FS 2020

Introduction to Quantum Chromodynamics

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Preface

These lecture notes are based on the handwritten notes by Dr. Vittorio Del Duca for the course *Introduction to Quantum Chromodynamics* taught during FS2019 and FS2020.

The course is based on the following textbooks

- Peskin & Schroeder, *An Introduction to Quantum Field Theory*
- Sterman, *An Introduction to Quantum Field Theory*
- Weinberg, *The Quantum Theory of Fields, Volume II*
- Quigg, *Gauge Theories of the Strong, Weak and Electromagnetic Interactions*
- Feynman, *Photon-Hadron Interactions*
- Muta, *Foundations of Quantum Chromodynamics*
- Field, *Applications of Perturbative QCD*
- Collins, *Foundations of Perturbative QCD*
- Ellis, Stirling & Webber, *QCD and Collider Physics*

Specifically for helicity amplitudes, recommended readings are

- Mangano & Parke, *Multiparton Amplitudes in Gauge Theories* Phys. Rep. 200 (1991) 301
- Dixon, *Calculating Scattering Amplitudes efficiently* (TASI 1995) arXiv:hep-ph/9601359
- Dixon, *A brief introduction to modern amplitude methods* (TASI 2013) arXiv:1310.5353

The lecture notes follow the conventions of Peskin & Schroeder for metrics and Feynman rules.

Contents

Chapter 1

Introduction

This lecture course provides an introduction to Quantum Chromodynamics (QCD). Although the presentation is not in chronological order, it does cover basically the decade which began with the deep inelastic scattering (DIS) experiments, the Parton Model and the establishment of QCD, as an asymptotically free gauge theory which describes the strong interactions among particles.

After a presentation of DIS, Bjorken's scaling and Feynman's Parton Model, we introduce QCD as a non-Abelian gauge theory, and present the renormalisation of its ultraviolet divergences, which occur in the short-distance behaviour of the quantum corrections to the quantities that we compute in field theory. We then discuss how to handle its infrared and mass divergences, which occur when the momentum of one gluon or two or more partons become vanishingly small, or when the momenta of two or more partons become collinear. In order to exhibit how those divergences arise in QCD, we make use of the exemplary cases of the production rate of $e^+e^- \to$ hadrons and DIS.

Chapter 2

Deep Inelastic Scattering

In the invitation (see slides), we stated that QCD is the correct theory to describe the strong interactions, and that it is necessary to describe the high-energy collisions at hadrons colliders like the LHC. We also outlined a historical introduction to the quark model and to hadron spectroscopy, which are based on an SU(3) flavour global symmetry, and argued that to describe the Δ^{++} puzzle and the $e^+e^- \to hadrons$ data, it was necessary to introduce a new internal symmetry: colour. QCD is in fact based on a gauge symmetry, colour, which we shall describe later, but its dynamical foundations rest upon the **parton model**, which was conceived in **deep inelastic scattering (DIS)**. We shall first describe the general properties of DIS, then we shall introduce the parton model.

In the 1960's, proton-proton collisions at $\sqrt{s} \sim 10$ GeV yielded large numbers of pions, distributed uniformly in the scattering angle and with a transverse momentum $P_{\perp\pi} \lesssim 300$ MeV.
The emerging picture was that the proton was a loose bound state of components (quarks?). In an elastic $p-p$ collision, the probability of producing a pion π with momentum $P_\perp$ was thought to be $\propto e^{-P_\perp^2}$ (it has actually a power-like fall-off); low $P_\perp$ meaning small momentum transfer $q^2 = t$: the proton was deemed to have too soft a structure: no high $P_\perp$'s.
However, at SLAC the experimenters were doing DIS: $ep \to eX$, shooting 20 GeV electrons on a fixed proton target. The total rate was comparable to the QED expectation of the proton as an elementary particle. However, they also observed events with a large scattering angle and large Q^2 as if a hard scattering was taking place.

2.1 Kinematics of Deep Inelastic Scattering

We consider the scattering process in figure 2.1. Here q is the space-like momentum of the virtual photon. The following relations hold,

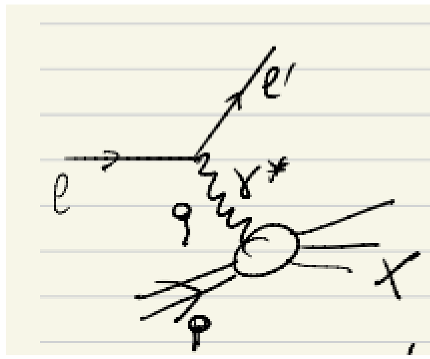


Figure 2.1: Deep inelastic scattering process of a lepton (ℓ) with a proton (P), yielding a lepton (ℓ') and unknown hadrons X.

$$\begin{aligned} \hat{t} &= q^2 = -Q^2, \\ (q+P)^2 &= W^2 = -Q^2 + 2q\cdot P + m_p^2. \end{aligned} \tag{2.1}$$

The centre-of-mass energy is

$$s = (P+\ell)^2 = m_p^2 + 2P\cdot\ell + m_\ell^2 \approx m_p^2 + 2P\cdot\ell, \tag{2.2}$$

where we suppose that s is high, thus we can ignore the electron mass.

- If $W^2 = (\ell - \ell' + P)^2 = m_p^2$ then $Q^2 = 2q\cdot P$, i.e. the collision is *elastic*.
- If $W^2 > m_p^2$ then $Q^2 < 2q\cdot P$, i.e. the collision is *inelastic*.

Suppose we don't know the interaction between the proton and the virtual photon. So the S-matrix element for DIS is

$$i\mathcal{M}(ep\to eX) = -ie\bar{u}(\ell')\gamma_\mu u(\ell)\frac{-i}{q^2}ie\int d^4x\, e^{iq\cdot x}\langle X|\, J^\mu(x)\,|P\rangle\,. \tag{2.3}$$

- $\bar{u}(\ell')\gamma^\mu u(\ell)$ is the leptonic current due to the electron-photon vertex of QED.
- $\langle X|\, J^\mu(x)\,|P\rangle$ is the hadronic current between the proton and the final state X, which features the (unknown) hadron dynamics.

2.2 Hadronic Tensor and Optical Theorem

Now, we introduce the **hadronic tensor**,

$$\tilde{W}^{\mu\nu} \equiv i\int d^4x\, e^{iq\cdot x}\langle P|\, T\big(J^\mu(x)J^\nu(0)\big)\,|P\rangle\,. \tag{2.4}$$

We see $\tilde{W}^{\mu\nu}$ is related to *forward Compton scattering* (depicted in figure 2.2). Indeed the matrix

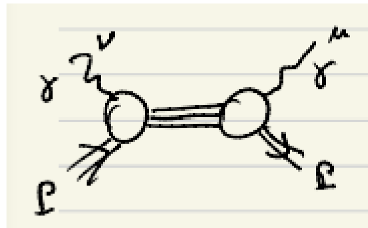


Figure 2.2: Forward Compton scattering.

element for Compton scattering at $q^2 = 0$ is

$$\begin{aligned} i\mathcal{M}(\gamma P \to \gamma P) &= (ie)^2 \epsilon^*_\mu(q)\epsilon_\nu(q) \int d^4x\, e^{iq\cdot x} \langle P| T(J^\mu(x)J^\nu(0)) |P\rangle \\ &= ie^2 \epsilon^*_\mu(q)\epsilon_\nu(q)\tilde{W}^{\mu\nu}(P,q). \end{aligned} \tag{2.5}$$

The **optical theorem**, which stems from the unitarity of the S-matrix, then relates forward Compton scattering to photo-production,

$$2\,\mathrm{Im}(\mathcal{M}(\gamma P \to \gamma P)) = \sum_n \int d\Pi_n |\mathcal{M}(\gamma P \to X_n)|^2, \tag{2.6}$$

where $d\Pi_n$ is a short-hand notation for the n-body phase space,

$$d\Pi_n \equiv \prod_{i=1}^{n} \frac{d^3k_i}{(2\pi)^3 2E_i} \left[(2\pi)^4 \delta^4\left(P + q - \sum_i k_i\right)\right]. \tag{2.7}$$

The matrix element for photon-production is

$$\begin{aligned} \mathcal{M}(\gamma P \to X) &= ie\epsilon^\mu(q) \int d^4x\, e^{iq\cdot x} \langle X| J_\mu(x) |P\rangle \\ &= ie\epsilon^\mu(q) \langle X| J_\mu(q) |P\rangle . \end{aligned} \tag{2.8}$$

The corresponding complex conjugated matrix element is

$$\begin{aligned} \mathcal{M}^*(\gamma P \to X) &= -ie\epsilon^{\mu *}(q) \int d^4x\, e^{-iq\cdot x} \langle P| J_\mu(x) |X\rangle \\ &= -ie\epsilon^{\mu *}(q) \langle P| J_\mu(-q) |X\rangle , \end{aligned} \tag{2.9}$$

where we assumed that the current is Hermitian, i.e. $J_\mu^\dagger = J_\mu$.
So in this case, the optical theorem implies that

$$\begin{aligned} 2\,\mathrm{Im}\,\mathcal{M}(\gamma P \to \gamma P) &= 2e^2 \epsilon^*_\mu(q)\epsilon_\nu(q)\,\mathrm{Im}\left(\tilde{W}^{\mu\nu}(P,q)\right) \\ &= e^2 \epsilon^*_\mu(q)\epsilon_\nu(q) \sum_n \int d\Pi_n \langle P| J^\mu(-q) |X_n\rangle \langle X_n| J^\nu(q) |P\rangle , \end{aligned} \tag{2.10}$$

which connects forward Compton scattering and photon-production through the hadronic tensor $\tilde{W}^{\mu\nu}(P,q)$.

For the relation between the hadronic tensor, Compton scattering and DIS, further reading is in Sterman, *An Introduction to Quantum Field Theory* and Muta, *Foundations of Quantum Chromodynamics*.

Now we may use $2\,\mathrm{Im}\,\tilde{W}^{\mu\nu}(P,q)$ to compute the DIS cross-section,

$$\begin{aligned}\sigma(eP\to eX) =&\frac{1}{2s}\int\frac{d^3\ell'}{(2\pi)^3 2E'}e^4\frac{1}{2}\sum_{spin}\left[\bar{u}(\ell)\gamma_\mu u(\ell')\bar{u}(\ell')\gamma_\nu u(\ell)\right]\\ &\cdot\frac{1}{(Q^2)^2}\sum_n\int d\Pi_n \left\langle P\right| J^\mu(-q)\left|X_n\right\rangle\left\langle X_n\right| J^\nu(q)\left|P\right\rangle ,\end{aligned}\tag{2.11}$$

where we summed (averaged) over the final (initial) electron spin.
Using the optical theorem, the DIS cross section becomes

$$\sigma(eP\to eX) =\frac{1}{2s}\int\frac{d^3\ell'}{(2\pi)^3 2E'}e^4\frac{1}{(Q^2)^2}L_{\mu\nu}2\,\mathrm{Im}\,\tilde{W}^{\mu\nu}(P,q),\tag{2.12}$$

where, neglecting the electron mass, the leptonic tensor is

$$\begin{aligned}L_{\mu\nu} =&\frac{1}{2}\sum_{spin}\left[\bar{u}(\ell)\gamma_\mu u(\ell')\bar{u}(\ell')\gamma_\nu u(\ell)\right]\\ =&\frac{1}{2}\mathrm{Tr}(\not{\ell}\gamma_\mu\not{\ell}'\gamma_\nu)\\ =&2(\ell_\mu\ell'_\nu+\ell_\nu\ell'_\mu-g_{\mu\nu}\ell\cdot\ell').\end{aligned}\tag{2.13}$$

Note that $\ell\cdot\ell' = q\cdot\ell' = -q\cdot\ell = \frac{-q^2}{2} = \frac{Q^2}{2}$, since, neglecting the electron mass, $\ell^2=\ell'^2=0$.
The leptonic tensor fulfils *current conservation*,

$$L_{\mu\nu}q^\mu = 0.\tag{2.14}$$

Indeed one can verify that

$$\begin{aligned}L_{\mu\nu}q^\mu =&2((q\cdot\ell)\ell'_\nu+(q\cdot\ell')\ell_\nu-q_\nu\frac{Q^2}{2})\\ =&Q^2(\ell_\nu-\ell'_\nu-q_\nu)\\ =&0\,,\end{aligned}\tag{2.15}$$

which is of course true, even including electron masses.
The hadronic tensor $\tilde{W}_{\mu\nu}$, which for convenience we normalise as $\tilde{W}_{\mu\nu}=2\pi m_p W_{\mu\nu}$, has the following general Lorentz structure in unpolarised DIS,

$$W_{\mu\nu} =V_1 g_{\mu\nu}+V_2P_\mu P_\nu+V_3(P_\mu q_\nu+P_\nu q_\mu)+V_4(P_\mu q_\nu-P_\nu q_\mu)+V_5 q_\mu q_\nu+V_6\epsilon_{\mu\nu\alpha\beta}P^\alpha q^\beta,\tag{2.16}$$

where the V's are all functions of P and q and $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor.
It can be shown ($\to$ Exercises) that using the parity invariance of QED, which implies that the leptonic tensor $L_{\mu\nu}$ is symmetric in $\mu\leftrightarrow\nu$, and thus so is the hadronic tensor $W_{\mu\nu}$, and using current conservation $\tilde{W}_{\mu\nu}q^\nu=0$, the Lorentz structure of $W_{\mu\nu}$ is constrained to be,

$$W_{\mu\nu} =-V_1(P,q)\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)+V_2(P,q)\left(P_\mu-q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu-q_\nu\frac{P\cdot q}{q^2}\right),\tag{2.17}$$

which we can conventionally write as

$$W_{\mu\nu} = - W_1(P,q)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2(P,q)}{m_p^2}\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right). \tag{2.18}$$

Then using the explicit expression for $L_{\mu\nu}$ and $\tilde{W}^{\mu\nu}$ we obtain

$$L_{\mu\nu}\,\mathrm{Im}\,\tilde{W}^{\mu\nu}(P,q) = 2\pi m_p \cdot 2\left[2\ell\cdot\ell'\,\mathrm{Im}\,W_1(P,q) + (2(P\cdot\ell)(P\cdot\ell') - m_p^2\ell\cdot\ell')\frac{\mathrm{Im}\,W_2(P,q)}{m_p^2}\right], \tag{2.19}$$

where we have used eq. (2.14). Now, writing the volume element as

$$d^3\ell' = dE' E'^2 d\Omega, \tag{2.20}$$

the differential DIS cross-section becomes

$$d\sigma(eP\to eX) = \frac{e^4}{16\pi^3}\frac{E'}{s(Q^2)^2}L_{\mu\nu}\,\mathrm{Im}\,\tilde{W}^{\mu\nu}(P,q)\,dE'd\Omega, \tag{2.21}$$

that is,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \alpha^2\frac{E'}{\pi s(Q^2)^2}L_{\mu\nu}\,\mathrm{Im}\,\tilde{W}^{\mu\nu}(P,q), \tag{2.22}$$

2.3 The Target frame

In the target rest frame, the momenta are

$$\ell = (E,0,0,E) \quad P = (m_p,0,0,0) \quad \ell' = (E', E'\sin\theta, 0, E'\cos\theta), \tag{2.23}$$

if we assume the initial electron moves in the z-direction. The kinematic invariants are

$$\begin{aligned} s = 2P\cdot\ell &= 2m_pE \\ P\cdot\ell' &= m_pE' \\ \ell\cdot\ell' &= EE'(1-\cos\theta), \end{aligned} \tag{2.24}$$

so the differential DIS cross-section becomes

$$\frac{d^2\sigma}{dE'd\Omega} = 4\alpha^2\frac{E'^2}{(Q^2)^2}\left[2\sin^2\left(\frac{\theta}{2}\right)\mathrm{Im}\,W_1(P,q) + \cos^2\left(\frac{\theta}{2}\right)\mathrm{Im}\,W_2(P,q)\right], \tag{2.25}$$

which agrees with the usual expression for the DIS cross-section in the target frame, as e.g. you can find in Quigg, *Gauge Theories of the Strong, Weak and Electromagnetic Interactions*, Field, *Applications of Perturbative QCD* or Feynman, *Photon-Hadron Interactions.*

2.4 Bjorken Scaling

In the target rest frame, $\nu = E - E'$ is the electron energy loss. The Lorentz invariant quantity,

$$y \equiv \frac{2P\cdot q}{2P\cdot\ell}, \tag{2.26}$$

is called fractional energy loss, since in the target rest frame, $y = \dfrac{q^0}{\ell^0} = \dfrac{E - E'}{E}$.
Then we can write the tensor contraction (2.19) as

$$L_{\mu\nu} \operatorname{Im} \tilde{W}^{\mu\nu} = 2\pi m_p \left[2Q^2 \operatorname{Im} W_1 + \left(s(s - m_p^2)(1 - y) - m_p^2 Q^2 \right) \frac{\operatorname{Im} W_2}{m_p^2} \right], \tag{2.27}$$

where since the fractional energy loss is $y = \dfrac{2P \cdot q}{2P \cdot \ell} = 1 - \dfrac{2P \cdot \ell'}{2P \cdot \ell}$ we have used that $2P \cdot \ell' = (s - m_p^2)(1 - y)$. Since $m_p^2 \ll Q^2$, s, we neglect the terms proportional to the proton mass in the numerator. Then

$$L_{\mu\nu} \operatorname{Im} \tilde{W}^{\mu\nu} = 2\pi m_p \left[2Q^2 \operatorname{Im} W_1 + s^2 (1 - y) \frac{\operatorname{Im} W_2}{m_p^2} \right], \tag{2.28}$$

In 1969, Bjorken discovers that the dimensionless combinations, $m_p W_1(P, q)$ and $\nu W_2(P, q)$, depend upon the kinematics invariants only through the ratio,

$$x = \frac{Q^2}{2m_p \nu} = \frac{Q^2}{2P \cdot q}, \tag{2.29}$$

with $Q^2, \nu \to \infty$, while keeping x fixed. This is called **Bjorken scaling**.
Then we can introduce the functions,

$$\begin{cases} F_1(x) \equiv m_p \operatorname{Im} W_1(P, q)\,, \\ F_2(x) \equiv \nu \operatorname{Im} W_2(P, q)\,, \end{cases} \tag{2.30}$$

Trading W_1 and W_2 for F_1 and F_2 in the tensor contraction (2.28),

$$L_{\mu\nu} \operatorname{Im} \tilde{W}^{\mu\nu} = 2\pi \left[2Q^2 F_1(x) + s^2 (1 - y) \frac{F_2(x)}{m_p \nu} \right], \tag{2.31}$$

and noting that $y = \dfrac{Q^2}{xs}$ and $m_p \nu = P \cdot q = \dfrac{sy}{2}$ we can also write it as

$$L_{\mu\nu} \operatorname{Im} \tilde{W}^{\mu\nu} = 2\pi \left[2xysF_1(x) + 2s \frac{1 - y}{y} F_2(x) \right]. \tag{2.32}$$

Noting that in the target rest frame,

$$\begin{cases} y = \dfrac{E - E'}{E} \\ x = \dfrac{Q^2}{2P \cdot q} = \dfrac{EE'(1 - \cos\theta)}{m_p(E - E')}, \end{cases} \tag{2.33}$$

we can trade the variables E' and θ for x and y using the Jacobian[1],

$$\left| \frac{\partial(x, y)}{\partial(E', \cos\theta)} \right| = \frac{2E'}{ys}, \tag{2.34}$$

which entails that

$$\mathrm{d}E' \mathrm{d}\Omega = 2\pi \mathrm{d}E' \mathrm{d}\cos\theta = \pi \frac{ys}{E'} \mathrm{d}x \mathrm{d}y, \tag{2.35}$$

[1] The reader is encouraged to check the validity of this Jacobian

and we rewrite the DIS cross section (2.22) as

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y} &= \frac{\alpha^2 y}{(Q^2)^2} L_{\mu\nu} \operatorname{Im} \tilde{W}^{\mu\nu} \\ &= 4\pi\alpha^2 \frac{s}{(Q^2)^2} \left[x y^2 F_1(x) + (1-y) F_2(x) \right] .\end{aligned} \tag{2.36}$$

The DIS cross section above makes no assumptions on the hadron dynamics. It assumes only Bjorken scaling as a fact.

2.5 Virtual Photon Polarisation

We may then understand better the relationship between the structure functions W_1 and W_2 (or F_1 and F_2) and the polarisation of the virtual photon γ^* by considering the $\gamma^* P$ cross section in the target rest frame ($\rightarrow$ Exercises),

$$\sigma_\lambda(\gamma^* P \to X) = \frac{4\pi^2\alpha}{|\vec{q}|} \epsilon_\mu^\lambda(q) \epsilon_\nu^{\lambda *}(q) \operatorname{Im} W^{\mu\nu}(P, q), \tag{2.37}$$

where λ is the virtual-photon polarisation. Note that although the cross section between on-shell real states is a Lorentz invariant, the one above is not: it depends on the frame in which it is computed.

Suppose that the scattering $\gamma^* \rightsquigarrow\leftarrow P$ is in the z-direction. In the target rest frame, $P = (m_p, \vec{0})$ and $q = (q^0, 0, 0, q^z)$. Take γ^* as transversely polarised, then the polarisation vector is

$$\epsilon_\perp^{\mu\pm}(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \qquad \text{with} \qquad \begin{cases} \epsilon_\perp^\pm \cdot \epsilon_\perp^{\mp *} = 0 \\ \epsilon_\perp^\pm \cdot \epsilon_\perp^{\pm *} = -1 . \end{cases} \tag{2.38}$$

Define the transverse $\gamma^* P$ cross section as

$$\sigma_T \equiv \frac{1}{2} \left[\sigma^+ + \sigma^- \right] . \tag{2.39}$$

Recall that the hadronic tensor is

$$\operatorname{Im} W^{\mu\nu}(P, q) = -\operatorname{Im} W_1 \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + \frac{\operatorname{Im} W_2}{m_p^2} \left(P^\mu - q^\mu \frac{P \cdot q}{q^2} \right) \left(P^\nu - q^\nu \frac{P \cdot q}{q^2} \right) . \tag{2.40}$$

Since $q \cdot \epsilon_\perp^\pm = P \cdot \epsilon_\perp^\pm = 0$, we have

$$\epsilon_\perp^{\mu\pm} \epsilon_\perp^{\nu\pm *} \operatorname{Im} W_{\mu\nu} = -\epsilon_\perp^\pm \cdot \epsilon_\perp^{\pm *} \operatorname{Im} W_1 = \operatorname{Im} W_1 , \tag{2.41}$$

so the transverse $\gamma^* P$ cross section is

$$\sigma_T = \frac{4\pi^2\alpha}{|\vec{q}|} \operatorname{Im} W_1 . \tag{2.42}$$

Now, $q^2 = q_0^2 - |\vec{q}|^2$ implies $|\vec{q}|^2 = q_0^2 - q^2 \stackrel{\text{lab}}{=} \nu^2 + Q^2$, In the scaling limit, we take $\nu \to \infty$ and $Q^2 \to \infty$ while keeping x fixed. Therefore we can assume that $\nu^2 \gg Q^2$, i.e.

$$\sigma_T(\gamma^* P) \stackrel{\text{scaling}}{\to} \frac{4\pi^2\alpha}{\nu} \operatorname{Im} W_1, \tag{2.43}$$

which in terms of F_1 is

$$\begin{aligned} \sigma_T(\gamma^* P) &\stackrel{\text{scaling}}{\to} \frac{4\pi^2\alpha}{P \cdot q} F_1(x) \\ &= 4\pi^2\alpha \frac{2x}{Q^2} F_1(x). \end{aligned} \tag{2.44}$$

So the transverse $\gamma^* P$ cross section is directly related to F_1.

The γ^* longitudinal polarisation must be a function of P^μ and q^μ. For $q = (q^0, 0, 0, q^z)$, one can choose

$$\epsilon^\mu_{\lambda=0}(q) = \frac{1}{\sqrt{Q^2}}(q^z, 0, 0, q^0)\,, \tag{2.45}$$

such that $q \cdot \epsilon_0 = 0$ and $\epsilon_0 \cdot \epsilon_0^* = 1$ (γ^* is spacelike). In general, one can parametrise the longitudinal polarisation as

$$\epsilon_0^\mu(q, P) = aP_\mu + bq_\mu, \tag{2.46}$$

with a, b constants. Since $q \cdot \epsilon_0 = 0$, then

$$\epsilon_0^\mu(q, P) = a\left(P_\mu - \frac{P \cdot q}{q^2} q_\mu\right). \tag{2.47}$$

Fixing the normalisation to be $\epsilon_0 \cdot \epsilon_0^* = 1$ yields

$$a^2 = \left(P^2 - \frac{(P \cdot q)^2}{q^2}\right)^{-1}. \tag{2.48}$$

Then the contraction with the hadronic tensor yields

$$\epsilon_0^\mu \epsilon_0^{\nu *} \operatorname{Im} W_{\mu\nu} = a^2 P^\mu P^\nu \operatorname{Im} W_{\mu\nu}, \tag{2.49}$$

since $q^\mu \operatorname{Im} W_{\mu\nu} = 0$. Now,

$$P^\mu P^\nu \operatorname{Im} W_{\mu\nu} = -\operatorname{Im} W_1 \left(P^2 - \frac{(P \cdot q)^2}{q^2}\right) + \left(P^2 - \frac{(P \cdot q)^2}{q^2}\right)^2 \frac{\operatorname{Im} W_2}{m_p^2}, \tag{2.50}$$

so clearly,

$$\epsilon_0^\mu \epsilon_0^{\nu *} \operatorname{Im} W_{\mu\nu} = -\operatorname{Im} W_1 + \left(P^2 - \frac{(P \cdot q)^2}{q^2}\right) \frac{\operatorname{Im} W_2}{m_p^2}, \tag{2.51}$$

and the longitudinal $\gamma^* P$ cross section becomes

$$\sigma_L(\gamma^* P) = \frac{4\pi^2\alpha}{|\vec{q}|}\left[-\operatorname{Im} W_1 + \left(P^2 - \frac{(P \cdot q)^2}{q^2}\right) \frac{\operatorname{Im} W_2}{m_p^2}\right]. \tag{2.52}$$

It is convenient to sum σ_T and σ_L,

$$\begin{aligned}\sigma_T + \sigma_L &= \frac{4\pi^2\alpha}{\sqrt{\nu^2+Q^2}}\left(P^2 - \frac{(P\cdot q)^2}{q^2}\right)\frac{\operatorname{Im} W_2}{m_p^2} \\ &= \frac{4\pi^2\alpha}{\sqrt{\nu^2+Q^2}}\left(1+\frac{\nu^2}{Q^2}\right)\operatorname{Im} W_2 \\ &= 4\pi^2\alpha\frac{\sqrt{\nu^2+Q^2}}{Q^2}\operatorname{Im} W_2.\end{aligned} \tag{2.53}$$

In the scaling limit,

$$\sigma_T + \sigma_L = 4\pi^2\alpha\frac{F_2(x)}{Q^2}, \tag{2.54}$$

which implies that

$$\sigma_L(\gamma^* P) = \frac{4\pi^2\alpha}{Q^2}\left[F_2(x) - 2xF_1(x)\right], \tag{2.55}$$

where we defined

$$F_L(x) = F_2(x) - 2xF_1(x) \tag{2.56}$$

The equation,

$$F_2(x) = 2xF_1(x), \tag{2.57}$$

is known as **Callan-Gross relation**, whose physical meaning we discuss in the next paragraph. Thus, we have found that the longitudinal $\gamma^* P$ cross section is proportional to the violations of the Callan-Gross relation.

Now, a spin–1/2 quark absorbs a transversely polarised photon. It cannot absorb a longitudinally polarised photon. Thus for a spin–1/2 quark, $\sigma_L = 0 \implies F_L = 0 \implies F_2 = 2xF_1$, so the Callan-Gross relation hints at a proton structure made of spin–1/2 quarks.
Conversely, a spin–0 quark absorbs a longitudinally polarised photon. It cannot absorb a transversely polarised photon. Thus for a spin–0 quark, $\sigma_T = 0 \implies F_1 = 0 \implies F_L = F_2$.
So, is $F_L = 0$ (spin–1/2 quarks), or $F_L = F_2$ (spin–0 quarks)? DIS data show that $F_L \ll F_2$.

2.6 DIS structure functions

Assuming Bjorken scaling, in eq. (2.36) we had computed the DIS cross section as a function of F_1 and F_2,

$$\frac{d\sigma}{dx dy} = 4\pi\alpha^2\frac{s}{(Q^2)^2}\left[xy^2F_1(x) + (1-y)F_2(x)\right]. \tag{2.58}$$

Using eq. (2.56), we can also write it as a function of F_1 and F_L,

$$\frac{d\sigma}{dx dy} = 4\pi\alpha^2\frac{s}{(Q^2)^2}\left\{x[1+(1-y)^2]F_1(x) + (1-y)F_L(x)\right\}, \tag{2.59}$$

where we see the contribution to DIS scattering of the transversely and longitudinally polarised photon, or we can write it as a function of F_2 and F_L,

$$\frac{d\sigma}{dx dy} = 2\pi\alpha^2\frac{s}{(Q^2)^2}\left\{[1+(1-y)^2]F_2(x) - y^2F_L(x)\right\}, \tag{2.60}$$

which states that, up to violations of the Callan-Gross relation, the DIS cross section is directly related to F_2.

Since the violations of the Callan-Gross relation are small $F_L \ll F_2$, another popular way is to express the scaling violations in terms of the ratio,

$$R(x) \equiv \frac{\sigma_L(\gamma^* P)}{\sigma_T(\gamma^* P)} = \frac{F_2(x) - 2xF_1(x)}{2xF_1(x)} = \frac{F_2(x)}{2xF_1(x)} - 1. \tag{2.61}$$

In that case the DIS cross section can be written as

$$\frac{\mathrm{d}\sigma(eP \to eX)}{\mathrm{d}x\mathrm{d}y} = 4\pi\alpha^2 \frac{s}{(Q^2)^2} \left[1 - y + \frac{y^2}{2} \frac{1}{1 + R(x)}\right] F_2(x). \tag{2.62}$$

In fig. 2.3, it is plotted the DIS cross section for neutral-current exchange as a function of Q^2. The hollow-square data come from fixed-target experiments. The solid circles and squares correspond to HERA data from electron-proton and positron-proton scattering, respectively. In addition to Bjorken scaling, i.e. the horizontal lines, the data display logarithmic scaling violations (i.e. almost straight non-horizontal lines), with a negative (positive) slope at larger (smaller) values of x. We comment on these in sec. 5.7. Further, we observe that for a given value of x, the solid circles and squares overlap, but for very large values of Q^2. That is because the neutral-current exchange is a combination of photon and Z-boson exchange, with the Z boson dominating at large values of Q^2. The electromagnetic interaction of the photon with a fermion is left-right symmetric. It implies that the cross section due to photon exchange is even under charge conjugation, and thus the same for electron-proton or positron-proton scattering. However, the Z boson has different couplings with a left- or a right-handed fermion. Accordingly, the parity of the cross section due to Z-boson exchange under charge conjugation is lost.

2.7 Feynman's Parton Model

Shortly after Bjorken discovered the scaling (2.30), Feynman introduced the **Parton model**, whose tenets are

- A proton is a loosely bound set of partons, including quarks.
- In DIS, the electron knocks a quark out of the proton.
- Take the infinite-momentum frame, where the proton is highly relativistic (i.e., neglect its mass). Then, if P is the proton momentum, $p = \zeta P$ is the parton momentum with $0 < \zeta < 1$ the *longitudinal momentum fraction*. If the proton is light-like, then also the partons are light-like.
- Neglect the parton tranverse momentum.
- If $f_i(\zeta)$ is the probability of finding a parton i within the proton, then the DIS cross-section is

$$d\sigma(e(\ell)p(P) \to e(\ell')X) = \int_0^1 d\zeta \sum_i f_i(\zeta) d\hat{\sigma}(e(\ell)q_i(\zeta P) \to e(\ell')X), \tag{2.63}$$

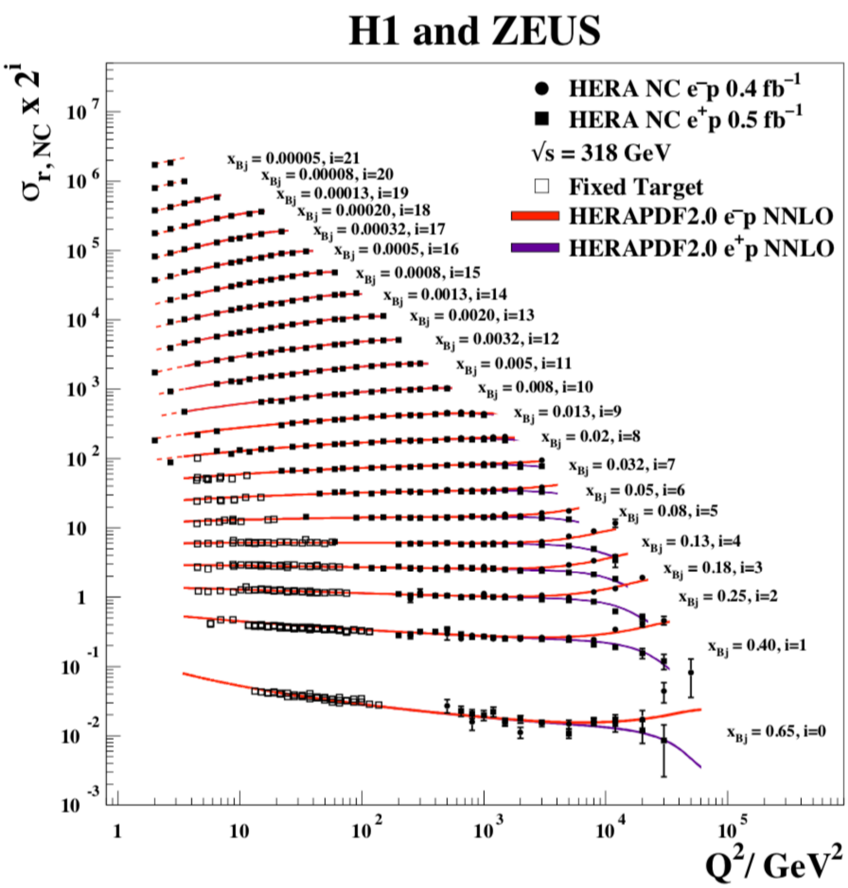


Figure 2.3: DIS cross section for neutral-current exchange as a function of Q^2, from H. Abramowicz *et al.* [H1 and ZEUS], "Combination of measurements of inclusive deep inelastic $e^{\pm}p$ scattering cross sections and QCD analysis of HERA data," Eur. Phys. J. C **75** (2015) no.12, 580 [arXiv:1506.06042 [hep-ex]].

where q_i is a constituent quark (or anti-quark) and $d\hat{\sigma}$ is the electron-quark elastic scattering cross-section.

Diagrammatically, this is

Since the virtual photon is spacelike, we can boost it to a frame where $E_\gamma = 0$. That defines the **Breit frame**. There the virtual photon momentum can be taken as: $q = (0, 0, 0, \sqrt{Q^2})$ with $q^2 = -Q^2$ as before. Suppose that $m_p^2 \ll Q^2$, so we can take the proton massless. Then the proton momentum can be taken as

$$P = \left(\frac{\sqrt{Q^2}}{2x}, 0, 0, -\frac{\sqrt{Q^2}}{2x} \right), \tag{2.64}$$

with Bjorken $x = \dfrac{Q^2}{2q \cdot P}$. Take now a parton of momentum,

$$p = \zeta P = \left(\frac{\zeta}{x} \frac{\sqrt{Q^2}}{2}, 0, 0, -\frac{\zeta}{x} \frac{\sqrt{Q^2}}{2} \right), \tag{2.65}$$

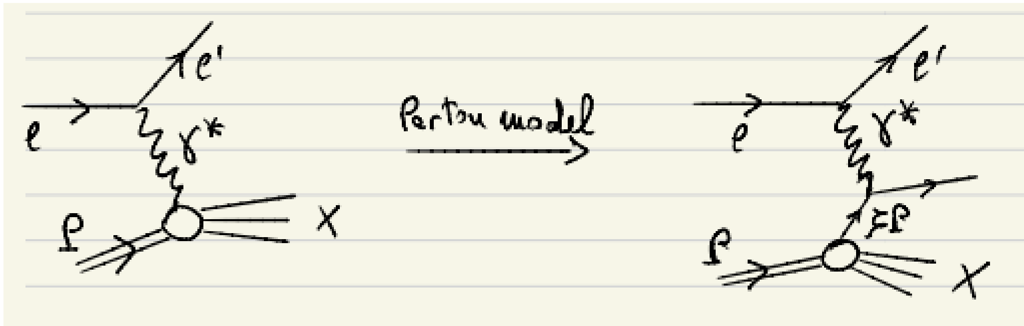


Figure 2.4: Feynman's parton model for DIS.

inside the proton. If that parton absorbs the photon, the resulting parton momentum is

$$p' = p + q = \left(\frac{\zeta}{x} \frac{\sqrt{Q^2}}{2}, 0, 0, \left(1 - \frac{\zeta}{2x}\right) \sqrt{Q^2} \right). \tag{2.66}$$

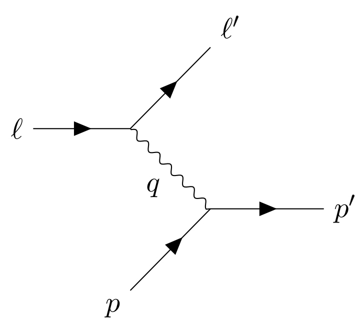
The mass-shell condition for the outgoing massless parton is

$$p'^2 = (p+q)^2 = 0 \Longrightarrow \frac{\zeta}{2x} = 1 - \frac{\zeta}{2x} \Longrightarrow \zeta = x, \tag{2.67}$$

i.e. the *parton momentum fraction equals the Bjorken x in the Breit frame.*
Then the incoming and outgoing parton respectively have momentum,

$$p = \left(\frac{\sqrt{Q^2}}{2}, 0, 0, -\frac{\sqrt{Q^2}}{2} \right) \qquad p' = \left(\frac{\sqrt{Q^2}}{2}, 0, 0, \frac{\sqrt{Q^2}}{2} \right). \tag{2.68}$$

In other words, the photon is absorbed by the parton, which changes direction. In addition, helicity is conserved for a massless quark. Thus, the parton spin is reversed in the interaction. This implies that the photon must have helicity ± 1, i.e. the photon is transversely polarized.
As we mentioned before, in the parton model we need the cross-section for $eq \to eq$ scattering. We state it here, and we shall derive it later.



The squared amplitude for $eq \to eq$ scattering summed (averaged) over final (initial) spins is

$$\frac{1}{4} \sum_{spin} |\mathcal{M}|^2 = \frac{8e^4 Q_i^2}{\hat{t}^2} \frac{\hat{s}^2 + \hat{u}^2}{4}, \tag{2.69}$$

with $\hat{u} = -\hat{s} - \hat{t}$, and Q_i is the quark charge. The partonic cross-section is

$$\begin{aligned}\frac{d\hat{\sigma}}{d\hat{t}} =& \frac{1}{16\pi\hat{s}^2}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 \\ =& \frac{2\pi\alpha^2 Q_i^2}{\hat{s}^2}\frac{\hat{s}^2 + (\hat{s}+\hat{t})^2}{\hat{t}^2}.\end{aligned} \tag{2.70}$$

Note that:

- The parton centre-of-mass energy (CME) $\hat{s}$ is related to the hadron CME s by
$$\hat{s} = (p+\ell)^2 = 2p\cdot\ell = 2\zeta P\cdot\ell = \zeta s, \tag{2.71}$$
as long as the proton is taken to be massless.
- The mass-shell condition for the outgoing massless quark is
$$p'^2 = 0 = (p+q)^2 = 2p\cdot q + q^2 = 2\zeta P\cdot q - Q^2 \longrightarrow \zeta = \frac{Q^2}{2P\cdot q}. \tag{2.72}$$
Since $x = \frac{Q^2}{2P\cdot q}$ we have that $\zeta = x$. (We made that statement in the Breit frame, but as we see it is a Lorentz-invariant statement).
- The momentum of the outgoing electron allows us to determine x and Q^2.

Using the parton model formula for the DIS cross section (2.63) and the explicit partonic cross section, we can write the DIS cross section as

$$\frac{d\sigma}{dxdQ^2} = \sum_i f_i(x) Q_i^2 \frac{2\pi\alpha^2}{(Q^2)^2}\left[1 + \left(1 - \frac{Q^2}{xs}\right)^2\right]\theta(xs - Q^2). \tag{2.73}$$

So the parton model predicts that apart from the QED-induced factor,

$$\frac{1}{(Q^2)^2}\left[1 + \left(1 - \frac{Q^2}{xs}\right)^2\right], \tag{2.74}$$

the DIS cross-section depends only on x. See also Peskin-Schroeder, *An Introduction to Quantum Field Theory*, fig 14.2, and Ellis-Stirling-Webber *QCD and Collider Physics*, fig 4.2. Using the fractional energy loss y (2.26), and the change of variables,

$$dxdQ^2 = \frac{dQ^2}{dy}dydx = xsdxdy, \tag{2.75}$$

we can write the DIS cross section as

$$\frac{d\sigma}{dxdy} = \sum_i x f_i(x) Q_i^2 \frac{2\pi\alpha^2 s}{(Q^2)^2}\left[1 + (1-y)^2\right], \tag{2.76}$$

i.e. up to the $\frac{1}{(Q^2)^2}$ term, the DIS cross-section factorises into x and y dependence. Further, comparing to eq. (2.60), we see that in the parton model, $F_L(x) = 0$, so the Callan-Gross relation holds, and

$$F_2(x) = \sum_i x f_i(x) Q_i^2\,. \tag{2.77}$$

Eq. (2.76) warrants the following comments:

- The x dependence is related to Bjorken scaling (the parton distributions f_i's do not depend on Q^2, at this level).
- The y dependence is related to the Callan-Gross relation, i.e. to the spin structure of electron-quark scattering. In fact, the y dependence of eq. (2.76) is a consequence of having considered in eq. (2.69) the scattering of electrons off spin–1/2 quarks. As we know from the discussion at the end of sec. 2.5, and from eq. (2.59). the scattering of electrons off spinless quarks would entail a different y dependence.

