Ex. 6 : The BFKL eigenvalue is

$$\omega(\nu, n) = \frac{\alpha_s C_A}{\pi} \chi_{\nu,n}$$

with $\chi_{\nu,n} = -\psi\left(\frac{|n|+1}{2} + i\nu\right) - \psi\left(\frac{|n|+1}{2} - i\nu\right) - 2\gamma_E$

Please check that using the recursive formula

$$\psi(1+x) = \psi(x) + \frac{1}{x}$$

you obtain an equation for $\chi_{\nu,n}$ recursive over n

$$\chi_{\nu,n+2} = \chi_{\nu,n} - \frac{|n|+1}{\nu^2 + \frac{(|n|+1)^2}{4}}$$

Iterating it, you can make $\chi_{\nu,n}$ explicit for the even and odd modes

$$\chi_{\nu,2m} = \chi_{\nu,0} - \sum_{j=0}^{|m|-1} \frac{1+2j}{\nu^2 + \frac{(1+2j)^2}{4}}$$

$$\chi_{\nu,2m+1} = \chi_{\nu,1} - \sum_{j=0}^{|m|-1} \frac{2(j+1)}{\nu^2+(j+1)^2}$$

For small ν, please check that using the expansion

$$\psi(1+x) - \psi(1) = \sum_{k=2}^{\infty} (-)^k \zeta_k \, x^{k-1} \qquad \text{for } x \ll 1$$

and the doubling formula,

$$2\psi(2x) = 2\ln 2 + \psi(x) + \psi\left(x+\tfrac{1}{2}\right)$$

you obtain $\chi_{\nu,n}$ as a power series in ν

$$\chi_{\nu,n} = 2 \sum_{k=0}^{\infty} a_{kn} \nu^{2k}$$

with

$$a_{00} = 2 \ln 2 \qquad a_{k0} = (-)^k \left(2^{2k+1} - 1\right) \zeta_{2k+1}$$

$$a_{01} = 0 \qquad a_{k1} = (-)^k \zeta_{2k+1} \qquad k \neq 0$$

So

$$\chi_{\nu,0} = 2\left(2 \ln 2 - 7 \zeta_3 \nu^2 + 31 \zeta_5 \nu^4 + \cdots\right)$$

$$\chi_{\nu,1} = 2\left(-\zeta_3 \nu^2 + \zeta_5 \nu^4 + \cdots\right)$$

For $n=0$, this is the usual expansion of the BFKL eigenvalue

The recursive equation for $\chi_{\nu n}$ allows us to obtain a formula for a_{kn} recursive over n

$$a_{k\,n+2} = a_{kn} + (-)^{k+1} \left(\frac{2}{|n|+1}\right)^{2k+1}$$

For example, for $r = 0, 1, 2$ the recursion for a_{rn} yields

$$a_{0,n+2} = a_{0n} - \frac{2}{|n|+1}$$

$$a_{1,n+2} = a_{1n} + \left(\frac{2}{|n|+1}\right)^3$$

$$a_{2,n+2} = a_{2n} - \left(\frac{2}{|n|+1}\right)^5 \quad \text{and so on}$$

one can solve them explicitly, and write

$$a_{0,2n} = a_{00} - \sum_{k=1}^{n} \frac{2}{2k-1} = 2\ln 2 - \sum_{k=1}^{n} \frac{2}{2k-1}$$

$$a_{1,2n} = a_{1,0} + \sum_{k=1}^{n} \left(\frac{2}{2k-1}\right)^3 = -7\zeta_3 + \sum_{k=1}^{n} \left(\frac{2}{2k-1}\right)^3$$

$$a_{2,2n} = a_{20} - \sum_{k=1}^{n} \left(\frac{2}{2k-1}\right)^5 = 31\zeta_5 + \sum_{k=1}^{n} \left(\frac{2}{2k-1}\right)^5$$

$$a_{0,2n+1} = a_{01} - \sum_{k=1}^{n} \frac{1}{k} = -\sum_{k=1}^{n} \frac{1}{k}$$

$$a_{1,2n+1} = a_{11} + \sum_{k=1}^{n} \left(\frac{1}{k}\right)^3 = -\zeta_3 + \sum_{k=1}^{n} \left(\frac{1}{k}\right)^3$$

$$a_{2,2n+1} = a_{21} - \sum_{k=1}^{n} \left(\frac{1}{k}\right)^5 = \zeta_5 + \sum_{k=1}^{n} \left(\frac{1}{k}\right)^5 \quad \text{and so on}$$

So, for exemple

$$\chi_{v,2} = 2\left(a_{02} + a_{12} v^2 + a_{22} v^4 + \cdots\right)$$

$$= 2\left(2(\ln 2 - 1) + (-7\zeta_3 + 8)v^2 + (31\zeta_5 - 32)v^4 + \cdots\right)$$