SYM amplitudes in the high-energy limit

Vittorio Del Duca

INFN LNF

CALC 2009

Dubna 11 July 2009
Bern-Dixon-Smirnov ansatz

an ansatz for MHV amplitudes in \( N=4 \) SYM

Bern Dixon Smirnov 05

\[
m_n = m_n^{(0)} \left[ 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \right]
\]

\[
= m_n^{(0)} \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + \text{Const}^{(l)} + E_n^{(l)}(\epsilon) \right) \right]
\]

coupling  \( a = \frac{\lambda}{8\pi^2} (4\pi e^{-\gamma})^\epsilon \)  \( \lambda = g^2 N \)  ‘t Hooft parameter

\[
f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \frac{l}{2} \hat{G}^{(l)} + \epsilon^2 f_2^{(l)}
\]

\[
E_n^{(l)}(\epsilon) = O(\epsilon)
\]

\( \hat{\gamma}_K^{(l)} \)  cusp anomalous dimension, known to all orders of \( a \)

Korchemsky Radyuskin 86
Beisert Eden Staudacher 06

\( \hat{G}^{(l)} \)  collinear anomalous dimension, known through \( O(a^4) \)

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 07
Brief history of **BDS ansatz**

**BDS ansatz** checked for the 3-loop 4-pt amplitude

2-loop 5-pt amplitude

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 06
Bern Czakon Kosower Roiban Smirnov 06
Brief history of BDS ansatz

BDS ansatz checked for the 3-loop 4-pt amplitude
2-loop 5-pt amplitude

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 06
Bern Czakon Kosower Roiban Smirnov 06

BDS ansatz shown to fail on 2-loop 6-pt amplitude

Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

Hints of break-up from strong-coupling expansion
hexagon Wilson loop
multi-Regge limit (?)

Alday Maldacena 07
Drummond Henn Korchemsky Sokatchev 07
Bartels Lipatov Sabio-Vera 08

Saturday, July 11, 2009
Brief history of BDS ansatz

BDS ansatz checked for the 3-loop 4-pt amplitude

2-loop 5-pt amplitude

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 06
Bern Czakon Kosower Roiban Smirnov 06

BDS ansatz shown to fail on 2-loop 6-pt amplitude

Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

Hints of break-up from strong-coupling expansion
hexagon Wilson loop
multi-Regge limit (?)

Alday Maldacena 07
Drummond Henn Korchemsky Sokatchev 07
Bartels Lipatov Sabio-Vera 08

The BDS ansatz implies an iteration formula
for the 2-loop $n$-pt amplitude $m_{n}^{(2)}$ (rescaled by the tree amplitude)

$$m_{n}^{(2)}(\epsilon) = \frac{1}{2} \left[ m_{n}^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) m_{n}^{(1)}(2\epsilon) + \text{Const}^{(2)} + \mathcal{O}(\epsilon)$$

Anastasiou Bern Dixon Kosower 03

The remainder function characterises the deviation from the ABDK/BDS iteration

$$R_{n}^{(2)} = m_{n}^{(2)}(\epsilon) - \frac{1}{2} \left[ m_{n}^{(1)}(\epsilon) \right]^2 - f^{(2)}(\epsilon) m_{n}^{(1)}(2\epsilon) - \text{Const}^{(2)}$$

Saturday, July 11, 2009
Why?
Why?

solid theory of the IR-divergent part

Mueller, Sen, Korchemsky, Radyuskin, Collins, Sterman, Magnea, ...
Why?

solid theory of the IR-divergent part

but not much known about the remainder function,
except for understanding why there shouldn’t be any for $n = 4, 5$

Alday Maldacena
Drummond Henn Korchemsky Sokatchev

Mueller, Sen, Korchemsky, Radyuskin,
Collins, Sterman, Magnea, ...
Why?

solid theory of the IR-divergent part

but not much known about the remainder function, apart from understanding why there shouldn’t be any for $n = 4, 5$

How?

What is the remainder function?
Why?

solid theory of the IR-divergent part

but not much known about the remainder function,
except from understanding why there shouldn’t be any for $n = 4, 5$

Mueller, Sen, Korchemsky, Radyuskin, Collins, Sterman, Magnea, ...

Alday Maldacena
Drummond Henn Korchemsky Sokatchev

How?

What is the remainder function?

we are trying to move forward analytically

Duhr Glover Smirnov VDD 09
**MHV amplitudes ↔ Wilson loops**

agreement between $n$-edged Wilson loop and $n$-point MHV amplitude, verified for

- $n$-edged 1-loop Wilson loop
  - Brandhuber Heslop Travaglini 07
- 6-edged 2-loop Wilson loop
  - Drummond Henn Korchemsky Sokatchev 07
  - Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
MHV amplitudes $\leftrightarrow$ Wilson loops

agreement between $n$-edged Wilson loop and $n$-point MHV amplitude, verified for Alday Maldacena 07

$n$-edged 1-loop Wilson loop
6-edged 2-loop Wilson loop

Brandhuber Heslop Travaglini 07
Drummond Henn Korchemsky Sokatchev 07
Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

7-edged & 8-edged 2-loop Wilson loops also computed (numerically)

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

if agreement holds up to 8-edged 2-loop Wilson loops, then $R_{7}^{(2)}$, $R_{8}^{(2)}$ are known numerically

$R_{n}^{(2)}$ unknown analytically, but functions of conformally-invariant cross-ratios

Drummond Henn Korchemsky Sokatchev 07
Ward identities & Wilson loops

$N=4$ SYM is invariant under $SO(2,4)$ conformal transformations
\textbf{Ward identities & Wilson loops}

- $N=4$ SYM is invariant under $SO(2,4)$ conformal transformations
- the \textbf{Wilson} loops fulfill conformal \textbf{Ward} identities
N=4 SYM is invariant under $SO(2,4)$ conformal transformations.

The Wilson loops fulfill conformal Ward identities.

The solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + $R$. 