2.8 The hadronic tensor in the parton model

Recall that we have introduced the hadronic tensor $\tilde{W}^{\mu\nu}$ in forward Compton scattering without making reference to the hadron dynamics,

$$\tilde{W}^{\mu\nu}(P,q) = i\int \mathrm{d}^4x\, e^{iq\cdot x}\,\langle P|\,T\{J^\mu(x)J^\nu(0)\}\,|P\rangle\,. \tag{2.78}$$

Now, we evaluate $\tilde{W}^{\mu\nu}$ for spin–1/2 quarks in the parton model, and we show how it is related to Bjorken scaling and to the Callan-Gross relation.

We change the proton matrix element into a sum of quark matrix elements,

$$\tilde{W}^{\mu\nu}(P,q) = i\int \mathrm{d}^4x\, e^{iq\cdot x}\int \mathrm{d}\zeta\sum_i f_i(\zeta)\frac{1}{\zeta}\,\langle q_i|\,T\{J^\mu(x)J^\nu(0)\}\,|q_i\rangle\,, \tag{2.79}$$

where the factor $\frac{1}{\zeta}$ comes from changing the flux factor $\frac{1}{2s}$ into $\frac{1}{2\hat{s}} = \frac{1}{2\zeta s}$ in the cross section. Thus, we must evaluate the diagrams depicted in figure 2.5.

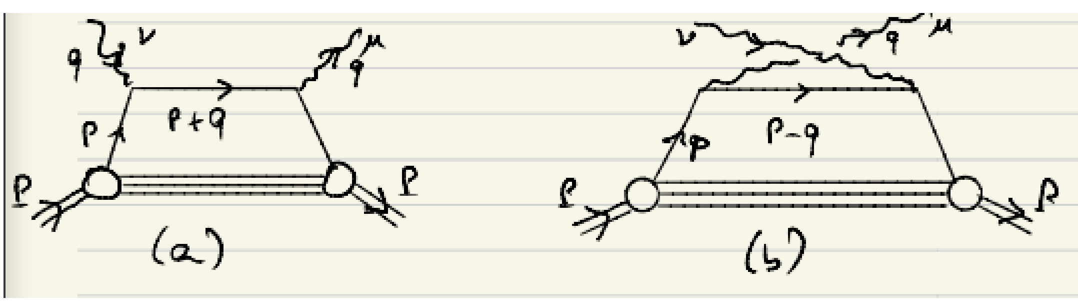


Figure 2.5: Forward Compton scattering in the parton model.

$$(a) \qquad \Longrightarrow \qquad i\int \mathrm{d}\zeta\sum_i f_i(\zeta)\frac{1}{\zeta}Q_i^2\bar{u}(p)\gamma^\mu\frac{i(\not{p}+\not{q})}{(p+q)^2+i\varepsilon}\gamma^\nu u(p), \tag{2.80}$$

and summing (averaging) over the final (initial) quark spins,

$$\tilde{W}^{\mu\nu}_{(a)}(P,q) = i\int \mathrm{d}\zeta\sum_i f_i(\zeta)\frac{1}{\zeta}Q_i^2\frac{1}{2}\,\mathrm{Tr}\left[\not{p}\gamma^\mu(\not{p}+\not{q})\gamma^\nu\right]\frac{i}{(p+q)^2+i\varepsilon}, \tag{2.81}$$

where

$$\mathrm{Tr}\left[\not{p}\gamma^\mu(\not{p}+\not{q})\gamma^\nu\right] = 4[p^\mu(p+q)^\nu + p^\nu(p+q)^\mu - g^{\mu\nu}p\cdot(p+q)]. \tag{2.82}$$

Now, unitarity and the Cutkosky rules imply that the discontinuity of the propagator is

$$\mathrm{Disc}\frac{1}{(p+q)^2+i\epsilon} = -2\pi i\delta((p+q)^2). \tag{2.83}$$

On the other hand, the Disc of the S-matrix across a cut on the real axis is

$$\mathrm{Disc}\mathcal{M}(s) = 2i\,\mathrm{Im}[\mathcal{M}(s+i\epsilon)]. \tag{2.84}$$

Thus,

$$\mathrm{Im}\,\frac{1}{(p+q)^2+i\epsilon} = -\pi\delta((p+q)^2), \tag{2.85}$$

and

$$\begin{aligned}\delta((p+q)^2) &= \delta(2p\cdot q - Q^2)\\ &= \delta(\zeta 2P\cdot q - Q^2)\\ &= \delta(\zeta\frac{Q^2}{x} - Q^2)\\ &= \frac{x}{Q^2}\delta(\zeta - x).\end{aligned} \tag{2.86}$$

So,

$$\mathrm{Im}\,\frac{1}{(p+q)^2+i\epsilon} = -\frac{\pi x}{Q^2}\delta(\zeta - x). \tag{2.87}$$

Now, to obtain $\mathrm{Im}\,\tilde{W}^{\mu\nu}_{(b)}$, we exchange q with $-q$ and μ with ν in $\mathrm{Im}\,\tilde{W}^{\mu\nu}_{(a)}$. But

$$\mathrm{Im}\,\frac{1}{(p-q)^2+i\epsilon} = -\pi\delta(2p\cdot q + Q^2) = 0, \tag{2.88}$$

because $2p\cdot q + Q^2 \neq 0$, so diagram (b) does not contribute to the imaginary part.

Hence, we get

$$\begin{aligned}\mathrm{Im}\,\tilde{W}^{\mu\nu}(P,q) &= \sum_i f_i(x)\frac{1}{x}Q_i^2\frac{\pi x}{Q^2}\left[4x^2P^\mu P^\nu + 2x(P^\mu q^\nu + P^\nu q^\mu) - g^{\mu\nu}Q^2\right]\\ &= \sum_i f_i(x)Q_i^2\frac{\pi}{Q^2}\left[\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)q^2 + 4x^2\left(P^\mu + \frac{q^\mu}{2x}\right)\left(P^\nu + \frac{q^\nu}{2x}\right)\right],\end{aligned} \tag{2.89}$$

where in the last step we added and subtracted $q^\mu q^\nu$ terms. But we know that

$$\mathrm{Im}\,\tilde{W}^{\mu\nu} = 2\pi m_p\left[-\mathrm{Im}\,W_1\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \frac{\mathrm{Im}\,W_2}{m_p^2}\left(P^\mu - q^\mu\frac{P\cdot q}{q^2}\right)\left(P^\nu - q^\nu\frac{P\cdot q}{q^2}\right)\right]. \tag{2.90}$$

Equating the two expressions above, we have

$$\begin{cases}2m_p\,\mathrm{Im}\,W_1(P,q) = \sum_i f_i(x)Q_i^2\\ \frac{1}{m_p}\,\mathrm{Im}\,W_2(P,q) = \frac{1}{P\cdot q}\sum_i x f_i(x)Q_i^2.\end{cases} \tag{2.91}$$

Using Bjorken scaling (2.30) and eq. (2.91), we obtain

$$\begin{cases}F_1(x) = \frac{1}{2}\sum_i f_i(x)Q_i^2\\ F_2(x) = \sum_i x f_i(x)Q_i^2,\end{cases} \tag{2.92}$$

which show that within the parton model the combinations $m_p W_1(P,q)$ and $\nu W_2(P,q)$ depend only on x, thus the parton model provides an explanation for Bjorken scaling. Further, eq. (2.92) entails, yet again, the Callan-Gross relation (2.57), which can be seen also as a relation between W_1 and W_2,

$$\mathrm{Im}\, W_2 = \frac{2m_p^2 x}{P \cdot q}\, \mathrm{Im}\, W_1 \,. \tag{2.93}$$

2.9 Proton Structure: Parton Densities and Sum Rules

We may use $F_2^p(x) = \sum_i x f_{i/p}(x) Q_i^2$, and thus the DIS cross section, to get information on the proton structure. Let us fix

$$f_{u/p} \equiv u(x) \qquad f_{d/p} \equiv d(x) \qquad f_{s/p} \equiv s(x) \qquad \text{etc.} \tag{2.94}$$

So,

$$F_2^p(x) = x \left\{ \frac{4}{9}[u(x) + \bar{u}(x)] + \frac{1}{9}[d(x) + \bar{d}(x)] + \frac{1}{9}[s(x) + \bar{s}(x)] + ... \right\}. \tag{2.95}$$

At the parton level, the isospin symmetry between protons and neutrons is a symmetry between u and d quarks. So,

$$f_{u/n} = d(x) \qquad f_{d/n} = u(x) \qquad f_{s/n} = s(x), \tag{2.96}$$

and

$$F_2^n(x) = x \left\{ \frac{4}{9}[d(x) + \bar{d}(x)] + \frac{1}{9}[u(x) + \bar{u}(x)] + \frac{1}{9}[s(x) + \bar{s}(x)] + ... \right\}. \tag{2.97}$$

Then,

$$\frac{F_2^n(x)}{F_2^p(x)} = \frac{u + \bar{u} + 4(d + \bar{d}) + ...}{4(u + \bar{u}) + d + \bar{d} + ..}. \tag{2.98}$$

If we neglect the strange-quark density, we get the bounds

$$\frac{1}{4} \le \frac{F_2^n}{F_2^p} \le 4, \tag{2.99}$$

which are in agreement with the experiment (see Quigg, fig. 7.8). Since for $x \to 0$, the sea quarks should dominate the parton distributions, we expect that for $x \to 0$ we have $u \approx \bar{u} \approx d \approx \bar{d}$, and so that

$$\lim_{x \to 0} \frac{F_2^n}{F_2^p} = 1, \tag{2.100}$$

which is indeed observed.

We can derive a few sum rules the parton densities must satisfy. Since protons and neutrons have no net strangeness, charm, ..., then

$$\int_0^1 \mathrm{d}x [s(x) - \bar{s}(x)] = \int_0^1 \mathrm{d}x [c(x) - \bar{c}(x)] = ... = 0. \tag{2.101}$$

The **electric charge sum rules** for protons and neutrons are, respectively,

$$\int_0^1 \mathrm{d}x \left[\frac{2}{3}[u(x) - \bar{u}(x)] - \frac{1}{3}[d(x) - \bar{d}(x)]\right] = 1, \tag{2.102}$$

$$\int_0^1 \mathrm{d}x \left[\frac{2}{3}[d(x) - \bar{d}(x)] - \frac{1}{3}[u(x) - \bar{u}(x)]\right] = 0, \tag{2.103}$$

which together entail that

$$\int_0^1 \mathrm{d}x \, [u(x) - \bar{u}(x)] = 2, \tag{2.104}$$

$$\int_0^1 \mathrm{d}x[d(x) - \bar{d}(x)] = 1. \tag{2.105}$$

These are also called baryon number sum rules, i.e. the baryon number and electric charge sum rules are equivalent.

Further, neglecting the strangeness, charm, ...,

$$\begin{aligned} I_G &= \int_0^1 \frac{\mathrm{d}x}{x}[F_2^p(x) - F_2^n(x)], \\ &= \frac{1}{3}\int \mathrm{d}x \left[u + \bar{u} - (d + \bar{d})\right], \\ &= \frac{1}{3}\int \mathrm{d}x[u - \bar{u} + 2\bar{u} - (d - \bar{d}) - 2\bar{d}], \\ &= \frac{1}{3} + \frac{2}{3}\int_0^1 \mathrm{d}x[\bar{u}(x) - \bar{d}(x)]. \end{aligned} \tag{2.106}$$

This is called the **Gottfried sum rule** and implies that if the sea is symmetric, $I_G = \frac{1}{3}$. However $I_G = 0.236 \pm 0.008$ (NMC Coll 1991) which points to a flavour asymmetry in the sea.

Further, the **momentum sum rule** states that the sum of the momenta of all the partons equals 1,

$$\sum_i \int \mathrm{d}x \, x f_i(x) = 1. \tag{2.107}$$

Neglecting strange, charm, ...,

$$\int \mathrm{d}x \, x(u + \bar{u} + d + \bar{d}) = 1. \tag{2.108}$$

Now,

$$F_2^p + F_2^n = \frac{5}{9}x(u + \bar{u} + d + \bar{d}) + ..., \tag{2.109}$$

so the momentum sum rule implies that

$$\frac{9}{5}\int \mathrm{d}x(F_2^p + F_2^n) = 1, \tag{2.110}$$

but the data tells us that

$$\frac{9}{5}\int \mathrm{d}x(F_2^p + F_2^n) \approx 0.45. \tag{2.111}$$

So about 55% of the proton momentum is not carried by u, d quarks and it is highly implausible that it can be carried by s, c, ..., which are not valence quarks. A plausible conclusion is that it is carried by gluons.

Combining lepton-hadron DIS data for neutral currents (that is, with exchange of a photon and/or a Z boson) with data for charged currents (i.e., with exchange of a W boson) it is possible to determine the parton distribution functions (PDF) for the individual partons. Originally, PDFs were determined in this way. Nowaday, DIS data are combined with data from other experiments, in such a way to determine the PDFs from global fits.

2.10 Helicity Amplitudes

To compute the DIS cross section in the parton model, in the first lecture, we used the square amplitude for $eq \to eq$ scattering summed (averaged) over final (initial) spins,

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{8e^4Q_i^2}{\hat{t}^2}\frac{\hat{s}^2+\hat{u}^2}{4}. \tag{2.112}$$

Now, we shall compute this expression starting from the helicity amplitudes of $e^+e^- \to \mu^+\mu^-$.

Let us introduce the conventions we use,

$$\underbrace{p^\pm = p^0 \pm p^3}_{\text{light-cone coordinates}}\quad , \qquad \underbrace{p_\perp = p_1 + ip_2}_{\text{complex transverse momentum}}\quad , \tag{2.113}$$

such that

$$2p\cdot q = p^+q^- + p^-q^+ - p_\perp q_\perp^* - p_\perp^* q_\perp, \tag{2.114}$$

and in particular,

$$p^2 = p^+p^- - p_\perp p_\perp^* \quad \underset{\text{light-like momentum}}{\longrightarrow} \quad p^+p^- = p_\perp p_\perp^*. \tag{2.115}$$

The Pauli matrices are then used in the following combinations,

$$\sigma^+ = \frac{1+\sigma^3}{2} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \sigma^- = \frac{1-\sigma^3}{2} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{2.116}$$

$$\sigma_\perp = \frac{\sigma^1+i\sigma^2}{2} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad \bar{\sigma}_\perp = \frac{\sigma^1-i\sigma^2}{2} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \tag{2.117}$$

such that,

$$\begin{cases} p\cdot\sigma = p^0\mathbb{I} - \vec{p}\cdot\vec{\sigma} = p^+\sigma^- + p^-\sigma^+ - p_\perp\bar{\sigma}_\perp - p_\perp^*\sigma_\perp = \begin{pmatrix}p^- & -p_\perp^*\\ -p_\perp & p^+\end{pmatrix}, \\[2ex] p\cdot\bar{\sigma} = p^0\mathbb{I} + \vec{p}\cdot\vec{\sigma} = p^+\sigma^+ + p^-\sigma^- + p_\perp\bar{\sigma}_\perp + p_\perp^*\sigma_\perp = \begin{pmatrix}p^+ & p_\perp^*\\ p_\perp & p^-\end{pmatrix}. \end{cases} \tag{2.118}$$

Note also that

$$p^2 = \det(p \cdot \sigma) = \det(p \cdot \bar{\sigma})\,, \qquad p^2 \mathbb{I} = (p \cdot \sigma)(p \cdot \bar{\sigma})\,. \tag{2.119}$$

In the $\longrightarrow$ Exercises, we have seen that for a fermion moving in the $+z$-direction, the spinor is represented as

$$u(p) = \begin{pmatrix} (\sqrt{p^+}\sigma^- + \sqrt{p^-}\sigma^+)\xi \\ (\sqrt{p^+}\sigma^+ + \sqrt{p^-}\sigma^-)\xi \end{pmatrix}, \tag{2.120}$$

where ξ is a 2-component spinor. In fact, for an arbitrary direction, we can take the spinors of positive energy u and negative energy v to be,

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\,\xi \\ \sqrt{p \cdot \bar{\sigma}}\,\xi \end{pmatrix}, \quad v(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\,\xi \\ -\sqrt{p \cdot \bar{\sigma}}\,\xi \end{pmatrix}, \tag{2.121}$$

where it is understood that for the square root of a matrix, we take the positive roots of the eigenvalues, $p^0 \pm |\vec{p}|$. In particular, for massless spinors with normalisation $u_L^\dagger u_L = u_R^\dagger u_R = 2E$, and $\hat{p}$ a unit vector in the direction of $\vec{p}$,

$$u(p) = \sqrt{2E} \begin{pmatrix} \dfrac{1 - \hat{p} \cdot \vec{\sigma}}{2}\,\xi \\ \dfrac{1 + \hat{p} \cdot \vec{\sigma}}{2}\,\xi \end{pmatrix}, \quad v(p) = \sqrt{2E} \begin{pmatrix} \dfrac{1 - \hat{p} \cdot \vec{\sigma}}{2}\,\xi \\ -\dfrac{1 + \hat{p} \cdot \vec{\sigma}}{2}\,\xi \end{pmatrix}. \tag{2.122}$$

We have also seen that ($\rightarrow$ Exercises) in terms of light-cone coordinates, the positive energy massless spinor is

$$u_L(p) = \begin{pmatrix} -\sqrt{p^-}e^{-i\phi_p} \\ \sqrt{p^+} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{-p_\perp^*}{\sqrt{p^+}} \\ \sqrt{p^+} \\ 0 \\ 0 \end{pmatrix}, \quad u_R(p) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p^+} \\ \sqrt{p^-}e^{i\phi_p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p^+} \\ \dfrac{p_\perp}{\sqrt{p^+}} \end{pmatrix}, \tag{2.123}$$

where we used the phase,

$$e^{i\phi_p} = \frac{p_\perp}{\sqrt{p^+p^-}} = \frac{p_\perp}{\sqrt{p_\perp p_\perp^*}} = \sqrt{\frac{p_\perp}{p_\perp^*}}. \tag{2.124}$$

Likewise, choosing the phase conventions of $v(p)$ above, the negative energy massless spinor can be taken as

$$v_L(p) = \begin{pmatrix} \sqrt{p^-} \\ -\sqrt{p^+}e^{i\phi_p} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{p^-} \\ -\dfrac{p_\perp}{\sqrt{p^-}} \\ 0 \\ 0 \end{pmatrix}, \quad v_R(p) = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{p^+}e^{-i\phi_p} \\ -\sqrt{p^-} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{p_\perp^*}{\sqrt{p^-}} \\ -\sqrt{p^-} \end{pmatrix}. \tag{2.125}$$

Eqs. (2.123) and (2.125) agree with eq.(2.122), as we shall see later with specific examples in the $\pm z$ directions.

Note that our conventions for the spinors are the ones used by Peskin-Schröder, *An Introduction to Quantum Field Theory*, chap. 3 and 5. There are other conventions used in the literature (see e.g. Dixon's TASI lectures).

We display the kinematics of $e^-(p)e^+(p') \longrightarrow \mu^-(k)\mu^+(k')$ scattering, where we take the electrons and muons to be massless, $p^2 = p'^2 = k^2 = k'^2 = 0$, and the collision axis to be the z axis.
The momenta are:

$$p = (E, 0, 0, E)\,, \qquad p' = (E, 0, 0, -E)\,, \qquad k = (E, \vec{k})\,, \qquad k' = (E, -\vec{k})\,, \tag{2.126}$$

with $|\vec{k}| = E$ and $\vec{k} \cdot \hat{z} = E\cos\theta$. Then the kinematic invariants are:

$$s = (p+p')^2 = 4E^2\,, \quad t = (p-k)^2 = -2E^2(1-\cos\theta)\,, \quad u = (p-k')^2 = -2E^2(1+\cos\theta)\,. \tag{2.127}$$

The goal is to compute the amplitude for $e^+e^- \longrightarrow \mu^+\mu^-$ at fixed helicities of the external states.

The amplitude for $e^+e^- \longrightarrow \mu^+\mu^-$,

$$\begin{aligned} i\mathcal{M}(e^+e^- \longrightarrow \mu^+\mu^-) &= \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\frac{-ig_{\mu\nu}}{q^2}\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k') \\ &= \frac{ie^2}{q^2}\bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^r(k)\gamma_\mu v^{r'}(k'), \end{aligned} \tag{2.128}$$

Now a right-handed electron in the $+z$-direction has spin up,

$$\xi = \begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \hat{p}\cdot\sigma \begin{pmatrix}1\\0\end{pmatrix} = \sigma_3 \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \tag{2.129}$$

so

$$u_R(p) = \sqrt{2E}\begin{pmatrix} \dfrac{1-\sigma_3}{2}\begin{pmatrix}1\\0\end{pmatrix} \\ \dfrac{1+\sigma_3}{2}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{2.130}$$

Since $p_\perp = 0$ and $p^+ = 2\text{E}$, in the light-cone representation (2.123) we obtain the same spinor $u_R(p)$.

A left-handed positron in the $-z$-direction has spin down,

$$\xi = \begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \hat{p}\cdot\sigma \begin{pmatrix}0\\1\end{pmatrix} = -\sigma_3 \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}, \tag{2.131}$$

and it corresponds to a right-handed negative energy electron,

$$v_R(p) = \sqrt{2E}\begin{pmatrix} \dfrac{1+\sigma_3}{2}\begin{pmatrix}0\\1\end{pmatrix} \\ -\dfrac{1-\sigma_3}{2}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}. \tag{2.132}$$

Since $p = (E, 0, 0, -E) \longrightarrow p_\perp = 0, p^- = 2E$, we obtain the same spinor $v_R(p)$ in the light-cone representation (2.125).

So the current for a right-handed electron and a left-handed positron is

$$\begin{aligned}
\bar{v}_R(p')\gamma^\mu u_R(p) &= 2E\,[0,0,0,-1]\begin{pmatrix}\bar{\sigma}^\mu & 0\\ 0 & \sigma^\mu\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\
&= 2E\,[0,-1]\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix}\\
&= -2E\,[0,1,i,0]. \qquad (2.133)
\end{aligned}$$

If we take the virtual photon γ^* with polarisation $+$ in the $+z$-direction, i.e., $\varepsilon_+ = \frac{1}{\sqrt{2}}(0,1,i,0)$, then,

$$\bar{v}_R(p')\gamma^\mu u_R(p) = -2E\sqrt{2}\varepsilon_+^\mu. \qquad (2.134)$$

In order to obtain the current for a right-handed muon μ^- and a left-handed antimuon μ^+, we may use in fact the current $\bar{v}_R(p')\gamma^\mu u_R(p)$ as a 4-vector and rotate it by the angle $-\theta$ in the $x-z$ plane (see figure 2.6). So the current becomes

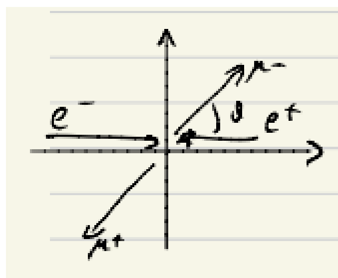


Figure 2.6: Rotation of the $\bar{v}_R(p')\gamma^\mu u_R(p)$ by an angle θ in the x-z plane.

$$\begin{aligned}
\bar{u}_R(k)\gamma^\mu v_R(k') &= [\bar{v}_R(k')\gamma^\mu u_R(k)]^*\\
&= -2E(0,\cos\theta,i,-\sin\theta)^*. \qquad (2.135)
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{M}(e_R^- e_L^+ \longrightarrow \mu_R^- \mu_L^+) &= \frac{e^2}{q^2}\bar{v}_R(p')\gamma^\mu u_R(p)\bar{u}_R(k)\gamma_\mu v_R(k')\\
&= \frac{e^2}{q^2}4E^2(-\cos\theta - 1)\\
&= -e^2(1+\cos\theta). \qquad (2.136)
\end{aligned}$$

In order to compute the other helicity amplitudes, we follow exactly the same steps as before but for a left-handed electron in the $+z$-direction and a right-handed positron in the $-z$-direction. A left-handed electron in the $+z$-direction has spin down, so the spinor is

$$u_L(p) = \sqrt{2E}\begin{pmatrix}\frac{1-\sigma_3}{2}\begin{pmatrix}0\\1\end{pmatrix}\\ \frac{1+\sigma_3}{2}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}. \qquad (2.137)$$

A right-handed positron in the $-z$-direction has spin up and it corresponds to a left-handed negative energy electron,

$$v_L(p) = \sqrt{2E} \begin{pmatrix} \frac{1+\sigma_3}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ -\frac{1-\sigma_3}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.138}$$

The current for a left-handed electron and a right-handed positron is:

$$\begin{aligned} \bar{v}_L(p')\gamma^\mu u_L(p) &= 2E\,[1,0,0,0] \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= 2E\,[1,0]\bar{\sigma}^\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= -2E\,[0,1,-i,0] \\ &= -2\sqrt{2}E\varepsilon_-^\mu, \end{aligned} \tag{2.139}$$

corresponding to a virtual photon γ^* with polarisation -. Then

$$\mathcal{M}(e_L^- e_R^+ \longrightarrow \mu_R^- \mu_L^+) = e^2(1-\cos\theta). \tag{2.140}$$

Finally, for a left-handed muon and a right-handed anti-muon $\bar{u}_L(k)\gamma^\mu v_L(k')$, rotate $\bar{v}_L(p')\gamma^\mu u_L(p)$ by $-\theta$,

$$\bar{u}_L(k)\gamma^\mu v_L(k') = -2E(0,\cos\theta, i, -\sin\theta), \tag{2.141}$$

by which we can get the remaining helicity amplitudes (Check!),

$$\begin{cases} \mathcal{M}(e_R^- e_L^+ \longrightarrow \mu_L^- \mu_R^+) = e^2(1-\cos\theta) \\ \mathcal{M}(e_L^- e_R^+ \longrightarrow \mu_L^- \mu_R^+) = -e^2(1+\cos\theta). \end{cases} \tag{2.142}$$

Here, we summarise the helicity scheme of $e^+e^- \to \mu^+\mu^-$ or $e^+e^- \to q\bar{q}$, with the goal to get the helicity structure of lepton-quark scattering by **crossing symmetry**, which implies that

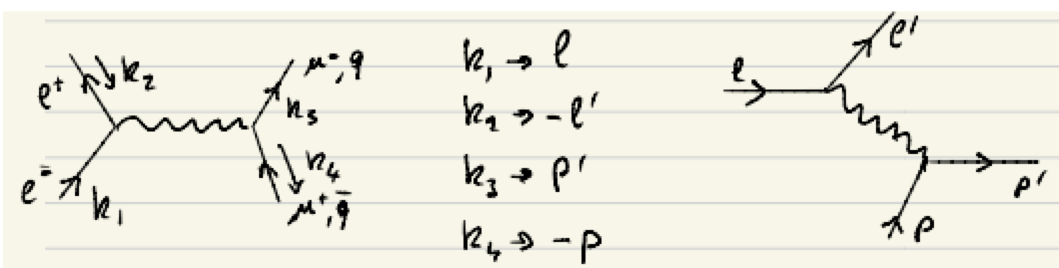


Figure 2.7: Crossing symmetry between $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to q\bar{q}$.

$$\begin{aligned} (k_1+k_2)^2 = s &\quad\to\quad (\ell-\ell')^2 = \hat{t} \\ (k_1-k_3)^2 = t &\quad\to\quad (\ell-p')^2 = \hat{u} \\ (k_1-k_4)^2 = u &\quad\to\quad (\ell+p)^2 = \hat{s}, \end{aligned}$$

where t and u are given in eq. (2.127).

The structure is summarised as follows,

$$\left.\begin{cases} e_R^- e_L^+ \to \mu_R^- \mu_L^+, q_R \bar{q}_L \\ e_L^- e_R^+ \to \mu_L^- \mu_R^+, q_L \bar{q}_R \end{cases}\right\} \sim u^2 \quad \to \quad \left.\begin{cases} e_R^- q_R \to e_R^- q_R \\ e_L^- q_L \to e_L^- q_L \end{cases}\right\} \sim \hat{s}^2 \tag{2.143}$$

$$\left.\begin{cases} e_R^- e_L^+ \to \mu_L^- \mu_R^+, q_L \bar{q}_R \\ e_L^- e_R^+ \to \mu_R^- \mu_L^+, q_R \bar{q}_L \end{cases}\right\} \sim t^2 \quad \to \quad \left.\begin{cases} e_R^- q_L \to e_R^- q_L \\ e_L^- q_R \to e_L^- q_R \end{cases}\right\} \sim \hat{u}^2, \tag{2.144}$$

so we see that in the squared amplitude for $eq \to eq$ scattering,

$$\frac{1}{4} \sum_{spin} |\mathcal{M}|^2 = \frac{8e^4 Q_i^2}{\hat{t}^2} \frac{\hat{s}^2 + \hat{u}^2}{4}, \tag{2.145}$$

the $\hat{s}^2$ term comes from the scattering of left(right)-handed electrons on left(right)-handed quarks, and the $\hat{u}^2$ term comes from the scattering of left(right)-handed electrons on right(left)-handed quarks. Thus, in the parton-model inspired DIS cross section,

$$\frac{d\sigma}{dxdy} = \frac{2\pi\alpha^2 s}{(Q^2)^2} \left[1 + (1-y)^2\right] \sum_i x f_i(x) Q_i^2 \,, \tag{2.146}$$

the term proportional to 1 stems from the scattering of left(right)-handed electrons on left(right)-handed quarks, while the the term proportional to $(1-y)^2$ from the scattering of left(right)-handed electrons on right(left)-handed quarks.

Chapter 3

The QCD Lagrangian

3.1 SU(N): Global symmetry and Colour Algebra

As we saw in the introductory slides, strong interactions were modelled by a triplet of quarks with an internal global symmetry, colour, or SU(3). Since there is no loss of generality in going from SU(N) to SU(3), we shall discuss SU(N) as a global symmetry. Take an N-plet of Dirac fields,

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}.$$

The Lagrangian,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{3.1}$$

is invariant under the **global symmetry**,

$$\psi(x) \to V\psi(x), \quad \text{with} \quad V \in \mathrm{SU(N)}. \tag{3.2}$$

V can be expressed in terms of the generators T^a of SU(N) with $a = 1, .., N^2 - 1$. The T^a's satisfy the algebra,

$$[T^a, T^b] = i f^{abc} T^c, \tag{3.3}$$

with f^{abc} the structure constants. In a representation r, the V's are represented in terms of traceless Hermitian generators t_r^a and of real parameters α^a, $V = \exp(i\alpha^a t_r^a)$. In fact,

$$VV^\dagger = 1 = \exp\Big(i\alpha^a (t_r^a - t_r^{a\dagger})\Big) \longrightarrow t_r^{a\dagger} = t_r^a \tag{3.4}$$

$$\det V = 1 \longrightarrow \ln\det V = \mathrm{Tr}\ln V = 0 \longrightarrow \mathrm{Tr}\, t^a = 0. \tag{3.5}$$

The normalisation of the generators is given by $\mathrm{Tr}\big(t_r^a t_r^b\big) = T_r \delta^{ab}$ (do not confuse this T_r with the trace operator) and the Casimir relation holds

$$\sum_a (t_r^a t_r^a)_{ij} = C_r \delta_{ij} \quad \text{with} \quad i,j = 1, ..., \dim(r) \quad \text{and} \quad C_r = T_r \frac{\dim(\text{group})}{\dim(r)}. \tag{3.6}$$

These properties and the ones of the colour or SU(N) algebra are discussed more thoroughly in the $\longrightarrow$ Exercises.

3.2 Abelian gauge Theory: U(1)

3.2.1 Local symmetry and the Comparator

We want to promote the global symmetry above to a **local symmetry** by making the α^a parameters space-time dependent, $\alpha^a(x)$, and

$$\psi(x) \to V(x)\psi(x), \quad \text{with} \quad V = \exp(i\alpha^a(x)t_r^a). \tag{3.7}$$

As it happens for an abelian gauge theory, like QED, the mass term $m\bar{\psi}\psi$ in the Lagrangian is invariant under the transformation, but the kinetic term $\bar{\psi}i\partial\!\!\!/\psi$ is not.

Let us review what happens in QED. The Lagrangian for the matter field is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{3.8}$$

with $\psi(x)$ the Dirac field. There is a global symmetry,

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x), \tag{3.9}$$

which implies a conservation law $\partial_\mu j^\mu(x) = 0$, with $j^\mu(x)$ given by Noether's theorem $j^\mu(x) = \bar{\psi}\gamma^\mu\psi$. If we promote the global symmetry to a local symmetry through the gauge transformation,

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x). \tag{3.10}$$

the derivative of ψ in the direction of n^μ,

$$n^\mu\partial_\mu\psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[\psi(x+\epsilon n) - \psi(x)], \tag{3.11}$$

is ill-defined under a gauge transformation because $\psi(x)$ and $\psi(x+\epsilon n)$ transform in different ways. One needs a field A_μ whose gauge transformation rule would involve a term $\partial_\mu\alpha(x)$ to cancel the $\partial_\mu\alpha$ term coming from $\partial_\mu\psi$. One can introduce a scalar function of two space-time points $U(y,x)$, the **comparator**, which under a gauge transformation,

$$U(y,x) \longrightarrow e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)} \quad \text{with} \quad U(x,x) = 1. \tag{3.12}$$

In general $U(y,x)$ can be written as a phase,

$$U(y,x) = e^{i\phi(y,x)} \quad \text{with} \quad \phi(x,x) = 0, \tag{3.13}$$

and we can require that $\phi(y,x)$ be a continuous function, with

$$\frac{\partial}{\partial y^\mu}\phi(y,x)|_{y=x} = -eA_\mu(x), \tag{3.14}$$

such that when we expand $\phi(y,x)$ in y about x,

$$U(x+\epsilon n, x) = 1 - ie\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2). \tag{3.15}$$

$A_\mu(x)$ is also called the **connection**. Now, under a gauge transformation,

$$U(x+\epsilon n, x) \longrightarrow e^{i\alpha(x+\epsilon n)}U(x+\epsilon n, x)e^{-i\alpha(x)},$$

replacing the explicit form of $U(x+\epsilon n, x)$ and expanding the exponentials, we get the transformation law of $A_\mu(x)$ (please check it!),

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x). \tag{3.16}$$

Under a gauge transformation, $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation rule,

$$\begin{aligned} \psi(y) &\longrightarrow e^{i\alpha(y)}\psi(y), \\ U(y,x)\psi(x) &\longrightarrow e^{i\alpha(y)}U(y,x)\cancel{e^{-i\alpha(x)}}\cancel{e^{i\alpha(x)}}\psi(x). \end{aligned}$$

Therefore, the **covariant derivative**,

$$n^\mu D_\mu \psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)], \tag{3.17}$$

is well defined under a gauge transformation. Expanding the terms on the right-hand side, we get (please check!)

$$D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi. \tag{3.18}$$

$D_\mu\psi$ is covariant under a gauge transformation, i.e., $D_\mu\psi \longrightarrow e^{i\alpha(x)}D_\mu\psi$ (please check!) and in particular (please check!),

$$[D_\mu, D_\nu]\psi \longrightarrow e^{i\alpha(x)}[D_\mu, D_\nu]\psi. \tag{3.19}$$

But

$$[D_\mu, D_\nu]\psi = ieF_{\mu\nu}\psi, \tag{3.20}$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ the **field strength tensor**. Note that $F_{\mu\nu}$ is not a differential operator, and that it is gauge invariant.

3.2.2 Wilson line and Wilson loop

A geometrical interpretation of $F_{\mu\nu}$ is obtained by using the comparator expanded to $\mathcal{O}(\epsilon^2)$ (please check!),

$$U(x+\epsilon n, x) = 1 - ie\epsilon n^\mu A_\mu - \frac{\epsilon^2}{2}e^2 n^\mu A_\mu n^\nu A_\nu - ie\frac{\epsilon^2}{2}n^\mu n^\nu \partial_\nu A_\mu + \mathcal{O}(\epsilon^3), \tag{3.21}$$

which can be written as

$$U(x+\epsilon n, x) = \exp\Big[-ie\epsilon n^\mu A_\mu(x+\frac{\epsilon}{2}n) + \mathcal{O}(\epsilon^3)\Big]. \tag{3.22}$$

Furthermore,

$$U(x,y) = U(y,x)^\dagger, \tag{3.23}$$

which implies that

$$\frac{\partial}{\partial x^\mu}\phi(y,x)|_{y=x} = -\frac{\partial}{\partial y^\mu}\phi(y,x)|_{y=x}. \tag{3.24}$$

So,

$$U(x, x+\epsilon n) = \exp\Big[ie\epsilon n^\mu A_\mu(x+\frac{\epsilon}{2}n) + \mathcal{O}(\epsilon^3)\Big]. \tag{3.25}$$

Now, one can consider a square in the $\hat{x} - \hat{y}$ plane, figure 3.1. This allows us to construct the

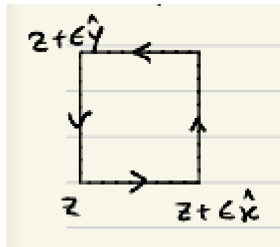


Figure 3.1: Infinitesimal square path in the $\hat{x} - \hat{y}$ plane.

quantity,

$$U(z) = U(z, z + \epsilon\hat{y})U(z + \epsilon\hat{y}, z + +\epsilon\hat{x} + \epsilon\hat{y})U(z + \epsilon\hat{x} + \epsilon\hat{y}, z + \epsilon\hat{x})U(z + \epsilon\hat{x}, z). \tag{3.26}$$

Using the expansion of the comparator above, we get

$$\begin{aligned} U(z) &= 1 - ie\epsilon^2(\partial_x A_y - \partial_y A_x) + \mathcal{O}(\epsilon^3) \\ &= 1 - ie\epsilon^2 F_{xy} + \mathcal{O}(\epsilon^3), \end{aligned} \tag{3.27}$$

where A_x is shorthand for $n^\mu A_\mu$ in the $\hat{x}$ direction. Since $U(y, x)$ transforms as in eq. (3.12), $U(z)$ is gauge invariant, and so are $F_{\mu\nu}$ and its derivatives.

From the gauge transformation of the comparator (3.12) for infinitesimal lengths, we have derived the gauge transformation (3.16). However, if we assume the transformation (3.16), we can generalise the comparator to finite lengths,

$$U_p(y, x) = \exp\left[-ie\int_x^y \mathrm{d}z^\mu A_\mu\right]. \tag{3.28}$$

This is called the **Wilson line**. The index p reminds us that $\int_x^y$ is a path-dependent line integral. Indeed, under a gauge transformation (please check!),

$$U_p(y, x) \longrightarrow e^{i\alpha(y)}U_p(y, x)e^{-i\alpha(x)}. \tag{3.29}$$

The Wilson line is a very useful quantity, which features in many studies of QCD.

For a closed path, we obtain the **Wilson loop**,

$$\begin{aligned} U_p(x, x) &= \exp\left[-ie\oint \mathrm{d}z^\mu A_\mu\right], \\ &= \exp\left[-i\frac{e}{2}\int \mathrm{d}\sigma^{\mu\nu}F_{\mu\nu}\right], \end{aligned} \tag{3.30}$$

where the second expression is obtained from Stokes' theorem, and generalises what we got on the square $\hat{x} - \hat{y}$.

The QED Lagrangian must then be formed by ψ and $D_\mu\psi$ and by $F_{\mu\nu}$ and its derivatives. The most general Lagrangian with operators of dimension 4 is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - C\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}, \tag{3.31}$$

however, because of the Levi-Civita tensor $\epsilon^{\alpha\beta\mu\nu}$ the last term is odd under both P and T symmetries. It is possible to add operators of dimension 5 (e.g. $\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi$), dimension 6 (e.g. $(\bar{\psi}\psi)^2$ or $(\bar{\psi}\gamma^5\psi)^2$), or higher dimensions, but these operators would correspond to non-renormalisable interactions.

3.3 Non-abelian Gauge Theory: SU(N)

3.3.1 Local symmetry

Now, we go back to a non-abelian gauge theory, with SU(N) as the group of the local symmetry. The gauge transformation is

$$\psi(x) \longrightarrow V(x)\psi(x) \quad \text{with} \quad V(x) = \exp(i\alpha^a(x)t_r^a). \tag{3.32}$$

The argument is the same as in the abelian case: One needs a field A_μ^a whose gauge transformation rule would involve a term $\partial_\mu\alpha^a(x)$ to cancel the $\partial_\mu\alpha^a(x)$ term coming from $\partial_\mu\psi$.

Just like in the abelian case, once can introduce a function of two space-time points, $U(y,x)$, which in the non-abelian case is matrix-valued, $U(y,x) \in$ SU(N), with $U(x,x) = \mathbb{I}$, such that under a gauge transformation,

$$U(y,x) \longrightarrow V(y)U(y,x)V^\dagger(x), \tag{3.33}$$

with $V(x) \in$ SU(N), a continuous function of $\alpha^a(x)$, and such that if we expand $U(y,x)$ in y about x,

$$U(x+\epsilon n, x) = \mathbb{I} + ig\epsilon n^\mu A_\mu^a t^a + \mathcal{O}(\epsilon^2). \tag{3.34}$$

The properties of the gauge transformations can then be derived like in the abelian case. We only outline them in this course as they will be examined in detail in the QFT II course. The gauge transformation of $U(x+\epsilon n, x)$ implies that the gauge field transforms as (please check!)

$$A_\mu^a t^a \longrightarrow V(x)(A_\mu^a t^a + \frac{i}{g}\partial_\mu)V^\dagger(x). \tag{3.35}$$

For an infinitesimal transformation,

$$V(x) = \mathbb{I} + i\alpha^a(x)t^a + \mathcal{O}(\alpha^2). \tag{3.36}$$

This implies that (please check!)

$$A_\mu^a \longrightarrow A_\mu^a + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A_\mu^b\alpha^c + \mathcal{O}(\alpha^2), \tag{3.37}$$

where the last term is due to the non-abelian nature of the gauge field. Extending the abelian definition of covariant derivative (3.17) implies that (please check!)

$$D_\mu = \partial_\mu - igA_\mu^a t^a, \tag{3.38}$$

such that under a gauge transformation (please check!),

$$n^\mu D_\mu\psi \longrightarrow V(x)n^\mu D_\mu\psi. \tag{3.39}$$

Note also that for a field α^a in the adjoint representation,

$$(D_\mu\alpha)_a \equiv \partial_\mu\alpha_a - igA^b_\mu(t^b)_{ac}\alpha_c \tag{3.40}$$
$$= \partial_\mu\alpha_a + gf^{abc}A^b_\mu\alpha_c, \tag{3.41}$$

therefore, under an infinitesimal gauge transformation,

$$A^a_\mu \longrightarrow A^a_\mu + \frac{1}{g}(D_\mu\alpha)^a + \mathcal{O}(\alpha^2). \tag{3.42}$$

Likewise, under a gauge transformation,

$$[D_\mu, D_\nu]\psi \longrightarrow V(x)[D_\mu, D_\nu]\psi, \tag{3.43}$$

where (please check!),

$$[D_\mu, D_\nu] \equiv -igF^a_{\mu\nu}t^a \quad \text{with} \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{3.44}$$

As in the abelian case, $[D_\mu, D_\nu]$ is not a differential operator. However, in the non-abelian case $F^a_{\mu\nu}$ has a quadratic term in the gauge field A^a_μ, and is not a gauge invariant. In fact (3.43) and $\psi \longrightarrow V\psi$ imply that

$$[D_\mu, D_\nu] \longrightarrow V(x)[D_\mu, D_\nu]V^\dagger(x). \tag{3.45}$$

For an infinitesimal transformation (please check!),

$$F^a_{\mu\nu}t^a \longrightarrow F^a_{\mu\nu}t^a + i\alpha^b F^a_{\mu\nu}[t^b, t^c], \tag{3.46}$$

hence,

$$F^a_{\mu\nu} \longrightarrow F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu}. \tag{3.47}$$

However,

$$\mathrm{Tr}\big[(F^a_{\mu\nu}t^a)^2\big] = \frac{1}{2}(F^a_{\mu\nu})^2, \tag{3.48}$$

is gauge invariant.

