# SYM amplitudes in the high-energy limit

Vittorio Del Duca

INFN LNF

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# Bern-Dixon-Smirnov ansatz

an ansatz for MHV amplitudes in *N*=4 SYM — Bern Dixon Smirnov 05

$$\begin{aligned} m_n &= m_n^{(0)} \left[ 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \right] \\ &= m_n^{(0)} \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right] \end{aligned}$$

coupling $a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}$ $\lambda = g^2 N$ 't Hooft parameter

$$f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \, \frac{l}{2} \, \hat{G}^{(l)} + \epsilon^2 \, f_2^{(l)} \qquad E_n^{(l)}(\epsilon) = O(\epsilon)$$

$\hat{\gamma}_K^{(l)}$ cusp anomalous dimension, known to all orders of *a* — Korchemsky Radyuskin 86, Beisert Eden Staudacher 06

$\hat{G}^{(l)}$ collinear anomalous dimension, known through O($a^4$) — Bern Dixon Smirnov 05, Cachazo Spradlin Volovich 07

# Brief history of BDS ansatz

BDS ansatz checked for the 3-loop 4-pt amplitude — Bern Dixon Smirnov 05

2-loop 5-pt amplitude — Cachazo Spradlin Volovich 06; Bern Czakon Kosower Roiban Smirnov 06

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Hints of break-up from strong-coupling expansion — Alday Maldacena 07

hexagon Wilson loop — Drummond Henn Korchemsky Sokatchev 07

multi-Regge limit (?) — Bartels Lipatov Sabio-Vera 08

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The BDS ansatz implies an iteration formula
for the 2-loop *n*-pt amplitude $m_n^{(2)}$ (rescaled by the tree amplitude)

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + \mathcal{O}(\epsilon)$$

Anastasiou Bern Dixon Kosower 03

The remainder function characterises the deviation from the ABDK/BDS iteration

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - Const^{(2)}$$

# Why ?

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Alday Maldacena
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What is the remainder function ?

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Alday Maldacena
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# How ?

What is the remainder function ?

we are trying to move forward analytically

Duhr Glover Smirnov VDD 09

# MHV amplitudes ⇔ Wilson loops

agreement between $n$-edged Wilson loop and $n$-point MHV amplitude, verified for

Alday Maldacena 07

$n$-edged 1-loop Wilson loop — Brandhuber Heslop Travaglini 07

6-edged 2-loop Wilson loop — Drummond Henn Korchemsky Sokatchev 07; Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

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Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

7-edged & 8-edged 2-loop Wilson loops also computed (numerically)

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

if agreement holds up to 8-edged 2-loop Wilson loops,
then $R_7^{(2)}, R_8^{(2)}$ are known numerically

$R_n^{(2)}$ unknown analytically,
but functions of conformally-invariant cross-ratios

Drummond Henn Korchemsky Sokatchev 07

# Ward identities & Wilson loops

Drummond Henn Korchemsky Sokatchev 07

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Drummond Henn Korchemsky Sokatchev 07

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Drummond Henn Korchemsky Sokatchev 07

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  for *n* ≥ 6, *R* is an unknown function of conformally invariant cross ratios
- for *n* = 6, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

with $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$

# Colour decomposition of the tree *n*-point amplitude

$$\mathcal{M}_n^{(0)} = 2^{n/2}\, g^{n-2} \sum_{S_n/Z_n} \mathrm{tr}(T^{d_1} \cdots T^{d_n})\, m_n^{(0)}(1,\ldots,n)$$

$m_n^{(0)}(1,2,\ldots,n)$ colour-stripped amplitude

MHV amplitude $$m_n^{(0)}(1,2,\ldots,n) = \frac{\langle p_i p_j \rangle^4}{\langle p_1 p_2 \rangle \cdots \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

# Regge factorisation of the 4-pt amplitude

colour-stripped 4-pt amplitude $g_1\, g_2 \to g_3\, g_4$ in the Regge limit $s \gg -t$

$$m_4(1,2,3,4) = s\,[g\, C(p_2,p_3,\tau)]\, \frac{1}{t} \left(\frac{-s}{\tau}\right)^{\alpha(t)} [g\, C(p_1,p_4,\tau)]$$

Glover VDD 08

$\alpha(t)$ Regge trajectory $C(p_2,p_3,\tau)$ coefficient function τ Regge-factorisation scale

$$\alpha(t) = \bar{g}^2 \bar{\alpha}^{(1)}(t) + \bar{g}^4 \bar{\alpha}^{(2)}(t) + \bar{g}^6 \bar{\alpha}^{(3)}(t) + O(\bar{g}^8) \qquad \bar{g}^2 = g^2 N c_\Gamma$$

$$C(p_i,p_j,\tau) = C^{(0)}(p_i,p_j) \left(1 + \bar{g}^2 \bar{C}^{(1)}(t,\tau) + \bar{g}^4 \bar{C}^{(2)}(t,\tau) + \bar{g}^6 \bar{C}^{(3)}(t,\tau) + \mathcal{O}(\bar{g}^8)\right)$$

$\bar{\alpha}^{(n)}(t)\,,\quad \bar{C}^{(n)}(t,\tau)$ are re-scaled loop coefficients

$$\bar{\alpha}^{(n)}(t) = \left(\frac{\mu^2}{-t}\right)^{n\epsilon} \alpha^{(n)}\,, \quad \bar{C}^{(n)}(t,\tau) = \left(\frac{\mu^2}{-t}\right)^{n\epsilon} C^{(n)}(t,\tau)$$

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Because the Regge limit is exponential in the Regge trajectory, one can use (the logarithm of) the BDS ansatz to obtain the Regge trajectory to all loops

Naculich Schnitzer 07
Drummond Korchemsky Sokatchev 07
Bartels Lipatov Sabio-Vera 08
Glover VDD 08

$$\alpha^{(l)}(\epsilon) = 2^{l-1} \alpha^{(1)}(l\epsilon) \left(\frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon\, \frac{l}{2}\, \hat{G}^{(l)}\right) + O(\epsilon) \qquad \alpha^{(1)}(\epsilon) = \frac{2}{\epsilon}$$

# Caveat

In QCD the standard Regge factorisation is on the colour-dressed amplitude

$$M_4(1,2,3,4) = s\left[ig\, f^{abe}\, C(p_2,p_3,\tau)\right] \frac{1}{t}\left(\frac{-s}{\tau}\right)^{\alpha(t)} \left[ig\, f^{cde}\, C(p_1,p_4,\tau)\right]$$

Kuraev Fadin Lipatov 76
Fadin Lipatov 93

but it is known to be only approximate

other colour structures occur at one loop  C.R. Schmidt VDD 98

# Regge factorisation of the 1-loop 4-pt amplitude

$$m_4^{(1)} = \bar{\alpha}^{(1)}(t)L + \; 2\bar{C}^{(1)}(t,\tau)$$

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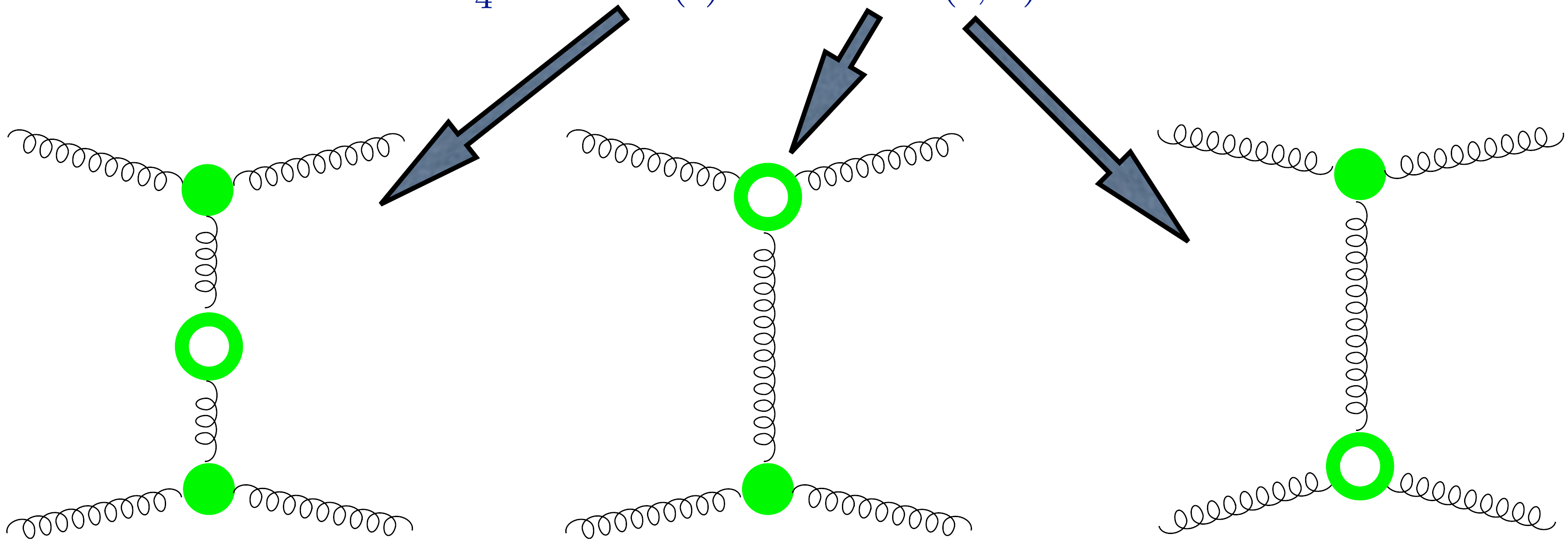