Drummond Henn Korchemsky Sokatchev 07
Ward identities & Wilson loops

- $N=4$ SYM is invariant under $SO(2,4)$ conformal transformations
- the Wilson loops fulfill conformal Ward identities
- the solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + $R$
- for $n = 4, 5$, $R$ is a constant
- for $n \geq 6$, $R$ is an unknown function of conformally invariant cross ratios
Ward identities & Wilson loops

- N=4 SYM is invariant under SO(2,4) conformal transformations
- the Wilson loops fulfill conformal Ward identities
- the solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + $R$
- for $n = 4, 5$, $R$ is a constant
- for $n \geq 6$, $R$ is an unknown function of conformally invariant cross ratios
- for $n = 6$, the conformally invariant cross ratios are

\[
\begin{align*}
  u_1 &= \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \\
  u_2 &= \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \\
  u_3 &= \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}
\end{align*}
\]

with $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$
Colour decomposition of the tree $n$-point amplitude

\[
\mathcal{M}_n^{(0)} = 2^{n/2} g^{n - 2} \sum_{S_n/Z_n} \text{tr}(T^{d_1} \cdots T^{d_n}) m_n^{(0)}(1, \ldots, n)
\]

$m_n^{(0)}(1, 2, \ldots, n)$ colour-stripped amplitude

MHV amplitude $m_n^{(0)}(1, 2, \ldots, n) = \frac{\langle p_i p_j \rangle^4}{\langle p_1 p_2 \rangle \cdots \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$
Regge factorisation of the 4-pt amplitude

colour-stripped 4-pt amplitude  \( g_1 g_2 \to g_3 g_4 \)  in the Regge limit  \( s \gg -t \)

\[
m_4(1, 2, 3, 4) = s \left[ g \, C(p_2, p_3, \tau) \right] \frac{1}{t} \left(-\frac{s}{\tau}\right)^{\alpha(t)} \left[ g \, C(p_1, p_4, \tau) \right]
\]

\( \alpha(t) \) Regge trajectory  \( C(p_2, p_3, \tau) \) coefficient function  \( \Gamma \) Regge-factorisation scale

\[
\alpha(t) = \bar{g}^2 \bar{\alpha}^{(1)}(t) + \bar{g}^4 \bar{\alpha}^{(2)}(t) + \bar{g}^6 \bar{\alpha}^{(3)}(t) + O(\bar{g}^8) \\
\bar{g}^2 = g^2 N c_G
\]

\[
C(p_i, p_j, \tau) = C^{(0)}(p_i, p_j) \left( 1 + \bar{g}^2 \bar{C}^{(1)}(t, \tau) + \bar{g}^4 \bar{C}^{(2)}(t, \tau) + \bar{g}^6 \bar{C}^{(3)}(t, \tau) + O(\bar{g}^8) \right)
\]

\( \bar{\alpha}^{(n)}(t), \quad \bar{C}^{(n)}(t, \tau) \) are re-scaled loop coefficients

\[
\bar{\alpha}^{(n)}(t) = \left( \frac{\mu}{-t} \right)^{ne} \alpha^{(n)}, \quad \bar{C}^{(n)}(t, \tau) = \left( \frac{\mu}{-t} \right)^{ne} C^{(n)}(t, \tau)
\]
Regge factorisation of the 4-pt amplitude

colour-stripped 4-pt amplitude \( g_1 g_2 \rightarrow g_3 g_4 \) in the Regge limit \( s \gg -t \)

\[
m_4(1, 2, 3, 4) = s \left[ g C(p_2, p_3, \tau) \right] \frac{1}{t} \left( \frac{-s}{\tau} \right)^{\alpha(t)} \left[ g C(p_1, p_4, \tau) \right]
\]

\( \alpha(t) \) Regge trajectory \( C(p_2, p_3, \tau) \) coefficient function \( T \) Regge-factorisation scale

\[
\alpha(t) = \bar{g}^2 \bar{\alpha}^{(1)}(t) + \bar{g}^4 \bar{\alpha}^{(2)}(t) + \bar{g}^6 \bar{\alpha}^{(3)}(t) + O(\bar{g}^8)
\]

\[
C(p_i, p_j, \tau) = C^{(0)}(p_i, p_j) \left(1 + \bar{g}^2 \bar{C}^{(1)}(t, \tau) + \bar{g}^4 \bar{C}^{(2)}(t, \tau) + \bar{g}^6 \bar{C}^{(3)}(t, \tau) + O(\bar{g}^8)\right)
\]

\( \bar{\alpha}^{(n)}(t), \quad \bar{C}^{(n)}(t, \tau) \) are re-scaled loop coefficients

\[
\bar{\alpha}^{(n)}(t) = \left( \frac{\mu^2}{-t} \right)^{n\epsilon} \alpha^{(n)}(t), \quad \bar{C}^{(n)}(t, \tau) = \left( \frac{\mu^2}{-t} \right)^{n\epsilon} C^{(n)}(t, \tau)
\]

Because the Regge limit is exponential in the Regge trajectory,
one can use (the logarithm of) the BDS ansatz to obtain
the Regge trajectory to all loops

\[
\alpha^{(l)}(\epsilon) = 2^{l-1} \alpha^{(1)}(l\epsilon) \left( \frac{\hat{\gamma}^{(l)}}{4} + \epsilon \frac{l}{2} \hat{G}^{(l)} \right) + O(\epsilon)
\]

\( \alpha^{(1)}(\epsilon) = \frac{2}{\epsilon} \)
Caveat

In QCD the standard Regge factorisation is on the colour-dressed amplitude

\[ M_4(1, 2, 3, 4) = s \left[ i g f^{a b e} C(p_2, p_3, \tau) \right] \frac{1}{t} \left( -\frac{s}{\tau} \right)^{\alpha(t)} \left[ i g f^{c d e} C(p_1, p_4, \tau) \right] \]

but it is known to be only approximate

other colour structures occur at one loop

C.R. Schmidt VDD 98

Kuraev Fadin Lipatov 76
Fadin Lipatov 93
Regge factorisation of the 1-loop 4-pt amplitude

\[ m_4^{(1)} = \bar{\alpha}^{(1)}(t)L + 2\bar{C}^{(1)}(t, \tau) \]
Regge factorisation of the 1-loop 4-pt amplitude

\[ m_4^{(1)} = \tilde{\alpha}^{(1)}(t)L + 2\tilde{C}^{(1)}(t, \tau) \]

valid to all orders in \( \varepsilon \)
Regge factorisation of the 1-loop 4-pt amplitude

\[ m_4^{(1)} = \bar{\alpha}^{(1)}(t) L + 2\bar{C}^{(1)}(t, \tau) \]

valid to all orders in \( \epsilon \)