In analogy with QED, we can take the **Yang-Mills Lagrangian** to be

$$\mathcal{L}_{YM} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2, \tag{3.49}$$

which is the most general P- and T-conserving Lagrangian with operators of dimension 4. It was derived in 1954 by Yang and Mills with local symmetry group SU(2), for the isospin symmetry between protons and neutrons. It would imply that the spin-1 gauge fields A^a_μ were massless. Since the vector mesons interacting with protons and neutrons were massive, the Yang-Mills model was shelved. It was revived as QCD with local group SU(3), for the colour symmetry, and as Glashow-Salam-Weinberg theory with local group SU(2) $\otimes$ U(1) for the electroweak isospin symmetry.

3.3.2 Wilson Line and Wilson Loop

In the abelian case, we introduced the Wilson line (3.28) where the integral is path-dependent. In the non-abelian case, the matrices do not commute at different space-time points. It is convenient to parametrise the path: $z(\tau)$ with $z(0) = x$ and $z(1) = y$, and introduce the **path ordering** of the matrices,

$$P(A^a_\mu(z(\tau_1))t^a A^b_\nu(z(\tau_2))t^b) = A^a_\mu(z(\tau_1))t^a A^b_\nu(z(\tau_2))t^b\theta(\tau_1 - \tau_2) + A^b_\nu(z(\tau_2))t^b A^a_\mu(z(\tau_1))t^a\theta(\tau_2 - \tau_1), \tag{3.50}$$

then the Wilson line can be defined as a path-ordered expansion of the exponential,

$$U_p(y, x, A) = P\left[\exp\left[ig\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}z^\mu}{\mathrm{d}\tau} A^a_\mu(z(\tau))t^a\right]\right]. \tag{3.51}$$

Thus defined, the Wilson line has the correct gauge transformation rule ($\longrightarrow$ Exercises),

$$U_p(y, x, A^V) = V(y)U_p(y, x, A)V^\dagger(x). \tag{3.52}$$

However, the Wilson line, closed to form a loop, is not an invariant in the non-abelian case,

$$U_p(x, x, A^V) = V(x)U_p(x, x, A)V^\dagger(x). \tag{3.53}$$

One can see this by considering an infinitesimal square in the $\hat{x} - \hat{y}$ plane. One obtains ($\longrightarrow$ Exercises)

$$U_p(z, z) = \mathbb{I} + ig\epsilon^2 F^a_{xy}t^a + \mathcal{O}(\epsilon^3), \tag{3.54}$$

and indeed, as we know, $F^a_{\mu\nu}t^a$ is not gauge invariant. However, the **Wilson loop**, defined as $\mathrm{Tr}\, U_p(z, z)$, is gauge invariant. In fact, exponentiating the infinitesimal shift above,

$$U_p(z, z) = \exp\left[ig\epsilon^2 F^a_{xy}t^a + \mathcal{O}(\epsilon^3)\right] \tag{3.55}$$

$$= \mathbb{I} + ig\epsilon^2 F^a_{xy}t^a - \frac{g^2}{2}\epsilon^4 F^a_{xy}F^b_{xy}t^a t^b + \mathcal{O}(\epsilon^5). \tag{3.56}$$

Using $\mathrm{Tr}(t^a) = 0$ and $\mathrm{Tr}\big(t^a t^b\big) = \frac{\delta^{ab}}{2}$, one finds

$$\mathrm{Tr}\, U_p(z, z) = N - \frac{g^2}{4}\epsilon^4 (F^a_{xy})^2 + \mathcal{O}(\epsilon^5). \tag{3.57}$$

3.4 Feynman Rules

We start by recalling the Yang-Mills (YM) Lagrangian,

$$\mathcal{L}_{YM} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2, \tag{3.58}$$

with $D_\mu = \partial_\mu - igA^a_\mu t^a$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$, where A^a_μ is the gluon field. Unfolding the $(F^a_{\mu\nu})^2$ term, we can spell out the interaction term and write the Lagrangian as

$$\mathcal{L}_{YM} = \mathcal{L}_{kin} + gA^a_\mu\bar{\psi}\gamma^\mu t^a\psi - gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \frac{g^2}{4}f^{abe}f^{cde}A^a_\mu A^b_\nu A^{\mu c}A^{\nu d}, \tag{3.59}$$

where $\mathcal{L}_{kin}$ is a shorthand for the kinetic terms of the Lagrangian.

Upon quantisation of the Lagrangian, one can derive the Feynman rules. We will not do it (it will be done in detail in the QFT II course). We will just list the Feynman rules here, for future use.

From the kinetic terms of the Lagrangian, we obtain the fermion and the gluon propagators, which are a straightforward extension of the Abelian case. The fermion propagator is

$$j \xleftarrow{\ k\ } i \qquad \longrightarrow \qquad D_{ij}(k) = \frac{i}{\not{k} - m + i\epsilon}\delta_{ij} \quad i,j = 1,\ldots,N. \tag{3.60}$$

Choosing a generalised Lorenz gauge, $G(A) = \partial_\mu A_a^\mu(x) - \omega_a(x)$, the gluon propagator is

$$\longrightarrow \quad D_{ab}^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\zeta)\frac{k^\mu k^\nu}{k^2}\right)\delta_{ab} \quad a,b = 1,\ldots,N^2-1\,, \tag{3.61}$$

where for $\zeta = 1$ we have the *Feynman gauge* and for $\zeta = 0$ we have the *Landau gauge*, however, note that ζ is an arbitrary parameter, and other values of ζ have been used in the literature. Further, note that, up to the colour index, $D_{ab}^{\mu\nu}$ equals the photon propagator. The gluon propagator can also be written in terms of transverse and longitudinal projectors, $D_T^{\mu\nu}$ and $D_L^{\mu\nu}$,

$$D_{ab}^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(D_T^{\mu\nu} + \zeta D_L^{\mu\nu}\right)\delta_{ab}, \tag{3.62}$$

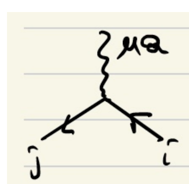
where

$$D_T^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \qquad D_L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}. \tag{3.63}$$

They are projectors, since

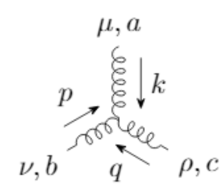
$$D_T^{\mu\nu} D_T^{\nu\lambda} = D_T^{\mu\lambda}, \qquad D_L^{\mu\nu} D_L^{\nu\lambda} = D_L^{\mu\lambda}, \qquad D_T^{\mu\nu} D_L^{\nu\lambda} = 0. \tag{3.64}$$

Looking at the interaction term of the Lagrangian, from $gA_\mu^a \bar{\psi}\gamma^\mu t^a \psi$ we obtain the fermion-gluon vertex, again a direct extension of the Abelian case,



$$\longrightarrow \qquad ig\gamma^\mu t_{ji}^a \tag{3.65}$$

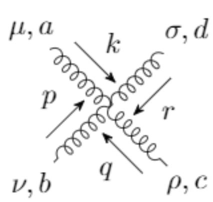
From the cubic term $-gf^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c}$, we get



$$\longrightarrow gf^{abc}[g^{\mu\rho}(q-k)^\nu + g^{\mu\nu}(k-p)^\rho + g^{\rho\nu}(p-q)^\mu]. \tag{3.66}$$

which fulfils Bose symmetry, since it is antisymmetric under the exchange of any two momenta and accordingly under the exchange of any two colour indices.

From the quartic term, $-\frac{g^2}{4} f^{abe} f^{cde} A_\mu^a A_\nu^b A^{\mu c} A^{\nu d}$ we obtain



$$\longrightarrow -ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})$$
$$+ f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})], \tag{3.67}$$

which is again Bose symmetric.

3.5 Ward Identity and Unitarity

With the Feynman rules we have, we can examine the question of the **Ward identity** and **unitarity**. In QED, the Ward identity for a general S-matrix implies that by replacing one photon polarisation with its momentum, the S-matrix vanishes (see figure 3.2). Singling out the photon

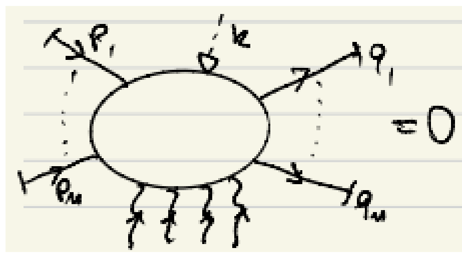


Figure 3.2: Ward Identity for a general S-matrix in QED. The perpendicular bars at the end of each line indicate on-shell conditions for the incoming and outgoing momenta.

momentum and polarisation, the amplitude is

$$A = A^\mu(k) \cdot \epsilon_\mu(k) \longrightarrow A^\mu(k) \cdot k_\mu = 0, \tag{3.68}$$

i.e., QED is invariant under $\epsilon_\mu(k) \longrightarrow \epsilon_\mu(k) + \tilde{\alpha}(k)k^\mu$, i.e., under $A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x)$, where $\tilde{\alpha}(x)$ is the Fourier transform of $\alpha(x)$.

So non-transverse polarisation states do not contribute to a scattering amplitude, and thus to its square. In order to state it more precisely, we take the amplitude squared,

$$\sum_\lambda |A_\mu \epsilon^\mu_\lambda|^2 = \sum_{\lambda=1,2} (\epsilon^\mu_{\perp\lambda} \epsilon^{\nu *}_{\perp\lambda}) A_\mu A^*_\nu, \tag{3.69}$$

where the sum is over the physical polarisations. Suppose the photon is in the z-direction $k^\mu = (k^0, 0, 0, k^0)$. Its transverse polarisations are

$$\epsilon_{\perp 1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \epsilon_{\perp 2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \tag{3.70}$$

Define the *polarisation tensor*,

$$U^{\mu\nu} = \sum_{\lambda=1,2} \epsilon^\mu_{\perp\lambda} \epsilon^{\nu *}_{\perp\lambda}, \tag{3.71}$$

whose entries are $U^{11} = U^{22} = 1$ and $U^{12} = U^{21} = 0$. The other entries are obviously zero.

The non-physical polarisations are the longitudinal polarisation and the time-like polarisation,

$$\epsilon^\mu_L(k) = \left(0, 0, 0, \frac{k^0}{|\vec{k}|}\right), \qquad \epsilon^\mu_t(k) = \left(\frac{k^0}{|\vec{k}|}, 0, 0, 0\right). \tag{3.72}$$

Set the vector $\bar{k}^\mu = (k^0, 0, 0, -k^0)$. Then k^μ and $\bar{k}^\mu$ single out light-cone directions. We can introduce light-cone polarisations and write

$$\epsilon^\mu_\pm(k) = \frac{\epsilon^\mu_t(k) \pm \epsilon^\mu_L(k)}{\sqrt{2}} = \frac{1}{\sqrt{2}|\vec{k}|}(k^0, 0, 0, \pm k^0), \tag{3.73}$$

i.e.

$$\epsilon^\mu_+(k) = \frac{k^\mu}{\sqrt{k \cdot \bar{k}}}, \qquad \epsilon^\mu_-(k) = \frac{\bar{k}^\mu}{\sqrt{k \cdot \bar{k}}}, \tag{3.74}$$

with $k \cdot \bar{k} = 2|\vec{k}|^2$. $\epsilon^\mu_+(k)$, and by extension k^μ, is often called *scalar* polarisation. Note that the four polarisations $\epsilon^\mu_\pm$ and $\epsilon^\mu_{\perp i}$ are orthonormal,

$$\epsilon_+ \cdot \epsilon^*_- = 1, \quad |\epsilon_+|^2 = |\epsilon_-|^2 = 0, \quad \epsilon_{\perp i} \cdot \epsilon^*_{\perp j} = -\delta_{ij}, \quad \epsilon_\pm \cdot \epsilon^*_{\perp i} = 0 \quad i, j = 1, 2, \tag{3.75}$$

where we represent in general the dot product as complex-valued, although the light-cone polarisations, $\epsilon_\pm$, are real.

Let us define the *non-physical polarisation tensor*,

$$V^{\mu\nu} = \epsilon^\mu_+ \epsilon^{\nu *}_- + \epsilon^\mu_- \epsilon^{\nu *}_+, \tag{3.76}$$

whose entries are $V^{00} = -V^{33} = 1$ and $V^{30} = V^{03} = 0$. The other entries are obviously zero. So,

$$V^{\mu\nu} - U^{\mu\nu} = \epsilon^\mu_+ \epsilon^{\nu *}_- + \epsilon^\mu_- \epsilon^{\nu *}_+ - \sum_{\lambda=1,2} \epsilon^\mu_{\perp\lambda} \epsilon^{\nu *}_{\perp\lambda} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = g^{\mu\nu}, \tag{3.77}$$

with

$$\epsilon^\mu_+ \epsilon^{\nu *}_- + \epsilon^\mu_- \epsilon^{\nu *}_+ = \frac{k^\mu \bar{k}^\nu + \bar{k}^\mu k^\nu}{k \cdot \bar{k}}, \tag{3.78}$$

thus,

$$\sum_{\lambda=1,2} \epsilon^\mu_{\perp\lambda} \epsilon^{\nu *}_{\perp\lambda} = -g^{\mu\nu} + \frac{k^\mu \bar{k}^\nu + \bar{k}^\mu k^\nu}{k \cdot \bar{k}} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}. \tag{3.79}$$

However, because of the Ward identity,

$$k^\mu A_\mu = 0 \quad \Rightarrow \quad k^0 A^0 - k^0 A^3 = 0 \quad \Rightarrow \quad A^0 = A^3, \tag{3.80}$$

thus,

$$\begin{aligned} \sum_{\lambda=1}^{2} |A_\mu \epsilon^\mu_\lambda|^2 &= |A_1|^2 + |A_2|^2 \\ &= -|A_0|^2 + |A_1|^2 + |A_2|^2 + |A_3|^2 \\ &= -g^{\mu\nu} A_\mu A^*_\nu. \end{aligned} \tag{3.81}$$

Because of the Ward identity, in QED we may replace the sum over the physical photon polarisations with $-g^{\mu\nu}$, because the non-physical polarisation states decouple in a scattering process. This is represented diagrammatically in fig. 3.3.

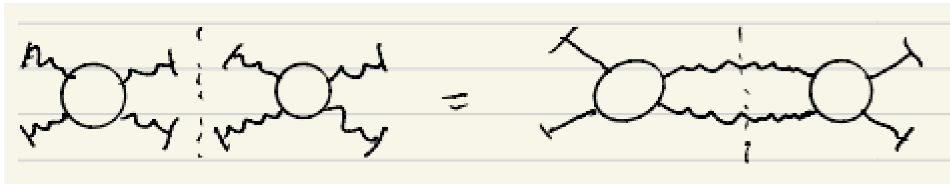
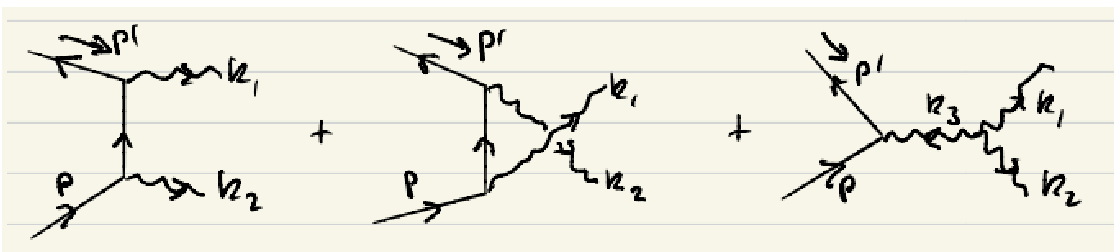


Figure 3.3: The sum over the physical polarisations (3.79) is replaced by (minus) the full sum (3.77).

Now, let us analyse how the Ward identity works with gluons in QCD, in the simple scattering process $q\bar{q} \to gg$,

$$ \qquad (3.82)$$



The abelian-like diagrams yield

$$iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) = (ig)^2\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\bar{v}(p')\left(\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m}\gamma^\nu t^b + \gamma^\nu t^b \frac{i}{\not{p} - \not{k}_1 - m}\gamma^\mu t^a\right)u(p). \tag{3.83}$$

The quarks are on-shell, as they should be in an amplitude, but let us not assume, for now, the transversality condition $k_i \cdot \epsilon(k_i) = 0$ on the on-shell gluons. In order to check the Ward identity, we replace $\epsilon^*_\nu(k_2)$ with $k_{2\nu}$, fig. 3.4,

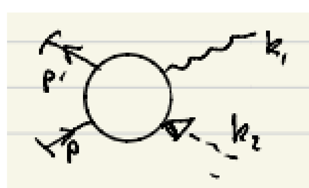


Figure 3.4: Replacing the polarisation of one gluon with a scalar polarisation.

In the abelian diagrams, we get

$$iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} = -g^2\epsilon^*_\mu(k_1)\bar{v}(p')\left(\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m}\not{k}_2 t^b + \not{k}_2 t^b \frac{i}{\not{k}_2 - \not{p}' - m}\gamma^\mu t^a\right)u(p), \tag{3.84}$$

where on the second term we used momentum conservation. Then we add and subtract $(\not{p} - m)$ to the first term, and $(\not{p}' + m)$ to the second term, and use the Dirac equations,

$$\begin{cases} (\not{p} - m)u(p) = 0, \\ \bar{v}(p)(\not{p}' + m) = 0, \end{cases} \tag{3.85}$$

and we get

$$
\begin{aligned}
iM_{1,2}^{\mu\nu}\epsilon_\mu^*(k_1)k_{2\nu} &= -g^2\epsilon_\mu^*(k_1)\bar{v}(p')\left(-i\gamma^\mu[t^a,t^b]\right)u(p)\\
&= -g^2 f^{abc}\epsilon_\mu^*(k_1)\bar{v}(p')\gamma^\mu t^c u(p). \qquad (3.86)
\end{aligned}
$$

In an abelian theory, the right-hand side would have vanished. Thus, as we know, in QED $M^\nu(k_2) \equiv M_{1,2}^{\mu\nu}\epsilon_\mu^*(k_1)$ is such that $M^\nu(k_1)\cdot k_{2,\nu} = 0$. In QCD, we also need the gluon self interaction,

$$
\begin{aligned}
iM_3^{\mu\nu}\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2) &= ig\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\bar{v}(p')\gamma_\rho t^c u(p)\\
&\times \frac{-i}{k_3^2}gf^{abc}\left[(k_2-k_1)^\rho g^{\mu\nu} + (k_1-k_3)^\nu g^{\mu\rho} + (k_3-k_2)^\mu g^{\nu\rho}\right]. \qquad (3.87)
\end{aligned}
$$

Now, as before, we replace $\epsilon_\nu^*(k_2)$ with $k_{2\nu}$,

$$
\begin{aligned}
iM_3^{\mu\nu}\epsilon_\mu^*(k_1)k_{2\nu} &= g\epsilon_\mu^*(k_1)\bar{v}(p')\gamma_\rho t^c u(p)\,\frac{1}{k_3^2}gf^{abc}\left[(k_2-k_1)^\rho k_2^\mu + (k_1-k_3)\cdot k_2 g^{\mu\rho} + (k_3-k_2)^\mu k_2^\rho\right]\\
&= g^2 f^{abc}\epsilon_\mu^*(k_1)\bar{v}(p')\gamma_\rho t^c u(p)\,\frac{1}{k_3^2}\left[(k_1+k_3)^\mu k_1^\rho + (k_3^2-k_1^2)g^{\mu\rho} - (k_1+k_3)^\rho k_3^\mu\right]\\
&= g^2 f^{abc}\epsilon_\mu^*(k_1)\frac{1}{k_3^2}\bar{v}(p')[\not{k}_1 k_1^\mu - \not{k}_3 k_3^\mu + (k_3^2-k_1^2)\gamma^\mu]t^c u(p), \qquad (3.88)
\end{aligned}
$$

where we used momentum conservation, $k_2 = -(k_1+k_3)$, for the 3-gluon vertex in the second step, and explicitly contracted γ^ρ in the last step. Now we use again momentum conservation $k_3 = -(p+p')$ and the Dirac equation $\bar{v}(p')(\not{p}+\not{p}')u(p) = 0$. Thus we get finally

$$
iM_3^{\mu\nu}\epsilon_\mu^*(k_1)k_{2\nu} = g^2 f^{abc}\epsilon_\mu^*(k_1)\frac{1}{k_3^2}\bar{v}(p')[\not{k}_1 k_1^\mu + (k_3^2-k_1^2)\gamma^\mu]t^c u(p). \qquad (3.89)
$$

We may distinguish three cases:

1. If the first gluon is real, i.e. it is on-shell $k_1^2 = 0$ and its polarisation is transverse $k_1\cdot\epsilon(k_1) = 0$, then

$$
iM_3^{\mu\nu}\epsilon_\mu^*(k_1)k_{2\nu} = g^2 f^{abc}\epsilon_\mu^*(k_1)\bar{v}(p')\gamma^\mu t^c u(p). \qquad (3.90)
$$

   This, summed to the abelian diagrams (3.86), yields

$$
i(M_{1,2}^{\mu\nu} + M_3^{\mu\nu})\epsilon_\mu^*(k_1)k_{2\nu} = 0. \qquad (3.91)
$$

   Diagrammatically this is represented by fig. 3.5,

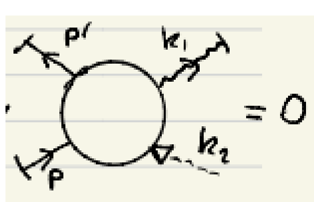


Figure 3.5: Fulfilment of the Ward identity in case 1.

   i.e., the Ward identity is fulfilled if one gluon polarisation is scalar and the other gluon is real (provided the couplings of the 3-gluon vertex and of the gluon-fermion vertex are equal. Analogously, fig. 3.5 holds if the couplings of the 3-gluon vertex and of the 4-gluon vertex are equal.)

2. If we replace $\epsilon_\mu(k_1) \to k_{1\mu}$, i.e., the polarisation of both gluons is scalar, then from the abelian diagrams and the 3-gluon vertex we get, respectively,

$$
\begin{aligned}
iM^{\mu\nu}_{1,2} k_{1\mu} k_{2\nu} &= -g^2 f^{abc} \bar{v}(p') \not{k}_1 t^c u(p), \\
iM^{\mu\nu}_{3} k_{1\mu} k_{2\nu} &= g^2 f^{abc} \bar{v}(p') \not{k}_1 t^c u(p),
\end{aligned}
\tag{3.92}
$$

such that, fig. 3.6,

$$
i(M^{\mu\nu}_{1,2} + M^{\mu\nu}_3) k_{1\mu} k_{2\nu} = 0, \tag{3.93}
$$

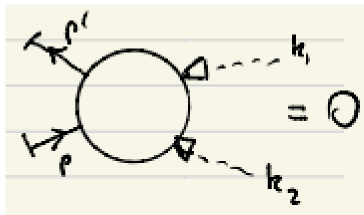


Figure 3.6: Ward identity if both gluons have a scalar polarisation.

i.e. the amplitude $q\bar{q} \to gg$ with production of two gluons with scalar polarisations vanishes. Properties 1. and 2. are generalisable to any number of gluons: if n gluons are produced, out of which m gluons with scalar polarisations, with $1 \le m \le n$, and the other $(n-m)$ gluons are real, then the amplitude vanishes.

3. If the first gluon is on-shell, $k_1^2 = 0$ but its polarisation is not transverse, then

$$
iM^{\mu\nu}_3 \epsilon^*_\mu(k_1) k_{2\nu} = g^2 f^{abc} \epsilon^*_\mu(k_1) \frac{1}{k_3^2} \bar{v}(p') [\not{k}_1 k_1^\mu + k_3^2 \gamma^\mu] t^c u(p), \tag{3.94}
$$

which, summed to the abelian diagrams (3.86), yields

$$
i(M^{\mu\nu}_{1,2} + M^{\mu\nu}_3) \epsilon^*_\mu(k_1) k_{2\nu} = g^2 f^{abc} \frac{k_1 \cdot \epsilon^*(k_1)}{k_3^2} \bar{v}(p') \not{k}_1 t^c u(p). \tag{3.95}
$$

In order to understand the physical relevance of this result, recall the light-cone polarisations, (3.73). As polarisations for the gluons of $q\bar{q} \to gg$, we choose $\epsilon^\mu_+(k_2)$, which is proportional to k_2^μ, and $\epsilon^\mu_-(k_1)$. Then

$$
i(M^{\mu\nu}_{1,2} + M^{\mu\nu}_3) \epsilon^*_{-\mu}(k_1) \epsilon^*_{+\nu}(k_2) = g^2 f^{abc} \frac{k_1 \cdot \epsilon^*_-(k_1)}{\sqrt{k_2 \cdot \bar{k}_2} k_3^2} \bar{v}(p') \not{k}_1 t^c u(p). \tag{3.96}
$$

But

$$
k_1 \cdot \epsilon^*_-(k_1) = \frac{k_1 \cdot \bar{k}_1}{\sqrt{k_1 \cdot \bar{k}_1}} = \sqrt{k_1 \cdot \bar{k}_1} = \sqrt{2} |\vec{k}_1|. \tag{3.97}
$$

Replacing it in the equation above, we obtain

$$
i(M^{\mu\nu}_{1,2} + M^{\mu\nu}_3) \epsilon^*_{-\mu}(k_1) \epsilon^*_{+\nu}(k_2) = g^2 f^{abc} \frac{|\vec{k}_1|}{|\vec{k}_2| k_3^2} \bar{v}(p') \not{k}_1 t^c u(p). \tag{3.98}
$$

Thus, the amplitude with the light-cone polarisations above does not vanish. Since eq. (3.79) holds and we have just shown that

$$
\frac{k_1^\mu \bar{k}_1^\rho + \bar{k}_1^\mu k_1^\rho}{k_1 \cdot \bar{k}_1} \frac{k_2^\nu \bar{k}_2^\sigma + \bar{k}_2^\nu k_2^\sigma}{k_2 \cdot \bar{k}_2} M_{\mu\nu} M^*_{\rho\sigma} \ne 0, \tag{3.99}
$$

this implies that we cannot substitute

$$\left(\sum_{\lambda=1,2} \epsilon^{*\mu}_{\perp\lambda}(k_1)\epsilon^{\rho}_{\perp\lambda}(k_1)\right)\left(\sum_{\lambda=1,2} \epsilon^{*\nu}_{\perp\lambda}(k_2)\epsilon^{\sigma}_{\perp\lambda}(k_2)\right) M_{\mu\nu}M^*_{\rho\sigma}$$

with $g^{\mu\rho}g^{\sigma\nu}M_{\mu\nu}M^*_{\rho\sigma}$, as we did in QED. So the optical theorem seems endangered (fig. 3.7)

$$2\,\mathrm{Im}(\mathcal{M}(a\to b)) \neq \sum_f \int \mathrm{d}\Pi_f \mathcal{M}(a\to f)\mathcal{M}^*(b\to f), \tag{3.100}$$

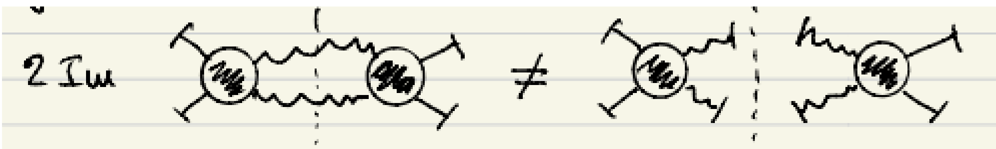
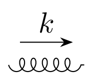


Figure 3.7: Naive picture of the optical theorem if the first gluon is on-shell but not transversely polarised.

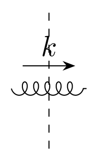
i.e., we can use $g^{\mu\nu}$ in the propagators on the left-hand side only if we keep all four polarisations on the right-hand side (even the non-physical ones).

3.6 Feynman Ghosts

Let us examine the optical theorem in more detail. The Cutkosky rule prescribes that we replace the gluon propagator with the cut propagator,



$$-i\frac{g_{\mu\nu}}{k^2} \Rightarrow$$

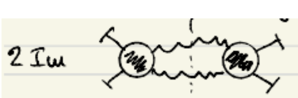


$$-ig_{\mu\nu}(-2\pi i)\delta(k^2). \tag{3.101}$$

So from the amplitude $iM^{\mu\nu}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$ of $q\bar{q}\to gg$ and the squared amplitude integrated over the final-state phase space,

$$\frac{1}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{-ig_{\mu\rho}}{k_1^2}\frac{-ig_{\nu\sigma}}{k_2^2}(iM^{\mu\nu})(iM'^{\rho\sigma}), \tag{3.102}$$

where $M'^{\rho\sigma}$ is the amplitude for $gg\to q\bar{q}$ and $1/2$ is a symmetry factor, we get the imaginary part using the Cutkosky rule,



$$= \frac{1}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}2\pi\delta(k_1^2)\frac{\mathrm{d}^4k_2}{(2\pi)^4}2\pi\delta(k_2^2)(iM^{\mu\nu})g_{\mu\rho}g_{\nu\sigma}(iM'^{\rho\sigma}) \tag{3.103}$$

$$= \frac{1}{2}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2E_2}(iM^{\mu\nu})g_{\mu\rho}g_{\nu\sigma}(iM'^{\rho\sigma}). \tag{3.104}$$

Recall that $g^{\mu\nu} = \epsilon_+^\mu \epsilon_-^{\nu *} + \epsilon_-^\mu \epsilon_+^{\nu *} - \sum_{\lambda=1,2} \epsilon_{\perp\lambda}^\mu \epsilon_{\perp\lambda}^\nu$. The pieces with transverse polarisations are what we expected to find. In addition, we know that

$$\begin{cases} iM_{\mu\nu}\epsilon_\perp^\mu(k_1)\epsilon_\pm^\nu(k_2) = 0 \\ iM_{\mu\nu}\epsilon_\pm^\mu(k_1)\epsilon_\perp^\nu(k_2) = 0 \\ iM_{\mu\nu}\epsilon_\pm^\mu(k_1)\epsilon_\pm^\nu(k_2) = 0, \end{cases} \tag{3.105}$$

so the only piece left over to analyse is

$$\mathcal{I}_{+-} = \frac{1}{2}\int \mathrm{d}\tilde{k}_1 \mathrm{d}\tilde{k}_2 [(iM_{\mu\nu}\epsilon_-^{\mu *}(k_1)\epsilon_+^{\nu *}(k_2))(iM'_{\rho\sigma}\epsilon_+^{\rho *}(k_1)\epsilon_-^{\sigma *}(k_2)) + (k_1 \longleftrightarrow k_2)], \tag{3.106}$$

where we used the short-hand notation $\mathrm{d}\tilde{k} = \dfrac{\mathrm{d}^3 k}{(2\pi)^3 2E}$. Using the explicit result we got for $iM_{\mu\nu}\epsilon_-^{\mu *}(k_1)\epsilon_+^{\nu *}(k_2)$, we can write

$$\mathcal{I}_{+-} = \frac{1}{2}\int \mathrm{d}\tilde{k}_1 \mathrm{d}\tilde{k}_2 \left[\left(g^2 f^{abc}\frac{|\vec{k}_1|}{|\vec{k}_2| k_3^2}\bar{v}(p')\not{k}_1 t^c u(p)\right)\left(-g^2 f^{abd}\frac{|\vec{k}_2|}{|\vec{k}_1| k_3^2}\bar{u}(q)\not{k}_2 t^d v(q')\right) + (k_1 \leftrightarrow k_2)\right]. \tag{3.107}$$

Using the Dirac equation

$$\bar{v}(p')(\not{p} + \not{p}')u(p) = \bar{v}(p')(\not{k}_1 + \not{k}_2)u(p) = 0,$$

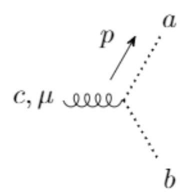
i.e.,

$$\bar{v}(p')\not{k}_1 u(p) = -\bar{v}(p')\not{k}_2 u(p),$$

we notice that the second contribution in eq. (3.107) equals the first. So we get

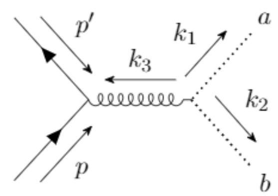
$$\mathcal{I}_{+-} = \int \mathrm{d}\tilde{k}_1 \mathrm{d}\tilde{k}_2 \left(g^2 f^{abc}\frac{\cancel{|\vec{k}_1|}}{\cancel{|\vec{k}_2|} k_3^2}\bar{v}(p')\not{k}_1 t^c u(p)\right)\left(-g^2 f^{abd}\frac{\cancel{|\vec{k}_2|}}{\cancel{|\vec{k}_1|} k_3^2}\bar{u}(q)\not{k}_2 t^d v(q')\right) \tag{3.108}$$

If we postulate the existence of a ghost particle, the **Feynman ghost**, which couples only to gluons and whose Feynman rule is



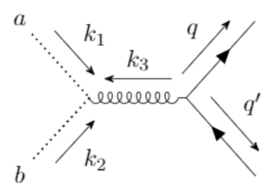
$$\sim -g f^{acb} p^\mu \tag{3.109}$$

where the ghost is a fermion which transforms with the adjoint of SU(N). Then, the amplitude for a $q\bar{q}$ pair annihilating into a pair of ghosts is



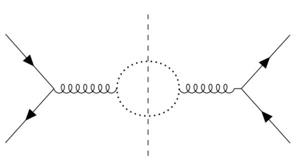
$$
\begin{aligned}
iM_{ghost} &= ig\bar{v}(p')\gamma_\mu t^c u(p)\frac{-i}{k_3^2}(gf^{abc}k_1^\mu) \\
&= g^2 f^{abc}\frac{1}{k_3^2}\bar{v}(p')\not{k}_1 t^c u(p).
\end{aligned} \tag{3.110}
$$

And likewise,



$$
\begin{aligned}
iM'_{ghost} &= ig\bar{u}(q)\gamma_\nu t^d v(q')\frac{-i}{k_3^2}(gf^{abd}(-k_2^\nu)) \\
&= -g^2 f^{abd}\frac{1}{k_3^2}\bar{u}(q)\not{k}_2 t^d v(q').
\end{aligned} \tag{3.111}
$$

In addition, for a ghost loop we get a factor (-1), the ghost being a fermion. Thus,



$$
= -\int \mathrm{d}\tilde{k}_1 \mathrm{d}\tilde{k}_2 \left(g^2 f^{abc}\frac{1}{k_3^2}\bar{v}(p')\not{k}_1 t^c u(p)\right)\left(-g^2 f^{abd}\frac{1}{k_3^2}\bar{u}(q)\not{k}_2 t^d v(q')\right).
$$

This means that such a contribution exactly cancels $\mathcal{I}_{+-}$, eq. (3.108). The optical theorem, fig. 3.8.

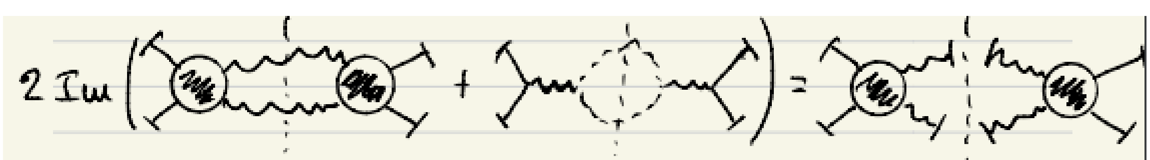


Figure 3.8: The optical theorem, where to the sum of all four polarisations on the left-hand side is added the creation of a ghost pair, in order to cancel the nonphysical degrees of freedom.

The bottom line of this section thus is that either we use the Cutkosky rule $\frac{-ig^{\mu\nu}}{k^2} \longrightarrow -ig^{\mu\nu}(-2\pi i)\delta(k^2)$ and include ghosts to cancel the nonphysical degrees of freedom, or we square the amplitude using only transverse polarisations.

3.7 Faddeev-Popov ghosts

In the previous section, we had to postulate the existence of a fictitious particle, that we termed the Feynman ghost (Feynman called it the "dopey" particle) in order to restore the optical theorem. A more formally correct procedure is to quantise the gluon field using the functional integral,

$$\int \mathcal{D}A\, e^{iS[A]} \quad \text{with} \quad S[A] = \int \mathrm{d}^4x \left[-\frac{1}{4}(F^a_{\mu\nu})^2\right]. \tag{3.112}$$

Like in QED, one finds an infinite set of equivalent gauge configurations that must be integrated out. This is done through the Faddeev-Popov procedure[1], which introduces some Grassmann fields $c_a(x)$, which belong to the adjoint of SU(N), and which couple to the gluon field through the term,

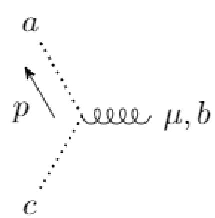
$$(D_\mu c)_a = \partial_\mu c_a + g f^{abc} A^b_\mu c^c. \tag{3.113}$$

The fields $c_a(x)$ are called the **Faddeev-Popov ghosts** and they induce an additional term in the quantised Lagrangian,

$$\mathcal{L}_{ghost} = -\bar{c}^a(\partial^2\delta^{ac} + g f^{abc}\partial^\mu A^b_\mu)c^c, \tag{3.114}$$

which leads to the Feynman rules for ghosts,

$$a \overset{\xrightarrow{p}}{\cdots\cdots} b \;=\; \frac{i\delta^{ab}}{p^2} \tag{3.115}$$



$$= -g f^{abc} p^\mu. \tag{3.116}$$

So we realise that the Faddeev-Popov ghosts are the "dopey particles" we introduced previously to restore unitarity in the optical theorem.

Finally, the quantised Yang-Mills Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\zeta}(\partial_\mu A^\mu_a)^2 - \bar{c}^a\partial^\mu(D_\mu c)^a, \tag{3.117}$$

where the third term on the right-hand side is the gauge-fixing term, also to be described in the QFT II course.

[1] This procedure will be studied in detail in the QFT II course.

Chapter 4

Renormalisation of Yang-Mills theory: UV Divergences

4.1 Renormalisation of the Yang-Mills Lagrangian

The quantised Yang-Mills Lagrangian, we displayed at the end of the previous chapter, contains only operators of dimension 4, like the QED Lagrangian. Thus, it can be renormalised, i.e. the ultraviolet (UV) divergences, that appear in the loop diagram for high values of the loop momentum, can be removed by a finite number of counterterms[1]. In addition, gauge invariance constrains the UV divergences to be no more than logarithmic. Using gauge invariance, the Yang-Mills Lagrangian has been proven to be renormalisable at all loops by 't-Hooft and Veltman.

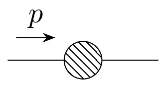
Like in QED, we write the Yang-Mills Lagrangian in terms of bare fields and spell out all the terms,

$$\begin{aligned}\mathcal{L}_{YM} =&\bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 + g_0 A^a_{\mu 0}\bar{\psi}_0 t^a\gamma^\mu\psi_0 - \frac{1}{4}(\partial_\mu A^a_{\nu 0} - \partial_\nu A^a_{\mu 0})^2 - g_0 f^{abc}(\partial_\mu A^a_{\nu 0})^2 A_0^{b\mu}A_0^{c\nu}\\ &- \frac{g_0^2}{4} f^{abc}f^{aed}A^b_{\mu 0}A^c_{\nu 0}A_0^{e\mu}A_0^{d\nu} - \frac{1}{2\zeta_0}(\partial_\mu A_0^{\mu a})^2 - \bar{c}^a_0\partial_\mu\partial^\mu c^a_0 - g_0 f^{abc}\bar{c}^a_0\partial_\mu A_0^{\mu b}c^c_0. \qquad (4.1)\end{aligned}$$

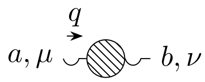
As in QED, we shall find that the gluon self-energy is transverse, thus the gauge fixing term is not renormalised $\frac{1}{2\zeta_0}(\partial_\mu A_0^{\mu a})^2 = \frac{1}{2\zeta}(\partial_\mu A^{\mu a})^2$, and therefore it can be ignored in the discussion that follows.

[1] Renormalisation will be studied in detail during the QFT II course.

