeq y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|$$



n-pt amplitude in quasi-multi-Regge kinematics

$$m_n(1,2,\dots,n) = s\left[g^2 A(p_2,p_3,p_4)\right] \frac{1}{t_{n-4}} \left(\frac{-s_{n-4}}{\tau}\right)^{\alpha(t_{n-4})} \left[g\, V(q_{n-4},q_{n-5},\kappa_{n-5})\right]$$

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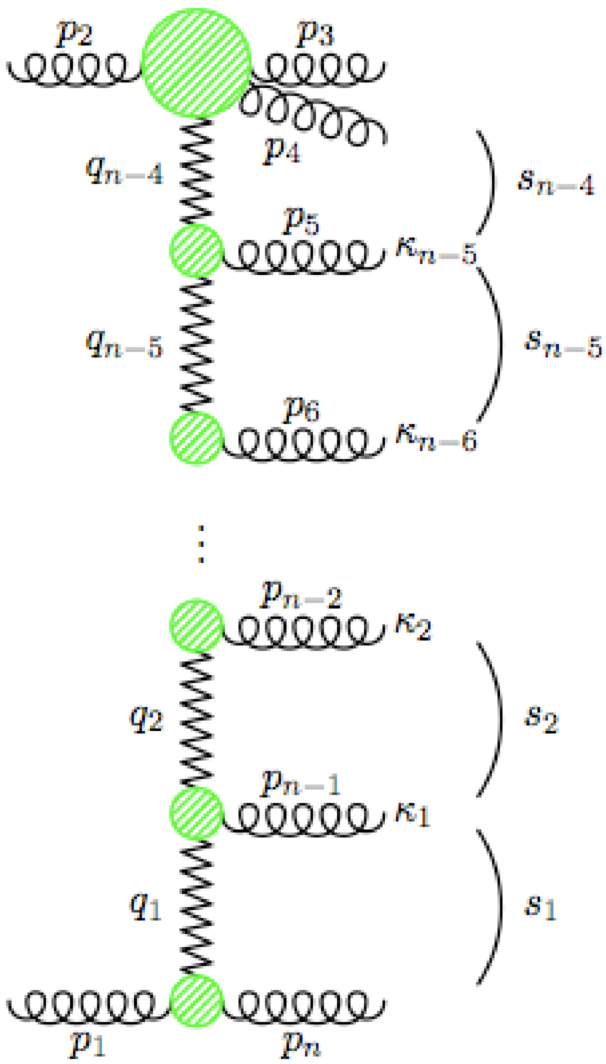
A new coefficient function $A(p_2,p_3,p_4,\tau)$ occurs already at n = 5, for which the BDS ansatz is fulfilled. Because no new coefficient functions appear for $n \geq 6$, a violation of the BDS ansatz cannot be found even in this case



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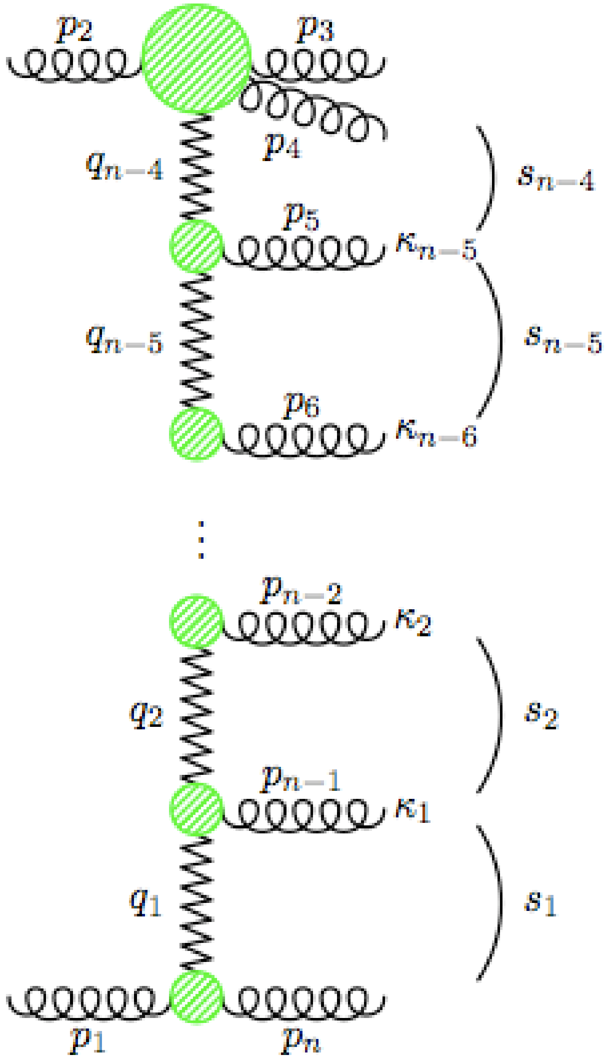

The same can be said for the quasi-multi-Regge kinematics

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the 3 conformally-invariant cross-ratios

$$u_1 = \frac{s_{12}\, s_{45}}{s_{345}\, s_{456}}, \qquad u_2 = \frac{s_{23}\, s_{56}}{s_{234}\, s_{456}}, \qquad u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}}$$

take the values

$$u_1 = 1 + \mathcal{O}\left(\frac{t}{s}\right), \qquad u_2 = \mathcal{O}\left(\frac{t}{s}\right), \qquad u_3 = \mathcal{O}\left(\frac{t}{s}\right)$$

like in the multi-Regge kinematics and in the collinear limit

More general quasi-multi-Regge kinematics

A necessary condition to see a violation of the BDS ansatz for the 2-loop 6-pt amplitude, is to go to a quasi-multi-Regge kinematics for which new coefficient functions appear for $n \geq 6$