I-loop coefficient function

\[
C^{(1)}(t, \tau) = \frac{\psi(1 + \epsilon) - 2\psi(-\epsilon) + \psi(1)}{\epsilon} - \frac{1}{\epsilon} \ln \frac{-t}{\tau}
\]

\[
= \frac{1}{\epsilon^2} \left( -2 - \epsilon \ln \frac{-t}{\tau} + 3 \sum_{n=1}^{\infty} \zeta_{2n} \epsilon^{2n} + \sum_{n=1}^{\infty} \zeta_{2n+1} \epsilon^{2n+1} \right)
\]
Factorisation of the 2-loop amplitude

\[ m_4^{(2)} = \frac{1}{2} \left( \tilde{\alpha}^{(1)}(t) \right)^2 L^2 \]

\[ + \left( \tilde{\alpha}^{(2)}(t) + 2 \bar{C}^{(1)}(t, \tau) \tilde{\alpha}^{(1)}(t) \right) L \]

\[ + 2 \bar{C}^{(2)}(t, \tau) + \left( \bar{C}^{(1)}(t, \tau) \right)^2 \]

valid to all orders in $\varepsilon$
Factorisation of the 2-loop amplitude

\[ m^{(2)}_4 = \frac{1}{2} \left( \bar{\alpha}^{(1)}(t) \right)^2 L^2 \]
\[ + \left( \bar{\alpha}^{(2)}(t) + 2 \bar{C}^{(1)}(t, \tau) \bar{\alpha}^{(1)}(t) \right) L \]
\[ + 2 \bar{C}^{(2)}(t, \tau) + \left( \bar{C}^{(1)}(t, \tau) \right)^2 \]

valid to all orders in \( \epsilon \)

a more efficient way of writing it

\[ m^{(2)}_4 = \frac{1}{2} \left( m^{(1)}_4 \right)^2 + \bar{\alpha}^{(2)}(t) L + 2 \bar{C}^{(2)}(t, \tau) - \left( \bar{C}^{(1)}(t, \tau) \right)^2 \]

where \( m^{(1)}_4 \) must be known at least through \( \mathcal{O}(\epsilon^2) \)
Factorisation of the 2-loop amplitude

\[ m_4^{(2)} = \frac{1}{2} \left( \bar{\alpha}^{(1)}(t) \right)^2 L^2 \]
\[ + \left( \bar{\alpha}^{(2)}(t) + 2 \bar{C}^{(1)}(t, \tau) \bar{\alpha}^{(1)}(t) \right) L \]
\[ + 2 \bar{C}^{(2)}(t, \tau) + \left( \bar{C}^{(1)}(t, \tau) \right)^2 \]

valid to all orders in \( \epsilon \)

a more efficient way of writing it

\[ m_4^{(2)} = \frac{1}{2} \left( m_4^{(1)} \right)^2 + \bar{\alpha}^{(2)}(t) L + 2 \bar{C}^{(2)}(t, \tau) - \left( \bar{C}^{(1)}(t, \tau) \right)^2 \]

where \( m_4^{(1)} \) must be known at least through \( \mathcal{O}(\epsilon^2) \)
by direct calculation from
the 2-loop 4-pt amplitude \( m_4^{(2)} \) to \( O(\varepsilon^2) \)
we get 2-loop trajectory

\[
\alpha^{(2)} = -\frac{2\zeta_2}{\varepsilon} - 2\zeta_3 - 8\zeta_4 \varepsilon + (36\zeta_2 \zeta_3 + 82\zeta_5) \varepsilon^2 + O(\varepsilon^3)
\]

2-loop coefficient function

\[
C^{(2)}(t, \tau) = \frac{1}{2} \left[ C^{(1)}(t, \tau) \right]^2 + \frac{\zeta_2}{\varepsilon^2} + \left( \zeta_3 + \zeta_2 \ln \frac{-t}{\tau} \right) \frac{1}{\varepsilon}
\]

\[
+ \left( \zeta_3 \ln \frac{-t}{\tau} - 19\zeta_4 \right) + \left( 4\zeta_4 \ln \frac{-t}{\tau} - 2\zeta_2 \zeta_3 - 39\zeta_5 \right) \varepsilon
\]

\[
- \left( 48\zeta_3^2 + \frac{1773}{8}\zeta_6 + (18\zeta_2 \zeta_3 + 41\zeta_5) \ln \frac{-t}{\tau} \right) \varepsilon^2 + O(\varepsilon^3)
\]

where \( C^{(1)}(t, \tau, \varepsilon) \) must be known at least through \( O(\varepsilon^2) \)
A similar factorisation holds also for QCD amplitudes. In that case, the 2-loop 4-parton amplitude $m_4^{(2)}$ yields the 2-loop trajectory

$$\alpha^{(2)} = C_A \left[ \beta_0 \frac{1}{\epsilon^2} + K \frac{2}{\epsilon} + C_A \left( \frac{404}{27} - 2\zeta_3 \right) - \frac{56}{27} N_F \right] + O(\epsilon)$$

$$\beta_0 = \frac{11}{3} C_A - \frac{2}{3} N_F$$

$$K = \left( \frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} N_F$$

maximal transcendentality:

$$\zeta_n, \ln^n, \epsilon^{-n}$$ have weight $n$ in transcendentality

$N=4$ SYM amplitudes, and quantities derived from them, are homogeneous polynomials of maximal transcendentality
the iteration formula for the 2-loop $n$-pt amplitude $m_n^{(2)}$

$$m_n^{(2)}(\epsilon) = \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 + \frac{2 G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) + 4 Const^{(2)} + O(\epsilon)$$

valid for $n = 4, 5$

$$f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2 \quad Const^{(2)} = -\frac{\zeta_2^2}{2}$$