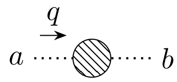
We rescale the fields in the **2-point functions**, given in the Källen-Lehmann representation,



$$= i\frac{Z_2}{\not{p} - m_0 + i\epsilon} + fin. \quad \Rightarrow \quad \psi_0 \equiv Z_2^{1/2}\psi \tag{4.2}$$



$$= i\frac{Z_3}{q^2 + i\epsilon} g^{\mu\nu}\delta_{ab} + fin. \quad \Rightarrow \quad A^\mu_{0a} \equiv Z_3^{1/2} A^\mu_a \tag{4.3}$$



$$= i\frac{Z_2^c}{q^2 + i\epsilon}\delta_{ab} + fin. \quad \Rightarrow \quad c^a_0 \equiv (Z_2^c)^{1/2} c^a. \tag{4.4}$$

Relations like $\psi_0 = Z_2^{1/2}\psi$ express the bare field ψ_0 in terms of the renormalised field ψ through the **counterterm** $Z_2 = 1 + \mathcal{O}(\alpha_s)$, which is defined as an expansion in the strong coupling constant α_s, but whose coefficients contain infinities (UV divergences). These divergences manifest themselves as poles in ϵ when we consider dimensional regularisation (DR) which has $d = 4 - 2\epsilon$ space-time dimensions. In terms of renormalised fields and counterterms, the Yang-Mills Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{YM} =& Z_2\bar{\psi}(i\not{\partial} - m_0)\psi + g_0 Z_2 Z_3^{1/2} A^a_\mu \bar{\psi} t^a \gamma^\mu \psi - \frac{1}{4} Z_3(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g_0 Z_3^{3/2} f^{abc}(\partial_\mu A^a_\nu) A^{\mu b} A^{\nu c} \\ &- \frac{g_0^2}{4} Z_3^2 f^{abc} f^{aed} A^b_\mu A^c_\nu A^{\mu e} A^{\nu d} - Z_2^c \bar{c}^a \partial_\mu \partial^\mu c^a - g_0 Z_3^{1/2} Z_2^c f^{abc} \bar{c}^a \partial_\mu A^{\mu b} c^c_0.\end{aligned} \tag{4.5}$$

We may write the counterterms as

$$Z_2 = 1 + \delta_2, \quad Z_3 = 1 + \delta_3, \quad Z_2^c = 1 + \delta_2^c, \tag{4.6}$$

where the δ's are the $\mathcal{O}(\alpha_s)$ terms which contain the UV ϵ poles. Then, as in QED, we may fix the bare mass in the fermion propagator and the bare coupling in the quark-gluon vertex in terms of renormalised ones,

$$Z_2 m_0 \equiv Z_2 Z_m m = (1 + \delta_2)(1 + \delta_m)m, \tag{4.7}$$

$$g_0 Z_2 Z_3^{1/2} \equiv g Z_1 = g(1 + \delta_1). \tag{4.8}$$

Likewise, we may express the coupling of the 3-gluon, 4-gluon and ghost-gluon vertex in terms of the renormalised ones,

$$g_0 Z_3^{3/2} \equiv g Z_1^{3g} = g(1 + \delta_1^{3g}), \tag{4.9}$$

$$g_0^2 Z_3^2 \equiv g^2 Z_1^{4g} = g^2(1 + \delta_1^{4g}), \tag{4.10}$$

$$g_0 Z_2^c Z_3^{1/2} \equiv g Z_1^c = g(1 + \delta_1^c). \tag{4.11}$$

Beside Z_2,Z_3 and Z_2^c, there are five counterterms, Z_1, Z_m, Z_1^{3g}, Z_1^{4g}, Z_1^c, and only two parameters, g_0 and m_0.

Note that eqs. (4.9) and (4.10) imply that $g_0 Z_1^{3g} Z_3^{1/2} = g Z_1^{4g}$, so we can write the relations among counterterms,

$$\frac{g}{g_0} = \frac{Z_2 Z_3^{1/2}}{Z_1} = \frac{Z_3^{3/2}}{Z_1^{3g}} = \frac{Z_2^c Z_3^{1/2}}{Z_1^c} = \frac{Z_1^{3g} Z_3^{1/2}}{Z_1^{4g}}.$$

This can be rewritten as

$$\frac{Z_2}{Z_1} = \frac{Z_3}{Z_1^{3g}} = \frac{Z_2^c}{Z_1^c} = \frac{Z_1^{3g}}{Z_1^{4g}}. \tag{4.12}$$

These are known as the **Slavnov-Taylor identities**, and are valid at all loop orders. They provide three more relations among counterterms. They can also be written as

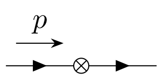
$$\delta_2 - \delta_1 = \delta_2^c - \delta_1^c = \delta_3 - \delta_1^{3g} = \delta_1^{3g} - \delta_1^{4g}, \tag{4.13}$$

which can also be checked from the explicit calculation of the one-loop relevant diagrams.

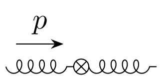
In terms of the expanded counterterms, the YM Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_{YM} =& \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2 - \bar{c}^a\partial^\mu D^{ab}_\mu c^b - \frac{1}{2\zeta}(\partial_\mu A^{\mu a})^2 + \bar{\psi}(i\delta_2\not{\partial} - (\delta_m + \delta_2)m)\psi + g\delta_1\bar{\psi}t^a\not{A}^a\psi \\ & - \frac{\delta_3}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g\delta_1^{3g} f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - g^2\frac{\delta_1^{4g}}{4} f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} \\ & - \delta_2^c\bar{c}^a\partial_\mu\partial^\mu c^a - g\delta_1^c f^{abc}\bar{c}^a\partial^\mu A^b_\mu c^c.\end{aligned} \tag{4.14}$$

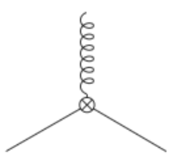
The Feynman rules for the counterterms are



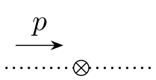
$$\sim i(\delta_2\not{p} - (\delta_2 + \delta_m)m)\delta_{ij} \tag{4.15}$$



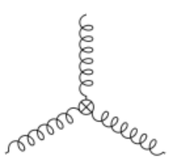
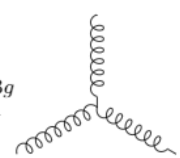
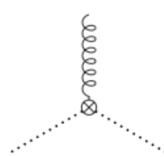
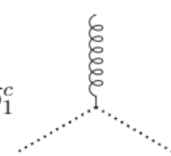
$$\sim -ip^2\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\delta_3\delta^{ab} \tag{4.16}$$



$$\sim igt^a\gamma^\mu\delta_1 \tag{4.17}$$



$$\sim ip^2\delta_2^c\delta^{ab} \tag{4.18}$$


$= \delta_1^{3g}$


$= \delta_1^c$

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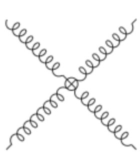
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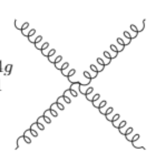

(4.20)

Notice that the first three diagrams are the same as in QED, up to the colour factors.

A quantum field theory is renormalisable if only a finite number of Green's functions diverge, although UV divergences can be found in those Green's functions at all orders of the coupling g. This implies that only a finite number of counterterms are needed to renormalise the theory. In particular, that quantum field theory is super-renormalisable if in those Green's functions UV divergences can be found only up to a certain order of g. Conversely, a quantum field theory is non-renormalisable if the number of Green's functions with UV divergences is not finite.

We want to examine the conditions under which QCD is renormalisable, as a function of the dimensions, $d = 4 - 2\epsilon$ of space-time in dimensional regularisation (DR). In order to keep the action, $S = \int d^d x \mathcal{L}$, dimensionless in natural units $\hbar = c = 1$, the mass dimension of the Lagrangian is $[\mathcal{L}] = \mu^d$. The kinetic terms of the gluon field, $(\partial A)^2$, implies that the mass dimension of A is $[A] = \mu^{\frac{d-2}{2}}$. Likewise, the kinetic terms of the fermion field, $\psi\partial\psi$, implies that $[\psi] = \mu^{\frac{d-1}{2}}$. Accordingly, the interaction term, $g\bar{\psi}A\psi$, implies that the mass dimension of the coupling constant is $[g] = \mu^{\frac{4-d}{2}}$.

From the **UV power counting** of loop diagrams in d-dimensional QCD, the superficial degree of UV divergence is ($\longrightarrow$ Exercises)

$$D = d + \frac{d-4}{2}V + \frac{d-4}{2}V_{4g} - \frac{d-1}{2}N_q - \frac{d-2}{2}N_g - \frac{d-2}{2}N_c\,, \tag{4.21}$$

where

- V = the total number of vertices of a loop diagram;
- V_{4g} = number of 4-gluon vertices;
- N_q = number of external quarks;
- N_g = number of external gluons;
- N_c = number of external ghosts.

It generalises the case of d-dimensional QED, for which the superficial degree of UV divergence is

$$D = d + \frac{d-4}{2}V - \frac{d-1}{2}N_e - \frac{d-2}{2}N_\gamma\,, \tag{4.22}$$

where

- N_e = number of external electrons and positrons;
- N_γ = number of external photons.

Note that the case $d = 4$ is special,

$$D = 4 - \frac{3}{2}N_q - N_g - N_c\,, \qquad \text{for QCD}\,, \tag{4.23}$$

$$D = 4 - \frac{3}{2}N_e - N_\gamma\,, \qquad \text{for QED}\,. \tag{4.24}$$

Increasing the number of N's the Green's functions will become finite, but the ones which have UV divergences will have it at all loops, since the superficial degree of UV divergence D does not depend on the number of vertices V, thus both QED and QCD are renormalisable in $d = 4$ dimensions.

For $d < 4$, the V terms in eqs. (4.21) and (4.22) have a negative yield, thus given a Green's function which at a given loop has UV divergences, it will become finite increasing the number of loops, and thus of vertices, so both QED and QCD are super-renormalisable in $d < 4$ dimensions.

For $d > 4$, the V terms in eqs. (4.21) and (4.22) have a positive yield, thus given a Green's function which at a given loop is finite, it will display UV divergences increasing the number of loops, and thus of vertices, so both QED and QCD are non-renormalisable in $d > 4$ dimensions.

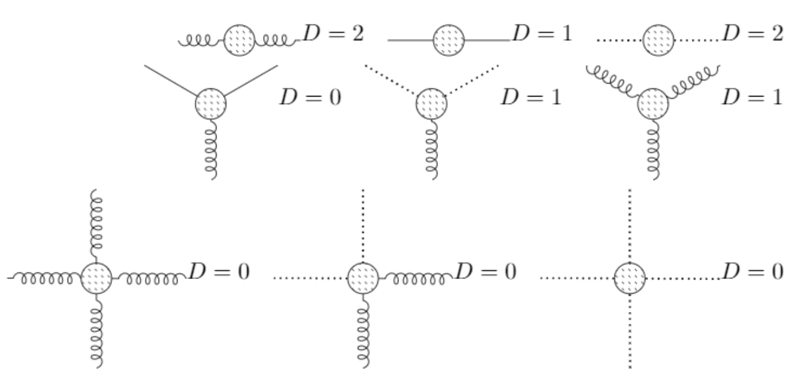
Thus, for

$$\left\{\begin{array}{l} d < 4 \\ d = 4 \\ d > 4 \end{array}\right\}, \quad \text{QED/QCD are} \quad \left\{\begin{array}{l} \text{super-renormalisable} \\ \text{renormalisable} \\ \text{non–renormalisable} \end{array}\right\}. \tag{4.25}$$

Recall that the mass dimension of the coupling constant is $[g] = \mu^{\frac{4-d}{2}}$, i.e. it equals the coefficient of the V terms with opposite sign, thus we can also say that for a coupling constant

$$g \text{ with} \left\{\begin{array}{l} \text{negative mass dimension} \\ \text{dimensionless} \\ \text{positive mass dimension} \end{array}\right\}, \quad \text{QED/QCD are} \quad \left\{\begin{array}{l} \text{super-renormalisable} \\ \text{renormalisable} \\ \text{non–renormalisable} \end{array}\right\}. \tag{4.26}$$

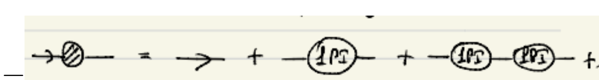
In QCD, the 1-particle irreducible (1PI) functions with $D \geq 0$ are



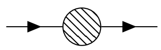
The degree is superficial, because the symmetries of the theory may lower it (so the values above represent an upper bound, thus the worst case).

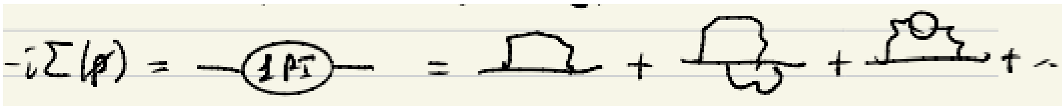
4.1.1 Quark Self-energy and Chiral Symmetry

For example, the full quark propagator is

$S(p) =$ 

$$\begin{aligned}
&= \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}(-i\Sigma(\not{p}))\frac{i}{\not{p} - m} + \dots \\
&= \frac{i}{\not{p} - m}\frac{1}{1 - \frac{\Sigma(\not{p})}{\not{p} - m}} \\
&= \frac{i}{\not{p} - m - \Sigma(\not{p})}.
\end{aligned} \tag{4.27}$$

So the full propagator has a simple pole at $\not{p} = m + \Sigma(\not{p})$, in agreement with the Källen-Lehmann representation  $= \frac{iZ_2}{\not{p}-m}$. The 1PI quark self-energy (D=1) is



.

We may expand it about $p = 0$,

$$-i\Sigma(\not{p}) = A_0 + A_1\not{p} + A_2 p^2 + ... \tag{4.28}$$

with

$$A_n = \frac{1}{n!}\frac{\mathrm{d}}{\mathrm{d}\not{p}^n}\left(-i\Sigma(\not{p})\right)\Big|_{p=0}. \tag{4.29}$$

But

$$\frac{\mathrm{d}}{\mathrm{d}\not{p}}\frac{1}{\not{p}+\not{k}-m} = -\frac{1}{(\not{p}+\not{k}-m)^2}, \tag{4.30}$$

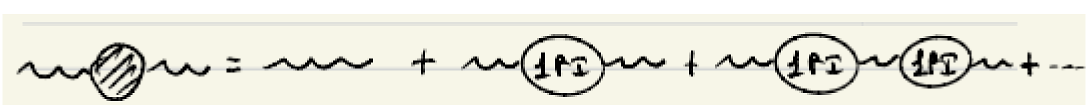
so each $\dfrac{\mathrm{d}}{\mathrm{d}\not{p}}$ yields one factor of k in the denominator, thus it lowers D by 1, so A_0 has $D = 1$, A_1 has $D = 0$, A_n with $n \geq 2$ has $D < 0$. Then $-i\Sigma(\not{p}) \propto c_0\Lambda + c_1\not{p}\log\Lambda +$ finite, with Λ a UV cutoff. However, this implies a mass shift $\delta m \propto \Lambda$ which is forbidden by the **chiral symmetry**: if the quark is massless $m = 0$, it stays massless after renormalisation, i.e., at most $\delta m \propto m\log\Lambda$, thus $A_0 = 0$ and the 1PI self-energy must be

$$-i\Sigma(\not{p}) \propto (c_0 m + c_1\not{p})\log\Lambda + finite. \tag{4.31}$$

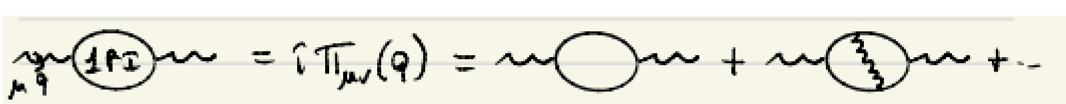
In other words, $D = 0$: chiral symmetry protects the quark self-energy from linear divergences.

4.1.2 The Gluon Self-energy and Gauge Symmetry

The full gluon propagator is



The 1PI self-energy ($D = 2$) is



the tensor structure of $\Pi_{\mu\nu}(q)$ is such that it may contain only $g_{\mu\nu}$ and $q_\mu q_\nu$,

$$\Pi_{\mu\nu}(q) = a g_{\mu\nu} + b q_\mu q_\nu. \tag{4.32}$$

But the Ward identity implies that

$$q_\mu \Pi^{\mu\nu}(q) = 0 \quad \Rightarrow \quad a q_\nu + b q^2 q_\nu = 0 \quad \Rightarrow \quad a = -b q^2, \tag{4.33}$$

so

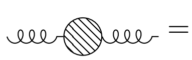
$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2). \tag{4.34}$$

Note that since $\Pi_{\mu\nu}(q)$ has only 1PI diagrams, it cannot have a pole at $q^2 = 0$. Using the projectors (3.63) introduced for the gluon propagator, we can write the 1PI self-energy as

$$i\Pi_{\mu\nu}(q) = i q^2 D_T^{\mu\nu}(q)\Pi(q^2), \tag{4.35}$$

with $\Pi(q^2)$ regular at $q^2 = 0$. We see that the 1PI self-energy, which naively could have behaved as Λ^2, is at most logarithmic in $\Lambda \Rightarrow D = 0$. In other words, the Ward identity, i.e., **gauge symmetry** protects the gluon self-energy from quadratic divergences.

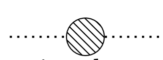
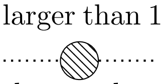
Finally, the full gluon propagator is ($\longrightarrow$ Exercises)



$$= \left[\frac{-i\left(g_{\mu\nu} - \dfrac{q_\mu q_\nu}{q^2}\right)}{q^2[1 - \Pi(q^2)]} - i\zeta \frac{q_\mu q_\nu}{(q^2)^2} \right] \delta_{ab}. \tag{4.36}$$

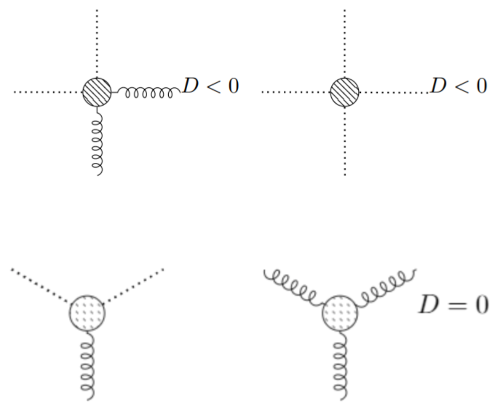
Note that

- only the transverse part is renormalised, however, since $\Pi(q^2)$ is regular at $q^2 = 0$, the full propagator still has a pole only at $q^2 = 0$, with residue $Z_3 = \dfrac{1}{1 - \Pi(0)}$, and the gluon stays massless;
- the longitudinal part is proportional to the gauge-fixing term ζ and is not renormalised. So, it must be $\zeta_0 = Z_3\zeta$. This implies that $\frac{1}{2\zeta_0}(\partial_\mu A_0^{\mu a})^2 = \frac{1}{2\zeta}(\partial_\mu A^{\mu a})^2$, so no counterterm needs be associated with the gauge-fixing part of the Lagrangian.

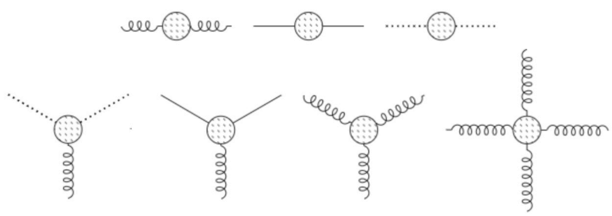
Finally the ghost self-energy ($D = 2$) is  But one of the two ghost-gluon vertices, associated with the external ghost lines, carries the external ghost momentum. So D can never be larger than 1. In fact, it turns out that the ghost self-energy has at most a logarithmic divergence, so  $D = 0$ ($\longrightarrow$ Exercises). Like for the fermion self-energy, the chiral symmetry prevents the massless ghost from acquiring a mass, and protects the ghost self-energy from having power-like divergences.

To recap, we have seen how the symmetries of the Lagrangian protect the quark, gluon and ghost self-energies from having power-like divergences. In SU(2)$\otimes$U(1), there is nothing to protect the Higgs-boson self-energy (which has $D = 2$) from having quadratic divergences. This is known as the fine-tuning problem (because one has to tune finely the terms in the coefficient of the quadratic divergences to make them vanish). Supersymmetry, and some New Physics models, provide the additional symmetry which protects the Higgs-boson self-energy from having quadratic divergences. Unfortunately, no evidence has been found in Nature for Supersymmetry or those New Physics models.

Let us examine the other 1PI functions which are potentially divergent. The 1PI functions with external ghost lines are suppressed by one power for each ghost-gluon vertex which has the momentum of an external ghost. Furthermore, also the 3-gluon vertex has an external gluon momentum. So



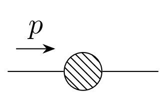
To recap, the 1PI functions with non-negative D are



They all have $D = 0$, so in the Yang-Mills Lagrangian, there are no more than logarithmic divergences.

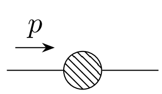
4.2 On-shell prescription

As we said above, the quark self-energy in the Källen-Lehmann representation is

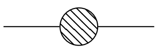


$$= i\frac{Z_2}{\not{p} - m_0 + i\epsilon} + finite \tag{4.37}$$

We absorb the residue Z_2 by writing the bare field ψ_0 in terms of the renormalised field ψ, $\psi_0 = Z_2^{1/2}\psi$. We can relate ψ_0 and the bare mass m_0 to the renormalised ones through the counterterms $Z_2 = 1 + \delta_2$ and $m_0 = Z_m m = (1 + \delta_m)m$. However, the operative definition of δ_2 and δ_m depends on the renormalisation prescription. One may choose to renormalise the quark self-energy at the on-shell point, $\not{p} = m$, e.g. choosing

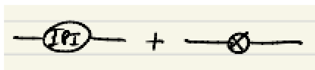


$$= \left.\frac{i}{\not{p} - m + i\epsilon}\right|_{\not{p}=m}, \tag{4.38}$$

in terms of the renormalised fields. This is the **on-shell prescription**. Since the full propagator  $= \dfrac{i}{\not p - m - \Sigma(\not p)}$ has a simple pole at $\not p = m + \Sigma(\not p)$, eq. (4.38) implies that

$$\Sigma(\not p)\Big|_{\not p=m} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}\Sigma}{\mathrm{d}\not p}\bigg|_{\not p=m} = 0. \tag{4.39}$$

Since the renormalised 1PI quark self-energy is

$-i\Sigma(\not p) =$ 

$$\begin{aligned}&= -i\Sigma_0(\not p) + i(\delta_2\not p - (\delta_2+\delta_m)m) \qquad (4.40)\\ &= -i\left(\Sigma_0(\not p)\Big|_{\not p=m} + (\not p - m)\,\frac{\mathrm{d}\Sigma_0}{\mathrm{d}\not p}\bigg|_{\not p=m} + \mathcal{O}\left((\not p-m)^2\right)\right) + i(\delta_2\not p - (\delta_2+\delta_m)m),\end{aligned}$$

the bare 1 PI quark self-energy must fulfil

$$\Sigma_0(\not p)\Big|_{\not p=m} = -m\delta_m \qquad \text{and} \qquad \frac{\mathrm{d}\Sigma_0}{\mathrm{d}\not p}\bigg|_{\not p=m} = \delta_2. \tag{4.41}$$

The bare 1 PI quark self-energy may be obtained from the one in QED by including the suitable colour factor. In DR, and in Feynman gauge $\zeta = 1$, one finds

$$-\,i\Sigma_0(\not p) = -ig^2 C_F\,\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\frac{md-(d-2)x\not p}{[-p^2x(1-x)+m^2(1-x)+\mu^2x]^{2-\frac{d}{2}}}, \tag{4.42}$$

where we keep a fictitious gluon mass μ^2 as an infrared regulator. Then expanding about $\epsilon = 2 - \dfrac{d}{2}$, we find ($\longrightarrow$ Exercises)

$$\delta_2 = -\frac{\alpha_s}{4\pi}C_F\delta_{ij}\left[\frac{1}{\epsilon} - \left(\log\left(\frac{m^2}{4\pi}\right) + \gamma_E\right) + finite + \mathcal{O}(\epsilon)\right] \tag{4.43}$$

$$\delta_m = -3\frac{\alpha_s}{4\pi}C_F\delta_{ij}\left[\frac{1}{\epsilon} - \left(\log\left(\frac{m^2}{4\pi}\right) + \gamma_E\right) + finite + \mathcal{O}(\epsilon)\right]. \tag{4.44}$$

Since the mass counterterm is $(Z_m - 1)m = m\delta_m$, as $m \to 0$, $m\delta_m$ vanishes in agreement with chiral symmetry; but δ_2 and δ_m display a mass logarithmic divergence. Thus, the on-shell prescription, i.e. choosing to renormalise the quark self-energy at the on-shell point, $\not p = m$, is valid as long as m does not vanish.

4.3 Off-shell prescription

For massless theories, it is convenient to use an off-shell prescription due to Georgi and Politzer, and renormalise the 1PI functions at a point outside the physical region. e.g. choosing

$$\Sigma(\not p)\ |_{p^2=-M^2} = 0, \tag{4.45}$$

where M^2 is space-like. The choice of the renormalisation scale M is arbitrary.

Taking eq. (4.42) in the limit $m \to 0$, we obtain the self-energy of the massless quark, in DR and Feynman gauge ($\zeta = 1$),

$$-i\Sigma_0(\not{p}) = ig^2 C_F \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}(d-2)\not{p}(-p^2)^{\frac{d}{2}-2}\int_0^1 \mathrm{d}x x^{\frac{d}{2}-1}(1-x)^{\frac{d}{2}-2}\,, \tag{4.46}$$

whose solution is

$$-i\Sigma_0(\not{p}) = ig^2 C_F \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(d-1)}(d-2)\not{p}(-p^2)^{\frac{d}{2}-2}\,. \tag{4.47}$$

At the renormalisation point, $p^2 = -M^2$, the counterterm may be defined as

$$\delta_2 = \left.\frac{\mathrm{d}\Sigma_0}{\mathrm{d}\not{p}}\right|_{p^2=-M^2}\,, \tag{4.48}$$

which can be easily computed using

$$\frac{\mathrm{d}}{\mathrm{d}\not{p}}\left(\not{p}(-p^2)^{\frac{d}{2}-2}\right) = [1-(d-4)](-p^2)^{\frac{d}{2}-2}.$$

which allows us to write the counterterm as

$$\delta_2 = -\frac{g^2}{(4\pi)^2}C_F\left(\frac{1}{2-\frac{d}{2}} - \log\left(M^2\right) + fin\right)\,. \tag{4.49}$$

Since we are interested in subtracting the UV divergences, the precise definition of the finite terms of δ_2 is immaterial: one may define δ_2 to equal just the pole part of $\frac{\mathrm{d}\Sigma_0}{\mathrm{d}\not{p}}$, as it is done in the Minimal Subtraction (MS) scheme, $\delta_2 = -\frac{\alpha_s}{4\pi}C_F\frac{1}{\epsilon}(M^2)^{-\epsilon}$, however, since the term

$$\begin{aligned}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}-2}} &= (4\pi)^\epsilon\Gamma(\epsilon) = (4\pi)^\epsilon\frac{\Gamma(1+\epsilon)}{\epsilon} \\ &= \frac{1}{\epsilon} + \log(4\pi) - \gamma_E + \mathcal{O}(\epsilon)\,,\end{aligned} \tag{4.50}$$

in eq. (4.46) is ubiquitous in loop computations in DR, it is more convenient to subtract

$$S_\epsilon = (4\pi)^\epsilon\Gamma(1+\epsilon) = 1 + \epsilon\left((4\pi) - \gamma_E\right) + \mathcal{O}(\epsilon^2)\,, \tag{4.51}$$

as well, and define

$$\delta_2 = -\frac{\alpha_s}{4\pi}C_F\frac{S_\epsilon}{\epsilon}(M^2)^{-\epsilon}\,, \tag{4.52}$$

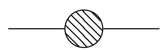
as it is done in the modified Minimal Subtraction ($\overline{\mathrm{MS}}$) scheme[2].

Since we are ultimately interested in computing finite quantities, to be compared with data, the procedure of subtracting the UV divergences may seem arbitrary, which yields a finite remainder dependent on the procedure. However, it is not so: physical observables do not have UV divergences. So, if those physical observables are in a one-to-one correspondence with a loop integral, that integral cannot have UV divergences. If they are the result of a combination of loop integrals, those integrals are renormalised through a UV renormalisation procedure, but the combination of the regulated integrals, i.e. of the finite remainders, does not depend on the procedure, to the desired loop accuracy.

[2]Note that in the literature S_ϵ is often defined also as $S_\epsilon = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$, which is equivalent to eq. (4.51) to $\mathcal{O}(\epsilon)$.

4.4 The Renormalisation Group Equations

We shall now work out the systematics of how Green's functions, or correlators, are renormalised in massless theories, under a change of scale. We shall do it first in a scalar theory like $\lambda\varphi^n$, which displays the basic features, and then straightforwardly extend it to QED and QCD. For a massless scalar field φ, the propagator in the Källen-Lehmann representation is



$$= i\frac{Z}{p^2 + i\epsilon} + fin. \tag{4.53}$$

A Green's function in terms of bare fields is,

$$G_0^{(n)}(x_1, .., x_n) = \langle\Omega|\, T\left(\varphi_0(x_1)...\varphi_0(x_n)\right)|\Omega\rangle\,, \tag{4.54}$$

depends on the bare coupling λ_0 and a UV cut-off Λ. The renormalised Green's function,

$$\begin{aligned} G^{(n)}(x_1, .., x_n) &= \langle\Omega|\, T\left(\varphi(x_1)...\varphi(x_n)\right)|\Omega\rangle \\ &= Z^{-\frac{n}{2}}\, G_0^{(n)}(x_1, .., x_n), \end{aligned} \tag{4.55}$$

depends on the renormalised coupling λ, defined at a scale M^2.

Let us do a change of scale $M \to M + \delta M$. Accordingly, the renormalised coupling λ and field φ change as

$$\lambda \longrightarrow \lambda + \delta\lambda\,, \qquad \varphi \longrightarrow (1 + \delta\eta)\varphi\,. \tag{4.56}$$

Under the shift $M \to M + \delta M$, the renormalised Green's function changes as

$$G^{(n)} \longrightarrow (1 + n\delta\eta)G^{(n)}\,, \tag{4.57}$$

but $G^{(n)} = G^{(n)}(\lambda, M)$, so the total differential can be written as

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial\lambda}\delta\lambda = n\delta\eta G^{(n)}. \tag{4.58}$$

Let us introduce as dimensionless parameters, the shift in the coupling β and the shift in the field strength γ,

$$\beta \equiv M\frac{\delta\lambda}{\delta M}\,, \qquad \gamma \equiv -M\frac{\delta\eta}{\delta M}. \tag{4.59}$$

Thus we get the **Callan-Symanzik equation**,

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right] G^{(n)}(x_1, ..., x_n; \lambda, M) = 0. \tag{4.60}$$

The renormalised Green's functions do not depend on the cut-off $\Lambda \Rightarrow \beta, \gamma$ do not depend on Λ. β, γ are dimensionless $\Rightarrow \beta, \gamma$ do not depend on M. This implies that they only depend on the renormalised coupling, $\beta = \beta(\lambda), \quad \gamma = \gamma(\lambda)$.

Repeating the procedure above in massless QCD (QED) we get the Callan-Symanzik (CS) equation,

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + n_q\gamma_2(g) + n_g\gamma_3(g)\right] G^{(n_q,n_g)}(x_1, ..., x_n; g, M) = 0\,, \tag{4.61}$$

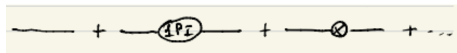
where n_q is the number of quark (electron) fields and n_g is the number of gluon (photon) fields.

The simplest CS equation is the one for the quark (electron) 2-point function,

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}+2\gamma_2(g)\right]G^{(2,0)}(p)=0, \tag{4.62}$$

where to $\mathcal{O}(\alpha_s)$,

$$G^{(2,0)}(p) =$$



$$= \frac{i}{\not p}+\frac{i}{\not p}(-i\Sigma_0(\not p))\frac{i}{\not p}+\frac{i}{\not p}i\delta_2\not p\frac{i}{\not p}. \tag{4.63}$$

We note that to $\mathcal{O}(\alpha_s)$, the β term does not contribute which gives the shift in the coupling, and is always smaller by at least one power of the coupling. The only dependence on M is in δ_2, so the CS equation becomes

$$-\frac{i}{\not p}M\frac{\partial}{\partial M}\delta_2+2\gamma_2(g)\frac{i}{\not p}=0, \tag{4.64}$$

i.e.,

$$\gamma_2(g)=\frac{1}{2}M\frac{\partial}{\partial M}\delta_2\,, \tag{4.65}$$

to the lowest order in α_s, which yields the shift in the quark (electron) field strength due to a change of scale. In eq. (4.49), we computed the counterterm for the massless quark field. In order to stress the role of the UV cut-off Λ, we rewrite it as

$$\delta_2=-\frac{\alpha_s}{4\pi}C_F\left(\log\left(\frac{\Lambda^2}{M^2}\right)+fin\right)\,, \tag{4.66}$$

such that

$$\gamma_2(g)=\frac{\alpha}{4\pi}C_F\,. \tag{4.67}$$

For a scalar field, analogously to eq. (4.65), we get

$$\gamma(\lambda)=\frac{1}{2}M\frac{\partial}{\partial M}\delta_z, \tag{4.68}$$

to the lowest order in λ.

It is possible to extend this beyond the lowest order, by considering the relation between bare and renormalised fields,

$$\varphi(p)=Z(M)^{-\frac{1}{2}}\varphi_0(p). \tag{4.69}$$

Under the change of scale $M\to M+\delta M$, the field strength changes as $\varphi\to(1+\delta\eta)\varphi$. In terms of the bare field, it will be $\varphi\to Z(M+\delta M)^{-\frac{1}{2}}\varphi_0$. So

$$\begin{aligned}\delta\eta\varphi &= [Z(M+\delta M)^{-\frac{1}{2}}-Z(M)^{-\frac{1}{2}}]\varphi_0,\\ &= \left[\left(\frac{Z(M+\delta M)}{Z(M)}\right)^{-\frac{1}{2}}-1\right]\varphi,\end{aligned} \tag{4.70}$$

i.e.,

$$\delta\eta = \left(1 + \frac{1}{Z}\frac{\partial Z}{\partial M}\delta M\right)^{-\frac{1}{2}} - 1,$$
$$= -\frac{1}{2}\frac{1}{Z}\frac{\partial Z}{\partial M}\delta M. \tag{4.71}$$

Therefore the shift in the field strength is

$$\gamma = -M\frac{\delta\eta}{\delta M} = \frac{1}{2}\frac{M}{Z}\frac{\partial Z}{\partial M}, \tag{4.72}$$

which, though, is valid to all orders in the coupling. Likewise, for an electron or a quark field ψ,

$$\gamma_2 = \frac{1}{2}\frac{M}{Z_2}\frac{\partial Z_2}{\partial M}, \tag{4.73}$$

to all orders in the coupling.

Analogously, the shift in the coupling, $\beta \equiv M\dfrac{\delta\lambda}{\delta M}$, keeps the bare Green's function G_0 untouched, which is independent of M. But $G_0 = G_0(\lambda_0, \Lambda)$, so we can also write the shift in the coupling as

$$\beta(\lambda) = M\frac{\partial\lambda}{\partial M}\bigg|_{\lambda_0,\Lambda}. \tag{4.74}$$

Also this relation is exact, i.e., valid to all orders in the coupling. In massless QCD (QED), it is

$$\beta(g) = M\frac{\partial g}{\partial M}\bigg|_{g_0,\Lambda}. \tag{4.75}$$

Now, we want to provide a solution of the CS equation for the 2-point function. For the sake of simplicity, we shall do it for a scalar 2-point function,

$G^{(2)}(p) =$ 

$$= \frac{i}{p^2} + \frac{i}{p^2}(-i\Sigma_0(p))\frac{i}{p^2} + \frac{i}{p^2}(i\delta_Z p^2)\frac{i}{p^2}. \tag{4.76}$$

with

$$\Sigma_0(p) = p^2\left(A\log\left(\frac{\Lambda^2}{-p^2}\right) + fin.\right),$$
$$\delta_Z = A\log\left(\frac{\Lambda^2}{M^2}\right) + fin.,$$

and A is a constant, such that

$$G^{(2)}(p) = \frac{i}{p^2} + \frac{i}{p^2}\left(ip^2 A\log\left(\frac{-p^2}{M^2}\right) + fin.\right)\frac{i}{p^2}. \tag{4.77}$$

In general, we can say that

$$G^{(2)}(p) = \frac{i}{p^2} f\left(\frac{-p^2}{M^2}\right), \tag{4.78}$$

which fulfills the CS equation

$$\left[M\frac{\partial}{\partial M}+\beta(\lambda)\frac{\partial}{\partial\lambda}+2\gamma(\lambda)\right]G^{(2)}(p)=0. \tag{4.79}$$

Setting $P=\sqrt{-p^2}$, we can trade $\partial/\partial M$ with $\partial/\partial P$ (which you may check by fixing $x=-p^2/M^2$), such that

$$M\frac{\partial}{\partial M}G^{(2)}(p)=-P\frac{\partial}{\partial P}G^{(2)}(p)-2G^{(2)}(p).$$

We rewrite the CS equation as

$$\left[P\frac{\partial}{\partial P}-\beta(\lambda)\frac{\partial}{\partial\lambda}+2-2\gamma(\lambda)\right]G^{(2)}(p)=0. \tag{4.80}$$

For a free-field theory, $\beta=\gamma=0$ and solving the CS equation one finds that (please check!) $G^{(2)}(p)=\frac{i}{p^2}$. Fixing $t=\log\left(\frac{P}{M}\right)=\frac{1}{2}\log\left(\frac{-p^2}{M^2}\right)$, the CS equation,

$$\left[\frac{\partial}{\partial t}-\beta(\lambda)\frac{\partial}{\partial\lambda}+2-2\gamma(\lambda)\right]G^{(2)}(p)=0, \tag{4.81}$$

has solution

$$G^{(2)}(P,\lambda)=\mathcal{G}(\bar\lambda(P,\lambda))\exp\left(-2\int_0^t \mathrm{d}t'[1-\gamma(\bar\lambda(P',\lambda))]\right), \tag{4.82}$$

where $\bar\lambda$ is the solution of the RGE,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar\lambda(P,\lambda)=\beta(\bar\lambda), \tag{4.83}$$

with boundary condition $\bar\lambda(M,\lambda)=\lambda$ (Please check the validity of the solution by differentiating it and setting $P=M$). $\bar\lambda$ is called the **running coupling constant**, and it is given in terms of the constant λ at a scale M and of the logarithms $\log(P/M)$. Pulling $\exp\left(-2\int dt\right)$ out of the solution for the 2-point function, we can also write

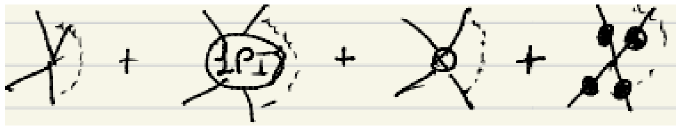
$$G^{(2)}(P,\lambda)=\frac{i}{p^2}\mathcal{G}(\bar\lambda(P,\lambda))\exp\left(2\int_0^t \mathrm{d}t'\gamma(\bar\lambda(P',\lambda))\right). \tag{4.84}$$

$\mathcal{G}(\bar\lambda)$ may be computed by matching the perturbative expansion of $G^{(2)}(p)$ in λ to the one of $G^{(2)}(P,\lambda)$ also in λ. The advantage of $G^{(2)}(P,\lambda)$ above is that it organises the perturbative series in terms of a running coupling and that it resums (large) logarithms $\log\frac{P}{M}$ into the exponential. For a fixed point, $\bar\lambda=\lambda^*$,

$$G^{(2)}(P,\lambda)=\frac{i}{p^2}G(\lambda^*)\left(\frac{p^2}{-M^2}\right)^{\gamma(\lambda^*)}, \tag{4.85}$$

so $\gamma(\lambda^*)$ modifies the scaling dimension of $G^{(2)}$, and it is thus also called **anomalous dimension**.

The running of $\bar\lambda$ in eq. (4.84) is determined by eq. (4.83). For a given theory, in order to understand how the coupling evolves, it is then paramount to compute the β function. Since the coupling occurs in the interaction vertex, we want to analyse how the coupling in an interaction vertex is modified under a change of scale, and in order to do that we need to determine the CS equation for the vertex. Let us consider then a general theory with a dimensionless coupling λ in an n-point vertex, with up to n different fields. For simplicity and without loss of generality, suppose the fields to be scalar. The Green's function of the vertex is

$$G^{(n)}(p_1, ..., p_n) =$$ 

$$= \left[-i\lambda - iB\log\left(\frac{\Lambda^2}{-p^2}\right) - i\delta_\lambda - i\lambda\sum_{i=1}^{n}\left(A_i\log\left(\frac{\Lambda^2}{-p_i^2}\right) - \delta_{Z_i}\right)\right]\prod_{j=1}^{n}\frac{i}{p_j^2}, \quad (4.86)$$

where the last term represents the 2-point functions of the n fields. The CS equation of the vertex is

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \sum_{i=1}^{n}\gamma_i(\lambda)\right]G^{(n)}(p_1, ..., p_n) = 0. \quad (4.87)$$

Since the only dependence on M is in δ_λ and δ_{Z_i}, we can write the CS equation as

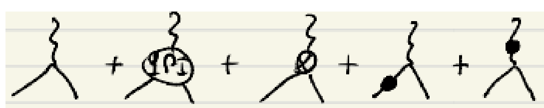
$$M\frac{\partial}{\partial M}\left(\delta_\lambda - \lambda\sum_j \delta_{Z_j}\right) + \beta(\lambda) + \lambda\sum_j \gamma_j(\lambda) = 0, \quad (4.88)$$

to the lowest order in λ. Using $\gamma(\lambda) = \frac{1}{2}M\frac{\partial}{\partial M}\delta_Z$ we obtain

$$\beta(\lambda) = M\frac{\partial}{\partial M}\left(-\delta_\lambda + \frac{\lambda}{2}\sum_i \delta_{Z_i}\right). \quad (4.89)$$

This yields the shift of the coupling due to a change of scale, to lowest order in λ.