valid to all orders in ε

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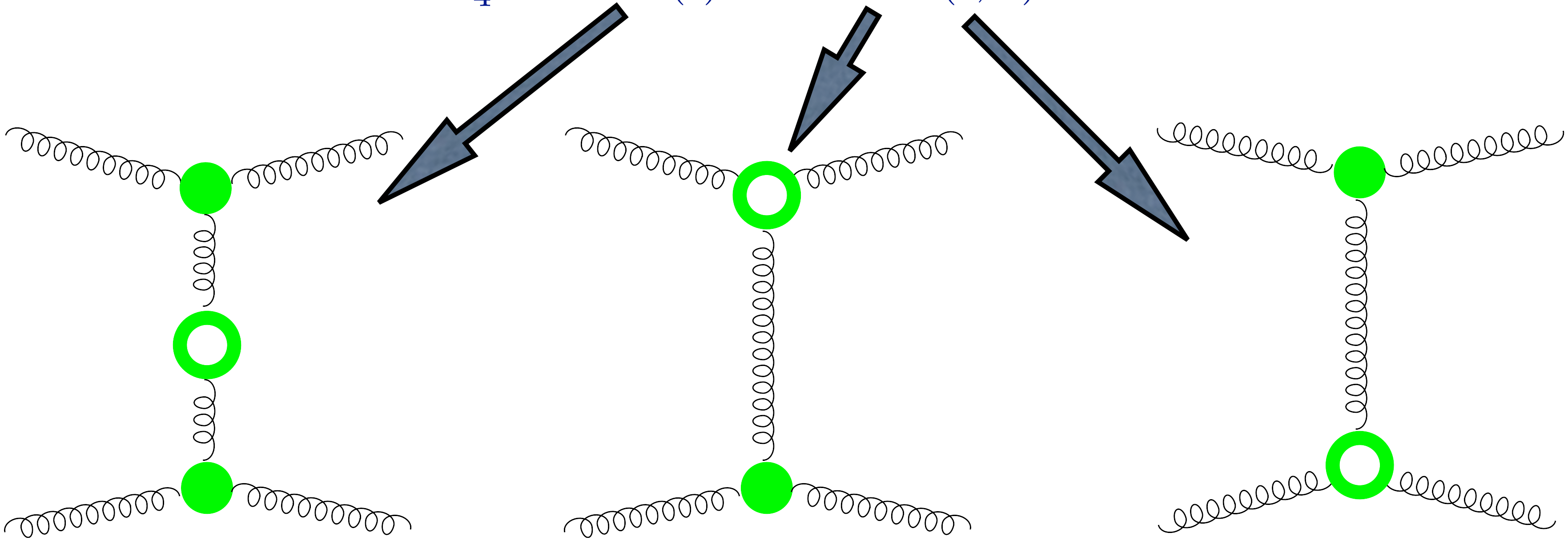
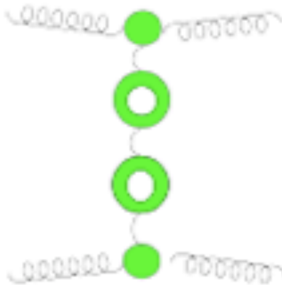


valid to all orders in ε

1-loop coefficient function

$$\begin{aligned} C^{(1)}(t,\tau) &= \frac{\psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1)}{\epsilon} - \frac{1}{\epsilon}\ln\frac{-t}{\tau} \\ &= \frac{1}{\epsilon^2}\left(-2 - \epsilon\ln\frac{-t}{\tau} + 3\sum_{n=1}^{\infty}\zeta_{2n}\,\epsilon^{2n} + \sum_{n=1}^{\infty}\zeta_{2n+1}\,\epsilon^{2n+1}\right) \end{aligned}$$

# Factorisation of the 2-loop amplitude



$$
\begin{aligned}
m_4^{(2)} &= \frac{1}{2}\left(\bar{\alpha}^{(1)}(t)\right)^2 L^2 \\
&+ \left(\bar{\alpha}^{(2)}(t) + 2\,\bar{C}^{(1)}(t,\tau)\bar{\alpha}^{(1)}(t)\right) L \\
&+ 2\,\bar{C}^{(2)}(t,\tau) + \left(\bar{C}^{(1)}(t,\tau)\right)^2
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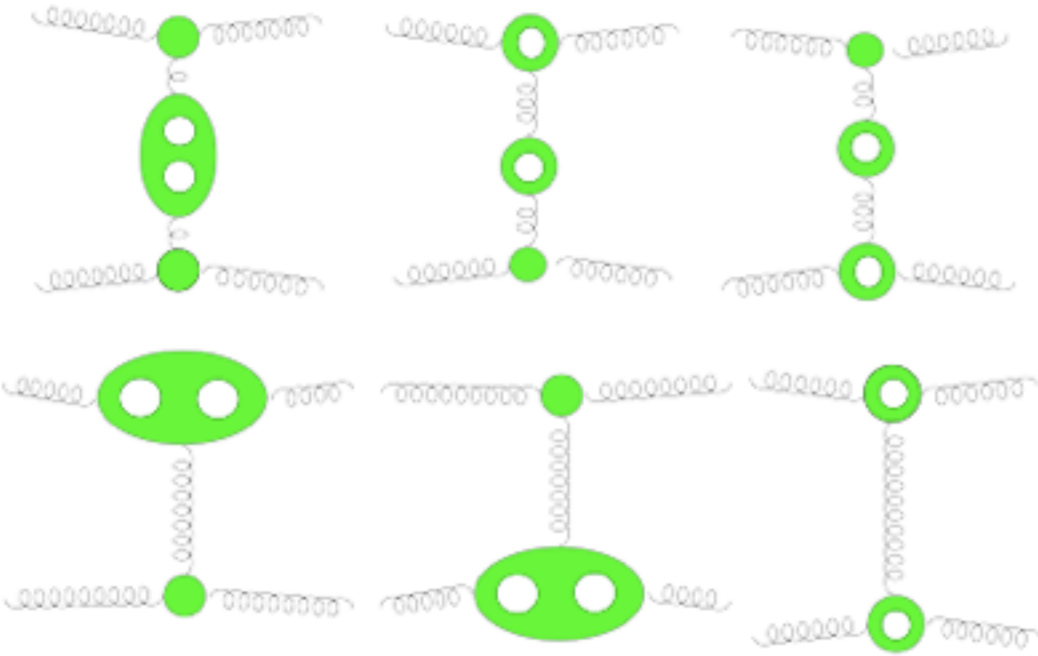
valid to all orders in ε