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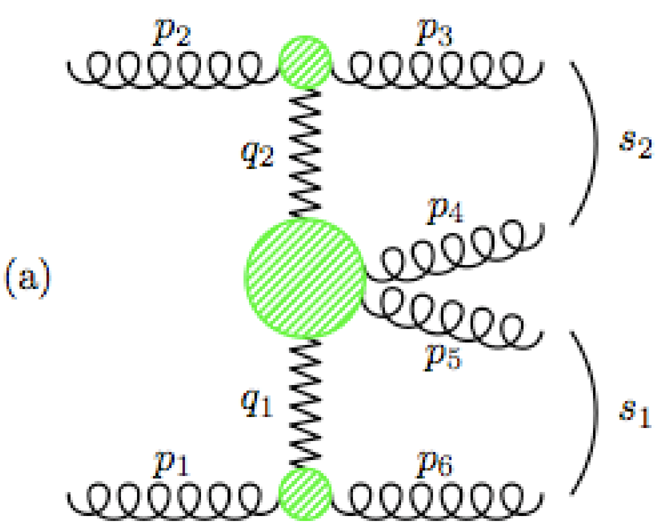

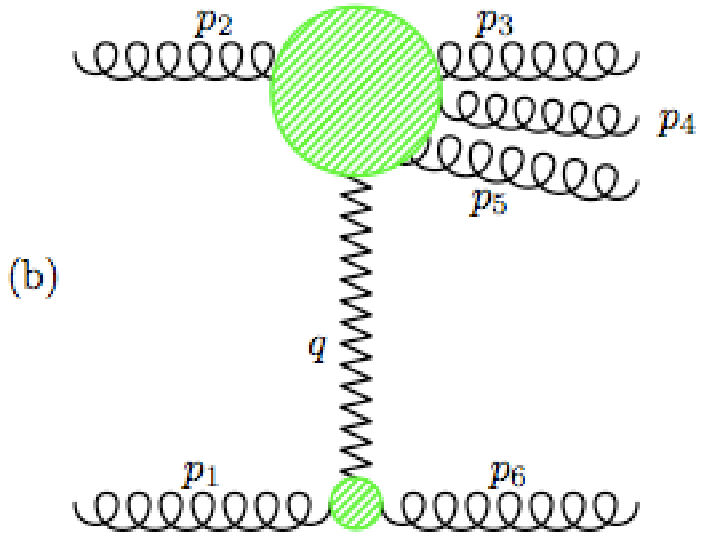

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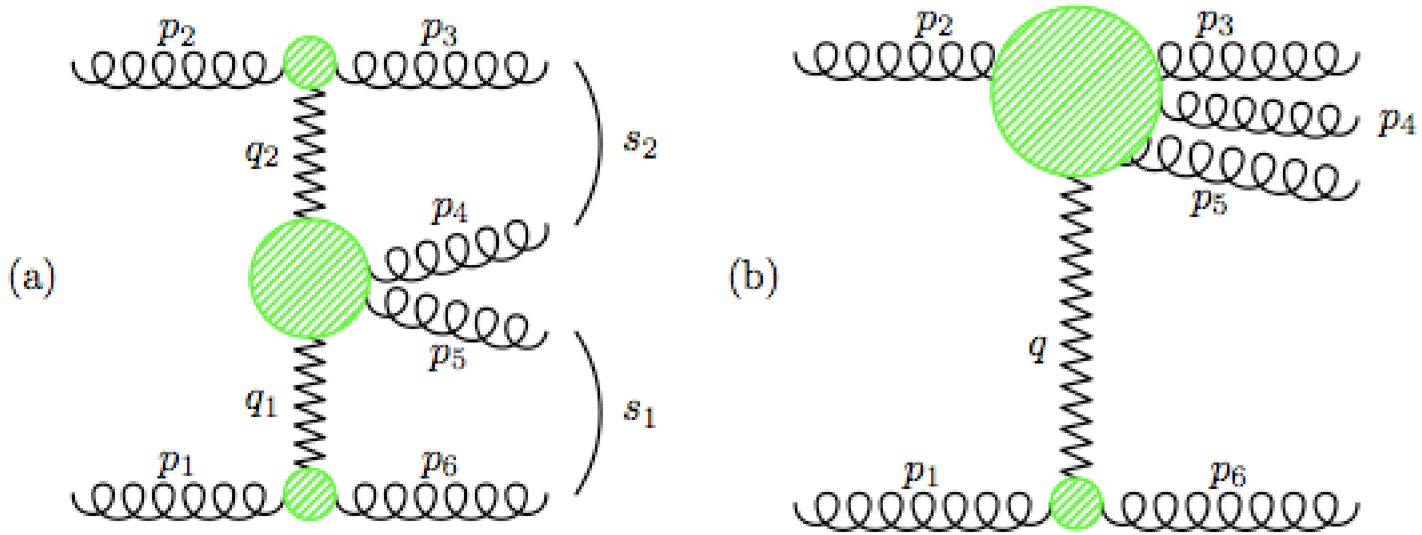

in both cases, the 3 conformally-invariant cross-ratios take values

$$u_1 = \mathcal{O}(1)\,, \qquad u_2 = \mathcal{O}(1)\,, \qquad u_3 = \mathcal{O}(1)$$

it remains to be seen if these kinematics harbour a violation of the BDS ansatz

Conclusions

- in multi-Regge kinematics, we have computed analytically the (6-2ε)-dim pentagon integral, and so the 1-loop 5-pt amplitude through $O(\varepsilon^2)$

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- the *l*-loop *n*-pt amplitude so built fulfils the BDS ansatz, thus any ansatz violation must be searched in less constraining (quasi-multi-Regge ?) kinematics