In massless QED, the Green's function of the electron-photon vertex is

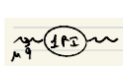
$$G^{(2,1)}(p_1, p_2, p_3) =$$ 

. (4.90)

The CS equation for the β function becomes

$$\beta(e) = M\frac{\partial}{\partial M}\left(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3\right), \quad (4.91)$$

where δ_3 is the countertem of the photon 2-point function, and δ_1 is the counterterm of the electron-photon vertex. δ_1 is gauge dependent, just like δ_2 (which we derived in the Feynman gauge). However, in QED the Ward identity implies that $\delta_2 = \delta_1$ to all orders of α. As we established in the previous lecture, only the transverse part of the photon 2-point function is renormalised, which does not depend explicitly on the gauge-fixing parameter ζ. Can it depend implicitly on ζ through its residue $Z_3 = \frac{1}{1-\Pi_3(0)}$? In other words, is the 1-PI photon self energy,



$=$ 

$$= i\Pi_{\mu\nu}(q) = iq^2 D_T^{\mu\nu}(q)\Pi(q^2), \quad (4.92)$$

gauge invariant? At one loop, it is trivially so, since the fermion bubble consists solely of fermion propagators. At two loops, it is not so obvious, since the photon propagator inside the bubble has a gauge-fixing term. As we shall see, the β function is directly related to physical quantities, so it must be gauge invariant, and thus so is δ_3.

The off-shell renormalisation prescription on the full gluon propagator is

$$\Pi(q^2)\big|_{q^2=-M^2} = 0 \quad \Rightarrow \quad \delta_3 = \Pi_0(q^2)\big|_{q^2=-M^2} \,. \tag{4.93}$$

At one loop,

$$\delta_3 = -\frac{4}{3}\frac{e^2}{(4\pi)^2}\log\left(\frac{\Lambda^2}{M^2}\right) + fin \,, \tag{4.94}$$

or

$$\delta_3 = -\frac{4}{3}\frac{e^2}{(4\pi)^2}\frac{S_\epsilon}{\epsilon} \qquad \text{in} \quad \overline{\text{MS}} \,, \tag{4.95}$$

so the β function becomes

$$\beta(e) = M\frac{\partial}{\partial M}\left(\frac{e}{2}\delta_3\right) = \frac{4}{3}\frac{e^3}{(4\pi)^2}. \tag{4.96}$$

Then the RGE for the running coupling $\bar{e}$ is

$$\frac{\mathrm{d}\bar{e}}{\mathrm{d}t} = \frac{4}{3}\frac{\bar{e}^3}{(4\pi)^2}, \tag{4.97}$$

for

$$t = \log\left(\frac{P}{M}\right) = \frac{1}{2}\log\left(\frac{-p^2}{M^2}\right) \,. \tag{4.98}$$

The solution is (please check!)

$$\frac{1}{\bar{e}^2} = \frac{1}{e^2} - \frac{1}{12\pi^2}\log\left(\frac{-p^2}{M^2}\right), \tag{4.99}$$

which can also be written as

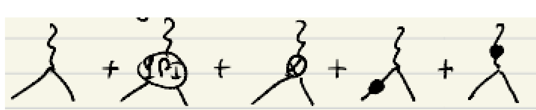
$$\bar{\alpha}(p) = \frac{\alpha}{1 - \dfrac{\alpha}{3\pi}\log\left(\dfrac{-p^2}{M^2}\right)}. \tag{4.100}$$

Thus, the QED coupling $\bar{\alpha}$ grows stronger at larger momenta, i.e., at smaller distances.

4.5 Renormalisation of the Yang-Mills Theory at One Loop

Just like in QED, in massless QCD the Green's function of the quark-gluon vertex is

$$G^{(2,1)}(p_1,p_2,p_3) = \tag{4.101}$$



,

thus the functional form of the CS equation for the β function is the same as in QED,

$$\beta(g) = gM\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right). \tag{4.102}$$

In order to compute the β function, at least at lowest order in α_s, we need to determine the counterterms δ_1, δ_2 and δ_3 at one loop. However, in QCD $\delta_1 \neq \delta_2$ and we do not expect that any counterterm be gauge invariant, only their combination to make the β function be so, as we shall see.

In eq. (4.47), we displayed the bare self-energy for a massless quark. Including in it the dependence on the gauge-fixing parameter ζ, that becomes

$$- i\Sigma_0(\not{p}) = ig^2 C_F \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} \zeta \not{p} (-p^2)^{\frac{d}{2}-2} + fin\,. \tag{4.103}$$

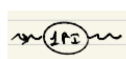
At the renormalisation point $p^2 = -M^2$, the counterterm is

$$\delta_2 = \left.\frac{\mathrm{d}\Sigma_0}{\mathrm{d}\not{p}}\right|_{p^2=-M^2} = -\frac{\alpha_s}{4\pi} C_F \Gamma\left(2-\frac{d}{2}\right) \zeta (M^2)^{\frac{d}{2}-2} + fin. \tag{4.104}$$

The precise definition of the finite terms of δ_2 is immaterial, since they do not depend on M. In $\overline{\mathrm{MS}}$,

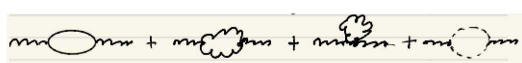
$$\delta_2 = -\frac{\alpha_s}{4\pi} C_F\, \zeta\, \frac{S_\epsilon}{\epsilon}\, (M^2)^{-\epsilon}\,. \tag{4.105}$$

Just like the photon self-energy in QED, the gluon self-energy is also transverse, because of the Ward identity. It can be written as

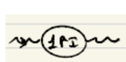


$$= iq^2 D_T^{\mu\nu}(q)\Pi(q^2)\delta_{ab}, \tag{4.106}$$

with $\Pi(q^2)$ regular as $q^2 \to 0$ and $D_T^{\mu\nu} = g^{\mu\nu} - \dfrac{q^\mu q^\nu}{q^2}$. At one loop, there are four diagrams which contribute to the gluon self-energy,



In particular, we are interested in the UV-divergent part, which in DR and including the gauge-fixing parameter ζ is ($\longrightarrow$ Exercises)



$$= i\frac{\alpha_s}{4\pi} q^2 D_T^{\mu\nu}(q)\delta^{ab} \frac{\Gamma\left(2-\frac{d}{2}\right)}{(-q^2)^{2-\frac{d}{2}}} \left[\left(\frac{13}{6} - \frac{\zeta}{2}\right) C_A - \frac{4}{3} T_f n_f\right], \tag{4.107}$$

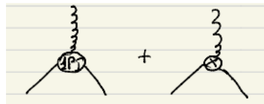
where n_f is the number of quarks circulating in the loop. We note that the matter part is obviously gauge independent, but the gauge part is not. So the counterterm is

$$\delta_3 = \Pi_0(q^2)\big|_{q^2=-M^2} = \frac{\alpha_s}{4\pi} \frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}} \left[\left(\frac{13}{6} - \frac{\zeta}{2}\right) C_A - \frac{4}{3} T_f n_f\right]. \tag{4.108}$$

In $\overline{\mathrm{MS}}$,

$$\delta_3 = \frac{\alpha_s}{4\pi} \frac{S_\epsilon}{\epsilon} (M^2)^{-\epsilon} \left[\left(\frac{13}{6} - \frac{\zeta}{2}\right) C_A - \frac{4}{3} T_f n_f\right]. \tag{4.109}$$

Finally, the renormalised quark-gluon interaction vertex is given by

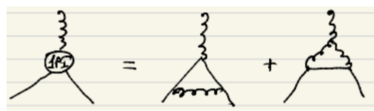


$$igt^a\Gamma^\mu(q^2) = igt^a\gamma^\mu\delta F_1(q^2) + igt^a\gamma^\mu\delta_1. \tag{4.110}$$

The counterterm is fixed by the off-shell prescription,

$$\Gamma^\mu\big|_{q^2=-M^2} = 0 \quad \Rightarrow \quad \delta_1 = -\,\delta F_1(q^2)\big|_{q^2=-M^2}\,. \tag{4.111}$$

The 1PI quark-gluon vertex at one loop is



$$. \tag{4.112}$$

The result of the one-loop computation is ($\longrightarrow$ Exercises)

$$igt^a\gamma^\mu\delta F_1(q^2) = i\frac{g^3}{(4\pi)^2}\left[\zeta C_F + \frac{3+\zeta}{4}C_A\right]t^a\gamma^\mu\frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}}, \tag{4.113}$$

thus,

$$\delta_1 = -\frac{\alpha_s}{4\pi}\left[\zeta C_F + \frac{3+\zeta}{4}C_A\right]\frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}}. \tag{4.114}$$

In $\overline{\mathrm{MS}}$,

$$\delta_1 = -\frac{\alpha_s}{4\pi}\frac{S_\epsilon}{\epsilon}\,(M^2)^{-\epsilon}\left[\zeta C_F + \frac{3+\zeta}{4}C_A\right]. \tag{4.115}$$

As expected, we see that $\delta_1 \neq \delta_2$. However, the "abelian" part of δ_1 equals δ_2.

The β function is

$$\beta(g) = gM\frac{\partial}{\partial M}\frac{g^2}{(4\pi)^2}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}}\left[-\zeta C_F + \zeta C_F + \frac{3+\zeta}{4}C_A + \frac{1}{2}\left[\left(\frac{13}{6}-\frac{\zeta}{2}\right)C_A - \frac{4}{3}T_f n_f\right]\right]. \tag{4.116}$$

Note that the gauge-fixing term cancels, and we get

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_A - \frac{4}{3}T_f n_f\right]. \tag{4.117}$$

Thus, the β function is negative!

Likewise, in $\overline{\mathrm{MS}}$ we get,

$$\beta(g) = gM\frac{\partial}{\partial M}\frac{g^2}{(4\pi)^2}\frac{S_\epsilon}{\epsilon}\,(M^2)^{-\epsilon}\frac{1}{2}\left[\frac{11}{3}C_A - \frac{4}{3}T_f n_f\right], \tag{4.118}$$

which agrees with eq. (4.117).

4.6 The β function of QCD and Asymptotic Freedom

In the previous lecture, we established that the β function of QCD at one loop is

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\beta_0\,, \tag{4.119}$$

with

$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f\,, \tag{4.120}$$

where we used that $T_f = 1/2$ and $C_A = N_c$. Since $N_c = 3$, as long as $n_f < 33/2$, the β function is negative. This has deep implications. The running of the coupling is determined by the RGE (4.83),

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{g} = \beta(\bar{g})\,, \qquad \text{with} \qquad t = \frac{1}{2}\log\left(\frac{-p^2}{M^2}\right)\,. \tag{4.121}$$

Substituting our expression for $\beta(\bar{g})$ above, we get

$$\frac{1}{\bar{g}^2} = \frac{1}{g^2} + \frac{\beta_0}{(4\pi)^2}\log\left(\frac{-p^2}{M^2}\right)\,, \tag{4.122}$$

which can be inverted. With $\alpha_s = \frac{g^2}{4\pi}$, it yields

$$\bar{\alpha}_s = \frac{\alpha_s}{1 + \frac{\beta_0}{4\pi}\alpha_s\log\left(\frac{-p^2}{M^2}\right)}. \tag{4.123}$$

Thus, the coupling becomes weaker at larger momenta (i.e., at smaller distances), implying that for $Q^2 \to \infty$, the interaction strength vanishes. This property is called **asymptotic freedom**. A summary of several measurements of the dependence of α_s on a scale Q^2 is provided in fig. 4.1.

If we define

$$\beta(\bar{\alpha}_s) = \frac{\mathrm{d}\bar{\alpha}_s}{\mathrm{d}\log\left(\frac{-p^2}{M^2}\right)}, \tag{4.124}$$

then $\beta(\bar{\alpha}_s) = \frac{\bar{g}}{4\pi}\beta(\bar{g})$. Using eq. (4.119), we obtain

$$\beta(\bar{\alpha}_s) = -\frac{\beta_0}{4\pi}\bar{\alpha}_s^2. \tag{4.125}$$

Note that in Sec. 4.1 we established that in $d = 4 - 2\epsilon$ dimensions the mass dimension of the bare coupling constant g_0 is M^ϵ, such that we can write $g_0 = \bar{g}M^\epsilon$, in terms of the renormalised dimensionless coupling $\bar{g}$. Then the β function (4.119) contains a term from the explicit differentiation of the mass dimension,

$$\beta(\bar{g}) = M\frac{\partial\bar{g}}{\partial M}\bigg|_{g_0,\Lambda} = -\epsilon\bar{g} - \frac{\beta_0}{(4\pi)^2}\bar{g}^3\,. \tag{4.126}$$

Likewise[3]

$$\beta(\bar{\alpha}_s) = M^2\frac{\partial\bar{\alpha}_s}{\partial M^2}\bigg|_{\alpha_s,\Lambda} = -2\epsilon\bar{\alpha}_s - \frac{\beta_0}{4\pi}\bar{\alpha}_s^2\,. \tag{4.127}$$

[3]Note that in the tutorials, $\beta(\bar{\alpha}_s)$ is defined as $\beta(\bar{\alpha}_s) = M\frac{\partial\bar{\alpha}_s}{\partial M}\big|_{\alpha_s,\Lambda}$, which differs from eq. (4.127) by a factor 2.

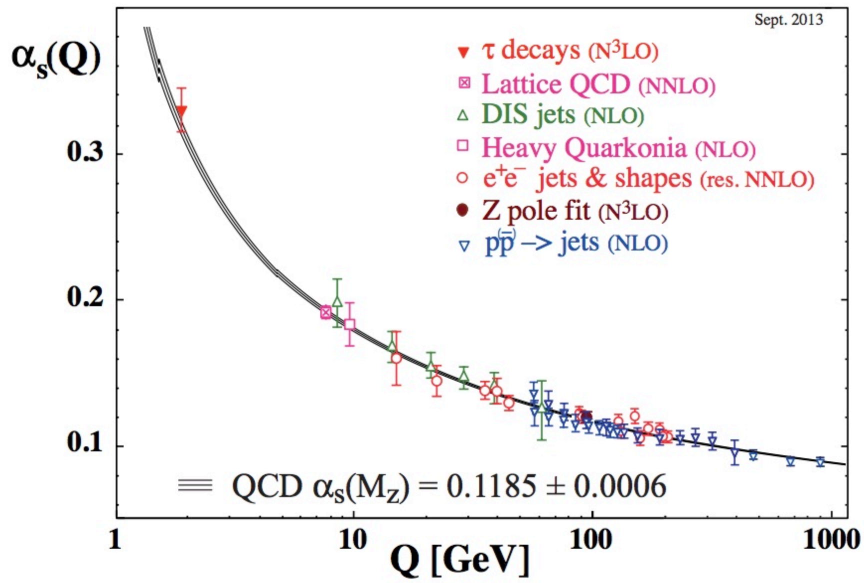


Figure 4.1: Running of α_s, from S. Bethke, G. Dissertori and G. Salam, "Quantum Chromodynamics", in: K.A. Olive et al. (Particle Data Group), Chin. Phys. C38 (2014) 090001.

Recall that in DIS, the interaction between the electron and the quark is electromagnetic. While this occurs, the strong interaction between the partons in the proton is neglected (Parton Model). The typical time of the electron-quark interaction is the inverse of the virtual photon energy. In the target frame it is

$$\tau_{eq} \sim q_0^{-1} = \frac{m}{q \cdot P} = \frac{2xm}{Q^2}, \tag{4.128}$$

while the typical interaction time of the partons in the proton, is the inverse of the proton mass, $\tau_p \sim \frac{1}{m}$. Since $\tau_{eq} \ll \tau_p$, i.e., the electron-quark scattering is very fast with respect to the time scales of the parton interactions within the proton, Bjorken scaling implies that during the time τ_{eq}, the parton interactions within the proton can be neglected, i.e., the partons can be treated as free.

When Bjorken scaling and the Parton Model were introduced, this was puzzling, since there were no known field theories in four dimensions featuring a coupling that vanishes at very large momenta (in QED and in the scalar φ^4-theory, the coupling grows with momentum transfer). Thus the exceptional importance of finding a β function with a negative sign. This points to QCD, i.e. an asymptotically free field theory, as the right candidate to model the strong interactions. Further impetus came from realising that the non-Abelian gauge theories are the only asymptotically free field theories in four dimensions.

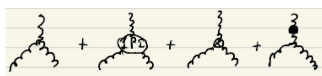
However, the correspondence between asymptotically free field theories and Bjorken scaling and the Parton Model cannot be exact, as the Parton Model completely ignores the interactions within the proton, while in an asymptotically free field theory, the coupling is still there, albeit small, at any large momentum transfer (it vanishes only asymptotically). Thus, one should see small deviations from the Bjorken scaling (known as *scaling violations*) which were in fact observed. We shall come back to this later.

4.6.1 β-function for Different Vertices

Let us go back now to the β function. In QED, there is only one interaction vertex, and the CS equation for that is the obvious candidate to compute the β function. However, in QCD, there are four interaction vertices. So beside the quark-gluon vertex, we could have used the CS equation for the 3-gluon vertex or the 4-gluon vertex, or even the CS equation for the ghost-gluon vertex. Of course, for the self-consistency of the theory, we expect to get the same β function and it is instructive and simple to show that in fact it is so.

The Green's function for the 3-gluon vertex is

$$G^{(3g)}(p_1, p_2, p_3) =$$



$$= g + gB\log\left(\frac{\Lambda^2}{-p^2}\right) + g\delta_i^{3g} + g\sum_{i=1}^{3}\left(A_i \log\left(\frac{\Lambda^2}{-p_i^2}\right) - \delta_3\right), \tag{4.129}$$

with the last term representing the full gluon propagators. The CS equation for the 3-gluon vertex is

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + 3\gamma_3(g)\right] G^{(3g)} = 0. \tag{4.130}$$

Since the only dependence on M is in δ_1^{3g} and δ_3, and using the leading-order relation for the γ function, we have

$$M\frac{\partial}{\partial M}\left(g\delta_1^{3g} - 3g\delta_3\right) + \beta(g) + \frac{3}{2}gM\frac{\partial}{\partial M}\delta_3 = 0, \tag{4.131}$$

which implies that

$$\beta(g) = gM\frac{\partial}{\partial M}\left(\frac{3}{2}\delta_3 - \delta_1^{3g}\right). \tag{4.132}$$

This is in agreement with the result we obtained in the previous lecture, for the β function of a generic vertex in scalar theory. Using the Slavnov-Taylor identity (4.13), $\delta_3 - \delta_1^{3g} = \delta_2 - \delta_1$, we see that the β function above is the same as the one from the quark-gluon vertex.

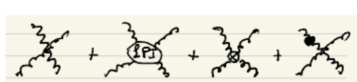
The Green's function and the CS equation for the ghost-gluon vertex are functionally like the ones for the quark-gluon vertex. Thus, for the β function we get

$$\beta(g) = gM\frac{\partial}{\partial M}\left(-\delta_1^c + \delta_2^c + \frac{1}{2}\delta_3\right). \tag{4.133}$$

Using the Slavnov-Taylor identity (4.13), $\delta_1^c - \delta_2^c = \delta_1 - \delta_2$, we obtain the usual β function.

The 4-gluon vertex is quadratic in the coupling, so we cannot readily apply the result for the β function of a generic vertex in a scalar theory, which is linear in the coupling. The Green's function for the 4-gluon vertex is

$$G^{(4g)}(p_1, p_2, p_3, p_4) =$$



$$= g^2 + g^2 B\log\left(\frac{\Lambda^2}{-p^2}\right) + g^2\delta_1^{4g} + g^2\sum_{i=1}^{4}\left(A_i \log\left(\frac{\Lambda^2}{-p_i^2}\right) - \delta_3\right). \tag{4.134}$$

The CS equation for the 4-gluon vertex is

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}+4\gamma_3(g)\right]G^{(4g)}=0. \tag{4.135}$$

Then we use the leading-order relation for the γ function and the fact that the only dependence on M is in δ_1^{4g} and δ_3. We obtain from the CS equation,

$$M\frac{\partial}{\partial M}\left(g^2\delta_1^{4g}-4g^2\delta_3\right)+2g\beta(g)+2g^2M\frac{\partial}{\partial M}\delta_3=0, \tag{4.136}$$

and thus

$$\beta(g)=gM\frac{\partial}{\partial M}\left(\delta_3-\frac{1}{2}\delta_1^{4g}\right). \tag{4.137}$$

Now we use the relation $g_0^2Z_3^2=g^2Z_1^{4g}$, to write another Slavnov-Taylor identity, which is not independent from the ones we already studied, eq. (4.12),

$$\frac{Z_2}{Z_1}=\sqrt{\frac{Z_3}{Z_1^{4g}}}, \tag{4.138}$$

which implies that

$$\frac{1}{2}\left(\delta_3-\delta_1^{4g}\right)=\delta_2-\delta_1. \tag{4.139}$$

So we obtain the usual β function.

4.6.2 Running Beyond One Loop

Beyond one loop, the RGE for the coupling may be written as

$$\frac{\mathrm{d}\bar{\alpha}_s(Q^2)}{\mathrm{d}\log\left(Q^2/M^2\right)}=-\frac{\beta_0}{(4\pi)}\alpha_s^2(Q^2)-\frac{\beta_1}{(4\pi)^2}\alpha_s^3(Q^2)-\frac{\beta_2}{(4\pi)^3}\alpha_s^4(Q^2)-\frac{\beta_3}{(4\pi)^4}\alpha_s^5(Q^2)-\frac{\beta_4}{(4\pi)^5}\alpha_s^6(Q^2)-... \tag{4.140}$$

The coefficients β_1, β_2, β_3, β_4 have been computed. For example, we know that

$$\beta_1=\frac{34}{3}C_A^2-\frac{20}{3}C_AT_Rn_f-4C_FT_Rn_f. \tag{4.141}$$

β_2, β_3 and β_4 can be found, e.g., in F. Herzog et al., JHEP 1702(2017)090, arXiv:1701.01404, where the 5-loop coefficient β_4 has been presented for the first time.

4.6.3 β function: Renormalisation Scheme Dependence

The coefficients β_0 and β_1 are renormalisation scheme independent. In order to see this, let us note that the running may depend on the chosen scheme. Let us suppose that under a change of scheme, the coupling changes as

$$g=Z_g(g')g'=g'\left(1+Z_0{g'}^2+Z_1{g'}^4+...\right). \tag{4.142}$$

For example, from the one-loop solution, we know that

$$Z_0 = \frac{1}{2} \frac{\beta_0}{(4\pi)^2} \log \left(\frac{Q^2}{M^2} \right). \tag{4.143}$$

Under a change of scheme, the β function,

$$\begin{aligned} \beta(g) = M \frac{\partial g}{\partial M} &= -\frac{\beta_0}{(4\pi)^2} g^3 - \frac{\beta_1}{(4\pi)^3} g^5 - \frac{\beta_2}{(4\pi)^4} g^7, \\ &\equiv -\tilde{\beta}_0 g^3 - \tilde{\beta}_1 g^5 - \tilde{\beta}_2 g^7, \end{aligned} \tag{4.144}$$

changes as $\beta(g) = \beta(g') \dfrac{\partial g}{\partial g'}$. Replacing the expansions of $\beta(g)$, $\beta(g')$ and $\dfrac{\partial g}{\partial g'}$ in equation (4.144), we have

$$\tilde{\beta}_0 g^3 + \tilde{\beta}_1 g^5 + \tilde{\beta}_2 g^7 + ... = \left(\tilde{\beta}'_0 g^{'3} + \tilde{\beta}'_1 g^{'5} + \tilde{\beta}'_2 g^{'7} + ... \right) \left(1 + 3Z_0 g^{'2} + 5Z_1 g^{'4} + ... \right). \tag{4.145}$$

Then we use eq. (4.142) to write the expansions,

$$\begin{aligned} g^3 &= g^{'3} \left(1 + 3Z_0 g^{'2} + 3(Z_1 + Z_0^2) g^{'4} + ... \right), \\ g^5 &= g^{'5} \left(1 + 5Z_0 g^{'2} + ... \right), \\ g^7 &= g^{'7}(1 + ...). \end{aligned}$$

Equating the coefficients of like powers of g', we obtain

$$\tilde{\beta}'_0 = \tilde{\beta}_0, \tag{4.146}$$

$$\tilde{\beta}'_1 = \tilde{\beta}_1, \tag{4.147}$$

$$\tilde{\beta}'_2 = \tilde{\beta}_2 + 2Z_0 \tilde{\beta}_1 + (3Z_0^2 - 2Z_1) \tilde{\beta}_0. \tag{4.148}$$

So we have shown that β_0 and β_1 are renormalisation-scheme independent. The scheme dependence shows up on the 3-loop coefficient β_2.

4.7 Renormalisation of Local Operators

So far, we have considered the renormalisation of massless theories. Now, we want to consider also mass operators in the Lagrangian. We shall do it, more generally, by considering the inclusion of local operators (which may be mass operators $m\bar{\psi}\psi$, or contact terms $\bar{\psi}\gamma^\mu(1-\gamma_5)\psi\bar{\psi}\gamma_\mu(1-\gamma_5)\psi$, or else) in a correlator. For the sake of simplicity, let us start again with a scalar theory, and consider an n-point Green's function, with the insertion of a local operator θ_0, which is renormalised as $\theta_0 = Z_\theta(M)\theta_M$. The renormalised Green's function shall be given by

$$\begin{aligned} G^{(n,1)}(p_1, ..., p_n; q) &= < \varphi(p_1)...\varphi(p_n)\theta_M(q) >, \\ &= Z^{-\frac{n}{2}} Z_\theta^{-1} < \varphi_0(p_1)...\varphi_0(p_n)\theta_0(q) > . \end{aligned} \tag{4.149}$$

Then, under a change of scale $M \to M + \delta M$, the renormalised coupling, field and local operator change respectively as,

$$\begin{cases} \lambda & \to \lambda + \delta\lambda \\ \varphi & \to (1+\delta\eta)\varphi \\ \theta & \to (1+\delta\tau)\theta \,. \end{cases}$$

The Green's function changes as $G^{(n,1)} \to (1 + n\delta\eta + \delta\tau)G^{(n,1)}$. But $G^{(n,1)} = G^{(n,1)}(\lambda, M)$, so the total differential becomes

$$\mathrm{d}G^{(n,1)} = \frac{\partial G^{(n,1)}}{\partial M}\delta M + \frac{\partial G^{(n,1)}}{\partial \lambda}\delta\lambda = (n\delta\eta + \delta\tau)G^{(n,1)}. \tag{4.150}$$

We introduce the β and γ functions like in eq. (4.59) and further a dimensionless shift in the local operator,

$$\gamma_\theta \equiv -M\frac{\delta\tau}{\delta M}, \tag{4.151}$$

and we generalise the Callan-Symanzik equation to be

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + \gamma_\theta(\lambda)\right] G^{(n,1)}(p_1, ..., p_n; q) = 0\,. \tag{4.152}$$

4.7.1 Anomalous Dimension of the Local Operator

Since the shift in the local operator is $\theta_M \to Z_\theta^{-1}(M+\delta M)\theta_0$, we can write it, just like for the field φ, as

$$\delta\tau\theta_M = \left[Z_\theta^{-1}(M+\delta M) - Z_\theta^{-1}(M)\right]\theta_0, \tag{4.153}$$

which implies that

$$\begin{aligned} \delta\tau &= \frac{Z_\theta^{-1}(M+\delta M)}{Z_\theta^{-1}(M)} - 1, \\ &= \left(1 + \frac{1}{Z_\theta}\frac{\partial Z_\theta}{\partial M}\delta M\right)^{-1} - 1, \\ &= -\frac{1}{Z_\theta}\frac{\partial Z_\theta}{\partial M}\delta M. \end{aligned}$$

So for the anomalous dimension of the local operator we have

$$\gamma_\theta = \frac{M}{Z_\theta}\frac{\partial Z_\theta}{\partial M}, \tag{4.154}$$

which is true to all orders in the coupling.

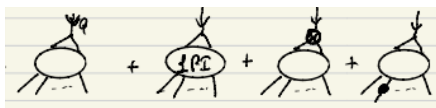
If there are m local operators θ^i, with $i = 1, ..., m$, with the same dimension and quantum numbers, the usually mix in the renormalisation procedure. Then $\theta_0^i = Z_\theta^{ij}(M)\theta_M^j$ and the CS equation becomes matrix-valued, with a shift matrix,

$$\gamma_\theta^{ij} = M(Z_\theta^{-1})_{ik}\frac{\partial(Z_\theta)_{kj}}{\partial M}\,. \tag{4.155}$$

Finally, the local operator associated to a conserved current cannot change normalisation, $Z_\theta(M) = 1$, which implies that $\gamma_\theta = 0$. For example, the quark number current, $\theta = \bar{\psi}\gamma^\mu\psi$, is conserved, since $\int \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi =$ equals the number of quarks, and thus $\gamma_\theta = 0$.

Anomalous Dimension of Local Operators at one Loop

Let the n-point Green's function of a local operator $G^{(n,1)}(p_1, ..., p_n; q) =< \varphi(p_1)...\varphi(p_n)\theta_M(q) >$ be written as

$G^{(n,1)}(p_1, ..., p_n; q) =$ 

$$= G^{(n,1)}_{tree}\left(1 + B\log\left(\frac{\Lambda^2}{-q^2}\right) + \delta_\theta + \sum_{i=1}^{n}\left(A_i\log\left(\frac{\Lambda^2}{-p_i^2}\right) - \delta_{Z_i}\right)\right). \qquad (4.156)$$

We suppose that at lowest order the local operator, and thus this Green's function, do not depend on the coupling. Since the only dependence on M is in δ_{Z_i} and δ_θ, the CS equation we just wrote becomes

$$M\frac{\partial}{\partial M}\left(\delta_\theta - \sum_i \delta_{Z_i}\right) + \sum_i \gamma_i(\lambda) + \gamma_\theta(\lambda) = 0, \qquad (4.157)$$

where we assumed that in general the n fields may be different. Using $\gamma_i(\lambda) = \frac{1}{2}M\frac{\partial}{\partial M}\delta_{Z_i}$, we obtain that at lowest order, the anomalous dimension of the local operator is

$$\gamma_\theta(\lambda) = M\frac{\partial}{\partial M}\left(\frac{1}{2}\sum_{i=1}^{n}\delta_{Z_i} - \delta_\theta\right). \qquad (4.158)$$

4.7.2 Renormalisation of a Mass Operator

We may consider the mass operator $\frac{1}{2}m^2\varphi_M^2$ as a local operator and as a perturbation of the massless Lagrangian renormalised at point M. We write the Green's function for the insertion of k mass operators as

$$G^{(n,k)}(p_1, ..., p_n; q_1, ..., q_k) =< \varphi(p_1)...\varphi(p_n)\varphi_M^2(q_1)...\varphi_M^2(q_k) >, \qquad (4.159)$$

which, with obvious generalisation, fulfils the CS equation,

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + k\gamma_{\varphi^2}(\lambda)\right)G^{(n,k)}(\{p_i\}, \{q_j\}, \lambda, M) = 0. \qquad (4.160)$$

Then, we can think of expanding an n-point Green's function, $G^{(n)}(p_1, ..., p_n)$ in m^2,

$$G^{(n)}(p_1, ..., p_n; \lambda, M, m) = \sum_{k=0}^{\infty}(m^2)^k G^{(n,k)}(p_1, ..., p_n; q_1, ..., q_k). \qquad (4.161)$$

The coefficient of $(m^2)^k$ fulfils the CS equation (4.160). Further, since it is a polynomial of degree k in m^2, because of Euler's formula on homogeneous polynomials of degree k,

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = k f(x_1, ..., x_n), \tag{4.162}$$

(in our case there is only one variable, m^2), we can write the CS equation (4.160) as

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + \gamma_{\varphi^2}(\lambda)m^2\frac{\partial}{\partial m^2}\right) G^{(n,k)}(\{p_i\},\{q_j\},\lambda,M) = 0, \tag{4.163}$$

and after summing the expansion, as a CS equation for $G^{(n)}$,

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + \gamma_{\varphi^2}(\lambda)m^2\frac{\partial}{\partial m^2}\right) G^{(n)}(\{p_i\};\lambda,M,m) = 0. \tag{4.164}$$

In general, any perturbation may be added to the massless Lagrangian $\mathcal{L}_M$ as an operator,

$$\mathcal{L} = \mathcal{L}_M + \sum_i C_i \theta^i_M(x)\,. \tag{4.165}$$

The ensuing Green's function fulfils the CS equation,

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + \sum_i \gamma_{\theta_i}(\lambda)C_i\frac{\partial}{\partial C_i}\right) G^{(n)}(\{p_i\};\lambda,M,\{C_i\}) = 0. \tag{4.166}$$

We can treat the C_i's as a set of coupling-like parameters, and $\beta_i = \gamma_{\theta_i}(\lambda)C_i$ as a set of β functions, which will obey the RGE,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{C}_i = \gamma_{\theta_i}(\bar{\lambda})\bar{C}_i\,, \qquad \text{with} \qquad t = \frac{1}{2}\log\left(\frac{-p^2}{M^2}\right)\,. \tag{4.167}$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{m}^2 = \gamma_{\varphi^2}(\bar{\lambda})\bar{m}^2. \tag{4.168}$$

All that we have stated about the scalar theory can be repeated for QCD. We add a mass term, $\mathcal{L}_m = -m(\bar{\psi}\psi)_M$, to the massless Lagrangian $\mathcal{L}_M$ through the mass operator, $(\bar{\psi}\psi)_M$, at a scale M. If we consider the Green's function of n (anti)quark fields (since (anti)quark fields occur in even numbers one could also set $n = 2n'$), it obeys the CS equation,

$$\left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + n\gamma_2(g) + \gamma_{\bar{q}q}(g)m\frac{\partial}{\partial m}\right) G^{(n)}(\{p_i\};g,M,m) = 0. \tag{4.169}$$

The RGE for the running mass is

$$\frac{\mathrm{d}\bar{m}}{\mathrm{d}t} = \gamma_{\bar{q}q}(\bar{g})\bar{m} \qquad \text{with} \quad \bar{m}(M) = m. \tag{4.170}$$

Running mass at one loop

Specifying the anomalous dimension of a local operator at one loop (4.158) to a mass operator, we obtain

$$\gamma_{\bar{q}q} = M\frac{\partial}{\partial M}\left(\delta_2 - \delta_{\bar{q}q}\right). \tag{4.171}$$

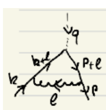
From eq. (4.104) we know that

$$\delta_2 = -\frac{\alpha_s}{4\pi}C_F\zeta\Gamma\left(2-\frac{d}{2}\right)(M^2)^{\frac{d}{2}-2}, \tag{4.172}$$

where ζ is the gauge-fixing parameter, and the analogous definition (4.177) in $\overline{\text{MS}}$. Let us define the insertion of a mass operator $\bar{\psi}\psi$ as

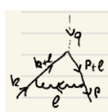
$$= 1 \quad \text{at} \quad q^2 = -M^2. \tag{4.173}$$

In order to compute $\delta_{\bar{q}q}$, we need to compute the one-loop corrections,



$$= \int\frac{\mathrm{d}^d\ell}{(2\pi)^d}igt^a\gamma^\mu i\frac{\not{p}+\not{\ell}}{(p+\ell)^2}1\,i\frac{\not{k}+\not{\ell}}{(k+\ell)^2}igt^a\gamma^\nu\frac{-i}{\ell^2}\left(g_{\mu\nu}-(1-\zeta)\frac{\ell^\mu\ell^\nu}{\ell^2}\right), \tag{4.174}$$

in an arbitrary gauge. Then, in order to compute the UV pole, it is enough to take $\ell \gg p, k$. One obtains ($\longrightarrow$ Exercises)



$$= g^2C_F\delta_{ij}\left[d-(1-\zeta)\right]\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(-q^2)^{2-\frac{d}{2}}}. \tag{4.175}$$

Thus, the counterterm is

$$\delta_{\bar{q}q} = -\left[d-(1-\zeta)\right]C_F\frac{\alpha_s}{4\pi}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}}. \tag{4.176}$$

In $\overline{\text{MS}}$,

$$\delta_{\bar{q}q} = -\left[d-(1-\zeta)\right]C_F\frac{\alpha_s}{4\pi}\frac{S_\epsilon}{\epsilon}(M^2)^{-\epsilon}. \tag{4.177}$$

As expected, in the anomalous dimension $\gamma_{\bar{q}q}$ the gauge-fixing terms cancel,

$$\begin{aligned}\gamma_{\bar{q}q} = M\frac{\partial}{\partial M}\left(\delta_2-\delta_{\bar{q}q}\right) &= (d-1)C_F\frac{\alpha_s}{4\pi}M\frac{\partial}{\partial M}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}}\\ &= 2(d-1)C_F\frac{\alpha_s}{4\pi}\left(\frac{d}{2}-2\right)\frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}}\\ &= -2(d-1)C_F\frac{\alpha_s}{4\pi}\frac{\Gamma\left(3-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}}\\ &\overset{d\to4}{=} -6C_F\frac{\alpha_s}{4\pi}.\end{aligned}$$

Thus, the RGE for the running mass yields

$$\frac{\mathrm{d}\bar{m}}{\mathrm{d}t} = \gamma_{\bar{q}q}(\bar{g})\bar{m} \tag{4.178}$$

$$= -6C_F\frac{\bar{\alpha}_s(Q^2)}{4\pi}\bar{m}. \tag{4.179}$$

with $t = \frac{1}{2}\log\left(\frac{-p^2}{M^2}\right)$ and $\bar{m}(M) = m$. Thus,

$$\frac{\mathrm{d}\bar{m}}{\bar{m}} = -\frac{6C_F}{4\pi}\bar{\alpha}_s(Q^2)\frac{1}{2}\,\mathrm{d}\log\left(\frac{-p^2}{M^2}\right). \tag{4.180}$$

In order to solve it, let us take the one-loop running of the coupling,

$$\frac{\mathrm{d}\bar{\alpha}_s}{\mathrm{d}\log\left(-p^2/M^2\right)} = -\frac{\beta_0}{4\pi}\bar{\alpha}_s^2 \quad \Rightarrow \quad \frac{\mathrm{d}\bar{\alpha}_s}{\bar{\alpha}_s} = -\frac{\beta_0}{4\pi}\bar{\alpha}_s\,\mathrm{d}\log\left(\frac{-p^2}{M^2}\right). \tag{4.181}$$

Replacing it in the RGE for the running mass, we get

$$\frac{\mathrm{d}\bar{m}}{\bar{m}} = \frac{3C_F}{\beta_0}\frac{\mathrm{d}\bar{\alpha}_s}{\bar{\alpha}_s}, \tag{4.182}$$

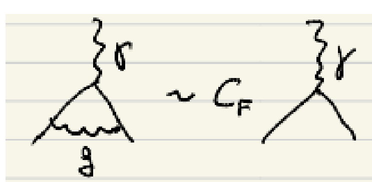
which can be readily integrated

$$\bar{m}(Q^2) = m\left(\frac{\alpha_s(Q^2)}{\alpha_s(M^2)}\right)^{\frac{3C_F}{\beta_0}}. \tag{4.183}$$

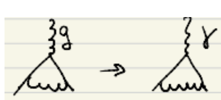
Note that for $Q^2 \gg M^2$, $\alpha_s(Q^2) < \alpha_s(M^2)$ and thus $\bar{m}(Q^2) < m$, i.e. the running mass decreases at larger momenta.

4.7.3 Non-renormalisation of the local operator of a conserved current

Finally, we may consider the one-loop gluon corrections to the quark-photon vertex. From the $\longrightarrow$ Exercises on the colour algebra,

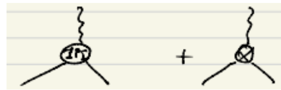


we expect that the one-loop corrections are just proportional to C_F. We can determine the UV-divergent part of the quark-photon vertex, by using the computation of the quark-gluon vertex ($\longrightarrow$ Exercises)



$$= ieQ_q\frac{g^2}{(4\pi)^2}C_F\frac{\Gamma\left(2-\frac{d}{2}\right)}{(-q^2)^{2-\frac{d}{2}}} + fin. \tag{4.184}$$

The renormalised quark-photon vertex is given by



$$\tag{4.185}$$

$$ieQ_q\Gamma^\mu(q^2) = ieQ_q\gamma^\mu\delta F_1^e(q^2) + ieQ_q\gamma^\mu\delta_1^e. \tag{4.186}$$

The counterterm is fixed by the off-shell prescription like we have done for the quark-gluon vertex,

$$\Gamma^\mu\big|_{q^2=-M^2} = 0 \quad \Rightarrow \quad \delta_1^e = -\,\delta F_1^e(q^2)\big|_{q^2=-M^2}\,. \tag{4.187}$$

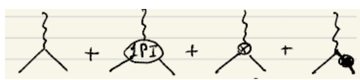
Thus for the counterterm we obtain

$$\delta_1^e = -\frac{g^2}{(4\pi)^2}C_F\frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}}, \tag{4.188}$$

which is equal to the counterterm δ_2 of the quark self-energy, $\delta_1^e = \delta_2$ (as expected, since we had seen in the previous lecture that the “Abelian” part of the counterterm δ_1 for the quark-gluon vertex equals δ_2).

Further, since there are no gluon corrections to the photon self-energy at one loop, we must have that $\delta_3 = 0$. Thus, the electric charge has no QCD corrections at $\mathcal{O}(\alpha_s)$: at one loop, gluons cannot make the electric charge renormalised.

More formally, we can write down the Green’s function for the quark-photon vertex,

$G^{(2q,\gamma)}(p_1,p_2,q) =$ 

$$= ieQ_q + ieQ_qB\log\left(\frac{\Lambda^2}{-q^2}\right) + ieQ_q\delta_1^e + ieQ_q\sum_{i=1}^{2}\left(A_i\log\left(\frac{\Lambda^2}{-p_i^2}\right) - \delta_2\right). \tag{4.189}$$

The CS equation is

$$\left[M\frac{\partial}{\partial M} + 2\gamma_2(g)\right]G^{(2q,\gamma)} = 0, \tag{4.190}$$

since at lowest order $\dfrac{\partial}{\partial g}G^{(2q,\gamma)} = 0$, i.e., the β function does not appear. Using the Green’s function above, the CS equation becomes

$$M\frac{\partial}{\partial M}\left(\delta_1^e - 2\delta_2\right) + \cancel{2}\frac{1}{\cancel{2}}M\frac{\partial}{\partial M}\delta_2 = 0. \tag{4.191}$$

And thus,

$$M\frac{\partial}{\partial M}(\delta_1^e - \delta_2) = 0, \tag{4.192}$$

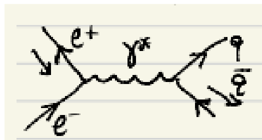
which is identically fulfilled since $\delta_1^e = \delta_2$.