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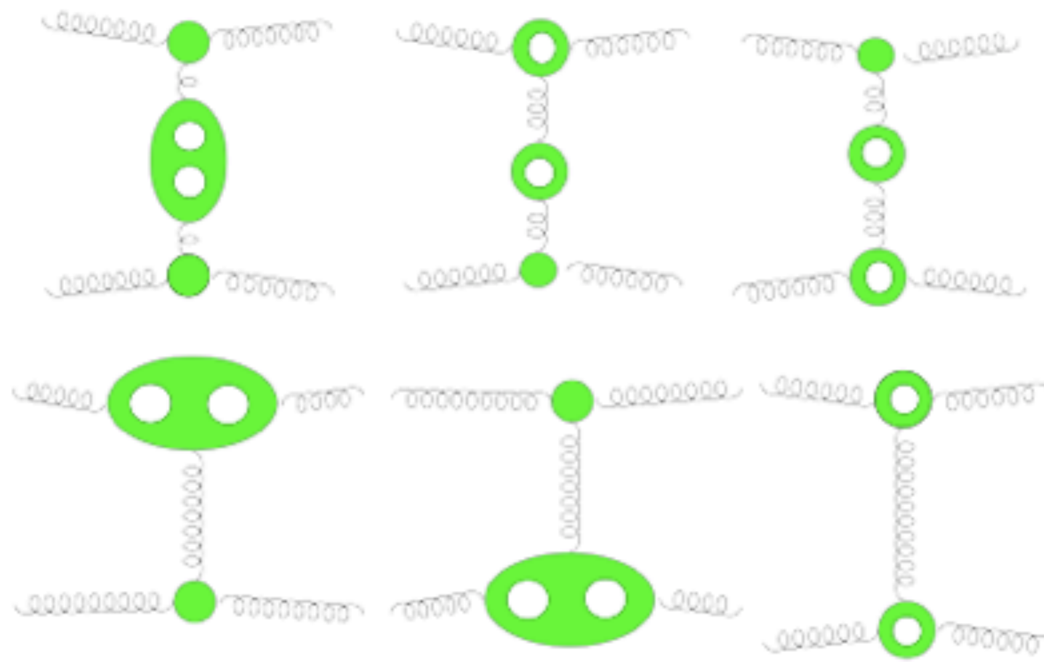

a more efficient way of writing it

$$
m_4^{(2)} = \frac{1}{2}\left(m_4^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t)L + 2\,\bar{C}^{(2)}(t,\tau) - \left(\bar{C}^{(1)}(t,\tau)\right)^2
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where $m_4^{(1)}$ must be known at least through $\mathcal{O}(\epsilon^2)$

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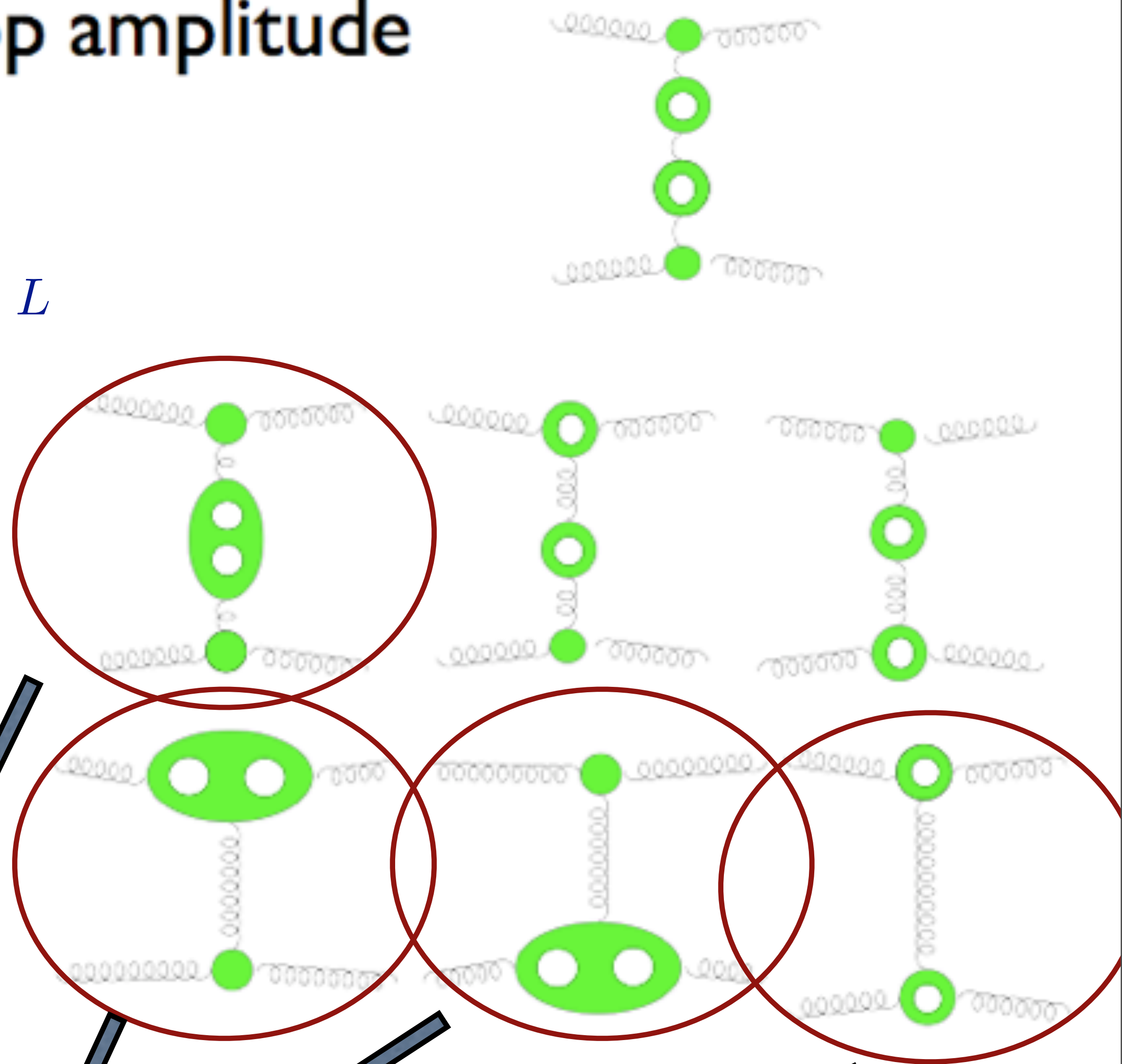

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by direct calculation from
the 2-loop 4-pt amplitude $m_4^{(2)}$ to $O(\varepsilon^2)$ Bern Dixon Smirnov 05
we get 2-loop trajectory

$$\alpha^{(2)} = -\frac{2\zeta_2}{\epsilon} - 2\zeta_3 - 8\zeta_4\epsilon + (36\zeta_2\zeta_3 + 82\zeta_5)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

2-loop coefficient function

$$\begin{aligned} C^{(2)}(t,\tau) &= \frac{1}{2}\left[C^{(1)}(t,\tau)\right]^2 + \frac{\zeta_2}{\epsilon^2} + \left(\zeta_3 + \zeta_2 \ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &+ \left(\zeta_3 \ln\frac{-t}{\tau} - 19\zeta_4\right) + \left(4\zeta_4 \ln\frac{-t}{\tau} - 2\zeta_2\zeta_3 - 39\zeta_5\right)\epsilon \\ &- \left(48\zeta_3^2 + \frac{1773}{8}\zeta_6 + (18\zeta_2\zeta_3 + 41\zeta_5)\ln\frac{-t}{\tau}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \end{aligned}$$

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where $C^{(1)}(t,\tau,\epsilon)$ must be known at least through $\mathcal{O}(\epsilon^2)$

A similar factorisation holds also for QCD amplitudes.
In that case, the 2-loop 4-parton amplitude $m_4^{(2)}$
yields the 2-loop trajectory

Fadin Fiore 95
Glover VDD 01

$$\alpha^{(2)} = C_A \left[ \beta_0 \frac{1}{\epsilon^2} + K \frac{2}{\epsilon} + C_A \left( \frac{404}{27} - 2\zeta_3 \right) - \frac{56}{27} N_F \right] + \mathcal{O}(\epsilon)$$

maximal trascendentality
Kotikov Lipatov 02

$$\beta_0 = \frac{11}{3} C_A - \frac{2}{3} N_F \qquad K = \left( \frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} N_F$$

maximal trascendentality:

$\zeta_n, \ \ln^n, \ \epsilon^{-n}$ have weight $n$ in trascendentality

$N$=4 SYM amplitudes, and quantities derived from them,
are homogeneous polynomials of maximal trascendentality