(we use a different normalisation from BDS)

$$G(\epsilon) = \frac{e^{-\gamma \epsilon} \Gamma(1 - 2\epsilon)}{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)} = 1 + O(\epsilon^2)$$
BDS ansatz and Regge limit

the iteration formula for the 2-loop $n$-pt amplitude $m_n^{(2)}$

$$m_n^{(2)}(\epsilon) = \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 + \frac{2 G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) + 4 \text{Const}^{(2)} + \mathcal{O}(\epsilon)$$

valid for $n = 4, 5$

$$f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2$$

$$\text{Const}^{(2)} = -\frac{\zeta_2^2}{2}$$

(we use a different normalisation from BDS)

$$G(\epsilon) = \frac{e^{-\gamma \epsilon} \Gamma(1 - 2\epsilon)}{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)} = 1 + \mathcal{O}(\epsilon^2)$$

from the iteration formula and Regge factorisation

we obtain iteration formulae for the Regge trajectory and the coefficient function

$$\alpha^{(2)}(\epsilon) = 2 f^{(2)}(\epsilon) \alpha^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

$$C^{(2)}(t, \tau, \epsilon) = \frac{1}{2} \left[ C^{(1)}(t, \tau, \epsilon) \right]^2 + \frac{2 G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon) C^{(1)}(t, \tau, 2\epsilon) + 2 \text{Const}^{(2)} + \mathcal{O}(\epsilon)$$

where $C^{(1)}(t, \tau, \epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$
the formulae for $n = 4$ implied by the BDS ansatz and by Regge factorisation differ in that

BDS: valid for arbitrary kinematics, but to $O(\varepsilon^0)$

Regge: valid to all orders in $\varepsilon$, but only in the Regge kinematics.

They overlap and agree in the Regge kinematics to $O(\varepsilon^0)$
Regge factorisation at 3 loops

\[ m_4^{(3)} = m_4^{(2)} m_4^{(1)} - \frac{1}{3} \left( m_4^{(1)} \right)^3 \]

\[ + \quad \bar{\alpha}^{(3)}(t)L + 2 \bar{C}^{(3)}(t, \tau) - 2 \bar{C}^{(2)}(t, \tau) \bar{C}^{(1)}(t, \tau) + \frac{2}{3} \left( \bar{C}^{(1)}(t, \tau) \right)^3 \]

with 3-loop trajectory

\[ \bar{\alpha}^{(3)} = \frac{44\zeta_4}{3\epsilon} + \frac{40}{3}\zeta_2\zeta_3 + 16\zeta_5 + \mathcal{O}(\epsilon) \]

3-loop coefficient function

\[ C^{(3)}(t, \tau) = C^{(2)}(t, \tau) C^{(1)}(t, \tau) - \frac{1}{3} \left[ C^{(1)}(t, \tau) \right]^3 \]

\[ - \quad \frac{44}{9} \frac{\zeta_4}{\epsilon^2} - \left( \frac{40}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5 + \frac{22}{3}\zeta_4 \ln \frac{-t}{\tau} \right) \frac{1}{\epsilon} \]

\[ + \quad \frac{3982}{27}\zeta_6 - \frac{68}{9}\zeta_3^2 - \left( 8\zeta_5 + \frac{20}{3}\zeta_2\zeta_3 \right) \ln \frac{-t}{\tau} + \mathcal{O}(\epsilon) \]

where \( C^{(1)}(t, \tau, \epsilon) \) must be known at least through \( \mathcal{O}(\epsilon^4) \)

\[ C^{(2)}(t, \tau, \epsilon) \quad \mathcal{O}(\epsilon^2) \]

Glover VDD 08

Saturday, July 11, 2009
BDS ansatz and 3-loop Regge factorisation

from BDS’s iteration formula for the 3-loop 4-point amplitude and Regge factorisation, we get iteration formulae for the 3-loop Regge trajectory and coefficient function

\[ \alpha^{(3)}(\epsilon) = 4 f^{(3)}(\epsilon) \alpha^{(1)}(3\epsilon) + \mathcal{O}(\epsilon) \]

\[ C^{(3)}(t, \tau, \epsilon) = C^{(2)}(t, \tau, \epsilon) C^{(1)}(t, \tau, \epsilon) - \frac{1}{3} \left[ C^{(1)}(t, \tau, \epsilon) \right]^3 \]

\[ + \frac{4 G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon) C^{(1)}(t, \tau, 3\epsilon) + 4 \text{Const}^{(3)} + \mathcal{O}(\epsilon) \]

with

\[ f^{(3)}(\epsilon) = \frac{11}{2} \zeta_4 + (6\zeta_5 + 5\zeta_2 \zeta_3)\epsilon + (c_1 \zeta_6 + c_2 \zeta_3^2)\epsilon^2 \]

\[ \text{Const}^{(3)} = \left( \frac{341}{216} + \frac{2}{9} c_1 \right) \zeta_6 + \left( -\frac{17}{9} + \frac{2}{9} c_2 \right) \zeta_3^2 \]

with \( c_1 \) and \( c_2 \) known constants (which drop out of the recursive formula above)
**BDS ansatz and 3-loop Regge factorisation**

From BDS’s iteration formula for the 3-loop 4-point amplitude and Regge factorisation, we get iteration formulae for the 3-loop Regge trajectory and coefficient function

\[
\alpha^{(3)}(\epsilon) = 4f^{(3)}(\epsilon)\alpha^{(1)}(3\epsilon) + O(\epsilon)
\]

\[
C^{(3)}(t, \tau, \epsilon) = C^{(2)}(t, \tau, \epsilon)C^{(1)}(t, \tau, \epsilon) - \frac{1}{3}\left[C^{(1)}(t, \tau, \epsilon)\right]^3
\]

\[
+ \frac{4G^3(\epsilon)}{G(3\epsilon)}f^{(3)}(\epsilon)C^{(1)}(t, \tau, 3\epsilon) + 4Const^{(3)} + O(\epsilon)
\]

With

\[
f^{(3)}(\epsilon) = \frac{11}{2}\zeta_4 + (6\zeta_5 + 5\zeta_2\zeta_3)\epsilon + (c_1\zeta_6 + c_2\zeta_3^2)\epsilon^2
\]

\[
Const^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2
\]

With \(c_1\) and \(c_2\) known constants (which drop out of the recursive formula above)

To \(O(\epsilon^0)\), the BDS iteration formulae above are in agreement with the Regge formulae of the previous slide
Regge factorisation is valid also for amplitudes with 5 or more points in generalised Regge limits.