Note that in the same way as we have examined the mass parameter, we can compute the insertion of a quark current $j^\mu = \bar{q}\gamma^\mu q$, which, being conserved, must have null anomalous dimensions. In order to do that, we must replace the mass operator insertion 1 with γ^μ. That is what we have done in computing the corrections to the quark-photon vertex. In fact, we found that $\delta_1^e = \delta_2$, and so the anomalous dimension of the quark current vanishes,

$$\gamma_{\bar{q}\gamma q} = M\frac{\partial}{\partial M}(\delta_2 - \delta_1^e) = 0. \tag{4.193}$$

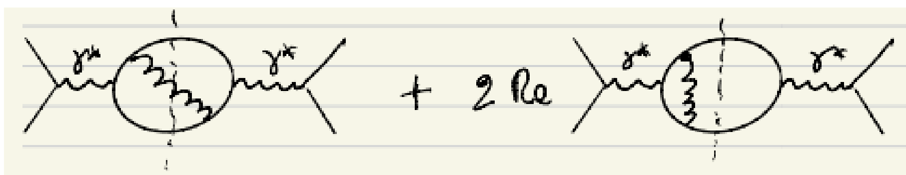
4.8 $e^+e^- \to$ hadrons

We shall consider now the cross section for $e^+e^- \to$ hadrons as an example of a physical quantity and we shall consider its behaviour under a change of renormalisation scale. Using the helicity amplitudes for $e^+e^- \to \mu^+\mu^-$ introduced previously, we can compute the cross section for $e^+e^- \to q\bar{q}$ ($\longrightarrow$ Exercises)



$$\sigma_{tot}(e^+e^- \to q\bar{q}) = \frac{4\pi}{3}\frac{\alpha^2}{s}N_c\sum_f Q_f^2, \tag{4.194}$$

where the factor N_c comes from the sum over colours. Let us include then the one-loop corrections,



Note that both real and virtual corrections contain infrared divergences, when the gluon becomes collinear and/or soft, and we shall come back to them. However, the divergences cancel out in the total rate, that we quote to $\mathcal{O}(\alpha_s)$ (please see Exercises),

$$\sigma_{tot} = \frac{4\pi}{3}\frac{\alpha^2}{s}N_c\sum_f Q_f^2\left(1+\frac{\alpha_s}{\pi}+\mathcal{O}(\alpha_s^2)\right). \tag{4.195}$$

The typical time of the $q\bar{q}$ pair creation is $\tau \sim \dfrac{1}{\sqrt{s}}$; on the other hand, the typical virtuality of the (anti)quark emitting the gluon is $p_\perp^2$. Thus the time scale of the gluon emission is $\tau \sim \dfrac{1}{p_{\perp g}}$. If $p_{\perp g} \ll \sqrt{s}$, the gluon will be emitted in the far future and cannot affect substantially the hard scattering (it only affects the details of the final states). If $p_{\perp g} \sim \sqrt{s}$, the gluon is hard and provides a finite correction to the leading order rate. So, only high-momentum gluons contribute to the total rate.

Since the total rate depends on α_s, we must specify a renormalisation scale. We are interested in the high-energy limit, where the quarks are treated as massless. We have already seen that in this case, the on-shell prescription leads to mass divergences. So we would rather renormalise α_s at a large scale. However, we shall choose that scale to be time-like. This leads to infrared divergences, which we shall discuss later. We write the total cross section as

$$\sigma_{tot}(e^+e^- \to q\bar{q}) = \sigma_0 f, \qquad \text{with} \quad \sigma_0 = \frac{4\pi}{3}\frac{\alpha^2}{s}N_c\sum_f Q_f^2\,, \tag{4.196}$$

and f is a dimensionless function. Since the quarks are taken as massless, the only scales are the centre-of-mass energy $\sqrt{s}$ and, because of the UV divergences, a renormalisation scale M, so

$f = f\left(\alpha_s, \frac{s}{M^2}\right)$. But the scale M is arbitrary, and physical observables, like σ_{tot}, do not depend on it. This leads naturally to the CS equation,

$$M\frac{\mathrm{d}}{\mathrm{d}M}f = \left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g}\right) f = 0, \tag{4.197}$$

without γ functions, since the external legs are on-shell and are not to be renormalised. Note that since f is a physical quantity, it is gauge independent, and so must be the β function.

Note that a physical quantity, f_{phys}, which involved the insertion of a mass operator, would obey a CS equation,

$$\left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + \gamma_{\bar{q}q}(g)m\frac{\partial}{\partial m}\right) f_{phys}(\{p_i\}; g, M, m) = 0. \tag{4.198}$$

Then, just like $\beta(g)$, also the anomalous dimension $\gamma_{\bar{q}q}$ of the mass operator must be gauge invariant.

The function f in eq. (4.196) can be expanded in the strong coupling constant α_s,

$$f\left(\alpha_s(M^2), \frac{s}{M^2}\right) = 1 + C_1\left(\frac{s}{M^2}\right)\alpha_s(M^2) + C_2\left(\frac{s}{M^2}\right)\alpha_s^2(M^2) + \dots. \tag{4.199}$$

The RGE for the coupling (4.121) has the solution (4.123), which can be expanded as

$$\bar{\alpha}_s(s) = \alpha_s\left(1 - \frac{\beta_0}{4\pi}\alpha_s \log\left(\frac{s}{M^2}\right) + \dots\right). \tag{4.200}$$

Since σ_{tot} is a physical observable, and thus an invariant under a change of scale M, so is f, which implies that

$$f\left(\alpha_s(M^2), \frac{s}{M^2}\right) = f\left(\bar{\alpha}_s(s), 1\right). \tag{4.201}$$

Thus we may take $M^2 = s$ in the expansion of f, and using eq. (4.200) we have

$$\begin{aligned} f\left(\bar{\alpha}_s(s), 1\right) &= 1 + C_1(1)\bar{\alpha}_s(s) + C_2(1)\bar{\alpha}_s^2(s) + \dots \\ &= 1 + C_1(1)\left(\alpha_s - \frac{\beta_0}{4\pi}\alpha_s^2 \log\left(\frac{s}{M^2}\right)\right) + C_2(1)\bar{\alpha}_s^2(s) + \mathcal{O}(\alpha_s^3) \\ &= 1 + C_1\left(\frac{s}{M^2}\right)\alpha_s + C_2\left(\frac{s}{M^2}\right)\alpha_s^2 + \mathcal{O}(\alpha_s^3). \end{aligned} \tag{4.202}$$

which entails that

$$\begin{aligned} C_1\left(\frac{s}{M^2}\right) &= C_1(1) = \frac{1}{\pi}, \\ C_2\left(\frac{s}{M^2}\right) &= C_2(1) - \frac{\beta_0}{4\pi}C_1 \log\left(\frac{s}{M^2}\right). \end{aligned} \tag{4.203}$$

Let us comment on the physical meaning of this result. In $e^+e^- \to$ hadrons, the quark-photon vertex occurs. We have seen that the conservation of the quark current implies that the anomalous dimension of the quark-photon vertex vanishes (4.193). Further, at one loop there are no gluon corrections to the photon self-energy δ_3 (non-renormalisation of the electric charge at one loop), which implies that the solution to the CS equation for $\sigma_{tot}(e^+e^- \to \text{hadrons})$ has a constant $\mathcal{O}(\alpha_s)$ coefficient, C_1. However, the expression for C_2 implies that beyond $\mathcal{O}(\alpha_s)$ the solution to the CS equation contains large logarithms, due to the running of the coupling.

The expansion $f = \sum_n C_n(1)\bar{\alpha}_s^n(s)$ converges much more quickly than $f = \sum_n C_n\left(\frac{s}{M^2}\right)\alpha_s^n(M^2)$ because the large $\log\left(s/M^2\right)$ have been resummed in $\bar{\alpha}_s(s)$, just like we saw for the solution (4.84) of the CS equation for the 2 point function. So yet another example that the CS equation allows for a better re-organisation of the terms in the perturbative series.

We can write the solution (4.122) for the running coupling as

$$\frac{1}{\alpha_s(s)} = \frac{1}{\alpha_s(M^2)} + \frac{\beta_0}{4\pi}\log\left(\frac{s}{M^2}\right). \tag{4.204}$$

Suppose that we take M^2 so small that $1/\alpha_s(M^2)$ becomes negligible. Then we define $M^2 = \Lambda_{QCD}^2$ as the scale where that approximate relation is an equality,

$$\frac{1}{\alpha_s(s)} = \frac{\beta_0}{4\pi}\log\left(\frac{s}{\Lambda_{QCD}^2}\right). \tag{4.205}$$

Λ_{QCD} is in the non-perturbative regime, and it is usually taken to be a few hundred MeV.

Example: Suppose we want to compute the R ratio,

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}. \tag{4.206}$$

After the $\mathcal{O}(\alpha_s)$ corrections, we can write

$$R(\alpha_s(s)) = R_0\left(1 + \frac{\alpha_s(s)}{\pi} + \mathcal{O}(\alpha_s^2)\right), \qquad \text{with} \quad R_0 = N_c\sum_f Q_f^2. \tag{4.207}$$

If we take $\Lambda_{QCD} = 300$ MeV and $\sqrt{s} = M_z \simeq 91$ GeV, i.e., we compute R at the Z_0 mass, then

$$\log\left(\frac{s}{\Lambda_{QCD}^2}\right) = 2\log\left(\frac{31}{0.3}\right) \simeq 11.5\,. \tag{4.208}$$

For 5 quark flavours, $\beta_0 = 23/3 \simeq 7.66$, then

$$\frac{\alpha_s(M_z^2)}{\pi} = \frac{4}{\beta_0\log\left(M_z^2/\Lambda_{QCD}^2\right)} \simeq 0.045, \tag{4.209}$$

which is close to the measured LEP value.

4.9 Couplings With Different Numbers of Light Quarks

In the example we have just considered, we have taken the R-ratio (4.206) with 5 flavours of light (i.e. massless) quarks. Why can we take the quarks as massless? We suppose that the quark mass m is much smaller than the square root of the center of mass energy $s = Q^2$. So we can take the mass as a perturbation, as we have done in sec. 4.7.2 with the mass operator, and write the R-ratio as an expansion in the quark mass,

$$R\left(\alpha_s(Q^2), m(Q^2)\right) = R\left(\alpha_s(Q^2)\right) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{m(Q^2)}{Q}\right)^k \left.\frac{\partial^k R\left(\alpha_s(Q^2), m(Q^2)\right)}{\partial m^k}\right|_{m=0}, \tag{4.210}$$

where $R\left(\alpha_s(Q^2)\right)$ is given in eq. (4.207), and we have used the shorthand $Q = \sqrt{Q^2}$. For $m \ll Q$, the derivative terms above are suppressed by powers of m/Q. The R-ratio is a physical observable, so it fulfils the CS equation (4.198), whose anomalous dimension $\gamma_{\bar{q}q}$, though, yields a running mass which varies at most with powers of $\log\left(Q^2\right)$: it cannot offset the power suppression of powers of m/Q (in fact, as we have seen in sec. 4.7.2, it leads to a further suppression). So we are fully justified in taking the quark as massless for $m \ll Q$. That is why for $Q \gg m_b \simeq 4.5$ GeV we consider $n_f = 5$ massless quarks, and thus the running of α_s for $Q \gg m_b$ is computed with $n_f = 5$. Conversely, for $Q < m_b$, the running of α_s should be computed with $n_f = 4$.

The top quark has mass $m_t \simeq 173$ GeV. At a scale Q much below 173 GeV, the effects of the top quark are suppressed by powers of Q/m_t and we neglect them. But when $Q > m_t$, the running of α_s should be computed with $n_f = 6$. At the scales $M^2 = m_b^2$ and $M^2 = m_t^2$, we add a heavy quark and the running changes slope, but we should require that in those points $\alpha_s(Q^2)$ be a continuous function of Q^2. We use eq. (4.204) for the running coupling with $M^2 = m_t^2$, and define $\bar{\alpha}_s^+(Q^2)$ in terms of β_0^+ (4.120) with $n_f = 6$ for $Q^2 > m_t^2$, and $\bar{\alpha}_s^-(Q^2)$ in terms of β_0^- with $n_f = 5$ for $Q^2 < m_t^2$. Then eq. (4.204) with Q^2 above and below m_t^2 implies that

$$\frac{1}{\alpha_s(m_t^2)} = \frac{1}{\bar{\alpha}_s^+(Q^2)} - \frac{\beta_0^+}{4\pi} \log\left(\frac{Q^2}{m_t^2}\right) = \frac{1}{\bar{\alpha}_s^-(Q^2)} - \frac{\beta_0^-}{4\pi} \log\left(\frac{Q^2}{m_t^2}\right). \tag{4.211}$$

But $\beta_0^+ - \beta_0^- = -2/3$, so

$$\frac{1}{\bar{\alpha}_s^+(Q^2)} = \frac{1}{\bar{\alpha}_s^-(Q^2)} - \frac{1}{6\pi} \log\left(\frac{Q^2}{m_t^2}\right), \tag{4.212}$$

i.e.

$$\bar{\alpha}_s^+(Q^2) = \frac{\bar{\alpha}_s^-(Q^2)}{1 - \frac{\bar{\alpha}_s^-(Q^2)}{6\pi} \log\left(\frac{Q^2}{m_t^2}\right)}, \tag{4.213}$$

thus the running changes slope at $Q^2 = m_t^2$ through a step-function in its derivative $\mathrm{d}\bar{\alpha}_s/\,\mathrm{d}t$, which develops a cusp in the running of α_s that makes it suddenly less steep.

Chapter 5

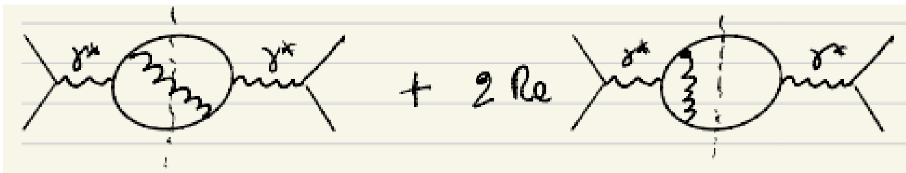
Infrared Divergences

In the previous chapter, we have dealt with the renormalisation of the UV divergences, which occur in the short-distance behaviour of the quantum corrections to the quantities that we compute in a field theory. In renormalisable theories, like QED and QCD in four dimensions, UV divergences may be consistently put aside through the renormalisation procedure, and are in a sense harmless. In this chapter, we shall deal with the behaviour of QED and QCD when infrared (IR) singularities occur. By IR singularities, we mean the soft divergences which occur when the momentum of one gluon or two or more partons become vanishingly small, and the mass divergences which occur when the momenta of two or more partons become collinear. Differently from UV divergences, the IR divergences are physical singularities, related to specific kinematic limits of particles involved in scattering processes, and cannot be put aside. They can be canceled, though, for suitably defined physical quantities, which are thus termed *infrared safe*.

We shall see that the operator product expansion (OPE), which as a short-distance expansion benefits from the renormalisation techniques that we have developed for the UV divergences in the previous chapter, is also very useful, as a light-cone expansion, to analyse the collinear divergences which occur in DIS. Although the collinear divergences are of a very different nature from the UV divergences, what they have in common and allows using for both an OPE is that they are localised, the UV divergences in the space-time and the collinear divergences on the light cone. Instead the soft divergences, which we will deal with anecdotally in simple specific cases, are not localised in space. Thus, their systematic analysis is more difficult, and beyond the scope of these lectures.

5.1 Final States in $e^+e^- \to$ hadrons

In the previous chapter, when we discussed the $\mathcal{O}(\alpha_s)$ corrections to the cross section for $e^+e^- \to$ hadrons, we said that both real and virtual corrections



contain IR divergences, when the gluon becomes collinear and/or soft. However, the divergences cancel out in the total rate. We argued that it is so because the typical time of the $q\bar{q}$ pair creation is $\tau \sim 1/\sqrt{s}$, while the time scale of the gluon emission is $\tau \sim 1/p_{\perp g}$. Thus, if $p_{\perp g} \ll \sqrt{s}$, the gluon is emitted in the far future and does not affect substantially the hard scattering, while, if $p_{\perp g} \sim \sqrt{s}$, the gluon is hard and provides finite corrections to the leading-order rate.

We should now substantiate those statements. Firstly, what about computing the $\mathcal{O}(\alpha_s)$ corrections to $e^+e^- \to q\bar{q}$, by just considering the $\mathcal{O}(\alpha_s)$ corrections to the quark-photon vertex? The quark-photon vertex behaves just like the electron-photon vertex that you studied in QFT I, in that it displays an IR divergence, when the loop momentum vanishes. In order to expose it, one can add a fictitious photon mass μ, like we did for the quark self-energy and write the vertex in the limit of small loop momentum k, then the $\mathcal{O}(\alpha_s)$ corrections to the electron-photon (or the $\mathcal{O}(\alpha_s)$ corrections to the quark-photon) vertex are (please see Exercises)

$$\delta F_1(q^2) = -\frac{\alpha}{2\pi}\Gamma(\epsilon)\left(\frac{q^2}{4\pi\mu^2}\right)^{\epsilon}\left(\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right) - 1\right), \tag{5.1}$$

where $\beta = \sqrt{1 - \frac{4m^2}{q^2}}$, $\epsilon = 2 - \frac{d}{2}$ and m is the quark (or electron) mass. The expansion in ϵ displays the IR logarithm $\log\left(q^2/\mu^2\right)$. Further, in the small mass limit,

$$\lim_{m\to 0}\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right) = \log\left(\frac{q^2}{m^2}\right), \tag{5.2}$$

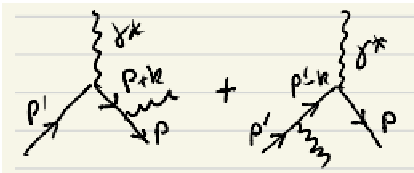
and the $\mathcal{O}(\alpha_s)$ corrections to the vertex become

$$\delta F_1(q^2) = -\frac{\alpha}{2\pi}\log\left(\frac{q^2}{m^2}\right)\left(\frac{1}{\epsilon} + \log\left(\frac{q^2}{\mu^2}\right)\right) + fin. \tag{5.3}$$

The second logarithm $\log\left(q^2/m^2\right)$ is due to a **mass singularity** or **collinear singularity**.

5.1.1 Soft-Gluon Emission

As we said in the introduction to this chapter, differently from the UV divergences, the IR divergences cannot be put aside through a renormalisation procedure: they are there to stay. Let us then examine the emission of soft gluons from the quark-photon vertex (or of soft photons from the electron-photon vertex),



$$i\mathcal{M} = \bar{u}(p) \left(igt^a\gamma^\nu \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(-ie\Gamma^\mu) + (-ie\Gamma^\mu)\frac{i(\not{p}'-\not{k}+m)}{(p'-k)^2-m^2}igt^a\gamma^\nu \right) u(p')\epsilon_\nu(k), \tag{5.4}$$

where, like for the vertex corrections, we keep a quark (or electron) mass. As $k \to 0$, we may just keep the $\mathcal{O}(k^0)$ term in the numerator, and simplify $(p+k)^2 - m^2 = 2p\cdot k$,

$$i\mathcal{M}_{soft} = g\bar{u}(p)\left[-t^a \not{\epsilon}(k) \frac{\not{p}+m}{2p\cdot k}(-ie\Gamma^\mu) + (-ie\Gamma^\mu)\frac{\not{p}'+m}{2p'\cdot k}t^a\not{\epsilon}(k)\right] u(p'). \tag{5.5}$$

Use Dirac equation to write

$$\bar{u}(p)\not{\epsilon}(k)(\not{p}+m) = \bar{u}(p)[(-\not{p}+m)\not{\epsilon} + 2p\cdot\epsilon] = \bar{u}(p)2p\cdot\epsilon. \tag{5.6}$$

Thus, the soft-gluon emission factorises

$$i\mathcal{M}_{soft} = -g\left(\frac{p\cdot\epsilon(k)}{p\cdot k} - \frac{p'\cdot\epsilon(k)}{p'\cdot k}\right)\bar{u}(p)t^a(-ie\Gamma^\mu)u(p'). \tag{5.7}$$

Now, we consider soft-gluon emission in $e^+e^- \to$ hadrons. In order to simplify matters, we just take the virtual photon creating a $q\bar{q}$ pair. Neglecting the soft gluon in the momentum conservation, $\delta^4(q-p-p'-k) \simeq \delta^4(q-p-p')$, the rate for soft-gluon emission factorises into the non-radiative rate times a soft gluon factor,

$$\mathrm{d}\sigma_{soft}(\gamma^* \to q\bar{q}g) = \mathrm{d}\sigma(\gamma^*\to q\bar{q})g^2\frac{\mathrm{Tr}(t^at^a)}{N_c}\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left|\frac{p\cdot\epsilon(k)}{p\cdot k} - \frac{p'\cdot\epsilon(k)}{p'\cdot k}\right|^2. \tag{5.8}$$

The averaging over colour is done because an N_c factor already appears in $\mathrm{d}\sigma(\gamma^*\to\bar{q}q)$. Then,

$$\left|\frac{p\cdot\epsilon(k)}{p\cdot k} - \frac{p'\cdot\epsilon(k)}{p'\cdot k}\right|^2 = \sum_i \epsilon^\mu_{\perp i}(k)\epsilon^{\nu *}_{\perp i}(k)\left(\frac{p_\mu}{p\cdot k} - \frac{p'_\mu}{p'\cdot k}\right)\left(\frac{p_\nu}{p\cdot k} - \frac{p'_\nu}{p'\cdot k}\right), \tag{5.9}$$

where the sum is over transverse polarisations. However, as we know in the case of one gluon only, we can use the Ward identity and replace $\sum_i \epsilon^\mu_{\perp i}(k)\epsilon^{\nu *}_{\perp i}(k) \to -g^{\mu\nu}$ and we obtain

$$\begin{aligned}\left|\frac{p\cdot\epsilon(k)}{p\cdot k} - \frac{p'\cdot\epsilon(k)}{p'\cdot k}\right|^2 &= -g^{\mu\nu}\left(\frac{p_\mu}{p\cdot k} - \frac{p'_\mu}{p'\cdot k}\right)\left(\frac{p_\nu}{p\cdot k} - \frac{p'_\nu}{p'\cdot k}\right) \\ &= \frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2}.\end{aligned} \tag{5.10}$$

(In case you are not convinced, you can check that you get the same result by using eq. (3.79)). So the rate for the soft-gluon emission becomes

$$\mathrm{d}\sigma_{soft}(\gamma^*\to q\bar{q}g) = \mathrm{d}\sigma(\gamma^*\to q\bar{q})g^2C_F\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left(\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2}\right). \tag{5.11}$$

The phase-space measure is

$$\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0} = \frac{\mathrm{d}k^0\, k^0}{8\pi^2}\frac{\mathrm{d}\varphi}{2\pi}\,\mathrm{d}\cos\theta\,. \tag{5.12}$$

As $k^\mu \to 0$, the $\bar{q}q$ pair yields two back-to-back particles. We may go to the rest frame of the $\bar{q}q$ pair, and approximate the kinematics as,

$$p = (p^0, 0, 0, \beta p^0), \qquad p' = (p^0, 0, 0, -\beta p^0), \tag{5.13}$$

with

$$\sqrt{1 - \frac{m^2}{p_0^2}} \simeq \sqrt{1 - \frac{4m^2}{q^2}} = \beta. \tag{5.14}$$

Then,

$$\begin{aligned} 2p\cdot p' &\simeq q^2 - 2m^2 = 4p_0^2\left(1 - \frac{2m^2}{q^2}\right), \\ p\cdot k &= p_0 k_0 (1 - \beta\cos\theta), \\ p'\cdot k &= p_0 k_0 (1 + \beta\cos\theta), \end{aligned} \tag{5.15}$$

where $(p + p')^2 \simeq q^2$. So,

$$\begin{aligned} \frac{2p\cdot p'}{p\cdot k p'\cdot k} &= \frac{4p_0^2\left(1 - \frac{2m^2}{q^2}\right)}{p_0^2 k_0^2 (1 - \beta^2\cos^2\theta)}, \\ &= \frac{4}{k_0^2}\left(1 - \frac{2m^2}{q^2}\right)\frac{1}{2}\left(\frac{1}{1 - \beta\cos\theta} + \frac{1}{1 + \beta\cos\theta}\right), \end{aligned} \tag{5.16}$$

after partial fractioning. Further,

$$\begin{aligned} \frac{m^2}{(p\cdot k)^2} &= \frac{m^2}{p_0^2 k_0^2 (1 - \beta\cos\theta)^2} = \frac{4m^2}{q^2 k_0^2 (1 - \beta\cos\theta)^2}, \\ \frac{m^2}{(p'\cdot k)^2} &= \frac{m^2}{p_0^2 k_0^2 (1 + \beta\cos\theta)^2} = \frac{4m^2}{q^2 k_0^2 (1 + \beta\cos\theta)^2}. \end{aligned} \tag{5.17}$$

So we can write the integral as,

$$\begin{aligned} &\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left(\frac{2p\cdot p'}{p\cdot k p'\cdot k} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2}\right) \\ &\quad = \frac{1}{8\pi^2} 4 \int_{\mu_R}^{\sqrt{q^2}} \frac{\mathrm{d}k^0}{k^0} \int_{-1}^{1} \mathrm{d}\cos\theta \left[\left(1 - \frac{2m^2}{q^2}\right)\frac{1}{2}\left(\frac{1}{1 - \beta\cos\theta} + \frac{1}{1 + \beta\cos\theta}\right)\right. \\ &\qquad\qquad \left. - \frac{m^2}{q^2}\frac{1}{(1 - \beta\cos\theta)^2} - \frac{m^2}{q^2}\frac{1}{(1 + \beta\cos\theta)^2}\right]. \end{aligned} \tag{5.18}$$

The integral over $\cos\theta$ is easily done, and the upper limit on k^0 is obtained from taking p and p' at rest, then $\mathrm{Max}(k^0) = \sqrt{q^2} - \mathcal{O}(m)$. Thus,

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left(\frac{2p\cdot p'}{p\cdot k p'\cdot k} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2}\right) = \frac{1}{2\pi^2}\frac{1}{2}\left(\log\left(\frac{q^2}{\mu_R^2}\right) + fin.\right)\left(\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right) - 1\right). \tag{5.19}$$

So the rate for soft-gluon emission becomes

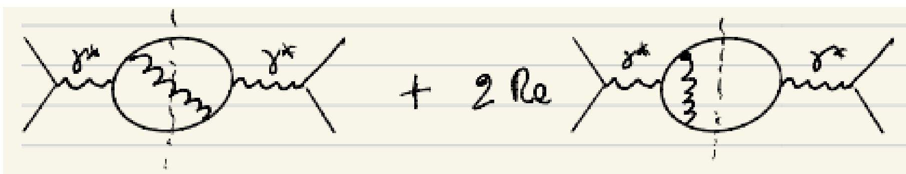
$$d\sigma_{soft}(\gamma^* \to q\bar{q}g) = d\sigma(\gamma^* \to q\bar{q})\frac{\alpha_s}{\pi}C_F\left(\log\left(\frac{q^2}{\mu_{IR}^2}\right) + fin.\right)\left(\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right) - 1\right), \tag{5.20}$$

which displays an IR logarithm, $\log\left(q^2/\mu_{IR}^2\right)$. Further, the dependence on β is exactly the same as for the electron-photon (quark-photon) vertex, and at $m \to 0$, we get a collinear logarithm, $\log\left(q^2/m^2\right)$. Had we worked out the soft-gluon emission in $d = 4 - 2\epsilon$ dimensions, we would have obtained,

$$d\sigma_{soft}(\gamma^* \to q\bar{q}g) = d\sigma(\gamma^* \to q\bar{q})\frac{\alpha_s}{\pi}C_F\Gamma(\epsilon)\left(\frac{q^2}{4\pi\mu_R^2}\right)^{\epsilon}\left(\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right) - 1\right). \tag{5.21}$$

So, if we take the sum of the $\mathcal{O}(\alpha_s)$ corrections to the quark-gluon vertex in the IR limit of small loop momentum and the rate for soft-gluon emission,

$$d\sigma_{soft}(\gamma^* \to q\bar{q}g) + 2\,\mathrm{Re}(\delta F_1(q^2))\,d\sigma(\gamma^* \to q\bar{q}), \tag{5.22}$$



,

we see that the IR logarithms cancel (in fact, we know the sum above when computed exactly for the total cross section $\sigma(e^+e^- \to \text{hadrons})$, has a finite residue: α_s/π).

5.2 Semiclassical Theory of Radiation

A few remarks are now in order. Including the leading piece of the $\mathcal{O}(\alpha)$ corrections to the electron-photon vertex, we may write

$$F_1(q^2) \simeq -ie\gamma^\mu\left(1 - \frac{\alpha}{2\pi}\log\left(\frac{q^2}{m^2}\right)\log\left(\frac{q^2}{\mu^2}\right)\right), \tag{5.23}$$

which is negative and, as μ^2 or m^2 vanishes, infinitely large. Beyond $\mathcal{O}(\alpha)$, this contribution exponentiates,

$$F_1(q^2) \simeq -ie\gamma^\mu\exp\left(-\frac{\alpha}{2\pi}\log\left(\frac{q^2}{m^2}\right)\log\left(\frac{q^2}{\mu^2}\right)\right), \tag{5.24}$$

and vanishes as μ^2 or m^2 vanishes.

We may think of it as yielding the probability of not emitting photons. As the photon energy goes to zero, so does that probability. In fact, this is an expected behaviour: from the classical theory of radiation, we know that the angular distribution of soft photons is the same as in the rate for soft-gluon emission (5.11). Also, we know that a charged particle moving in a force field radiates; the radiation intensity per unit frequency, $I(\omega)$, becomes independent of the frequency ω of the emitted radiation, as $\omega \to 0$ (see section 15 of Jackson's "Classical Electrodynamics"),

$$\lim_{\omega\to 0}\frac{dI}{d\omega} = 0. \tag{5.25}$$

In a semiclassical approach to the theory of radiation, by the correspondence principle, we interpret $I(\omega)$ as the mean number $n(\omega)$ of photons emitted times the energy $\hbar\omega$ of the single photon,

$$I(\omega) = \hbar\omega n(\omega). \tag{5.26}$$

Then, $I(0) \simeq \hbar\omega n(\omega)$, and we see that the number of photons emitted in the interval, $\omega_0 \leq \omega \leq \omega_1$, diverges logarithmically as $\omega_0 \to 0$,

$$\int_{\omega_0}^{\omega_1} \mathrm{d}\omega\, n(\omega) \simeq \frac{I(0)}{\hbar} \log\left(\frac{\omega_1}{\omega_0}\right). \tag{5.27}$$

Hence, we may expect that the probability of emitting infinitely many photons is finite as $\omega \to 0$ and conversely, that the probability of emitting any finite number of photons vanishes as $\omega \to 0$.

In particular, in the semiclassical theory of radiation, it was known that the probability of scattering of an electron in a Coulomb field with emission of a single photon in the interval $\omega_0 \leq \omega \leq \omega_1$ diverges as $\omega_0 \to 0$, a result known as *infrared catastrophe.* This problem was solved by Bloch and Nordsieck in 1938, in the way described above. This is mirrored by the IR behaviour we have observed.

In the quantum theory, we may say that if Δ is the energy resolution of our detector, photons with energy smaller than Δ are not observable. Then, the probability of creating a photon with energy below Δ is given by,

$$\mathrm{d}\sigma_{soft}(\gamma^* \to q\bar{q}\gamma)|_{k^0<\Delta} = \mathrm{d}\sigma(\gamma^* \to q\bar{q})\frac{\alpha}{\pi}\Gamma(\epsilon)\left(\frac{\Delta^2}{4\pi\mu_R^2}\right)^{\epsilon}\left(\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right)-1\right), \tag{5.28}$$

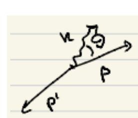
and the probability of not observing a photon with energy below Δ is given by,

$$\begin{aligned}(1+2\,\mathrm{Re}(\delta F_1(q^2))\sigma(\gamma^* \to q\bar{q}) + \int \mathrm{d}\sigma_{soft}(\gamma^* \to q\bar{q}\gamma)\Big|_{k^0<\Delta} \\ = \sigma(\gamma^* \to q\bar{q})\left(1+\frac{\alpha}{\pi}\log\frac{\Delta^2}{q^2}\left(\frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right)-1\right)\right),\end{aligned} \tag{5.29}$$

where the $\mathcal{O}(\alpha)$ correction is finite, although negative, since $\log\left(\Delta^2/q^2\right) < 0$. For many soft photons, the $\mathcal{O}(\alpha)$ term in eq. (5.29) exponentiates, and vanishes as $\Delta \to 0$. Thus, we see that in QED the Born cross section, or any low-order correction to it, may be a very good approximation.

5.3 Jets in $e^+e^- \to$ hadrons

In order to continue our study of the final-state structure, and understand better the nature of the collinear singularities, we introduce the definition of a **jet**, as a cluster of particles whose invariant mass does not exceed a fixed value J^2. For a 3-particle final state,



we have 3 jets if all invariants $s_{pp'}, s_{pk}, s_{p'k}$ are larger than J^2; we have 2 jets if at least one of them is smaller than J^2. We introduce the dimensionless parameter, $y = J^2/s$, where $s = (p + p' + k)^2$, and we study the 2-jet phase space. Let us suppose that the quarks are massless and that

$$s_{pk} < J^2 \quad \Rightarrow \quad 2p \cdot k = 2p^0 k^0 (1 - \cos\theta) < ys. \tag{5.30}$$

Let us suppose that the gluon energy is negligible. Then,

$$p^0 \simeq \frac{\sqrt{s}}{2} \quad \Rightarrow \quad k^0 (1 - \cos\theta) < y\sqrt{s}. \tag{5.31}$$

Then we have two options,

$$(i) \begin{cases} k_0 < y\sqrt{s} \\ -1 \leq \cos\theta \leq 1 \end{cases} \quad , \quad (ii) \begin{cases} k_0 > y\sqrt{s} \\ 1 - \dfrac{y\sqrt{s}}{k^0} < \cos\theta < 1 \end{cases} \tag{5.32}$$

(i) corresponds to a soft gluon, (ii) corresponds to a hard collinear gluon. Of course, $s_{p'k} < J^2$ yields the same contributions. Let us recall from the previous lecture that the rate for soft-gluon emission, in the massless quark case, is

$$\mathrm{d}\sigma_{soft}(\gamma^* \to q\bar{q}g) = \mathrm{d}\sigma(\gamma^* \to q\bar{q}) \frac{2\alpha_s}{\pi} C_F \int_0^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_{-1}^{1} \frac{\mathrm{d}\cos\theta}{1 - \cos^2\theta}, \tag{5.33}$$

which features a soft singularity as $k^0 \to 0$ and collinear singularities as $\theta \to 0, \pi$. Since the integral yields the same result in the two hemispheres, we can define the $\mathcal{O}(\alpha_s)$ corrections to the 2-jet rate by doubling the outcome for one of the two hemispheres. Fixing the Born cross section as $\sigma_0 = \sigma(e^+e^- \to q\bar{q})$,

$$\begin{aligned} &\frac{1}{\sigma_0}\sigma_{2-jet} \qquad\qquad (5.34) \\ &= \frac{4\alpha_s}{\pi} C_F \left[\int_0^{y\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_0^1 \frac{\mathrm{d}\cos\theta}{1 - \cos^2\theta} + \int_{y\sqrt{s}}^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_{1-\frac{y\sqrt{s}}{k^0}}^{1} \frac{\mathrm{d}\cos\theta}{1 - \cos^2\theta} - \int_0^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_0^1 \frac{\mathrm{d}\cos\theta}{1 - \cos^2\theta} \right], \end{aligned}$$

where the last term is the virtual contribution of the quark-photon vertex, in the soft-gluon and massless quark approximation. We may combine the first and last terms and obtain,

$$\frac{1}{\sigma_0}\sigma_{2-jet} = \frac{4\alpha_s}{\pi} C_F \left[-\int_{y\sqrt{s}}^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_0^1 \frac{\mathrm{d}\cos\theta}{1 - \cos^2\theta} + \int_{y\sqrt{s}}^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_{1-\frac{y\sqrt{s}}{k^0}}^{1} \frac{\mathrm{d}\cos\theta}{1 - \cos^2\theta} \right], \tag{5.35}$$

and combining these terms,

$$\frac{1}{\sigma_0}\sigma_{2-jet} = -\frac{4\alpha_s}{\pi} C_F \int_{y\sqrt{s}}^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_0^{1-\frac{y\sqrt{s}}{k^0}} \frac{\mathrm{d}\cos\theta}{1 - \cos^2\theta}. \tag{5.36}$$

Now

$$\begin{aligned} \int_0^{1-\frac{y\sqrt{s}}{k^0}} \frac{\mathrm{d}\cos\theta}{1 - \cos^2\theta} &= \frac{1}{2} \int_0^{1-\frac{y\sqrt{s}}{k^0}} \mathrm{d}\cos\theta \left(\frac{1}{1 - \cos\theta} + \frac{1}{1 + \cos\theta} \right) \\ &= -\frac{1}{2} \log\left(\frac{y\sqrt{s}}{k^0} \right) + fin. \end{aligned} \tag{5.37}$$

So,

$$
\begin{aligned}
\frac{1}{\sigma_0}\sigma_{2-jet} &= \frac{2\alpha_s}{\pi} C_F \int_{y\sqrt{s}}^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \left(\log\left(\frac{y\sqrt{s}}{k^0}\right) + fin. \right) \\
&= -\frac{\alpha_s}{\pi} C_F \log^2(y) + \mathcal{O}(\log(y)). \qquad (5.38)
\end{aligned}
$$

Thus, in the soft-gluon approximation, the 2-jet rate is

$$
\sigma_{2-jet} = \sigma_0 \left(1 - \frac{\alpha_s}{\pi} C_F \log^2(y) + \mathcal{O}(\log(y)) + \mathcal{O}(\alpha_s^2) \right). \qquad (5.39)
$$

Since,

$$
\sigma_{tot} = \sigma_{2-jet} + \sigma_{3-jet} + \sigma_{4-jet} + ..., \qquad (5.40)
$$

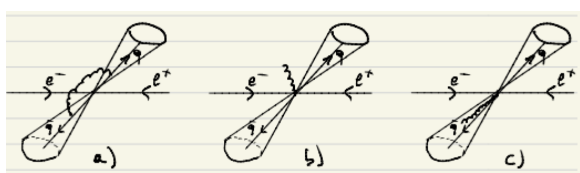
we may get the 3-jet rate by subtraction,

$$
\sigma_{3-jet} = \sigma_0 \frac{\alpha_s}{\pi} C_F \log^2(y) + \mathcal{O}(\log(y)) + \mathcal{O}(\alpha_s^2). \qquad (5.41)
$$

As y becomes smaller, $\sigma_{3-jet} > \sigma_{2-jet}$; this signals that the approximation is not good enough and higher-order terms are necessary. Eventually, as $y \to 0$, the logarithms $\log(y)$ must be resummed.

5.3.1 Sterman-Weinberg Jets

We now give an alternative definition of jets, perhaps less practical, but physically more clear and in fact, historically the first definition of a jet: an event contributes to the 2-jet cross section if we can find two cones of opening angle δ that contain all the energy of the event, excluding at most a fraction ϵ of the total energy (Sterman-Weinberg). We distinguish three contributions,



a) the virtual cross section contributes to the jet cross section, irrespective of the value of ϵ and δ; b) soft-gluon emission contributes to the jet cross section, when the energy of the soft gluon is $k^0 < \epsilon\sqrt{s}$; c) collinear-gluon emission contributes when $k^0 > \epsilon\sqrt{s}$, but the emission angle with respect to the (anti)quark is less than δ.
The contributions are

$$
\sigma_a = -\sigma_0 \frac{2\alpha_s}{\pi} C_F \int_0^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_{-1}^{1} \frac{\mathrm{d}\cos\theta}{1-\cos^2\theta}, \qquad (5.42)
$$

$$
\sigma_b = \sigma_0 \frac{2\alpha_s}{\pi} C_F \int_0^{\epsilon\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_{-1}^{1} \frac{\mathrm{d}\cos\theta}{1-\cos^2\theta}, \qquad (5.43)
$$

$$
\sigma_c = \sigma_0 \frac{2\alpha_s}{\pi} C_F \int_{\epsilon\sqrt{s}}^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \left[\int_{\cos\delta}^{1} \frac{\mathrm{d}\cos\theta}{1-\cos^2\theta} + \int_{-1}^{\cos(\pi-\delta)} \frac{\mathrm{d}\cos\theta}{1-\cos^2\theta} \right]. \qquad (5.44)
$$

Note that for $\epsilon = 1$, or for $\epsilon = 0$ and $\theta = 0, \pi$, $\sigma_b + \sigma_c = -\sigma_a$.

Then the 2-jet cross section is

$$\begin{aligned}\sigma_{2-jet} &= \sigma_0 + \sigma_a + \sigma_b + \sigma_c \\ &= \sigma_0 \left(1 - \frac{2\alpha_s}{\pi} C_F \int_{\epsilon\sqrt{s}}^{\sqrt{s}} \frac{\mathrm{d}k^0}{k^0} \int_{\cos(\pi-\delta)}^{\cos(\delta)} \frac{\mathrm{d}\cos\theta}{1-\cos^2\theta} \right) \\ &= \sigma_0 \left(1 - \frac{4\alpha_s}{\pi} C_F \log(\epsilon) \log\left(\frac{\delta}{2}\right) \right), \end{aligned} \tag{5.45}$$

after expanding $\cos\delta \simeq 1 - \frac{\delta^2}{2} + ...$

As you may have noted, there is an overlapping region in phase space, where the gluon is both soft and collinear. If we regulate the divergences in dimensional regularisation, in $d = 4-2\epsilon$ dimensions, the IR divergences will show up as $1/\epsilon$ poles. In particular, the emission of a gluon which is both soft and collinear will yield a $1/\epsilon^2$ divergence in the phase-space integral, while a gluon which is soft but emitted at a wide angle, or a gluon which is collinear but hard will yield $1/\epsilon$ divergences. As we will see, a very important feature of the IR emissions is that they are **universal**, i.e., they do not depend on the specific scattering where they occur. This is crucial when we compute IR-safe cross sections, and wish to organise the cancellation of the divergences in an efficient way.