# BDS ansatz and Regge limit

the iteration formula for the 2-loop *n*-pt amplitude $m_n^{(2)}$

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + 4\,Const^{(2)} + \mathcal{O}(\epsilon)$$

Anastasiou Bern Dixon Kosower 03

valid for $n = 4, 5$

$$f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2 \qquad Const^{(2)} = -\frac{\zeta_2^2}{2}$$

(we use a different normalisation from BDS) $\quad G(\epsilon) = \dfrac{e^{-\gamma\epsilon}\,\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)} = 1 + \mathcal{O}(\epsilon^2)$

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from the iteration formula and Regge factorisation
we obtain iteration formulae for the Regge trajectory and the coefficient function

$$\alpha^{(2)}(\epsilon) = 2\,f^{(2)}(\epsilon)\,\alpha^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

$$C^{(2)}(t,\tau,\epsilon) = \frac{1}{2}\left[C^{(1)}(t,\tau,\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, C^{(1)}(t,\tau,2\epsilon) + 2\,Const^{(2)} + \mathcal{O}(\epsilon)$$

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where $C^{(1)}(t,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

the formulae for $n = 4$ implied by
the BDS ansatz and by Regge factorisation differ in that
BDS: valid for arbitrary kinematics, but to $O(\varepsilon^0)$
Regge: valid to all orders in $\varepsilon$, but only in the Regge kinematics.
They overlap and agree in the Regge kinematics to $O(\varepsilon^0)$

# Regge factorisation at 3 loops

$$m_4^{(3)} = m_4^{(2)} m_4^{(1)} - \frac{1}{3}\left(m_4^{(1)}\right)^3 + \bar{\alpha}^{(3)}(t) L + 2\,\bar{C}^{(3)}(t,\tau) - 2\,\bar{C}^{(2)}(t,\tau)\,\bar{C}^{(1)}(t,\tau) + \frac{2}{3}\left(\bar{C}^{(1)}(t,\tau)\right)^3$$

valid to all orders in ε

with 3-loop trajectory

$$\alpha^{(3)} = \frac{44\zeta_4}{3\epsilon} + \frac{40}{3}\zeta_2\zeta_3 + 16\zeta_5 + \mathcal{O}(\epsilon)$$

3-loop coefficient function

$$\begin{aligned} C^{(3)}(t,\tau) &= C^{(2)}(t,\tau)\,C^{(1)}(t,\tau) - \frac{1}{3}\left[C^{(1)}(t,\tau)\right]^3 \\ &- \frac{44}{9}\frac{\zeta_4}{\epsilon^2} - \left(\frac{40}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5 + \frac{22}{3}\zeta_4 \ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &+ \frac{3982}{27}\zeta_6 - \frac{68}{9}\zeta_3^2 - \left(8\zeta_5 + \frac{20}{3}\zeta_2\zeta_3\right)\ln\frac{-t}{\tau} + \mathcal{O}(\epsilon) \end{aligned}$$

Glover VDD 08

where $C^{(1)}(t,\tau,\epsilon)$ must be known at least through $\mathcal{O}(\epsilon^4)$

$C^{(2)}(t,\tau,\epsilon)$ $\mathcal{O}(\epsilon^2)$

# BDS ansatz and 3-loop Regge factorisation

from BDS's iteration formula for the 3-loop 4-point amplitude and Regge factorisation, we get iteration formulae for the 3-loop Regge trajectory and coefficient function

$$\alpha^{(3)}(\epsilon) = 4\, f^{(3)}(\epsilon)\, \alpha^{(1)}(3\epsilon) + \mathcal{O}(\epsilon)$$

$$\begin{aligned} C^{(3)}(t,\tau,\epsilon) &= C^{(2)}(t,\tau,\epsilon)\, C^{(1)}(t,\tau,\epsilon) - \frac{1}{3}\left[C^{(1)}(t,\tau,\epsilon)\right]^3 \\ &+ \frac{4\, G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, C^{(1)}(t,\tau,3\epsilon) + 4\, Const^{(3)} + \mathcal{O}(\epsilon) \end{aligned}$$

with $$f^{(3)}(\epsilon) = \frac{11}{2}\zeta_4 + (6\zeta_5 + 5\zeta_2\zeta_3)\epsilon + (c_1\zeta_6 + c_2\zeta_3^2)\epsilon^2$$

$$Const^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2$$

with $c_1$ and $c_2$ known constants (which drop out of the recursive formula above)

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To $\mathcal{O}(\epsilon^0)$, the BDS iteration formulae above are in agreement with the Regge formulae of the previous slide

Regge factorisation is valid also for amplitudes with 5 or more points in generalised Regge limits.

The strategy is to use the modular form of the amplitudes dictated by high-energy factorisation, to obtain information on $n$-point amplitudes in terms of building blocks derived from $m$-point amplitudes, with $m < n$

# Regge factorisation of the 5-pt amplitude

5-pt amplitude $g_1 g_2 \to g_3 g_4 g_5$ in the multi-Regge limit $s \gg s_1, s_2 \gg -t_1, -t_2$

$$m_5 = s\,[g\,C(p_2,p_3,\tau)]\,\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa,\tau)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_5,\tau)]$$

$V$ is gluon-production vertex; $\kappa = |p_T|^2$ of central gluon

# Regge factorisation of the 5-pt amplitude

5-pt amplitude $g_1 g_2 \to g_3 g_4 g_5$ in the multi-Regge limit $s \gg s_1, s_2 \gg -t_1, -t_2$

$$m_5 = s\,[g\,C(p_2,p_3,\tau)]\,\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa,\tau)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_5,\tau)]$$

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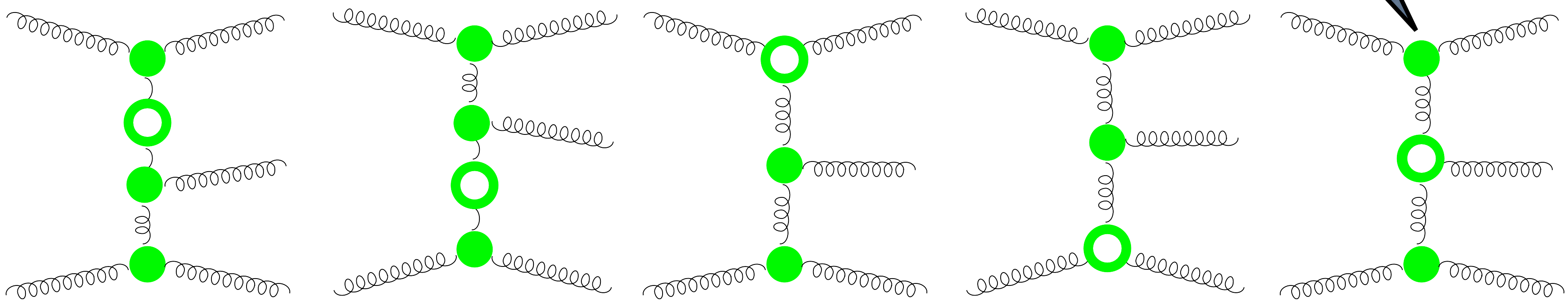
1 loop $m_5^{(1)} = \bar{\alpha}^{(1)}(t_1)L_1 + \bar{\alpha}^{(1)}(t_2)L_2 + \bar{C}^{(1)}(t_1,\tau) + \bar{C}^{(1)}(t_2,\tau) + \bar{V}^{(1)}(t_1,t_2,\kappa,\tau)$