The strategy is to use the modular form of the amplitudes dictated by high-energy factorisation, to obtain information on $n$-point amplitudes in terms of building blocks derived from $m$-point amplitudes, with $m < n$.
Regge factorisation of the 5-pt amplitude

5-pt amplitude $g_1 g_2 \rightarrow g_3 g_4 g_5$ in the multi-Regge limit $s \gg s_1, s_2 \gg -t_1, -t_2$

\[ m_5 = s \left[ g C(p_2, p_3, \tau) \right] \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g V(q_2, q_1, \kappa, \tau) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g C(p_1, p_5, \tau) \right] \]

\[ V \text{ is gluon-production vertex;} \quad \kappa = |p_T|^2 \text{ of central gluon} \]
Regge factorisation of the 5-pt amplitude

5-pt amplitude $g_1 g_2 \to g_3 g_4 g_5$ in the multi-Regge limit $s \gg s_1, s_2 \gg -t_1, -t_2$

$m_5 = s \left[ g \, C(p_2, p_3, \tau) \right] \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g \, V(q_2, q_1, \kappa, \tau) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g \, C(p_1, p_5, \tau) \right]$

$V$ is gluon-production vertex; $\kappa = |p_T|^2$ of central gluon

1 loop $m_5^{(1)} = \bar{\alpha}^{(1)}(t_1) L_1 + \bar{\alpha}^{(1)}(t_2) L_2 + \bar{C}^{(1)}(t_1, \tau) + \bar{C}^{(1)}(t_2, \tau) + \bar{V}^{(1)}(t_1, t_2, \kappa, \tau)$
Regge factorisation of the 5-pt amplitude

5-pt amplitude $g_1 g_2 \rightarrow g_3 g_4 g_5$ in the multi-Regge limit $s \gg s_1, s_2 \gg -t_1, -t_2$

$$m_5 = s \left[ g\, C(p_2, p_3, \tau) \right] \frac{1}{t_2} \left( -\frac{s_2}{\tau} \right)^{\alpha(t_2)} \left[ g\, V(q_2, q_1, \kappa, \tau) \right] \frac{1}{t_1} \left( -\frac{s_1}{\tau} \right)^{\alpha(t_1)} \left[ g\, C(p_1, p_5, \tau) \right]$$

$V$ is gluon-production vertex; $\kappa = |p_T|^2$ of central gluon

1 loop

$$m_5^{(1)} = \bar{\alpha}^{(1)}(t_1)L_1 + \bar{\alpha}^{(1)}(t_2)L_2 + \bar{C}^{(1)}(t_1, \tau) + \bar{C}^{(1)}(t_2, \tau) + \bar{V}^{(1)}(t_1, t_2, \kappa, \tau)$$

2 loops

$$m_5^{(2)} = \frac{1}{2} \left( m_5^{(1)} \right)^2 + \bar{\alpha}^{(2)}(t_1)L_1 + \bar{\alpha}^{(2)}(t_2)L_2$$

$$+ \bar{C}^{(2)}(t_1, \tau) + \bar{V}^{(2)}(t_1, t_2, \kappa, \tau) + \bar{C}^{(2)}(t_2, \tau)$$

$$- \frac{1}{2} \left( \bar{C}^{(1)}(t_1, \tau) \right)^2 - \frac{1}{2} \left( \bar{V}^{(1)}(t_1, t_2, \kappa, \tau) \right)^2 - \frac{1}{2} \left( \bar{C}^{(1)}(t_2, \tau) \right)^2$$

where $m_5^{(1)}$ must be known at least through $O(\epsilon^2)$
BDS ansatz and Regge limit for the 5-pt amplitude

Using the BDS and Regge 2-loop iteration formula for the 5-pt amplitude $m_5^{(2)}$ and the iteration formulae for the trajectory and the coefficient functions, one obtains a 2-loop iteration formula for the gluon-production vertex

$$ V^{(2)}(t_1, t_2, \kappa, \tau, \epsilon) = \frac{1}{2} \left[ V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon) \right]^2 + \frac{2 G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon) V^{(1)}(t_1, t_2, \kappa, \tau, 2\epsilon) + \mathcal{O}(\epsilon) $$

where $V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$
BDS ansatz and Regge limit for the 5-pt amplitude

Using the BDS and Regge 2-loop iteration formula for the 5-pt amplitude \( m_5^{(2)} \) and the iteration formulae for the trajectory and the coefficient functions, one obtains a 2-loop iteration formula for the gluon-production vertex

\[
V^{(2)}(t_1, t_2, \kappa, \tau, \epsilon) = \frac{1}{2} \left[ V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon) \right]^2 + \frac{2 G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon) V^{(1)}(t_1, t_2, \kappa, \tau, 2\epsilon) + \mathcal{O}(\epsilon)
\]

where \( V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon) \) must be known through \( \mathcal{O}(\epsilon^2) \)

Similarly, at 3 loops

\[
V^{(3)}(t_1, t_2, \kappa, \tau, \epsilon) = V^{(2)}(t_1, t_2, \kappa, \tau, \epsilon) V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon) - \frac{1}{3} \left[ V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon) \right]^3
\]

\[
+ \frac{4 G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon) V^{(1)}(t_1, t_2, \kappa, \tau, 3\epsilon) + \mathcal{O}(\epsilon)
\]

where \( V^{(1)}(t_1, t_2, \kappa, \tau, \epsilon) \) must be known through \( \mathcal{O}(\epsilon^4) \)

\( V^{(2)}(t_1, t_2, \kappa, \tau, \epsilon) \) \( \mathcal{O}(\epsilon^2) \)
1-loop 5-pt amplitude

\[ m_5^{(1)} = -\frac{1}{4} \sum_{\text{cyclic}} s_{12} s_{23} I_4^{1m}(1, 2, 3, 45, \epsilon) - \frac{\epsilon}{2} \epsilon_{1234} I_5^{6-2\epsilon}(\epsilon) \]

\[ \epsilon_{1234} = \text{tr}[\gamma_5 k_1 k_2 k_3 k_4] \]

one-mass boxes known to all orders in \( \epsilon \)

(6-2\( \epsilon \))-dim pentagon IR finite, but irreducible, and unknown analytically