5.4 Infrared-safe Cross Sections

In the previous lecture, we have seen that the soft and collinear singularities cancel in the total cross section for $e^+e^- \to$ hadrons. In fact, this is a general feature of the scattering of non-coloured partons, so leptons and photons, provided the production cross section is defined *inclusively*, i.e. the final state F is produced, F being an ensemble of jets and/or heavy particles, allowing though for the production of additional radiation X. We call the inclusive cross sections defined so, **infrared safe**. To be concrete, suppose that we want to consider the inclusive cross section for $e^+e^- \to$ 3 jets. In the measurement, we count in all events with at least 3 jets, which pass our selection criterion, supposing that in any event there is additional undetected radiation. In the computation,

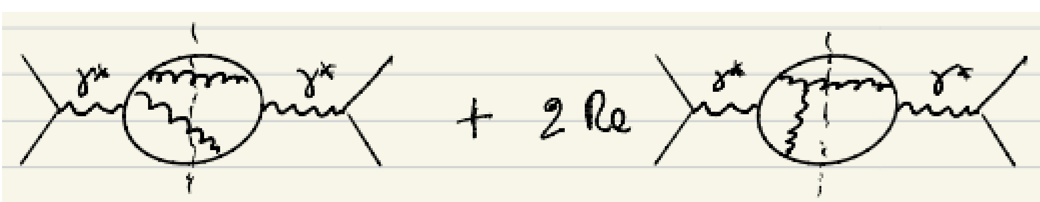


Figure 5.1: $e^+e^- \to 3$ jets: $\mathcal{O}(\alpha_s)$ corrections.

we consider final states with 3 particles that pass our selection criterion and form 3 jets; final states with 4 particles which either cluster into 3 jets or form 4 jets; final states with 5 particles..., and so on. To each order in α_s one must include the appropriate virtual corrections, in order to cancel the IR divergences.

When the scattering includes coloured partons, i.e. quarks or gluons, in the initial state, IR divergences still cancel in the final state for inclusive cross sections. However, mass singularities

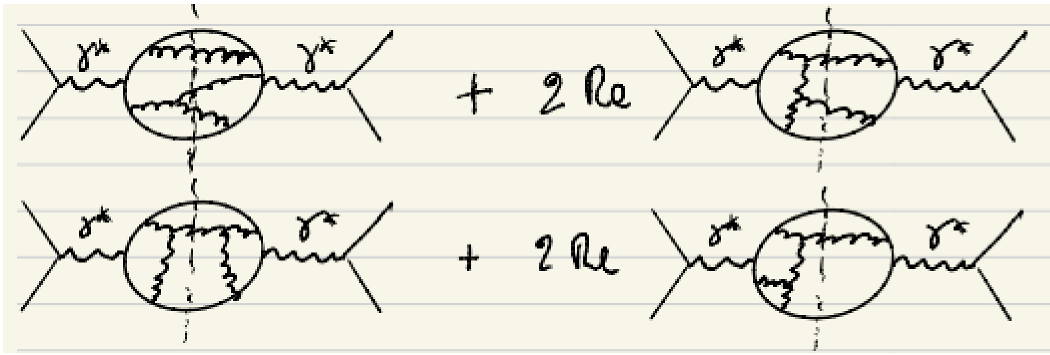


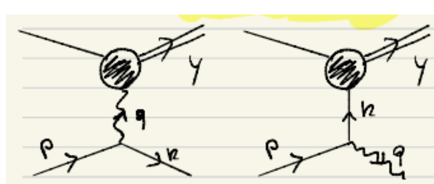
Figure 5.2: $e^+e^- \to 3$ jets: $\mathcal{O}(\alpha_s^2)$ corrections.

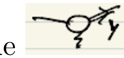
survive, which are associated to the collinear emission from the initial-state partons. We want to consider then the parton evolution associated to the collinear emissions in QCD. As the same pattern of collinear emissions occurs also in massless QED, it is easier to consider it first in this case.

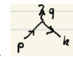
5.5 Parton Evolution in QED

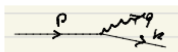
In sec. 4.2, we saw that the quark (electron) self-energy, computed in an on-shell renormalisation scheme, displays a mass logarithmic divergence when the quark (electron) mass vanishes. In sec. 5.1, we saw that the same occurs with the virtual corrections to the quark-gluon (electron-photon) vertex, and with the emission of a soft gluon (photon) out of that vertex. That mass logarithmic divergence, of $\mathcal{O}\left(\alpha \log\left(\frac{q^2}{m^2}\right)\right)$, will yield contributions of $\mathcal{O}\left(\alpha^n \log^n\left(\frac{q^2}{m^2}\right)\right)$ in the case of the emission of n collinear photons. We wish to examine how this happens.

5.5.1 Weiszacker-Williams Photon Approximation



Consider the collinear splitting on an electron on its way to a high momentum-transfer scattering, displayed by the shaded blob. The mass divergence occurs when the exchanged photon or electron is almost on shell, i.e. when the denominator of the exchanged propagator q^2 or k^2 vanishes. Let us see how to deal with the polarisation of the exchanged particle in the numerator of the propagator. In the electron case (diagram on the right-hand side), the numerator of the propagator $\not{k} = \sum_s u^s(k)\bar{u}^s(k)$ will be made of on-shell spinors $u^s(k)$ when $k^2 \to 0$, as we shall see later. In the photon case (diagram on the left-hand side), the numerator of the propagator is given by eq. (3.77), $g_{\mu\nu} = \epsilon^+_\mu \epsilon^{-*}_\nu + \epsilon^-_\mu \epsilon^{+*}_\nu - \sum_i \epsilon^i_{\perp\mu} \epsilon^{i*}_{\perp\nu}$, thus in principle one must take into account all the photon polarisations. But $\epsilon^+_\mu(q) = \dfrac{q^\mu}{\sqrt{2}\,|\vec{q}|}$, so the contraction of $\epsilon^+_\mu(q)$ with the amplitude 

yields zero by the Ward identity, and the contraction of $\epsilon^{+*}_\nu(q)$ with  yields zero by the Dirac equation, $q = p - k \quad \Rightarrow \quad \bar{u}(p)\not{q}u(k) = 0$, thus we can consider only transverse polarisations in the propagator $\dfrac{-ig^{\mu\nu}}{q^2} = \dfrac{i\sum_i \epsilon^i_{\perp\mu}\epsilon^{i*}_{\perp\nu}}{q^2}$, as if the exchanged photon were real.

Consider the vertex , where the photon is virtual, but almost on shell. We take the momentum $p = (p, 0, 0, p)$ for the incoming electron, and the parametrise the momentum of the outgoing electron as $k = ((1-z)p, -\vec{p}_\perp, k^z)$. The requirement that the outgoing electron be on shell $k^2 = 0$ implies that

$$k^z = (1-z)p - \frac{p^2_\perp}{2(1-z)p} + \mathcal{O}(p^4_\perp), \tag{5.46}$$

i.e.,

$$k = \left((1-z)p, -\vec{p}_\perp, (1-z)p - \frac{p^2_\perp}{2(1-z)p}\right). \tag{5.47}$$

By momentum conservation, the photon momentum is

$$q = p - k = \left(zp, \vec{p}_\perp, zp + \frac{p^2_\perp}{2(1-z)p}\right). \tag{5.48}$$

Note that the photon momentum is off-shell by

$$\begin{aligned} q^2 &= -p^2_\perp - \frac{z}{1-z}p^2_\perp + \mathcal{O}(p^4_\perp), \\ &= -\frac{p^2_\perp}{1-z} + \mathcal{O}(p^4_\perp). \end{aligned}$$

Note also that the photon virtuality is space-like, as it is appropriate to an initial-state collinear emission.

On the massless electron line, helicity is conserved, so we may consider the amplitude,

$$i\mathcal{M} = \bar{u}_L(k)(-ie\gamma^\mu)u_L(p)\epsilon^*_{\perp\mu}(q), \tag{5.49}$$

As we have seen in the $\longrightarrow$ Exercises, in a chiral basis $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$, we can write $u_L(k) = \sqrt{k^+}\begin{pmatrix} \xi(k) \\ 0 \end{pmatrix}$, with $\xi(k)$ a 2-spinor $\xi(k) = \begin{pmatrix} -\frac{k^*_\perp}{k^+} \\ 1 \end{pmatrix}$. So

$$\bar{u}_L(k)\gamma^\mu u_L(p) = \sqrt{k^+}\sqrt{p^+}\begin{pmatrix} 0 & \xi^+(k) \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \xi(p) \\ 0 \end{pmatrix}, \tag{5.50}$$

$$= \sqrt{k^+p^+}\xi^+(k)\bar{\sigma}^\mu\xi(p). \tag{5.51}$$

The transverse polarisation has no zero component, so

$$\xi^+(k)\bar{\sigma}^\mu\xi(p)\epsilon^*_{\perp\mu}(q) = \xi^+(k)\sigma^i\xi(p)\epsilon^{i*}_\perp(q). \tag{5.52}$$

So the amplitude is

$$i\mathcal{M} = (-ie)\sqrt{2p}\sqrt{2(1-z)p}\,\xi^+(k)\sigma^i\xi(p)\epsilon_\perp^{i*}(q). \tag{5.53}$$

For our parametrisation,

$$\xi(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \xi(k) = \begin{pmatrix} -\dfrac{p_\perp^*}{2(1-z)p} \\ 1 \end{pmatrix}. \tag{5.54}$$

The transverse polarisation must be such that $\epsilon_\perp(q)\cdot q = 0$ and $\epsilon_\perp^*\cdot\epsilon_\perp = -1$, so we can parametrise the polarisation as

$$\epsilon^\mu_{\perp R,L}(q) = \frac{1}{\sqrt{2}}\left(0, 1, \pm i, \epsilon^z\right). \tag{5.55}$$

Then,

$$\epsilon_{\perp L}\cdot q = 0 \quad\Rightarrow\quad -(p_1 - ip_2) - \epsilon^z(zp + \mathcal{O}(p_\perp^2)) = 0, \tag{5.56}$$

and so,

$$\epsilon^z = -\frac{p_\perp^*}{zp} + \mathcal{O}(p_\perp^3). \tag{5.57}$$

Thus, we get

$$\epsilon^{\mu*}_{\perp L}(q) = \frac{1}{\sqrt{2}}\left(0, 1, i, -\frac{p_\perp}{zp}\right). \tag{5.58}$$

and analogously,

$$\epsilon^{\mu*}_{\perp R}(q) = \frac{1}{\sqrt{2}}\left(0, 1, -i, -\frac{p_\perp^*}{zp}\right). \tag{5.59}$$

Now, let us consider a R-handed photon,

$$\xi^+(k)\sigma^i\xi(p)\epsilon^{i*}_{\perp R}(q) = \frac{1}{\sqrt{2}}\xi^+(k)\left(\sigma^1 - i\sigma^2 - \frac{p_\perp^*}{zp}\sigma^3\right)\xi(p), \tag{5.60}$$

but

$$\sigma^1 - i\sigma^2 = 2\bar{\sigma}_\perp = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \bar{\sigma}_\perp\xi(p) = 0,$$

so,

$$\begin{aligned}\xi^+(k)\sigma^i\xi(p)\epsilon^{i*}_{\perp R}(q) &= -\frac{1}{\sqrt{2}}\frac{p_\perp^*}{zp}\begin{pmatrix} -\dfrac{p_\perp}{2(1-z)p} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{p_\perp^*}{\sqrt{2}zp},\end{aligned} \tag{5.61}$$

Thus,

$$\begin{aligned}\bar{u}_L(k)\gamma^\mu u_L(p)\epsilon^*_{\perp R\mu}(q) &= \sqrt{2p}\sqrt{2(1-z)p}\frac{p_\perp^*}{\sqrt{2}zp} \\ &= \frac{\sqrt{2(1-z)}}{z}p_\perp^*.\end{aligned} \tag{5.62}$$

Finally, the amplitude is

$$i\mathcal{M}(e_L^- \to e_L^-\gamma_R) = -ie\frac{\sqrt{2(1-z)}}{z}p_\perp^*. \tag{5.63}$$

For a L-handed photon,

$$\xi^+(k)\sigma^i\xi(p)\epsilon^{i*}_{\perp L}(q) = \frac{1}{\sqrt{2}}\xi^+(k)\left(\sigma^1 + i\sigma^2 - \frac{p_\perp}{zp}\sigma^3\right)\xi(p), \tag{5.64}$$

with

$$\sigma^1 + i\sigma^2 = 2\sigma_\perp = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So,

$$\left(\sigma^1 + i\sigma^2 - \frac{p_\perp}{zp}\sigma^3\right)\xi(p) = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{p_\perp}{zp}\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \tag{5.65}$$

and thus,

$$\begin{aligned} \xi^+(k)\sigma^i\xi(p)\epsilon^{i*}_{\perp L}(q) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{p_\perp}{2(1-z)p} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ \frac{p_\perp}{zp} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\frac{p_\perp}{z(1-z)p}. \end{aligned} \tag{5.66}$$

So

$$\bar{u}_L(k)\gamma^\mu u_L(p)\epsilon^*_{\perp L\mu}(q) = \sqrt{2p}\sqrt{2(1-z)p}\frac{1}{\sqrt{2}}\frac{p_\perp}{z(1-z)p}, \tag{5.67}$$

and the amplitude is

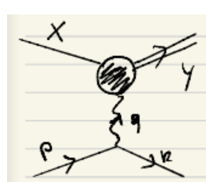
$$i\mathcal{M}(e^-_L \to e^-_L\gamma_L) = -ie\frac{\sqrt{2}}{z\sqrt{1-z}}p_\perp\,. \tag{5.68}$$

By parity invariance, we obtain the same result if all the helicities are reversed. The squared amplitude, averaged over initial helicities, is

$$\frac{1}{2}\sum_{hel}\left|\mathcal{M}(e^- \to e^-\gamma)\right|^2 = \frac{2e^2p_\perp^2}{z(1-z)}\frac{1+(1-z)^2}{z}, \tag{5.69}$$

where the factor 1 in the numerator of the second fraction comes from $e^-_L \to e^-_L\gamma_L$ and $e^-_R \to e^-_R\gamma_R$, while the $(1-z)^2$ factor comes from $e^-_L \to e^-_L\gamma_R$ and $e^-_R \to e^-_R\gamma_L$.

Now we can compute the cross section for the scattering $eX \to eY$. Let us assume that X be massless, and take its momentum as $p_X = (E_X, 0, 0, -E_X)$. Then



$$\sigma(eX \to eY) = \frac{1}{2s}\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\int \mathrm{d}\Pi_Y|\mathcal{M}(eX \to eY)|^2, \tag{5.70}$$

where $s = 2p\cdot p_X = 4pE_X$, and where $\mathrm{d}\Pi_Y$ is short-hand for the Y phase space,

$$\mathrm{d}\Pi_Y \equiv \prod_i \frac{\mathrm{d}^3k_{y_i}}{(2\pi)^3 2E_{y_i}}(2\pi)^4\delta^4\left(\sum_i P_{y_i} + k - p_X - p\right). \tag{5.71}$$

The amplitude in eq. (5.70) factorises as

$$|\mathcal{M}(eX \to eY)|^2 = \frac{1}{2}\sum_{hel} |\mathcal{M}(e^- \to e^-\gamma)|^2 \frac{1}{(q^2)^2} |\mathcal{M}(\gamma X \to Y)|^2. \tag{5.72}$$

Now, $\mathrm{d}^3k = \mathrm{d}^2p_\perp\, p\, \mathrm{d}z = \pi p\, \mathrm{d}p_\perp^2\, \mathrm{d}z$, and we can spell out σ,

$$\sigma(eX \to eY) = \frac{1}{2\cdot 4pE_X} \int \frac{\pi p\, \mathrm{d}z\, \mathrm{d}p_\perp^2}{16\pi^3(1-z)p} \left[\frac{1}{2}\sum_{hel} |\mathcal{M}(e^- \to e^-\gamma)|^2\right] \frac{(1-z)^2}{(p_\perp^2)^2} \int \mathrm{d}\Pi_Y |\mathcal{M}(\gamma X \to Y)|^2. \tag{5.73}$$

Note that $s_{\gamma X} = 2q\cdot p_X + \mathcal{O}(p_\perp^2) = 4pE_X z + \mathcal{O}(p_\perp^2)$. Neglecting the $\mathcal{O}(p_\perp^2)$ contributions, we can set

$$\sigma(\gamma X \to Y) = \frac{1}{2s_{\gamma X}} \int \mathrm{d}\Pi_Y |\mathcal{M}(\gamma X \to Y)|^2, \tag{5.74}$$

such that

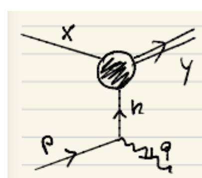
$$\sigma(eX \to eY) = \int \frac{\mathrm{d}z\, \mathrm{d}p_\perp^2}{16\pi^2} \frac{z(1-z)}{(p_\perp^2)^2} \left[\frac{1}{2}\sum_{hel} |\mathcal{M}(e^- \to e^-\gamma)|^2\right] \sigma(\gamma X \to Y). \tag{5.75}$$

Now we can substitute the amplitude and we obtain

$$\sigma(eX \to eY) = \frac{\alpha}{2\pi} \int_{m^2}^{s} \frac{\mathrm{d}p_\perp^2}{p_\perp^2} \int_0^1 \mathrm{d}z \frac{1+(1-z)^2}{z} \sigma(\gamma X \to Y), \tag{5.76}$$

which exhibits a logarithmic mass singularity. This is also known as **Weizsäcker-Williams** (WW) photon **approximation**.

The cross section for the $eX \to \gamma Y$ scattering can be computed accordingly.



Parametrising the momenta of the incoming electron and of the X state as for the $eX \to eY$ scattering, we take the photon momentum as

$$q = (zp, \vec{p}_\perp, q^z), \tag{5.77}$$

and the on-shell condition, $q^2 = 0$, yields

$$q = \left(zp, \vec{p}_\perp, zp - \frac{p_\perp^2}{2zp}\right). \tag{5.78}$$

By momentum conservation, the outgoing electron momentum is

$$k = p - q = \left((1-z)p, -\vec{p}_\perp, (1-z)p + \frac{p_\perp^2}{2zp}\right), \tag{5.79}$$

which is slightly off-shell,

$$k^2 = -\frac{p_\perp^2}{z} + \mathcal{O}(p_\perp^4). \tag{5.80}$$

However, to lowest order, the spinor $\xi(k)$ and the transverse polarisation $\epsilon_\perp(q)$ do not change, so the amplitudes $\mathcal{M}(e^-_L \to e^-_L\gamma_R)$ and $\mathcal{M}(e^-_L \to e^-_L\gamma_L)$ are the same as the ones computed in eqs. (5.63) and (5.68) for $\sigma(eX \to eY)$. The cross section is

$$\sigma(eX \to \gamma Y) = \frac{1}{2\cdot 4pE_X}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q^0}\int \mathrm{d}\Pi_Y |\mathcal{M}(eX \to \gamma Y)|^2, \tag{5.81}$$

with

$$|\mathcal{M}(eX \to \gamma Y)|^2 = \frac{1}{2}\sum_{hel}|\mathcal{M}(e^- \to e^-\gamma)|^2 \frac{1}{(k^2)^2}|\mathcal{M}(eX \to Y)|^2. \tag{5.82}$$

Note that $s_{eX} = 2k\cdot p_X + \mathcal{O}(p_\perp^2) = 4(1-z)pE_X + \mathcal{O}(p_\perp^2)$. Neglecting the $\mathcal{O}(p_\perp^2)$ contributions, we can set

$$\sigma(eX \to Y) = \frac{1}{2s_{eX}}\int \mathrm{d}\Pi_Y |\mathcal{M}(eX \to Y)|^2, \tag{5.83}$$

such that

$$\sigma(eX \to \gamma Y) = \int \frac{\pi p\,\mathrm{d}z\,\mathrm{d}p_\perp^2}{16\pi^3 zp}\frac{1}{2}\sum_{hel}|\mathcal{M}(e^- \to e^-\gamma)|^2\frac{z^2}{(p_\perp^2)^2}(1-z)\sigma(eX \to Y). \tag{5.84}$$

We substitute the amplitude, and obtain the same result as for the WW photon approximation,

$$\sigma(eX \to \gamma Y) = \frac{\alpha}{2\pi}\int_{m^2}^{s}\frac{\mathrm{d}p_\perp^2}{p_\perp^2}\int_0^1 \mathrm{d}z\frac{1+(1-z)^2}{z}\sigma(eX \to Y). \tag{5.85}$$

Because $(1-z)$ is the momentum fraction carried by the electron, we set $1-z=x$ and we may interpret,

$$f_e^{(1)}(x) = \frac{\alpha}{2\pi}\log\left(\frac{s}{m^2}\right)\frac{1+x^2}{1-x}, \tag{5.86}$$

as the distribution of an electron within an electron. At zeroth order, one has only the electron which has not interacted, $f_e^{(0)}(x) = \delta(1-x)$, so we would write the distribution as

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi}\log\left(\frac{s}{m^2}\right)\frac{1+x^2}{1-x}. \tag{5.87}$$

However, this expression is divergent as $x \to 1$, which corresponds to the emission of soft photons, and does not conserve probability, which is expressed by

$$\int_0^1 \mathrm{d}x f_e(x) = 1. \tag{5.88}$$

We lack the contribution of the virtual photons; they must contribute a piece with the same final state as the Born term, and yield a negative contribution, which lowers probability. Thus we write

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi}\log\left(\frac{s}{m^2}\right)\left(\frac{1+x^2}{1-x} - A\delta(1-x)\right). \tag{5.89}$$

We must choose A by requiring that $\int_0^1 \mathrm{d}x f_e(x) = 1$. However, in $\displaystyle\int_0^1 \mathrm{d}x\frac{1+x^2}{1-x}$ we would still have a divergence at $x=1$, so we must define the integrand as a distribution. Namely, for a smooth function $f(x)$ we require that

$$\int_0^1 \mathrm{d}x\frac{f(x)}{(1-x)_+} \equiv \int_0^1 \mathrm{d}x\frac{f(x)-f(1)}{1-x}. \tag{5.90}$$

Then

$$\int \mathrm{d}x \frac{1+x^2}{(1-x)_+} = \int_0^1 \mathrm{d}x \frac{x^2-1}{1-x} = -\int_0^1 \mathrm{d}x(1+x) = -\frac{3}{2}, \tag{5.91}$$

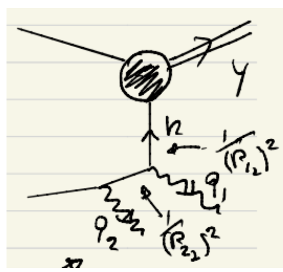
and the requirement that $\int_0^1 \mathrm{d}x f_e(x) = 1$ yields $A = -\frac{3}{2}$, so the electron distribution is

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi} \log\left(\frac{s}{m^2}\right)\left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right), \tag{5.92}$$

which is properly normalised, but still singular as $x \to 1$. We may expect then that the region $x \to 1$ is highly sensitive to the emission of many collinear photons.

5.5.2 Splitting Functions in QED

We may consider then a double emission, and we suppose that $p_{2\perp} \ll p_{1\perp}$.



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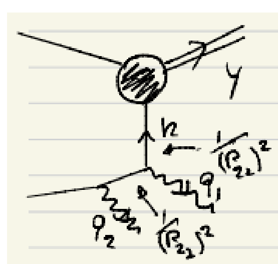
Then the relevant denominators are displayed in the diagram above. From the integral over the transverse momenta, we obtain

$$\left(\frac{\alpha}{2\pi}\right)^2 \int_{m^2}^{s} \frac{\mathrm{d}p_{1\perp}^2}{p_{1\perp}^2} \int_{m^2}^{p_{1\perp}^2} \frac{\mathrm{d}p_{2\perp}^2}{p_{2\perp}^2} = \frac{1}{2}\left(\frac{\alpha}{2\pi}\right)^2 \log^2\left(\frac{s}{m^2}\right). \tag{5.93}$$

If conversely, $p_{1\perp} \ll p_{2\perp}$,

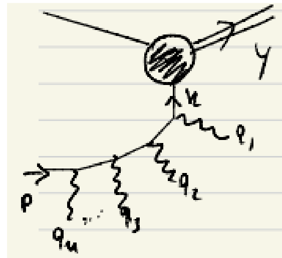


,

we obtain

$$\left(\frac{\alpha}{2\pi}\right)^2 \int_{m^2}^{s} \frac{\mathrm{d}p_{2\perp}^2}{p_{2\perp}^2} \int_{0}^{p_{2\perp}^2} \frac{\mathrm{d}p_{1\perp}^2}{p_{2\perp}^2} = \left(\frac{\alpha}{2\pi}\right)^2 \log\left(\frac{s}{m^2}\right). \tag{5.94}$$

We see that the double logarithm, and the largest contribution, corresponds to a strong ordering of the momenta and so of the virtualities. This can be generalised to a multiple emission,



,

where $p_{1\perp} \gg p_{2\perp} \gg ... \gg p_{n\perp}$, which yields the contribution,

$$\frac{1}{n!}\left(\frac{\alpha}{2\pi}\right)^n \log^n\left(\frac{s}{m^2}\right). \tag{5.95}$$

We may think of probing the electron distribution at larger and larger momenta, and thus at smaller and smaller distances, since $\Delta r \sim p_\perp^{-1}$ and so we may of probing how the distribution $f_e(x, Q^2)$ changes as we increase Q^2.

Analogously, in the $eX \to eY$ scattering, we may interpret

$$\frac{\alpha}{2\pi}\frac{\mathrm{d}p_\perp^2}{p_\perp^2}\frac{1+(1-z)^2}{z}, \tag{5.96}$$

as the probability for an electron to emit a photon, which carries a fraction z of its momentum, such that the photon distribution is given by

$$\begin{aligned} f_\gamma(x, Q^2 + \Delta Q^2) &= f_\gamma(x, Q^2) + \frac{\alpha}{2\pi}\frac{\Delta Q^2}{Q^2}\int_0^1 \mathrm{d}x' \int_0^1 \mathrm{d}z P^{(0)}_{\gamma\leftarrow e}(z) f_e(x', Q^2)\delta(x'z - x) \\ &= f_\gamma(x, Q^2) + \frac{\alpha}{2\pi}\frac{\Delta Q^2}{Q^2}\int_x^1 \frac{\mathrm{d}z}{z} P^{(0)}_{\gamma\leftarrow e}(z) f_e\left(\frac{x}{z}, Q^2\right), \end{aligned} \tag{5.97}$$

where we introduced the splitting function,

$$P_{\gamma\leftarrow e}(z) = \frac{\alpha}{2\pi}P^{(0)}_{\gamma\leftarrow e}(z) \qquad \text{with} \qquad P^{(0)}_{\gamma\leftarrow e}(z) = \frac{1+(1-z)^2}{z}. \tag{5.98}$$

Note that we integrate over the momentum fraction x' of the electron distribution and the momentum fraction z of the photon emission, with the constraint that the product $x'z$ must yield the momentum fraction x of the photon distribution.

We say that a function f is the **convolution**,

$$f = f_1 \otimes f_2, \tag{5.99}$$

of the functions f_1 and f_2 if

$$f(x) = \int \mathrm{d}x_1\, \mathrm{d}x_2 f_1(x_1) f_2(x_2)\delta(x - x_1 x_2). \tag{5.100}$$

Then the photon distribution is a convolution,

$$f_\gamma(Q^2 + \Delta Q^2) = f_\gamma(Q^2) + \frac{\alpha}{2\pi}\frac{\Delta Q^2}{Q^2}P^{(0)}_{\gamma\leftarrow e} \otimes f_e. \tag{5.101}$$

We may differentiate the equation for the photon distribution (5.97) and get an integral-differential equation,

$$Q^2 \frac{\partial f_\gamma(x,Q^2)}{\partial Q^2} = \frac{\alpha}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} P^{(0)}_{\gamma\leftarrow e}(z) f_e\left(\frac{x}{z}, Q^2\right), \tag{5.102}$$

which describes the evolution in Q^2 of the photon distribution. As a convolution,

$$Q^2 \frac{\partial f_\gamma(Q^2)}{\partial Q^2} = \frac{\alpha}{2\pi} P^{(0)}_{\gamma\leftarrow e} \otimes f_e. \tag{5.103}$$

The convolution notation is often very convenient for formal manipulations.

Likewise, in the scattering $eX \to \gamma Y$ that we examined previously, we can write an evolution equation for the electron distribution,

$$Q^2 \frac{\partial f_e(x,Q^2)}{\partial Q^2} = \frac{\alpha}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} P^{(0)}_{e\leftarrow e}(z) f_e\left(\frac{x}{z}, Q^2\right), \tag{5.104}$$

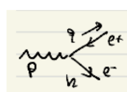
with splitting function,

$$P^{(0)}_{e\leftarrow e}(z) = \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x). \tag{5.105}$$

The initial conditions on the distributions above may be given requiring that

$$\begin{cases} f_e(x,m^2) = \delta(1-x), \\ f_\gamma(x,m^2) = 0. \end{cases} \tag{5.106}$$

To complete the **evolution equations** we take the splitting of a photon into an e^+e^- pair,



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We parametrise the momenta as $p = (p, 0, 0, p)$ and $k = ((1-z)p, -p_\perp, k^z)$. The requirement that the outgoing electron is on shell, $k^2 = 0$, implies that

$$k = \left((1-z)p, -p_\perp, (1-z)p - \frac{p_\perp^2}{2(1-z)p}\right), \tag{5.107}$$

such that the exchanged propagator of momentum,

$$q = p - k = \left(zp, p_\perp, zp + \frac{p_\perp^2}{2(1-z)p}\right), \tag{5.108}$$

has virtuality,

$$q^2 = -\frac{p_\perp^2}{1-z}. \tag{5.109}$$

We must compute the amplitude,

$$i\mathcal{M} = \bar{u}_L(k)(-ie\gamma^\mu)v_L(p)\epsilon_{\perp\mu}(q), \tag{5.110}$$

with spinors,

$$u_L(k) = \sqrt{k^+} \begin{pmatrix} \xi(k) \\ 0 \end{pmatrix} \qquad \text{with} \qquad \xi(k) = \begin{pmatrix} -\dfrac{k_\perp^*}{k^+} \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{p_\perp^*}{2(1-z)p} \\ 1 \end{pmatrix}, \tag{5.111}$$

$$v_L(q) = \sqrt{q^+} \begin{pmatrix} \xi^*(k) \\ 0 \end{pmatrix} \qquad \text{with} \qquad \xi(q) = \begin{pmatrix} -\dfrac{q_\perp^*}{q^+} \\ 1 \end{pmatrix} = \begin{pmatrix} -\dfrac{p_\perp^*}{2zp} \\ 1 \end{pmatrix}. \tag{5.112}$$

Note that

$$\begin{aligned} \bar{u}_L(k)\gamma^\mu v_L(p)\epsilon_{\perp\mu}(q) &= \sqrt{2zp}\sqrt{2(1-z)p}\,\xi^+(q)\bar{\sigma}^\mu\xi(k)\epsilon_{\perp\mu}(p) \\ &= \sqrt{2zp}\sqrt{2(1-z)p}\,\xi^+(q)\sigma^i\xi(k)\epsilon_\perp^i(p). \end{aligned} \tag{5.113}$$

For a L-handed photon, $\epsilon_{\perp L}^\mu(p) = \frac{1}{\sqrt{2}}(0,1,-i,0)$, so

$$\begin{aligned} \xi^+(q)\sigma^i\xi(k)\epsilon_{\perp L}^i(p) &= \frac{1}{\sqrt{2}}\xi^+(q)(\sigma^1 - i\sigma^2)\xi(k) \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\dfrac{p_\perp}{2zp} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{p_\perp^*}{2(1-z)p} \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \frac{p_\perp^*}{(1-z)p}. \end{aligned} \tag{5.114}$$

For a R-handed photon, $\epsilon_{\perp R}^\mu(p) = \frac{1}{\sqrt{2}}(0,1,i,0)$, one gets

$$\xi^+(q)\sigma^i\xi(k)\epsilon_{\perp R}^i(p) = -\frac{1}{\sqrt{2}}\frac{p_\perp}{zp}, \tag{5.115}$$

and we obtain the amplitudes,

$$i\mathcal{M}(\gamma_L \to e_L^- e_R^+) = -ie\frac{\sqrt{2z(1-z)}}{1-z}p_\perp^*, \tag{5.116}$$

$$i\mathcal{M}(\gamma_R \to e_L^- e_R^+) = ie\frac{\sqrt{2z(1-z)}}{z}p_\perp. \tag{5.117}$$

By parity invariance, we obtain the same result if all the helicities are reversed. The squared amplitude, averaged over initial helicities, is

$$\frac{1}{2}\sum_{hel}|\mathcal{M}(\gamma \to e^- e^+)|^2 = \frac{2e^2p_\perp^2}{z(1-z)}(z^2 + (1-z)^2), \tag{5.118}$$

and repeating the same procedure as above, we may interpret it in terms of a splitting function,

$$P_{e\leftarrow\gamma}(z,\alpha) = \frac{\alpha}{2\pi}P_{e\leftarrow\gamma}^{(0)}(z) \qquad \text{with} \qquad P_{e\leftarrow\gamma}^{(0)}(z) = z^2 + (1-z)^2. \tag{5.119}$$

In order to conserve probability, we must consider, though, also the option of the photon not to split,

$$\int_0^1 \mathrm{d}z \left[z^2 + (1-z)^2\right] = \frac{2}{3}, \tag{5.120}$$

and we obtain the splitting function,

$$P_{\gamma\leftarrow\gamma}(z,\alpha)=\frac{\alpha}{2\pi}P^{(0)}_{\gamma\leftarrow\gamma}(z)\qquad\text{with}\qquad P^{(0)}_{\gamma\leftarrow\gamma}(z)=-\frac{2}{3}\delta(1-z)\,. \tag{5.121}$$

This completes the set of **QED splitting functions**,

$$P^{(0)}_{e\leftarrow e}(z)=\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z), \tag{5.122}$$

$$P^{(0)}_{\gamma\leftarrow e}(z)=\frac{1+(1-z)^2}{z}, \tag{5.123}$$

$$P^{(0)}_{e\leftarrow\gamma}(z)=z^2+(1-z)^2, \tag{5.124}$$

$$P^{(0)}_{\gamma\leftarrow\gamma}(z)=-\frac{2}{3}\delta(1-z), \tag{5.125}$$

of the **Gribov-Lipatov evolution equations**,

$$Q^2\frac{\partial f_e(x,Q^2)}{\partial Q^2}=\frac{\alpha}{2\pi}\int_x^1\frac{\mathrm{d}z}{z}\left[P^{(0)}_{e\leftarrow e}(z)f_e\left(\frac{x}{z},Q^2\right)+P^{(0)}_{e\leftarrow\gamma}(z)f_\gamma\left(\frac{x}{z},Q^2\right)\right]\,, \tag{5.126}$$

$$Q^2\frac{\partial f_\gamma(x,Q^2)}{\partial Q^2}=\frac{\alpha}{2\pi}\int_x^1\frac{\mathrm{d}z}{z}\left\{P^{(0)}_{\gamma\leftarrow\gamma}(z)f_\gamma\left(\frac{x}{z},Q^2\right)+P^{(0)}_{\gamma\leftarrow e}(z)\left[f_e\left(\frac{x}{z},Q^2\right)+f_{\bar e}\left(\frac{x}{z},Q^2\right)\right]\right\}, \tag{5.127}$$

$$\tag{5.128}$$

where, the splitting functions being even in the electric charge, we used the fact that the splitting functions for the electron and the positron are the same, $P_{\gamma\leftarrow\bar e}(z,\alpha)=P_{\gamma\leftarrow e}(z,\alpha)$. Further, the evolution equation for the positron has the same functional form as for the electron,

$$Q^2\frac{\partial f_{\bar e}(x,Q^2)}{\partial Q^2}=\frac{\alpha}{2\pi}\int_x^1\frac{\mathrm{d}z}{z}\left[P^{(0)}_{e\leftarrow e}(z)f_{\bar e}\left(\frac{x}{z},Q^2\right)+P^{(0)}_{e\leftarrow\gamma}(z)f_\gamma\left(\frac{x}{z},Q^2\right)\right]\,, \tag{5.129}$$

where we used the charge parity,

$$P_{\bar e\leftarrow\bar e}(z,\alpha)=P_{e\leftarrow e}(z,\alpha), \tag{5.130}$$

$$P_{\bar e\leftarrow\gamma}(z,\alpha)=P_{e\leftarrow\gamma}(z,\alpha)\,. \tag{5.131}$$

When integrated, the Gribov-Lipatov evolution equations yield the electron, positron and photon distributions as a function of Q^2, with $f_i(x,m^2)$ with $i=e,\bar e,\gamma$ as initial conditions.

Writing the evolution equations as

$$\begin{aligned}&f_e(x,Q^2+\Delta Q^2) &(5.132)\\&\quad=f_e(x,Q^2)+\frac{\alpha}{2\pi}\frac{\Delta Q^2}{Q^2}\int_x^1\frac{\mathrm{d}z}{z}\left[P^{(0)}_{e\leftarrow e}(z)f_e\left(\frac{x}{z},Q^2\right)+P^{(0)}_{e\leftarrow\gamma}(z)f_\gamma\left(\frac{x}{z},Q^2\right)\right]\,,\end{aligned}$$

$$\begin{aligned}&f_{\bar e}(x,Q^2+\Delta Q^2) &(5.133)\\&\quad=f_{\bar e}(x,Q^2)+\frac{\alpha}{2\pi}\frac{\Delta Q^2}{Q^2}\int_x^1\frac{\mathrm{d}z}{z}\left[P^{(0)}_{e\leftarrow e}(z)f_{\bar e}\left(\frac{x}{z},Q^2\right)+P^{(0)}_{e\leftarrow\gamma}(z)f_\gamma\left(\frac{x}{z},Q^2\right)\right]\,,\end{aligned}$$

$$\begin{aligned}&f_\gamma(x,Q^2+\Delta Q^2) &(5.134)\\&\quad=f_\gamma(x,Q^2)+\frac{\alpha}{2\pi}\frac{\Delta Q^2}{Q^2}\int_x^1\frac{\mathrm{d}z}{z}\left\{P^{(0)}_{\gamma\leftarrow\gamma}(z)f_\gamma\left(\frac{x}{z},Q^2\right)+P^{(0)}_{\gamma\leftarrow e}(z)\left[f_e\left(\frac{x}{z},Q^2\right)+f_{\bar e}\left(\frac{x}{z},Q^2\right)\right]\right\},\end{aligned}$$

we find that using the conservation of probability,

$$\int_0^1 \mathrm{d}z P^{(0)}_{e\leftarrow e}(z) = 0\,, \tag{5.135}$$

with $P^{(0)}_{e\leftarrow e}(z)$ as in eq. (5.122), the net electron number is conserved,

$$\int_0^1 \mathrm{d}x \left[f_e(x,Q^2) - f_{\bar{e}}(x,Q^2)\right] = \text{const.}\,, \tag{5.136}$$

i.e. the integral (5.136) does not depend on Q^2. Likewise, we find that the total momentum is conserved, i.e. it does not depend on Q^2,

$$\int_0^1 \mathrm{d}x\, x \left[f_e(x,Q^2) + f_{\bar{e}}(x,Q^2) + f_\gamma(x,Q^2)\right] = \text{const.}\,, \tag{5.137}$$

provided that (please check!)

$$\int_0^1 \mathrm{d}z\, z \left(P^{(0)}_{e\leftarrow e}(z) + P^{(0)}_{\gamma\leftarrow e}(z)\right) = 0\,, \tag{5.138}$$

$$\int_0^1 \mathrm{d}z\, z \left(P^{(0)}_{\gamma\leftarrow \gamma}(z) + 2P^{(0)}_{e\leftarrow \gamma}(z)\right) = 0\,. \tag{5.139}$$

Note that in the splitting beyond the leading order, also the $P_{e\leftarrow\bar{e}}(z,\alpha)$ splitting function can occur, with $P_{e\leftarrow\bar{e}}(z,\alpha) = P_{\bar{e}\leftarrow e}(z,\alpha)$ by charge parity. Then the Gribov-Lipatov equations for the electron (5.126) and the positron (5.129) must be generalised accordingly (see Tutorial).

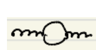
5.6 Parton Evolution in QCD

5.6.1 Splitting Functions in QCD

The procedure displayed above for the evolution equations in QED can be followed likewise for the evolution equations in QCD, but for the fact that one cannot simply get rid of the light-cone polarisations in the gluon propagator through the Ward identity, so it is best to use a physical gauge. The outcome for the splitting functions is then the same as in QED, if one includes the colour factors. From the quark self-energy, we know that

$$\rightarrow \quad \sum_a \left(t^a t^a\right)_{ij} = C_F \delta_{ij}. \tag{5.140}$$

Thus, summing (averaging) over the final (initial) colours, the colour factor for the splitting is C_F. Analogously, from the gluon self-energy,



$$\rightarrow \quad \mathrm{Tr}\left(t^a t^b\right) = T_F \delta^{ab}. \tag{5.141}$$

So summing (averaging) over the final (initial) colours, the colour factor for the splitting is T_F. Thus, we obtain the **QCD splitting functions**,

$$P_{qq}^{(0)}(z) = C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right), \tag{5.142}$$

$$P_{gq}^{(0)}(z) = C_F\frac{1+(1-z)^2}{z}, \tag{5.143}$$

$$P_{qg}^{(0)}(z) = T_F\left[z^2+(1-z)^2\right]. \tag{5.144}$$

Like in QED, we must consider the case for which the gluon does not split into a $q\bar{q}$ pair, in order to conserve probability. Thus, we obtain

$$P_{gg}^{f(0)} = -\frac{2}{3}T_F n_f\delta(1-z), \tag{5.145}$$

where we have considered the splitting of the gluon into n_f quark flavours. But the gluon can also emit gluons, so the full P_{gg} splitting function is (see Tutorial)

$$P_{gg}^{(0)}(z) = 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \left(\frac{11}{6}C_A - \frac{2}{3}T_F n_f\right)\delta(1-z)\,. \tag{5.146}$$

Because of charge conjugation invariance and SU(n_f) flavour symmetry, the splitting functions P_{qg} and P_{gq} are independent of the quark flavour and are the same for quarks and antiquarks,

$$P_{q_i g} = P_{\bar{q}_i g} = P_{qg}, \tag{5.147}$$

$$P_{g q_i} = P_{g\bar{q}_i} = P_{gq}. \tag{5.148}$$

If we write the splitting functions as expansions in α_s,

$$P_{q_i g}(z) = \frac{\alpha_s}{2\pi}P_{q_i g}^{(0)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{q_i g}^{(1)}(z) + ..., \tag{5.149}$$

these properties are true to all orders in α_s.