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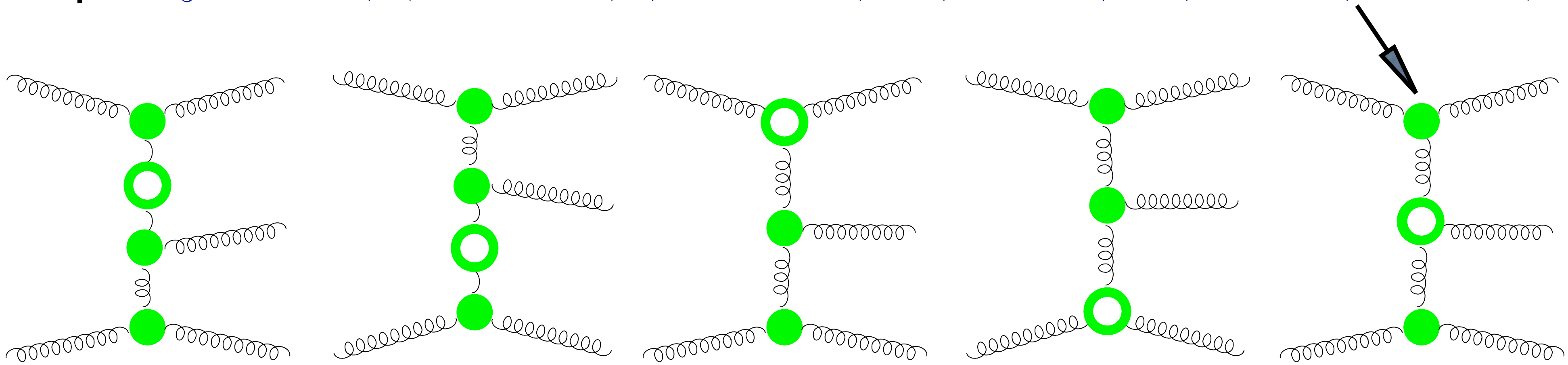

2 loops
$$\begin{aligned} m_5^{(2)} &= \frac{1}{2}\left(m_5^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t_1)L_1 + \bar{\alpha}^{(2)}(t_2)L_2 \\ &+ \bar{C}^{(2)}(t_1,\tau) + \bar{V}^{(2)}(t_1,t_2,\kappa,\tau) + \bar{C}^{(2)}(t_2,\tau) \\ &- \frac{1}{2}\left(\bar{C}^{(1)}(t_1,\tau)\right)^2 - \frac{1}{2}\left(\bar{V}^{(1)}(t_1,t_2,\kappa,\tau)\right)^2 - \frac{1}{2}\left(\bar{C}^{(1)}(t_2,\tau)\right)^2 \end{aligned}$$

where $m_5^{(1)}$ must be known at least through $\mathcal{O}(\epsilon^2)$

# BDS ansatz and Regge limit for the 5-pt amplitude

Using the BDS and Regge 2-loop iteration formula for the 5-pt amplitude $m_5^{(2)}$ and the iteration formulae for the trajectory and the coefficient functions, one obtains a 2-loop iteration formula for the gluon-production vertex

$$V^{(2)}(t_1, t_2, \kappa, \tau, \epsilon) = \frac{1}{2}\left[V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, V^{(1)}(t_1, t_2, \kappa, \tau, 2\epsilon) + \mathcal{O}(\epsilon)$$

Duhr Glover VDD 08

where $V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

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Similarly, at 3 loops

$$\begin{aligned} V^{(3)}(t_1,t_2,\kappa,\tau,\epsilon) &= V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon)V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon) - \frac{1}{3}\left[V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)\right]^3 \\ &+ \frac{4\,G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\,V^{(1)}(t_1,t_2,\kappa,\tau,3\epsilon) + \mathcal{O}(\epsilon) \end{aligned}$$

where $V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^4)$

$V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon)$ $\mathcal{O}(\epsilon^2)$

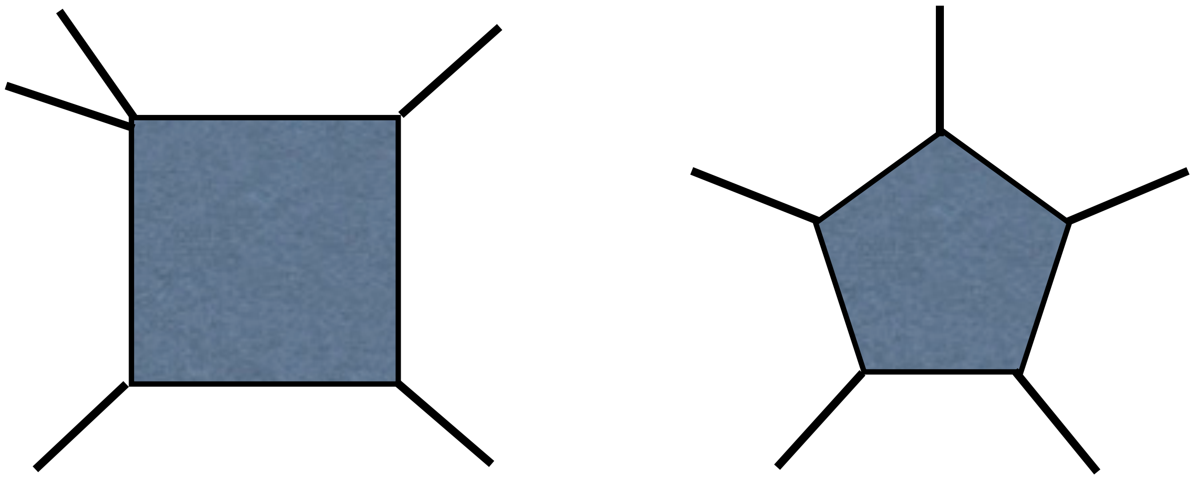
# 1-loop 5-pt amplitude

Bern Dixon Dunbar Kosower 97

$$m_5^{(1)} = -\frac{1}{4}\sum_{\text{cyclic}} s_{12}\, s_{23}\, I_4^{1m}(1,2,3,45,\epsilon) - \frac{\epsilon}{2}\,\epsilon_{1234}\, I_5^{6-2\epsilon}(\epsilon)$$

parity-even and $O(\varepsilon^{-2})$ parity-odd and $O(\varepsilon)$

$$\epsilon_{1234} = \mathrm{tr}[\gamma_5 \not{k}_1 \not{k}_2 \not{k}_3 \not{k}_4]$$



one-mass boxes known to all orders in ε

(6-2ε)-dim pentagon IR finite, but irreducible, and unknown analytically

1-loop 5-pt amplitude computed through $O(\varepsilon^2)$ numerically

Cachazo Spradlin Volovich 06

Bern Czakon Kosower Roiban Smirnov 06

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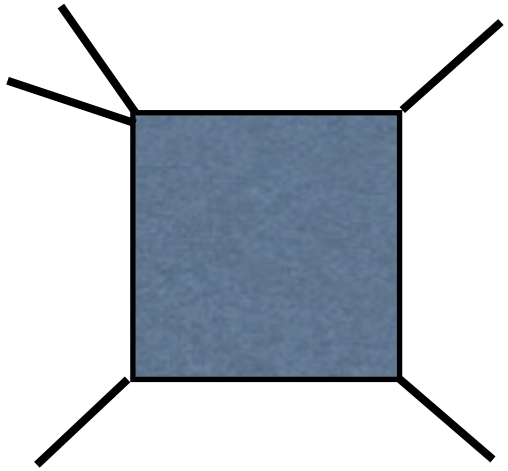

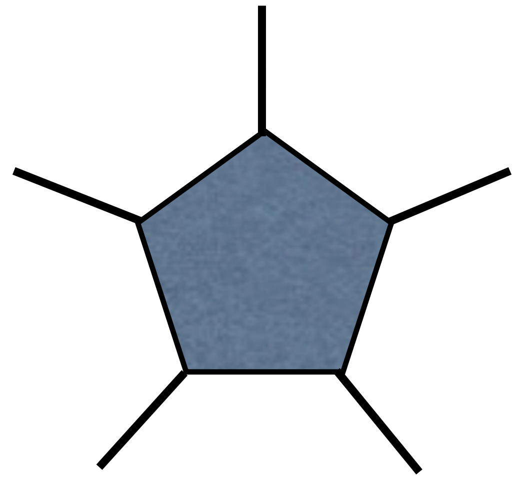

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in multi-Regge kinematics, we have computed analytically Duhr Glover Smirnov VDD 09
the 1-loop 5-pt amplitude to all orders in ε, expanded through $O(\varepsilon^2)$

# Regge factorisation of the 6-pt amplitude

6-pt amplitude $g_1 g_2 \to g_3 g_4 g_5 g_6$

in the multi-Regge limit $y_3 \gg y_4 \gg y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq | p_{5\perp}| \simeq | p_{6\perp}|$

$$s \gg s_1, s_2, s_3 \gg -t_1, -t_2, -t_3$$

$$m_6 = s\,[g\,C(p_2,p_3,\tau)]\;\frac{1}{t_3}\left(\frac{-s_3}{\tau}\right)^{\alpha(t_3)}\;[g\,V(q_2,q_3,\kappa_2,\tau)]$$
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no new vertices or coefficient functions appear, wrt $n = 5$