I-loop 5-pt amplitude computed through \( O(\epsilon^2) \) numerically

Saturday, July 11, 2009
\[ m_5^{(1)} = -\frac{1}{4} \sum_{\text{cyclic}} s_{12} s_{23} I_{4}^{1m}(1, 2, 3, 45, \epsilon) - \frac{\epsilon}{2} \epsilon_{1234} I_{5}^{6-2\epsilon}(\epsilon) \]

\[ \epsilon_{1234} = \text{tr}[\gamma_5 k_1 k_2 k_3 k_4] \]

parity-even and \( O(\epsilon^2) \)

parity-odd and \( O(\epsilon) \)

one-mass boxes known to all orders in \( \epsilon \)

\( (6-2\epsilon)\)-dim pentagon IR finite, but irreducible, and unknown analytically

\[ \text{1-loop 5-pt amplitude computed through } O(\epsilon^2) \text{ numerically} \]

\[ m_5^{(1)} = -\frac{1}{4} \sum_{\text{cyclic}} s_{12} s_{23} I_{4}^{1m}(1, 2, 3, 45, \epsilon) - \frac{\epsilon}{2} \epsilon_{1234} I_{5}^{6-2\epsilon}(\epsilon) \]

\[ \epsilon_{1234} = \text{tr}[\gamma_5 k_1 k_2 k_3 k_4] \]

Bern Dixon Dunbar Kosower 97

Cachazo Spradlin Volovich 06
Bern Czakon Kosower Roiban Smirnov 06

Duhr Glover Smirnov VDD 09

in multi-Regge kinematics, we have computed analytically the 1-loop 5-pt amplitude to all orders in \( \epsilon \), expanded through \( O(\epsilon^2) \)
Regge factorisation of the 6-pt amplitude

6-pt amplitude $g_1 g_2 \rightarrow g_3 g_4 g_5 g_6$

in the multi-Regge limit $y_3 \gg y_4 \gg y_5 \gg y_6; \quad |p_3\perp| \simeq |p_4\perp| \simeq |p_5\perp| \simeq |p_6\perp|$

$s \gg s_1, s_2, s_3 \gg -t_1, -t_2, -t_3$

$$m_6 = s \left[ gC(p_2, p_3, \tau) \right] \frac{1}{t_3} \left( \frac{-s_3}{\tau} \right)^{\alpha(t_3)} \left[ gV(q_2, q_3, \kappa_2, \tau) \right]$$

$$\times \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ gV(q_1, q_2, \kappa_1, \tau) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ gC(p_1, p_6, \tau) \right]$$
Regge factorisation of the 6-pt amplitude

6-pt amplitude \( g_1 g_2 \rightarrow g_3 g_4 g_5 g_6 \)
in the multi-Regge limit \( y_3 \gg y_4 \gg y_5 \gg y_6; \quad |p_{3\perp}| \approx |p_{4\perp}| \approx |p_{5\perp}| \approx |p_{6\perp}| \)

\[
m_6 = s \left[ g \, C(p_2, p_3, \tau) \right] \frac{1}{t_3} \left( \frac{-s_3}{\tau} \right)^{\alpha(t_3)} \left[ g \, V(q_2, q_3, \kappa_2, \tau) \right] \\
\times \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g \, V(q_1, q_2, \kappa_1, \tau) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g \, C(p_1, p_6, \tau) \right]
\]

no new vertices or coefficient functions appear, wrt \( n = 5 \)

The \( l \)-loop 6-pt amplitude can then be assembled using the \( l \)-loop trajectories, gluon-production vertices and coefficient functions, which can be determined through the \( l \)-loop 4-pt and 5-pt amplitudes
Regge factorisation of the 6-pt amplitude

6-pt amplitude \( g_1 g_2 \to g_3 g_4 g_5 g_6 \)
in the multi-Regge limit \( y_3 \gg y_4 \gg y_5 \gg y_6; \mid p_{3\perp} \mid \simeq \mid p_{4\perp} \mid \simeq \mid p_{5\perp} \mid \simeq \mid p_{6\perp} \mid \)
\[ s \gg s_1, s_2, s_3 \gg -t_1, -t_2, -t_3 \]

\[ m_6 = s \left[ g C(p_2, p_3, \tau) \right] \frac{1}{t_3} \left( \frac{-s_3}{\tau} \right)^{\alpha(t_3)} \left[ g V(q_2, q_3, \kappa_2, \tau) \right] \]
\[ \times \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g V(q_1, q_2, \kappa_1, \tau) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g C(p_1, p_6, \tau) \right] \]

no new vertices or coefficient functions appear, wrt \( n = 5 \)

The \( l \)-loop 6-pt amplitude can then be assembled using the \( l \)-loop trajectories, gluon-production vertices and coefficient functions, which can be determined through the \( l \)-loop 4-pt and 5-pt amplitudes

Thus, also the \( l \)-loop BDS iterative formula for \( n = 6 \) will be fulfilled

the multi-Regge limit is not able to detect
the BDS-ansatz violation for \( n = 6 \)
the remainder function of the 6-pt amplitude depends on
3 conformally-invariant cross-ratios

\[ R_6^{(2)} = R_6^{(2)}(u_1, u_2, u_3) \]

\[ u_1 = \frac{s_{12} s_{45}}{s_{345} s_{456}}, \quad u_2 = \frac{s_{23} s_{56}}{s_{234} s_{456}}, \quad u_3 = \frac{s_{34} s_{61}}{s_{234} s_{345}} \]
the remainder function of the 6-pt amplitude depends on

\[ R^{(2)}_6 = R^{(2)}_6(u_1, u_2, u_3) \]