The *l*-loop 6-pt amplitude can then be assembled using the *l*-loop trajectories, gluon-production vertices and coefficient functions, which can be determined through the *l*-loop 4-pt and 5-pt amplitudes

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The $l$-loop 6-pt amplitude can then be assembled using the $l$-loop trajectories, gluon-production vertices and coefficient functions, which can be determined through the $l$-loop 4-pt and 5-pt amplitudes

Thus, also the $l$-loop BDS iterative formula for $n = 6$ will be fulfilled

$\Longrightarrow$ the multi-Regge limit is not able to detect the BDS-ansatz violation for $n = 6$

# Remainder function

the remainder function of the 6-pt amplitude depends on
3 conformally-invariant cross-ratios

Drummond Henn Korchemsky Sokatchev 07

$$R_6^{(2)} = R_6^{(2)}(u_1.u_2, u_3)$$

$$u_1 = \frac{s_{12}\, s_{45}}{s_{345}\, s_{456}}\,, \qquad u_2 = \frac{s_{23}\, s_{56}}{s_{234}\, s_{456}}\,, \qquad u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}}$$

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in the multi-Regge kinematics

$$u_1 = 1 + \mathcal{O}\left(\frac{t}{s}\right)\,, \qquad u_2 = \mathcal{O}\left(\frac{t}{s}\right)\,, \qquad u_3 = \mathcal{O}\left(\frac{t}{s}\right)$$

like in the collinear limit

# 1-loop 6-pt amplitude

- computed through $O(\varepsilon^2)$ numerically

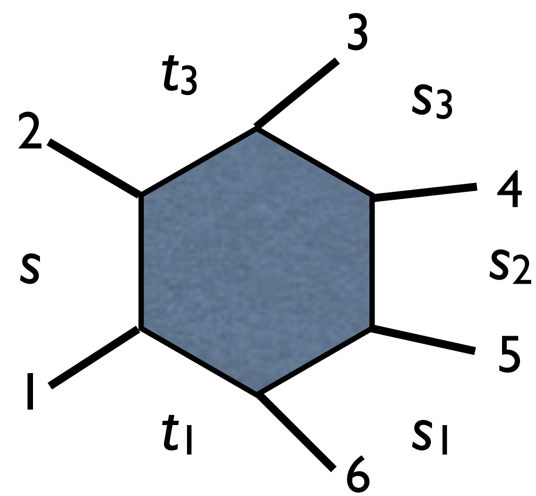
  even Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

  odd Cachazo Spradlin Volovich 08

through $O(\varepsilon^0)$, it is given in terms of $1m$ and $2me$ boxes

at $O(\varepsilon)$ a hexagon occurs in the even part

$$s \equiv s_{12}, \quad t_3 \equiv s_{23}, \quad s_3 \equiv s_{34}, \quad s_2 \equiv s_{45}, \quad s_1 \equiv s_{56}, \quad t_1 \equiv s_{61}$$



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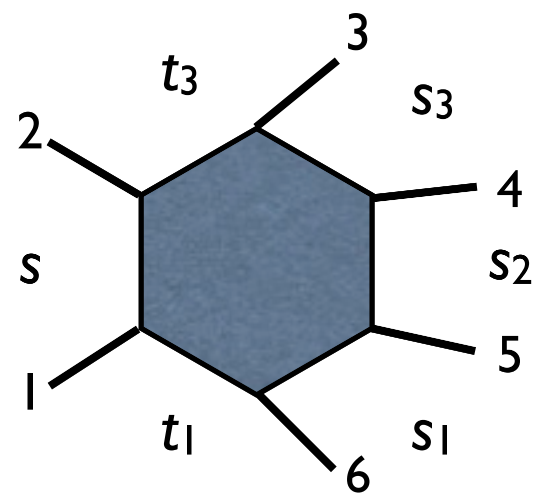
- multi-Regge kinematics (in Euclidean region)

$$-s \gg -s_1, -s_2, -s_3 \gg -t_1, -t_2, -t_3$$

$$s_1 \to \lambda^2 s_1, \quad s_2 \to \lambda^2 s_2, \quad s_3 \to \lambda^2 s_3, \quad t_1 \to \lambda^3 t_1, \quad t_3 \to \lambda^3 t_3, \qquad \lambda \ll 1$$



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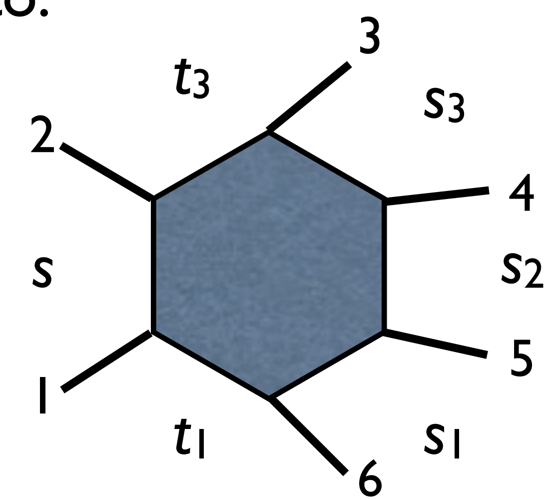
- to all orders in ε, the hexagon integral is reduced to:
  triple sums in NDIM,
  3-fold integrals through Mellin-Barnes



# Regge factorisation of the $n$-pt amplitude

$$m_n(1,2,\ldots,n) = s\left[g\,C(p_2,p_3)\right]\frac{1}{t_{n-3}}\left(\frac{-s_{n-3}}{\tau}\right)^{\alpha(t_{n-3})}\left[g\,V(q_{n-3},q_{n-4},\kappa_{n-4})\right]$$
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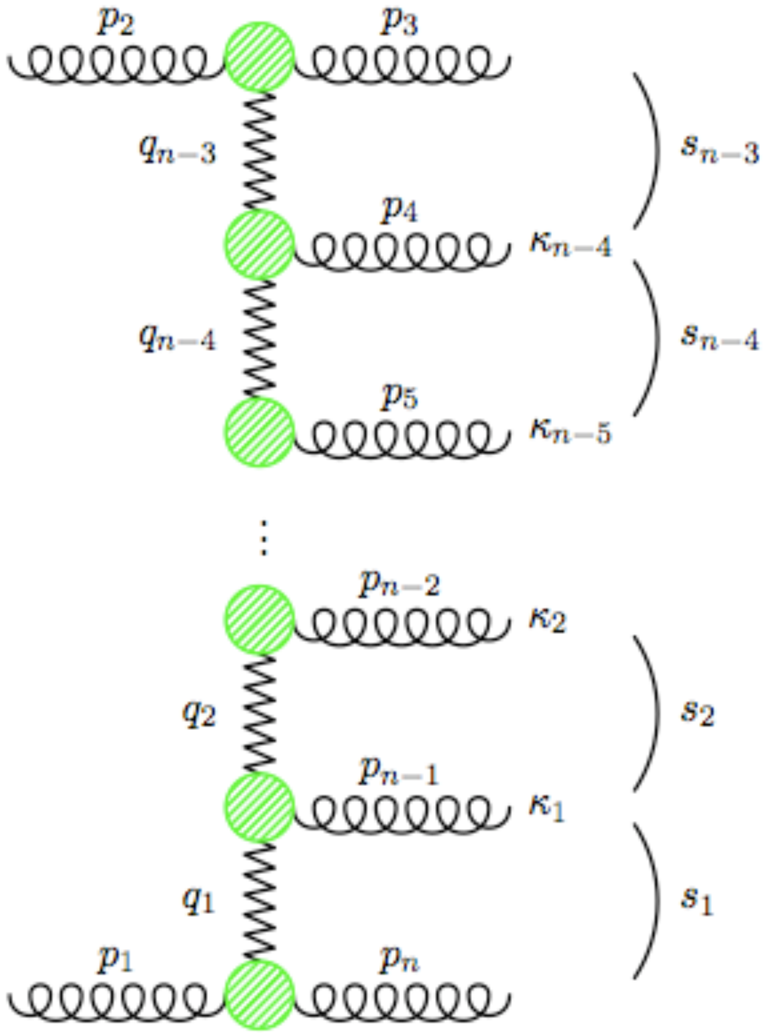
$n$-pt amplitude in the multi-Regge limit

$$y_3 \gg y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|$$

$$s \gg s_1, s_2, \ldots, s_{n-3} \gg -t_1, -t_2 \ldots, -t_{n-3}$$



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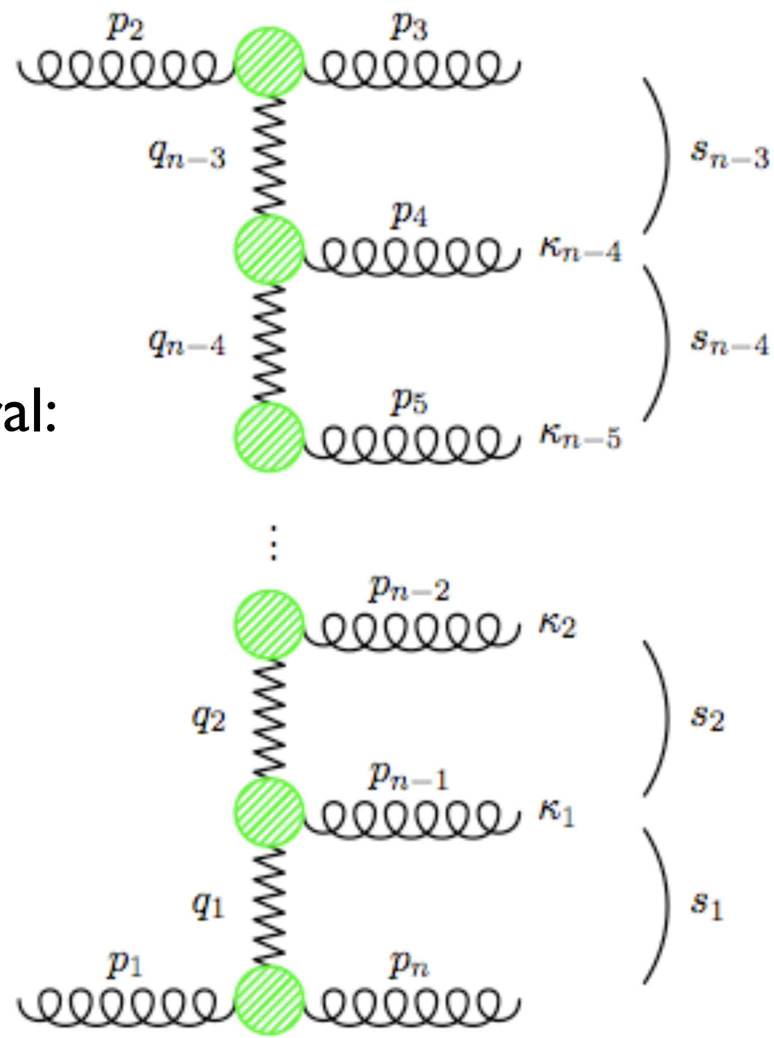


What we said for $n$ = 6 can be repeated in general: the $l$-loop $n$-pt amplitude can be assembled using the $l$-loop trajectories, vertices and coefficient functions, determined through the $l$-loop 4-pt and 5-pt amplitudes

⇨ no violation of the BDS ansatz can be found in the multi-Regge limit

To have a chance to detect the violation of the BDS ansatz for the 2-loop 6-pt amplitude, that we see in arbitrary kinematics, we must relax the strong-ordering constraints of the multi-Regge kinematics

# $n$-pt amplitude in quasi-multi-Regge kinematics

$$m_n(1,2,\ldots,n) = s\left[g^2\, A(p_2,p_3,p_4)\right]\, \frac{1}{t_{n-4}} \left(\frac{-s_{n-4}}{\tau}\right)^{\alpha(t_{n-4})} \left[g\, V(q_{n-4},q_{n-5},\kappa_{n-5})\right]$$
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quasi-multi-Regge kinematics

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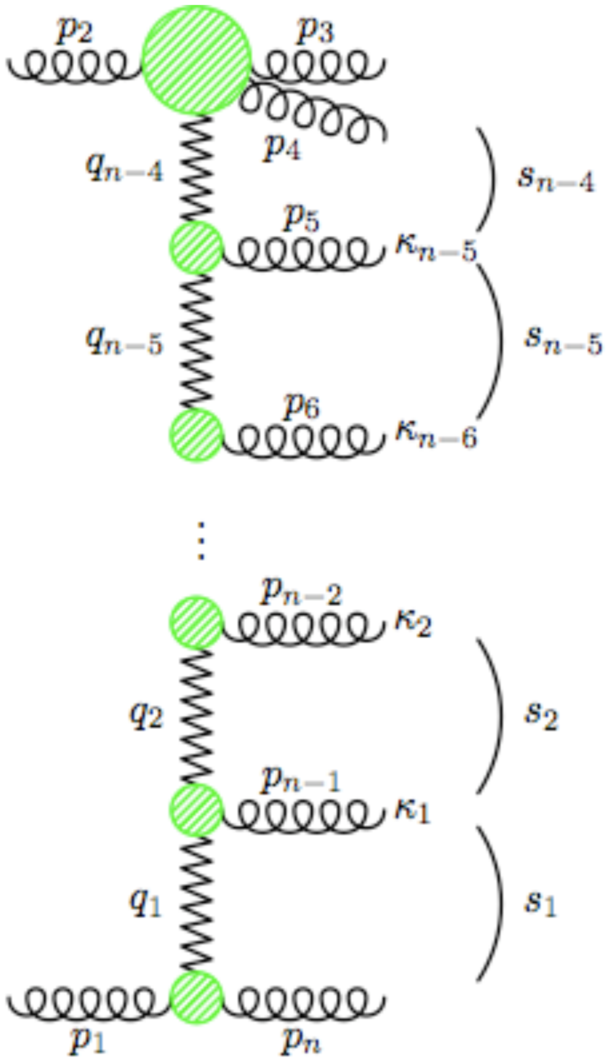

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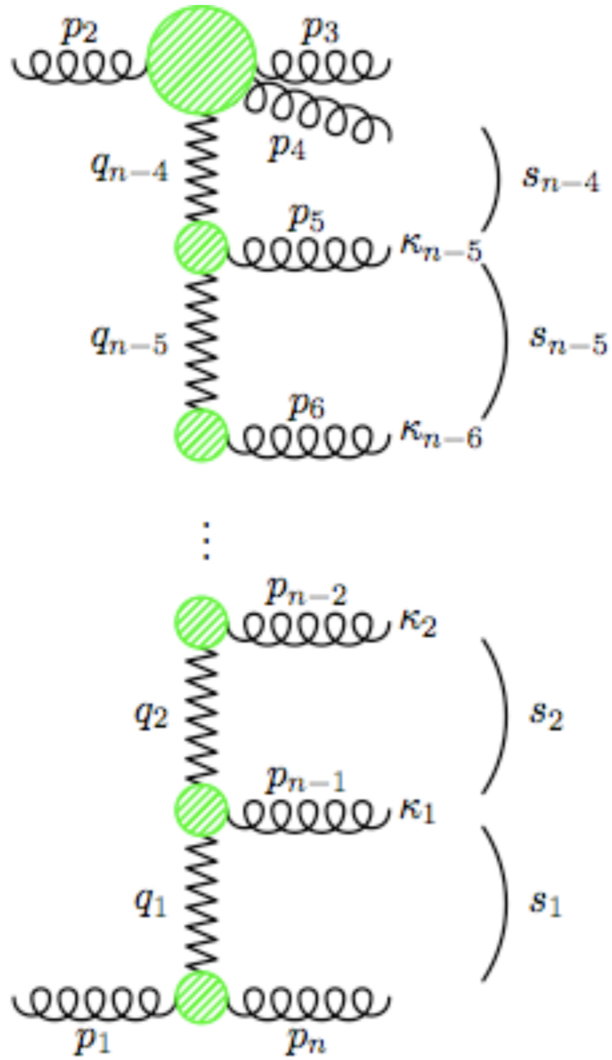