3 conformally-invariant cross-ratios

\[ u_1 = \frac{s_{12} s_{45}}{s_{345} s_{456}}, \quad u_2 = \frac{s_{23} s_{56}}{s_{234} s_{456}}, \quad u_3 = \frac{s_{34} s_{61}}{s_{234} s_{345}} \]

in the multi-Regge kinematics

\[ u_1 = 1 + O\left(\frac{t}{s}\right), \quad u_2 = O\left(\frac{t}{s}\right), \quad u_3 = O\left(\frac{t}{s}\right) \]

like in the collinear limit
1-loop 6-pt amplitude

computed through $O(\varepsilon^2)$ numerically

even Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
odd Cachazo Spradlin Volovich 08

through $O(\varepsilon^0)$, it is given in terms of 1m and 2me boxes
at $O(\varepsilon)$ a hexagon occurs in the even part

$$s \equiv s_{12}, \quad t_3 \equiv s_{23}, \quad s_3 \equiv s_{34}, \quad s_2 \equiv s_{45}, \quad s_1 \equiv s_{56}, \quad t_1 \equiv s_{61}$$
1-loop 6-pt amplitude

computed through $O(\varepsilon^2)$ numerically

**even** Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

**odd** Cachazo Spradlin Volovich 08

through $O(\varepsilon^0)$, it is given in terms of $1m$ and $2me$ boxes

at $O(\varepsilon)$ a hexagon occurs in the even part

$$s \equiv s_{12}, \quad t_3 \equiv s_{23}, \quad s_3 \equiv s_{34}, \quad s_2 \equiv s_{45}, \quad s_1 \equiv s_{56}, \quad t_1 \equiv s_{61}$$

multi-Regge kinematics (in Euclidean region)

$$-s \gg -s_1, -s_2, -s_3 \gg -t_1, -t_2, -t_3$$

$$s_1 \rightarrow \lambda^2 s_1, \quad s_2 \rightarrow \lambda^2 s_2, \quad s_3 \rightarrow \lambda^2 s_3, \quad t_1 \rightarrow \lambda^3 t_1, \quad t_3 \rightarrow \lambda^3 t_3, \quad \lambda \ll 1$$
1-loop 6-pt amplitude

computed through $O(\varepsilon^2)$ numerically

even Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
odd Cachazo Spradlin Volovich 08

through $O(\varepsilon^0)$, it is given in terms of $1m$ and $2me$ boxes
at $O(\varepsilon)$ a hexagon occurs in the even part

$$s \equiv s_{12}, \quad t_3 \equiv s_{23}, \quad s_3 \equiv s_{34}, \quad s_2 \equiv s_{45}, \quad s_1 \equiv s_{56}, \quad t_1 \equiv s_{61}$$

multi-Regge kinematics (in Euclidean region)

$$-s \gg -s_1, -s_2, -s_3 \gg -t_1, -t_2, -t_3$$

$$s_1 \to \lambda^2 s_1, \quad s_2 \to \lambda^2 s_2, \quad s_3 \to \lambda^2 s_3, \quad t_1 \to \lambda^3 t_1, \quad t_3 \to \lambda^3 t_3, \quad \lambda \ll 1$$

to all orders in $\varepsilon$, the hexagon integral is reduced to:
triple sums in NDIM,
3-fold integrals through Mellin-Barnes
Regge factorisation of the $n$-pt amplitude

$$m_n(1,2,\ldots,n) = s \left[ g \ C(p_2,p_3) \right] \frac{1}{t_{n-3}} \left( \frac{-s_{n-3}}{\tau} \right)^{\alpha(t_{n-3})} \left[ g \ V(q_{n-3},q_{n-4},\kappa_{n-4}) \right]$$

$$\cdots \times \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g \ V(q_2,q_1,\kappa_1) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g \ C(p_1,p_n) \right]$$

$n$-pt amplitude in the multi-Regge limit

$y_3 \gg y_4 \gg \cdots \gg y_n$; $|p_3\perp| \simeq |p_4\perp| \cdots \simeq |p_n\perp|$ 

$s \gg s_1, s_2, \ldots, s_{n-3} \gg -t_1, -t_2 \ldots, -t_{n-3}$
**Regge factorisation of the n-pt amplitude**

\[
m_n(1, 2, \ldots, n) = s \left[ g C(p_2, p_3) \right] \frac{1}{t_{n-3}} \left( \frac{-s_{n-3}}{\tau} \right)^{\alpha(t_{n-3})} \left[ g V(q_{n-3}, q_{n-4}, \kappa_{n-4}) \right] \\
\cdots \times \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g V(q_2, q_1, \kappa_1) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g C(p_1, p_n) \right]
\]

**n-pt amplitude in the multi-Regge limit**

\( y_3 \gg y_4 \gg \cdots \gg y_n; \quad |p_3 \perp| \simeq |p_4 \perp| \cdots \simeq |p_n \perp| \)

\( s \gg s_1, s_2, \ldots, s_{n-3} \gg -t_1, -t_2 \ldots, -t_{n-3} \)

What we said for \( n = 6 \) can be repeated in general: the \( l \)-loop \( n \)-pt amplitude can be assembled using the \( l \)-loop trajectories, vertices and coefficient functions, determined through the \( l \)-loop 4-pt and 5-pt amplitudes

---

\[
\text{no violation of the BDS ansatz can be found in the multi-Regge limit}
\]
To have a chance to detect the violation of the BDS ansatz for the 2-loop 6-pt amplitude, that we see in arbitrary kinematics, we must relax the strong-ordering constraints of the multi-Regge kinematics
$m_n(1, 2, \ldots, n) = s \left[ g^2 A(p_2, p_3, p_4) \right] \frac{1}{t_{n-4}} \left( \frac{-s_{n-4}}{\tau} \right)^{\alpha(t_{n-4})} \left[ g V(q_{n-4}, q_{n-5}, \kappa_{n-5}) \right]$

\[ \cdots \times \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g V(q_2, q_1, \kappa_1) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g C(p_1, p_n) \right] \]

quasi-multi-Regge kinematics

$y_3 \sim y_4 \gg \cdots \gg y_n; \quad |p_3\perp| \sim |p_4\perp| \cdots \sim |p_n\perp|$
$n$-pt amplitude in quasi-multi-Regge kinematics

$$m_n(1, 2, \ldots, n) = s \left[ g^2 A(p_2, p_3, p_4) \right] \frac{1}{t_{n-4}} \left( \frac{-s_{n-4}}{\tau} \right)^{\alpha(t_{n-4})} \left[ g V(q_{n-4}, q_{n-5}, \kappa_{n-5}) \right]$$

$$\cdots \times \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g V(q_2, q_1, \kappa_1) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g C(p_1, p_n) \right]$$

quasi-multi-Regge kinematics

$$y_3 \simeq y_4 \gg \cdots \gg y_n; \quad |p_3\perp| \simeq |p_4\perp| \cdots \simeq |p_n\perp|$$