A new coefficient function $A(p_2,p_3,p_4,\tau)$
occurs already at $n$ = 5,
for which the BDS ansatz is fulfilled.
Because no new coefficient functions
appear for $n \geq 6$, a violation of the
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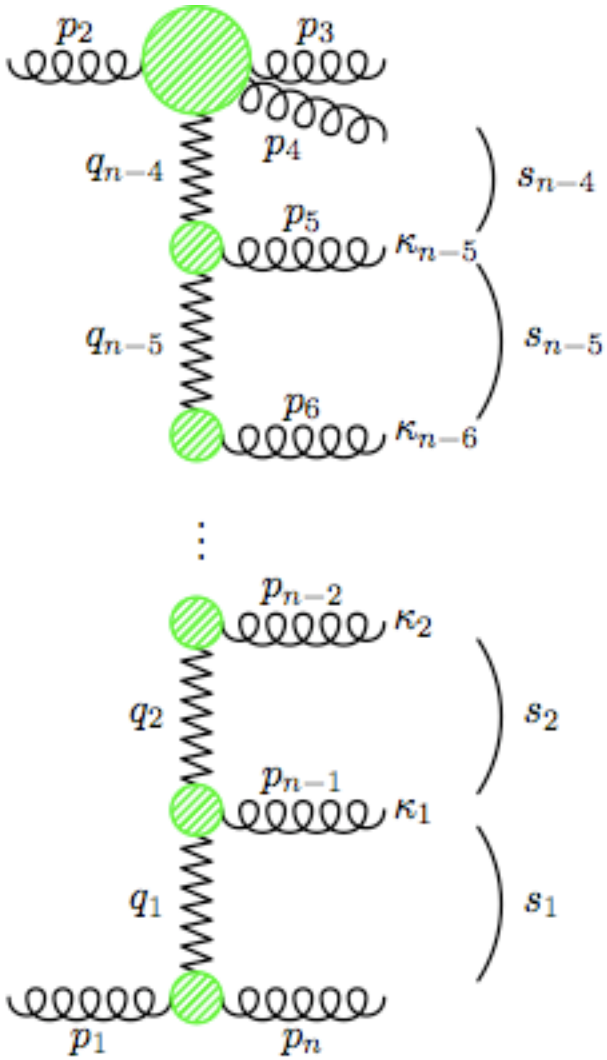


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The same can be said for the
quasi-multi-Regge kinematics

$$y_3 \simeq y_4 \gg \cdots \gg y_{n-1} \simeq y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|$$

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take the values

$$u_1 = 1 + \mathcal{O}\left(\frac{t}{s}\right), \qquad u_2 = \mathcal{O}\left(\frac{t}{s}\right), \qquad u_3 = \mathcal{O}\left(\frac{t}{s}\right)$$

like in the multi-Regge kinematics and in the collinear limit

# More general quasi-multi-Regge kinematics

A necessary condition to see a violation of the BDS ansatz for the 2-loop 6-pt amplitude, is to go to a quasi-multi-Regge kinematics for which new coefficient functions appear for $n \geq 6$

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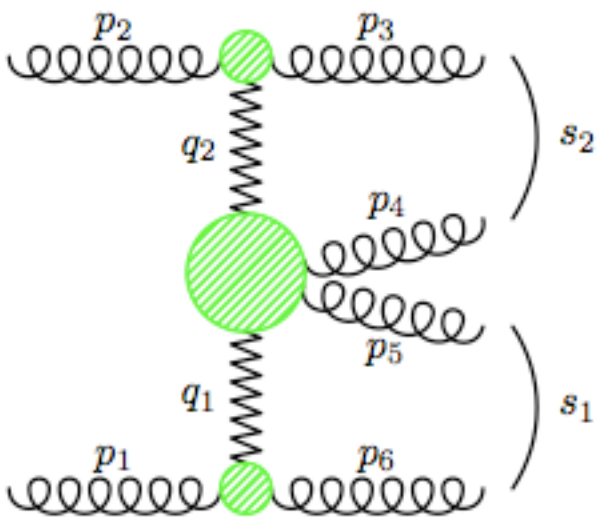
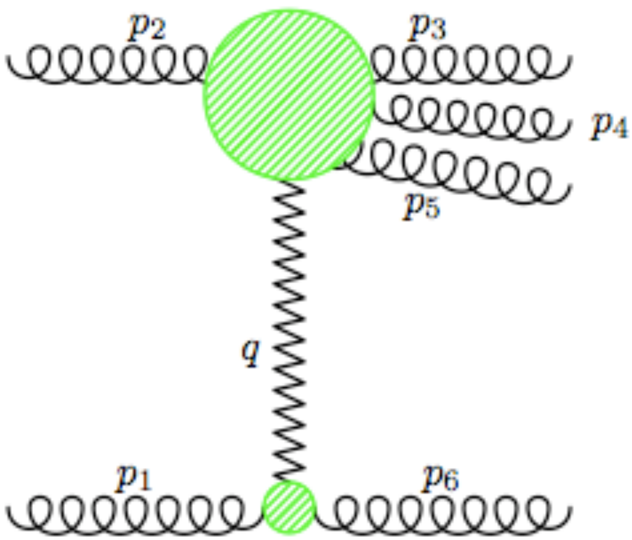
two such quasi-multi-Regge kinematics are

(a)




(b)



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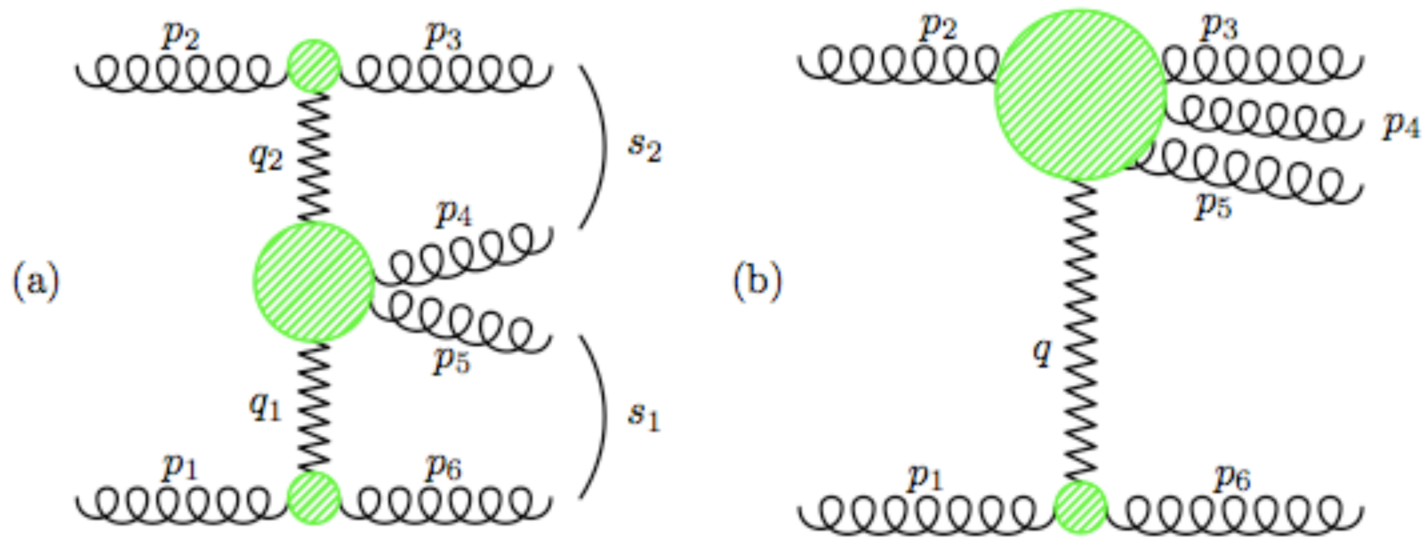
A necessary condition to see a violation of the BDS ansatz for the 2-loop 6-pt amplitude, is to go to a quasi-multi-Regge kinematics for which new coefficient functions appear for $n \geq 6$

two such quasi-multi-Regge kinematics are




in both cases, the 3 conformally-invariant cross-ratios take values

$$u_1 = \mathcal{O}(1)\,, \qquad u_2 = \mathcal{O}(1)\,, \qquad u_3 = \mathcal{O}(1)$$

it remains to be seen if these kinematics harbour a violation of the BDS ansatz

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Duhr's talk on Friday

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- the $l$-loop $n$-pt amplitude so built fulfils the BDS ansatz, thus any ansatz violation must be searched in less constraining (quasi-multi-Regge ?) kinematics