A new coefficient function $A(p_2, p_3, p_4, \tau)$ occurs already at $n = 5,$ for which the BDS ansatz is fulfilled. Because no new coefficient functions appear for $n \geq 6,$ a violation of the BDS ansatz cannot be found even in this case.
$n$-pt amplitude in quasi-multi-$\text{Regge}$ kinematics

$$m_n(1, 2, \ldots, n) = s \left[ g^2 A(p_2, p_3, p_4) \right] \frac{1}{t_{n-4}} \left( \frac{-s_{n-4}}{\tau} \right)^{\alpha(t_{n-4})} \left[ g V(q_{n-4}, q_{n-5}, \kappa_{n-5}) \right]$$

$$\ldots \times \frac{1}{t_2} \left( \frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[ g V(q_2, q_1, \kappa_1) \right] \frac{1}{t_1} \left( \frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[ g C(p_1, p_n) \right]$$

quasi-multi-Regge kinematics

$$y_3 \simeq y_4 \gg \cdots \gg y_n; \quad |p_3 \perp| \simeq |p_4 \perp| \cdots \simeq |p_n \perp|$$

A new coefficient function $A(p_2, p_3, p_4, \tau)$ occurs already at $n = 5$, for which the BDS ansatz is fulfilled. Because no new coefficient functions appear for $n \geq 6$, a violation of the BDS ansatz cannot be found even in this case.

The same can be said for the quasi-multi-Regge kinematics

$$y_3 \simeq y_4 \gg \cdots \gg y_{n-1} \simeq y_n; \quad |p_3 \perp| \simeq |p_4 \perp| \cdots \simeq |p_n \perp|$$
in the quasi-multi-Regge kinematics

\[ y_3 \simeq y_4 \gg \cdots \gg y_n; \quad |p_{3 \perp}| \simeq |p_{4 \perp}| \cdots \simeq |p_{n \perp}| \]

the 3 conformally-invariant cross-ratios

\[ u_1 = \frac{s_{12} s_{45}}{s_{345} s_{456}}, \quad u_2 = \frac{s_{23} s_{56}}{s_{234} s_{456}}, \quad u_3 = \frac{s_{34} s_{61}}{s_{234} s_{345}} \]

take the values

\[ u_1 = 1 + \mathcal{O}\left(\frac{t}{s}\right), \quad u_2 = \mathcal{O}\left(\frac{t}{s}\right), \quad u_3 = \mathcal{O}\left(\frac{t}{s}\right) \]

like in the multi-Regge kinematics and in the collinear limit
More general quasi-multi-Regge kinematics

A necessary condition to see a violation of the BDS ansatz for the 2-loop 6-pt amplitude, is to go to a quasi-multi-Regge kinematics for which new coefficient functions appear for $n \geq 6$. 
More general quasi-multi-Regge kinematics

A necessary condition to see a violation of the BDS ansatz for the 2-loop 6-pt amplitude, is to go to a quasi-multi-Regge kinematics for which new coefficient functions appear for $n \geq 6$

two such quasi-multi-Regge kinematics are
More general quasi-multi-Regge kinematics

A necessary condition to see a violation of the BDS ansatz for the 2-loop 6-pt amplitude, is to go to a quasi-multi-Regge kinematics for which new coefficient functions appear for \( n \geq 6 \)

two such quasi-multi-Regge kinematics are

\[
\begin{align*}
\text{in both cases, the 3 conformally-invariant cross-ratios take values} \\
u_1 &= O(1), \quad u_2 = O(1), \quad u_3 = O(1)
\end{align*}
\]

it remains to be seen if these kinematics harbour a violation of the BDS ansatz
Conclusions

in multi-Regge kinematics, we have computed analytically the (6-2$\varepsilon$)-dim pentagon integral, and so the 1-loop 5-pt amplitude through $O(\varepsilon^2)$

Duhr’s talk on Friday
Conclusions

In multi-Regge kinematics, we have computed analytically the $(6-2\varepsilon)$-dim pentagon integral, and so the 1-loop 5-pt amplitude through $O(\varepsilon^2)$ by the ABDK/BDS iteration, we can get the 2-loop 5-pt amplitude up to finite terms.

Duhr’s talk on Friday
Conclusions

in multi-Regge kinematics, we have computed analytically the \((6-2\epsilon)\)-dim pentagon integral, and so the 1-loop 5-pt amplitude through \(O(\epsilon^2)\).

by the ABDK/BDS iteration, we can get the 2-loop 5-pt amplitude up to finite terms.

by the Regge factorisation, the 1-loop 5-pt amplitude allows us to extract the 1-loop gluon-production vertex through \(O(\epsilon^2)\), and by the ABDK/BDS iteration the 2-loop gluon-production vertex up to finite terms.

Duhr’s talk on Friday
Conclusions

in multi-Regge kinematics, we have computed analytically the $(6-2\varepsilon)$-dim pentagon integral, and so the 1-loop 5-pt amplitude through $O(\varepsilon^2)$

by the ABDK/BDS iteration, we can get the 2-loop 5-pt amplitude up to finite terms

by the Regge factorisation, the 1-loop 5-pt amplitude allows us to extract the 1-loop gluon-production vertex through $O(\varepsilon^2)$, and by the ABDK/BDS iteration the 2-loop gluon-production vertex up to finite terms

by the factorisation of the $l$-loop $n$-pt amplitude in multi-Regge kinematics, we can build the amplitude in terms of $l$-loop coefficient functions and gluon-production vertices

Duhr’s talk on Friday
Conclusions

in multi-Regge kinematics, we have computed analytically the $(6-2\varepsilon)$-dim pentagon integral, and so the 1-loop 5-pt amplitude through $O(\varepsilon^2)$

by the ABDK/BDS iteration, we can get the 2-loop 5-pt amplitude up to finite terms

by the Regge factorisation, the 1-loop 5-pt amplitude allows us to extract the 1-loop gluon-production vertex through $O(\varepsilon^2)$, and by the ABDK/BDS iteration the 2-loop gluon-production vertex up to finite terms

by the factorisation of the $l$-loop $n$-pt amplitude in multi-Regge kinematics, we can build the amplitude in terms of $l$-loop coefficient functions and gluon-production vertices

the $l$-loop $n$-pt amplitude so built fulfils the BDS ansatz, thus any ansatz violation must be searched in less constraining (quasi-multi-Regge?) kinematics

Duhr’s talk on Friday