The splitting function $P_{q_i q_j}$ is diagonal only to leading order,

$$P_{q_i q_j}(z) = \frac{\alpha_s}{2\pi}\delta_{ij}P_{qq}^{(0)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{q_i q_j}^{(1)}(z) + ... \tag{5.150}$$

Further, beyond leading order also the splitting function $P_{q_i\bar{q}_j}$ appears. However, because of charge conjugation invariance,

$$P_{q_i q_j} = P_{\bar{q}_i\bar{q}_j}, \tag{5.151}$$

$$P_{q_i\bar{q}_j} = P_{\bar{q}_i q_j}, \tag{5.152}$$

hold to all order in α_s.

5.6.2 DGLAP Evolution Equations in QCD

The splitting functions drive the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi **(DGLAP) evolution equations**, which to leading order in α_s are

$$Q^2 \frac{\partial q(x,Q^2)}{\partial Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} \left[P_{qq}^{(0)}(z) q\left(\frac{x}{z}, Q^2\right) + P_{qg}^{(0)}(z) g\left(\frac{x}{z}, Q^2\right) \right], \tag{5.153}$$

$$Q^2 \frac{\partial g(x,Q^2)}{\partial Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} \left\{ P_{gg}^{(0)}(z) g\left(\frac{x}{z}, Q^2\right) + P_{gq}^{(0)}(z) \sum_q \left[q\left(\frac{x}{z}, Q^2\right) + \bar{q}\left(\frac{x}{z}, Q^2\right) \right] \right\}, \tag{5.154}$$

where the sum is over the quark flavours and for brevity we fix $f_q(x) \equiv q(x)$ and $f_g(x) \equiv g(x)$. Just like for the positron in QED, the evolution equation for the antiquark has the same functional form as for the quark,

$$Q^2 \frac{\partial \bar{q}(x,Q^2)}{\partial Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} \left[P_{qq}^{(0)}(z) \bar{q}\left(\frac{x}{z}, Q^2\right) + P_{qg}^{(0)}(z) g\left(\frac{x}{z}, Q^2\right) \right]. \tag{5.155}$$

These are $(2n_f + 1)$ evolution equations, where n_f is the number of quark flavours. Of course, they may also be grouped in matrix form, in the $(2n_f + 1)$-dimensional space of quarks, antiquarks and gluons. That is more convenient when one goes beyond the leading order,

$$Q^2 \frac{\partial}{\partial Q^2} \begin{pmatrix} \{q_i(x,Q^2)\} \\ \{\bar{q}_i(x,Q^2)\} \\ g(x,Q^2) \end{pmatrix} = \int_x^1 \frac{\mathrm{d}z}{z} \begin{pmatrix} P_{q_i q_j} & P_{q_i \bar{q}_j} & P_{q_i g} \\ P_{\bar{q}_i q_j} & P_{\bar{q}_i \bar{q}_j} & P_{\bar{q}_i g} \\ P_{g q_i} & P_{g \bar{q}_i} & P_{gg} \end{pmatrix} \begin{pmatrix} \{q_j\left(x/z, Q^2\right)\} \\ \{\bar{q}_j\left(x/z, Q^2\right)\} \\ g\left(x/z, Q^2\right) \end{pmatrix}, \tag{5.156}$$

where, using the charge conjugation invariance and SU(n_f) flavour symmetry, the splitting function matrix simplifies to

$$\begin{pmatrix} P_{q_i q_j} & P_{q_i \bar{q}_j} & P_{qg} \\ P_{q_i \bar{q}_j} & P_{q_i q_j} & P_{qg} \\ P_{gq} & P_{gq} & P_{gg} \end{pmatrix}. \tag{5.157}$$

Just like the RGE for the β function yields how the coupling evolves with a change of scale, the DGLAP equations tell us how the parton distributions evolve with a change of scale, and together with the RGE for the β function they are the most important equations in QCD.

Just like in QED at leading order, the net quark number is conserved,

$$\int_0^1 \mathrm{d}x \left[q(x,Q^2) - \bar{q}(x,Q^2) \right] = \text{const.}, \tag{5.158}$$

since

$$\int_0^1 \mathrm{d}z\, P_{qq}^{(0)}(z) = 0, \tag{5.159}$$

while the total momentum is conserved,

$$\int_0^1 \mathrm{d}x\, x \left[\sum_{i=1}^{n_f} \left(q_i(x,Q^2) + \bar{q}_i(x,Q^2) \right) + g(x,Q^2) \right] = \text{const.}, \tag{5.160}$$

provided that (please check!)

$$\int_0^1 \mathrm{d}z\, z \left(P_{qq}^{(0)}(z) + P_{gq}^{(0)}(z) \right) = 0, \tag{5.161}$$

$$\int_0^1 \mathrm{d}z\, z \left(P_{gg}^{(0)}(z) + 2n_f P_{qg}^{(0)}(z) \right) = 0. \tag{5.162}$$

5.6.3 Factorisation of Collinear Singularities

Just like for the electron distribution in QED,

$$f_e^{(1)}(x) = \delta(1-x) + \log\left(\frac{s}{m^2}\right) P_{e\leftarrow e}(x,\alpha), \tag{5.163}$$

we may write the quark distributions as

$$q^{(1)}(x) = \delta(1-x) + \log\left(\frac{Q^2}{\Lambda^2}\right) P_{qq}(x,\alpha_s), \tag{5.164}$$

where for the lower limit of integration in $k_\perp^2$, we use an infrared cut-off Λ^2 instead of the electron mass. As we know, the cut-off Λ^2 signals that in $\int \mathrm{d}p_\perp^2/p_\perp^2$ we have collinear divergences as $p_\perp^2 \to 0$, and we know that we cannot get rid of those divergences combining real and virtual contributions.

In sec. 2.6, we wrote the DIS cross section as,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}y} = 2\pi\alpha^2 \frac{s}{(Q^2)^2}\{\left[1+(1-y^2)\right] F_2(x) - y^2 F_L(x)\}, \tag{5.165}$$

where the structure function $F_2(x)$ was given by

$$F_2(x) = \sum_q x\, e_q^2\, q(x). \tag{5.166}$$

From the DGLAP equation, the evolution of the quark distribution is

$$Q^2 \frac{\partial q(x,Q^2)}{\partial Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}\zeta}{\zeta} P_{qq}^{(0)}\left(\frac{x}{\zeta}\right) q(\zeta,Q^2), \tag{5.167}$$

where we used the variable change $\zeta = x/z$ in order to swap the variables of P_{qq}, and f_q and we neglected the $P_{qg}f_g$ contribution, so we get a homogeneous equation in $q(x,Q^2)$.

Integrating the evolution of the quark distribution, we may write the corrections to the structure function as,

$$F_2(x,Q^2) = \sum_q x\, e_q^2 \left[q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}\zeta}{\zeta}\left[\log\left(\frac{Q^2}{\Lambda^2}\right) P_{qq}^{(0)}\left(\frac{x}{\zeta}\right) + C\left(\frac{x}{\zeta}\right)\right] q_0(\zeta)\right], \tag{5.168}$$

where the bare distribution $q_0(x)$ is the $q(x)$ of the Parton model formula, which is to be determined by the measurements, and $C\left(x/\zeta\right)$ keeps the finite non-collinear-enhanced corrections into account. We may think of absorbing the collinear singularities into $q_0(x)$ at a factorisation scale μ_F and define a "renormalised" distribution $q(x,\mu_F^2)$,

$$q(x,\mu_F^2) = q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}\zeta}{\zeta}\left[\log\left(\frac{\mu_F^2}{\Lambda^2}\right) P_{qq}^{(0)}\left(\frac{x}{\zeta}\right) + C\left(\frac{x}{\zeta}\right)\right] q_0(\zeta), \tag{5.169}$$

such that

$$F_2(x,Q^2) = \sum_q x\, e_q^2 \int_x^1 \frac{\mathrm{d}\zeta}{\zeta} q(\zeta,\mu_F^2)\left[\delta\left(1-\frac{x}{\zeta}\right) + \frac{\alpha_s}{2\pi}\log\left(\frac{Q^2}{\mu_F^2}\right) P_{qq}^{(0)}\left(\frac{x}{\zeta}\right)\right]. \tag{5.170}$$

The distribution $q(\zeta, \mu_F^2)$ cannot be computed, since it contains collinear singularities, but it may be determined from DIS data, since

$$F_2(x, Q^2) = \sum_q x\, e_q^2\, q(x, Q^2). \tag{5.171}$$

Note that we have defined $q(x, \mu_F^2)$ in such a way as to absorb completely the finite terms $C\,(x/\zeta)$ of $F_2(x, Q^2)$. $F_2(x, Q^2)$ being a physical quantity, it cannot depend on the arbitrary scale μ_F^2, i.e.

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} F_2(x, Q^2) = 0. \tag{5.172}$$

In fact, if we differentiate the right-hand side of the equation for $F_2(x, Q^2)$ above, we obtain that eq. (5.172) implies that

$$\mu_F^2 \frac{\partial q(x, \mu_F^2)}{\partial \mu_F^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}\zeta}{\zeta} P_{qq}^{(0)}\left(\frac{x}{\zeta}\right) q(\zeta, \mu_F^2), \tag{5.173}$$

which is the evolution equation of the quark PDF. So eq. (5.172) is identically fulfilled.

Now that we understand the basic features of the evolution of the parton distributions, we consider the full evolution equation of the quark distribution, including the P_{qg} splitting function,

$$Q^2 \frac{\partial q(x, Q^2)}{\partial Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}\zeta}{\zeta} \left(P_{qq}^{(0)}\left(\frac{x}{\zeta}\right) q(\zeta, Q^2) + P_{qg}^{(0)}\left(\frac{x}{\zeta}\right) g(\zeta, Q^2) \right). \tag{5.174}$$

Integrating the evolution of the quark distribution, the corrections to the structure functions are

$$\begin{aligned} F_2(x, Q^2) = \sum_q x\, e_q^2 \Bigg[q_0(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}\zeta}{\zeta} \Bigg\{ &\left[\log\left(\frac{Q^2}{\Lambda^2}\right) P_{qq}^{(0)}\left(\frac{x}{\zeta}\right) + C_q\left(\frac{x}{\zeta}\right) \right] q_0(\zeta) \\ &+ \left[\log\left(\frac{Q^2}{\Lambda^2}\right) P_{qg}^{(0)}\left(\frac{x}{\zeta}\right) + C_g\left(\frac{x}{\zeta}\right) \right] g_0(\zeta) \Bigg\} \Bigg], \end{aligned} \tag{5.175}$$

where C_q and C_g account for the finite corrections.

We define the distribution $q(x, \mu_F^2)$ at the scale μ_F as

$$\begin{aligned} q(x, \mu_F^2) = q_0(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}\zeta}{\zeta} \Bigg\{ &\left[\log\left(\frac{\mu_F^2}{\Lambda^2}\right) P_{qq}^{(0)}\left(\frac{x}{\zeta}\right) + C'_q\left(\frac{x}{\zeta}\right) \right] q_0(\zeta) \\ &+ \left[\log\left(\frac{\mu_F^2}{\Lambda^2}\right) P_{qg}^{(0)}\left(\frac{x}{\zeta}\right) + C'_g\left(\frac{x}{\zeta}\right) \right] g_0(\zeta) \Bigg\}, \end{aligned} \tag{5.176}$$

where the finite contributions C'_q and C'_g need not to be the same as C_q and C_g. In fact, there is an arbitrariness in how we define the finite parts when we absorb the collinear singularities in the distribution $q(x, \mu_F^2)$. By specifying the finite parts, we define the factorisation scheme we use. Then $F_2(x, Q^2)$ becomes

$$\begin{aligned} F_2(x, Q^2) = \sum_q x\, e_q^2 \int_x^1 \frac{\mathrm{d}\zeta}{\zeta} \Bigg\{ &q(\zeta, \mu_F^2) \left[\delta\left(1 - \frac{x}{\zeta}\right) + \frac{\alpha_s}{2\pi} \left(\log\left(\frac{Q^2}{\mu_F^2}\right) P_{qq}^{(0)}\left(\frac{x}{\zeta}\right) + (C_q - C'_q) \right) \right] \\ &+ g(\zeta, \mu_F^2) \frac{\alpha_s}{2\pi} \left(\log\left(\frac{Q^2}{\mu_F^2}\right) P_{qg}^{(0)}\left(\frac{x}{\zeta}\right) + (C_g - C'_g) \right) \Bigg\}. \end{aligned} \tag{5.177}$$

5.6.4 DIS and $\overline{\mathrm{MS}}$ Schemes

In the $\overline{\mathrm{MS}}$ scheme, in the quark distribution, we absorb only the pole and the usual $\log(4\pi) - \gamma_E$, which appears in any one-loop computation. Thus $C'_q = C'_g = \log(4\pi) - \gamma_E$ and we define,

$$C_q - C'_q = C_q^{\overline{\mathrm{MS}}}, \tag{5.178}$$

$$C_g - C'_g = C_g^{\overline{\mathrm{MS}}}. \tag{5.179}$$

Previously, we used implicitly the DIS scheme, for which $C'_q = C_q$ and $C'_g = C_g$. As we have seen, the DIS scheme is chosen in such a way that

$$F_2(x, Q^2) = \sum_q x e_q^2 q(x, Q^2). \tag{5.180}$$

5.6.5 Singlet and Non-singlet Distributions

In convolution notation, the DGLAP evolution equations are

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \begin{pmatrix} \{q_i(\mu_F^2)\} \\ \{\bar{q}_i(\mu_F^2)\} \\ g(\mu_F^2) \end{pmatrix} = \begin{pmatrix} P_{q_i q_j} & P_{q_i \bar{q}_j} & P_{qg} \\ P_{q_i \bar{q}_j} & P_{q_i q_j} & P_{qg} \\ P_{gq} & P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \{q_j\left(\mu_F^2\right)\} \\ \{\bar{q}_j\left(\mu_F^2\right)\} \\ g\left(\mu_F^2\right) \end{pmatrix}. \tag{5.181}$$

SU(n_f) flavour symmetry implies that we can write the $P_{q_i q_j}$ splitting functions in terms of flavour singlet (S) and non-singlet (V) quantities,

$$P_{q_i q_j} = \delta_{ij} P_{qq}^V + P_{qq}^S, \tag{5.182}$$

$$P_{q_i \bar{q}_j} = \delta_{ij} P_{q\bar{q}}^V + P_{q\bar{q}}^S. \tag{5.183}$$

As we know, at leading order,

$$P_{q\bar{q}}^{V(0)} = P_{qq}^{S(0)} = P_{q\bar{q}}^{S(0)} = 0. \tag{5.184}$$

At NLO, $P_{q\bar{q}}^V$, P_{qq}^S and $P_{q\bar{q}}^S$ contribute, however,

$$P_{qq}^{S(1)} = P_{q\bar{q}}^{S(1)}. \tag{5.185}$$

Note that if you take the difference of the evolution equations for two different quark flavours (please see Exercises),

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2}(q_i - q_j) = P_{qq}^V \otimes (q_i - q_j) + P_{q\bar{q}}^V \otimes (\bar{q}_i - \bar{q}_j). \tag{5.186}$$

At leading order we obtain a homogeneous equation,

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2}(q_i - q_j) = \frac{\alpha_s}{2\pi} P_{qq}^{(0)} \otimes (q_i - q_j), \tag{5.187}$$

and of course the same is true for two antiquarks. Since $P_{qq}^{V(0)} = P_{qq}^{(0)}$ is a non-singlet splitting function, $q_i - q_j$ is termed a **non-singlet** distribution, at leading order.

However, a non-singlet distribution to all orders of α_s is $q_i^+ - q_j^+$, where $q_i^+ = q_i + \bar{q}_i$. It yields the homogeneous equation (please see Exercises),

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2}(q_i^+ - q_j^+) = (P_{qq}^V + P_{q\bar{q}}^V) \otimes (q_i^+ - q_j^+), \tag{5.188}$$

while the distribution $q_i^- = q_i - \bar{q}_i$ yields the equation,

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} q_i^- = (P_{qq}^V - P_{q\bar{q}}^V) \otimes q_i^- + (P_{qq}^S - P_{q\bar{q}}^S) \otimes \sum_{j=1}^{n_f} q_j^-, \tag{5.189}$$

which is homogeneous at NLO.

Finally the **singlet** combination,

$$\Sigma(x, \mu_F^2) \equiv \sum_{i=1}^{n_f} q_i^+ = \sum_i (q_i + \bar{q}_i), \tag{5.190}$$

yields an evolution equation which is coupled to that of the gluon distribution (please see Exercises),

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \begin{pmatrix} \Sigma(\mu_F^2) \\ g(\mu_F^2) \end{pmatrix} = \begin{pmatrix} P_{qq}^\Sigma & 2n_f P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma(\mu_F^2) \\ g(\mu_F^2) \end{pmatrix}, \tag{5.191}$$

where

$$P_{qq}^\Sigma = P_{qq}^V + P_{q\bar{q}}^V + n_f(P_{qq}^S + P_{q\bar{q}}^S). \tag{5.192}$$

It is convenient to introduce the 5 combinations,

$$\begin{aligned}
1^+ &\equiv u^+ - d^+, \\
2^+ &\equiv (u^+ - s^+) + (d^+ - s^+), \\
3^+ &\equiv (u^+ - c^+) + (d^+ - c^+) + (s^+ - c^+), \\
4^+ &\equiv (u^+ - b^+) + (d^+ - b^+) + (s^+ - b^+) + (c^+ - b^+), \\
5^+ &\equiv (u^+ - t^+) + (d^+ - t^+) + (s^+ - t^+) + (c^+ - t^+) + (b^+ - t^+),
\end{aligned} \tag{5.193}$$

which, as we have seen, are non-singlet distributions to all orders in α_s. Then knowing these 5 combinations, plus the 6 distributions q_i^- and the singlet distribution Σ, one can solve the DGLAP evolution equations for the 12 individual quark and antiquark distributions.

5.7 Evolution in Moment Space

In order to solve the evolution equations, it is convenient to take the Mellin transform of the parton distributions, which are called **moments**,

$$q(N, Q^2) = \int_0^1 \mathrm{d}x\, x^{N-1} q(x, Q^2). \tag{5.194}$$

The moment of a splitting function is called anomalous dimension,

$$\gamma_{ij}(N, \alpha_s) = \int_0^1 \mathrm{d}x\, x^{N-1} P_{ij}(x, \alpha_s). \tag{5.195}$$

The advantage is that by taking the Mellin transform of the evolution equation, the convolution is reduced to a simple product. Let us show it, at leading order, on the non-singlet distribution $V = q_i - q_j$,

$$
\begin{aligned}
Q^2 \frac{\partial}{\partial Q^2} V(N, Q^2) &= \frac{\alpha_s}{2\pi} \int_0^1 \mathrm{d}x\, x^{N-1} \int_x^1 \frac{\mathrm{d}z}{z} P_{qq}^{(0)}(z) V\left(\frac{x}{z}, Q^2\right) \\
&= \frac{\alpha_s}{2\pi} \int_0^1 \mathrm{d}\left(\frac{x}{z}\right) \left(\frac{x}{z}\right)^{N-1} V\left(\frac{x}{z}, Q^2\right) \int_0^1 \mathrm{d}z\, z^{N-1} P_{qq}^{(0)}(z),
\end{aligned}
\tag{5.196}
$$

i.e.

$$
Q^2 \frac{\partial}{\partial Q^2} V(N, Q^2) = \frac{\alpha_s}{2\pi} \gamma_{qq}^{(0)}(N) V(N, Q^2)\,. \tag{5.197}
$$

At fixed α_s, it is easy to solve this equation,

$$
V(N, Q^2) = V(N, \mu^2) \left(\frac{Q^2}{\mu^2}\right)^{\frac{\alpha_s}{2\pi}\gamma_{qq}^{(0)}(N)}. \tag{5.198}
$$

Since the distribution V is dimensionless, we might expect that it is scale independent. Instead, we see that its moment scales like powers of Q^2, hence the term “anomalous dimension” for $\gamma_{qq}(N)$. If we include the one-loop running of α_s (4.205), with β_0 as in eq. (4.120), we can write the evolution equation (5.197) as

$$
\frac{\partial \log V(N, Q^2)}{\partial \log(Q^2/\Lambda^2)} = \frac{2}{\beta_0 \log\left(Q^2/\Lambda^2\right)} \gamma_{qq}^{(0)}(N), \tag{5.199}
$$

which integrates to

$$
\begin{aligned}
V(N, Q^2) &= V(N, \mu^2) \left(\frac{\log\left(Q^2/\Lambda^2\right)}{\log\left(\mu^2/\Lambda^2\right)}\right)^{\frac{2\gamma_{qq}^{(0)}}{\beta_0}} \\
&= V(N, \mu^2) \left(\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\right)^{\frac{2\gamma_{qq}^{(0)}}{\beta_0}},
\end{aligned}
\tag{5.200}
$$

which will yield the scaling violations of the parton distributions, and so of $F_2(x, Q^2)$.

The usual distribution in x is obtained by taking the inverse Mellin transform,

$$
V(x, Q^2) = \int_C \frac{\mathrm{d}N}{2\pi i} x^{-N} V(N, Q^2), \tag{5.201}
$$

where the integration contour C in the complex N plane is parallel to the imaginary axis and to the right of all the poles of the integrand. The inverse transform is usually done numerically.

For the non-singlet distribution $V = q_i - q_j$, we need to compute the anomalous dimension $\gamma_{qq}^{(0)}(N)$, using eq. (5.195) and the splitting function (5.142). We need the integrals,

$$
\begin{aligned}
&\int_0^1 \mathrm{d}z\, z^{N-1} = \frac{1}{N}, \\
&\int_0^1 \mathrm{d}z\, \frac{z^{N-1}}{(1-z)_+} = \int_0^1 \mathrm{d}z\, \frac{z^{N-1} - 1}{1-z} = -\sum_{i=0}^{N-2} \int_0^1 \mathrm{d}z\, z^i = -\sum_{i=1}^{N-1} \frac{1}{i},
\end{aligned}
\tag{5.202}
$$

where $\sum_{i=1}^{N-1}\frac{1}{i}=H_{N-1}$ is the harmonic number, and we get (please check!)

$$\gamma_{qq}^{(0)}(N)=C_F\left(\frac{3}{2}+\frac{1}{N(N+1)}-2\sum_{i=1}^{N}\frac{1}{i}\right). \tag{5.203}$$

Note that

$$\gamma_{qq}^{(0)}(1)=0=\int_0^1 \mathrm{d}z P_{qq}^{(0)}(z),\quad \text{i.e. conservation of probability,} \tag{5.204}$$

$$\gamma_{qq}^{(0)}(2)=-\frac{4}{3}C_F, \tag{5.205}$$

$$\gamma_{qq}^{(0)}(3)=-\frac{25}{12}C_F\,. \tag{5.206}$$

The harmonic series diverges logarithmically,

$$\lim_{N\to\infty}(H_N-\log(N))=\gamma_E, \tag{5.207}$$

then as $N\to\infty$,

$$\gamma_{qq}^{(0)}(N)\sim -2C_F\log(N)\,. \tag{5.208}$$

At leading order, the other splitting functions are given in eqs. (5.143), (5.144) and (5.146). Using eq. (5.195), the corresponding anomalous dimensions are (please check!)

$$\gamma_{qg}^{(0)}(N)=T_F\frac{N^2+N+2}{N(N+1)(N+2)}, \tag{5.209}$$

$$\gamma_{gq}^{(0)}(N)=C_F\frac{N^2+N+2}{N(N^2-1)}, \tag{5.210}$$

$$\gamma_{gg}^{(0)}(N)=2C_A\left(\frac{1}{N(N-1)}+\frac{1}{(N+1)(N+2)}+\frac{11}{12}-\sum_{i=1}^{N}\frac{1}{i}\right)-\frac{2}{3}T_f n_f. \tag{5.211}$$

From the Mellin transform, we see that moments at large N are most relevant for the $x\to 1$ region. Using eq. (5.208), as $N\to\infty$ the moment of the non-singlet distribution (5.200) behaves as

$$\begin{aligned}V(N,Q^2)&=V(N,\mu^2)\left(\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\right)^{\frac{-4C_F\log(N)}{\beta_0}}\\&=V(N,\mu^2)N^{-\frac{4C_F}{\beta_0}\log\left(\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\right)}\,.\end{aligned} \tag{5.212}$$

Now, suppose that as $x\to 1$, the non-singlet distribution behaves as $V(x,Q^2)\sim(1-x)^{a-1}$. Its Mellin transform is

$$\int_0^1 \mathrm{d}x\, x^{N-1}(1-x)^{a-1}=\frac{\Gamma(N)\Gamma(a)}{\Gamma(N+a)}\xrightarrow{N\to\infty}\frac{(a-1)!}{N^a}. \tag{5.213}$$

Then using eq. (5.212), we see that in the $x\to 1$ limit, the non-singlet distribution behaves as

$$V(x,Q^2)\simeq V(x,\mu^2)(1-x)^{\frac{4C_F}{\beta_0}\log\left(\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\right)-1}\,. \tag{5.214}$$

Since the one-loop running of α_s is given by eq. (4.205), the exponent in eq. (5.214) grows like $\log\left(\log\left(\frac{Q^2}{\Lambda^2}\right)\right)$, thus as $x \to 1$ the non-singlet distribution $V(x, Q^2)$ decreases as the scale increases. Likewise, one can show that as $x \to 0$, $V(x, Q^2)$ increases as Q^2 grows. In fact, the scaling violations, $\frac{\partial F_2}{\partial \log Q^2}$, are negative (positive) at large (small) x, as we see in fig. 2.3.

In moment space, the evolution equation (5.191) for the singlet and the gluon distribution is, at leading order,

$$\frac{\partial}{\partial \log Q^2}\begin{pmatrix}\Sigma(N,Q^2)\\ g(N,Q^2)\end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix}\gamma_{qq}^{(0)} & 2n_f\gamma_{qg}^{(0)}\\ \gamma_{gq}^{(0)} & \gamma_{gg}^{(0)}\end{pmatrix}\begin{pmatrix}\Sigma(N,Q^2)\\ g(N,Q^2)\end{pmatrix}, \tag{5.215}$$

with the entries in the anomalous dimension matrix given by eqs. (5.203), (5.209), (5.210) and (5.211). Requiring that $\det(\mathbf{M} - \gamma\mathbb{I}) = 0$, we find the eigenvalues,

$$\det\begin{vmatrix}\gamma_{qq}^{(0)} - \gamma & 2n_f\gamma_{qg}^{(0)}\\ \gamma_{gq}^{(0)} & \gamma_{gg}^{(0)} - \gamma\end{vmatrix} = 0 \qquad \Rightarrow \qquad (\gamma_{gg} - \gamma)(\gamma_{qq} - \gamma) - 2n_f\gamma_{qg}\gamma_{gq} = 0. \tag{5.216}$$

So

$$\gamma_\pm = \frac{1}{2}\left[\gamma_{gg} + \gamma_{qq} \pm \sqrt{(\gamma_{gg} - \gamma_{qq})^2 + 8n_f\gamma_{qg}\gamma_{gq}}\right]. \tag{5.217}$$

Let us take, for example, the second moment, $N = 2$. Then the evolution equation becomes

$$\frac{\partial}{\partial \log Q^2}\begin{pmatrix}\Sigma(2,Q^2)\\ g(2,Q^2)\end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix}-\frac{4}{3}C_F & \frac{n_f}{3}\\ \frac{4}{3}C_F & -\frac{n_f}{3}\end{pmatrix}\begin{pmatrix}\Sigma(2,Q^2)\\ g(2,Q^2)\end{pmatrix}, \tag{5.218}$$

where we have taken $T_F = 1/2$. The eigenvalues are

$$\begin{cases}\gamma^+(2) = 0,\\ \gamma^-(2) = -\left(\frac{4}{3}C_F + \frac{n_f}{3}\right).\end{cases} \tag{5.219}$$

The eigenvectors are

$$v_+ = \begin{pmatrix}1\\ \frac{4C_F}{n_f}\end{pmatrix}, \qquad v_- = \begin{pmatrix}1\\ -1\end{pmatrix}. \tag{5.220}$$

In the eigenvector basis, $Q\begin{pmatrix}\Sigma\\ g\end{pmatrix}$, i.e.

$$\begin{pmatrix}1 & 1\\ \frac{4C_F}{n_f} & -1\end{pmatrix}\begin{pmatrix}\Sigma(2,\mu_F^2)\\ g(2,\mu_F^2)\end{pmatrix} = \begin{pmatrix}\Sigma + g\\ \frac{4C_F}{n_f}\Sigma - g\end{pmatrix}, \tag{5.221}$$

the anomalous dimension matrix M is diagonal, $\gamma\mathbb{I} = QMQ^{-1}$,

$$\frac{\partial}{\partial \log Q^2}\begin{pmatrix}\Sigma + g\\ \frac{4C_F}{n_f}\Sigma - g\end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix}0 & 0\\ 0 & -\left(\frac{4}{3}C_F + \frac{n_f}{3}\right)\end{pmatrix}\begin{pmatrix}\Sigma + g\\ \frac{4C_F}{n_f}\Sigma - g\end{pmatrix}. \tag{5.222}$$

So the eigenvector,

$$O^+(2) = \Sigma(2,Q^2) + g(2,Q^2), \tag{5.223}$$

is independent of Q^2. But

$$O^+(2) = \int_0^1 \mathrm{d}x\, x \left[\Sigma(x,Q^2) + g(x,Q^2)\right],$$
$$= \int_0^1 \mathrm{d}x\, x \left[\sum_i (q_i + \bar{q}_i)(x,Q^2) + g(x,Q^2)\right], \tag{5.224}$$

i.e. $O^+(2)$ is the total momentum and consistently it does not depend on the scale Q^2.

The evolution equation for the other eigenvector,

$$O^-(2,Q^2) = \frac{4C_F}{n_f}\Sigma(2,Q^2) - g(2,Q^2), \tag{5.225}$$

has solution,

$$O^-(2,Q^2) = O^-(2,\mu^2)\left(\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\right)^{\frac{2\gamma^-(2)}{\beta_0}}. \tag{5.226}$$

Since the argument behaves like $\log\left(Q^2/\Lambda^2\right)$ and the exponent,

$$\frac{2\gamma^-(2)}{\beta_0} = -2\frac{\frac{4}{3}C_F + \frac{n_f}{3}}{\frac{11}{3}N_c - \frac{2}{3}n_f}, \tag{5.227}$$

is negative, $O^-(2,Q^2)$ vanishes asymptotically as $Q^2 \to \infty$. So

$$\frac{\Sigma(2,Q^2)}{g(2,Q^2)} \to \frac{n_f}{4C_F} = \frac{3}{16}n_f, \tag{5.228}$$

as the scale $Q^2 \to \infty$. But $\Sigma(2,Q^2)$ $\left(g(2,Q^2)\right)$ is the momentum fraction carried by the quarks (gluons) and they are in the relation,

$$p_g \sim \frac{16}{16+3n_f}, \qquad p_\Sigma \sim \frac{3n_f}{16+3n_f}, \tag{5.229}$$

which for $n_f = 6$ is roughly 1/2 each, which is experimentally observed.

5.8 Operator Product Expansion

In sec. 2.2, using unitarity and the optical theorem, we related the DIS cross section to forward Compton scattering, and thus to the hadronic tensor (2.4), expressed as the matrix element between proton states of the product of two electromagnetic currents. Other examples are the hadronic tensor of $e^+e^- \to$ hadrons, expressed as the matrix element between vacuum states of the product of two currents; or the product of currents in the low-energy (Fermi) approximation of weak interactions. All of these examples display a product of currents where, as we have seen in sec. 2.8 for the hadronic tensor $\tilde{W}^{\mu\nu}$, one wants to separate the parts of Feynman diagrams which carry a large momentum (transfer, in the case of DIS) and can be computed perturbatively in an asymptotically free field theory, from the parts which carry a small momentum (transfer) and must be extracted from experimental measurements.

If the operator product $J^\mu(x)J^\nu(0)$ were analytic in x^μ, its Fourier transform would decrease exponentially as $q^\mu \to \infty$. The leading terms in the limit $q^\mu \to \infty$ stem from the singularities of the operator product $J^\mu(x)J^\nu(0)$ as $x^\mu \to 0$. Let us then consider in general the product $O_1(x)O_2(y)$ of two operators, taken at different points in space-time. In 1969, Wilson conjectured that as $x \to y$ the singular part of the product $O_1(x)O_2(y)$ is given by a sum over local operators,

$$O_1(x)O_2(y) \to \sum_n C_{12}^n(x-y)O_n(y), \tag{5.230}$$

where $C_{12}^n(x-y)$ are singular functions. Using translation invariance, we can just write this **operator product expansion** (OPE) as

$$O_1(x)O_2(0) \to \sum_n C_{12}^n(x)O_n(0). \tag{5.231}$$

Dimensional analysis hints that $C_{12}^n(x)$ behaves like $x^{d_n-d_1-d_2}$ as $x \to 0$, if $d_n = [O_n]$ is the dimensionality of O_n in powers of the mass. Since d_n increases adding more fields or derivatives to O_n, the strength of the singularity of $C_{12}^n(x)$ decreases for operators O_n of increasing complexity.

5.8.1 Renormalisation and CS Equations of the Coefficient Functions

The OPE is just about operators. Thus, in applying it to a matrix element $\langle\alpha|\, O_1(x)O_2(0)\,|\beta\rangle$ it will yield the same functions C_{12}^n for any states $|\alpha\rangle$ and $|\beta\rangle$. In particular, if we take the product of two operators in a Green's function,

$$G_{12}(x; y_1, ..., y_m) =< O_1(x)O_2(0)\varphi(y_1)...\varphi(y_m) >, \tag{5.232}$$

taken in the limit $x \to 0$, and with fields at fixed points $y_1, \dots, y_m$ away from the origin, the OPE is independent of the fields $\varphi(y_1)...\varphi(y_m)$. So we can expand the Green's function for small x through the OPE,

$$G_{12}(x; y_1, ..., y_m) = \sum_n C_{12}^n(x)G_n(y_1, ..., y_m)\,, \tag{5.233}$$

with

$$G_n(y_1, ..., y_m) =< O_n(0)\varphi(y_1)...\varphi(y_m) > \,. \tag{5.234}$$

So the dependence on x is all in the functions $C_{12}^n(x)$. In momentum space, we are interested in the behaviour as $q \to \infty$. The Fourier transform of the operator product is

$$\int \mathrm{d}^4x\, e^{iq\cdot x}O_1(x)O_2(0) \to \sum_n C_{12}^n(q)O_n(0), \tag{5.235}$$

where $C_{12}^n(q)$ are functions of q^μ that as $q \to \infty$ decrease more and more rapidly for more and more complicated terms in the series.

Besides the naive dimensional argument that the functions $C_{12}^n(x)$ behave like $x^{d_n-d_1-d_2}$ as $x \to 0$, we expect that we have renormalisation effects since the functions $C_{12}^n(x)$ will be multiplied by scale-dependent renormalisation constants. It is then convenient to write the OPE at some scale M,

$$O_1(x)|_M\; O_2(0)|_M \to \sum_n C_{12}^n(x, M)\; O_n(0)|_M\,, \tag{5.236}$$

where also the functions C_{12}^n depend on M, since they must offset the operator rescalings.

In the renormalisation of local or mass operators of sec. 4.7, we considered the Green's function and the related CS equation for the insertion of one or more local operators. The Green's function of the product,

$$G_{12}(x; y_1, ..., y_m; M) =< O_1(x)O_2(0)\varphi(y_1)...\varphi(y_m) >|_M, \tag{5.237}$$

must then obey the CS equation (4.152),

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + m\gamma + \gamma_1 + \gamma_2\right] G_{12}(x; y_1, ..., y_m; M) = 0\,, \tag{5.238}$$

where γ_1 and γ_2 are the anomalous dimensions of the local operators O_1 and O_2. Further, in the OPE the Green's function of each operator $O_n(0)|_M$,

$$G_n(y_1, ..., y_m; M) =< O_n(0)\varphi(y_1)...\varphi(y_m) >|_M, \tag{5.239}$$

will obey the CS equation,

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + m\gamma + \gamma_n\right] G_n(y_1, ..., y_m; M) = 0\,. \tag{5.240}$$

Inserting then the OPE for G_{12} (5.233) in the CS equation for G_{12} (5.238), we obtain

$$\sum_n \left[\left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g}\right) C_{12}^n G_n + C_{12}^n \left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g}\right) G_n + (m\gamma + \gamma_1 + \gamma_2) C_{12}^n G_n\right] = 0. \tag{5.241}$$

Using then the CS equation for G_n (5.240), this becomes

$$\sum_n \left[\left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g}\right) C_{12}^n G_n + (\gamma_1 + \gamma_2 - \gamma_n) C_{12}^n G_n\right] = 0\,. \tag{5.242}$$

Since the Green's functions G_n are linearly independent, this implies a CS equation for the coefficient functions C_{12}^n,

$$\left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + \gamma_1 + \gamma_2 - \gamma_n\right) C_{12}^n(x; M) = 0. \tag{5.243}$$

From the dimensional analysis, we can write

$$C_{12}^n(x; M) = \frac{1}{|x|^{d_1+d_2-d_n}} \bar{C}(xM), \tag{5.244}$$

with $\bar{C}(xM)$ a dimensionless function. In analogy with the solution (4.84) of the CS equation for the 2-point function, we write $\bar{C}(xM)$ as

$$\bar{C}(xM) = C\left(\bar{g}\left(\frac{1}{x}\right)\right) \exp\left(\int_{\frac{1}{x}}^{M} \mathrm{d}\log(M')(\gamma_1 + \gamma_2 - \gamma_n)\right), \tag{5.245}$$

with $C(\bar{g})$ a dimensionless function of the running coupling at a scale $1/x$. Using the one-loop running (4.205), and parametrising the anomalous dimensions as $\gamma = \dfrac{\alpha_s}{2\pi}\gamma^{(0)}$, we can write the exponential as

$$\exp\left(\int_{\frac{1}{x^2}}^{M^2} \frac{\mathrm{d}\log\left(M'^2\right)}{2} \frac{2}{\beta_0 \log\left(M'^2/\Lambda^2\right)} (\gamma_1^{(0)} + \gamma_2^{(0)} - \gamma_n^{(0)})\right), \tag{5.246}$$

so that

$$\bar{C}(xM) = C\left(\bar{g}\left(\frac{1}{x}\right)\right)\left(\frac{\log\left(M^2/\Lambda^2\right)}{\log\left(1/(x^2\Lambda^2)\right)}\right)^{\frac{\gamma_1^{(0)}+\gamma_2^{(0)}-\gamma_n^{(0)}}{\beta_0}} . \tag{5.247}$$

Note that this is in agreement with the running mass at one loop (4.183) of the mass operator in section 4.7.2, which had anomalous dimension, $\gamma_{qq} = \frac{\alpha_s}{2\pi}\gamma_{qq}^{(0)} = -3C_F\frac{\alpha_s}{2\pi}$.

As outlined in sec. 4.7, if there is a set of local operators O^i, with the same dimension and quantum numbers, they usually mix in the renormalisation procedure. Then $O_0^i = Z_O^{ij}(M)O_M^j$, and the anomalous dimension becomes a matrix, eq. (4.155), and the CS equation for the Green's function becomes matrix valued. If that is the case for the operators O_n of the OPE, the CS equation for G_n becomes

$$\left[\delta_{np}\left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + m\gamma\right) + \gamma_{np}\right]G_p(y_1, ..., y_m; M) = 0. \tag{5.248}$$

Repeating the same derivation as above, we obtain a matrix-valued CS equation for the functions C_{12}^n,

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + \gamma_1 + \gamma_2\right]C_{12}^n(x; M) - C_{12}^k(x; M)\gamma_{kn} = 0, \tag{5.249}$$

and even more in general also the operators O_1 and O_2 might be part of sets of local operators with the same dimension and quantum numbers. However, if like for the hadronic tensor in DIS, the operator product is the product of two (conserved) currents, $J^\mu(x)J^\nu(0)$, their anomalous dimensions vanish and the CS equation for the functions C_{12}^n becomes

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g}\right]C_{12}^n(x; M) - C_{12}^k(x; M)\gamma_{kn} = 0. \tag{5.250}$$

5.8.2 OPE of the Hadronic Tensor of DIS

Let us now evaluate the hadronic tensor (2.4),

$$\tilde{W}^{\mu\nu} = i\int \mathrm{d}^4x\, e^{iq\cdot x}\langle P|\, T\big(J^\mu(x)J^\nu(0)\big)\,|P\rangle\ ,$$

as the OPE of the two currents. The leading contributions come from products of currents with quarks of the same flavour and we will then sum over the flavours of the quark currents, weighted by the quark square electric charge (later we will comment on the OPE of currents with quarks of different flavours). Thus to leading order, we consider the product,

$$\bar{\psi}\gamma^\mu\psi(x)\bar{\psi}\gamma^\nu\psi(0) = \bar{\psi}(x)\gamma^\mu\overbracket{\psi(x)\bar{\psi}}(0)\gamma^\nu\psi(0) + \bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(0)\gamma^\nu\psi(0) + ... \tag{5.251}$$

The contractions yield singular terms as $x \to 0$.

Taking the Fourier transform of the contraction,

$$\int \mathrm{d}^4x\, e^{iq\cdot x}\bar{\psi}(x)\gamma^\mu\overbracket{\psi(x)\bar{\psi}}(0)\gamma^\nu\psi(0) = \bar{\psi}\gamma^\mu i\frac{i\partial\!\!\!/ + q\!\!\!/}{(i\partial + q)^2}\gamma^\nu\psi, \tag{5.252}$$

where the derivative acts on the quark field to the right. This contribution has the structure of the diagram on the left-hand side of fig. 2.5. In the short-distance limit, $q^\mu \to \infty$, we can expand

$$\frac{1}{(i\partial + q)^2} = -\frac{1}{\partial^2 - 2iq\cdot\partial + Q^2} = -\frac{1}{Q^2}\sum_{n=0}^{\infty}\left(\frac{2iq\cdot\partial - \partial^2}{Q^2}\right)^n, \tag{5.253}$$

where

$$\frac{2iq\cdot\partial}{Q^2} \to \frac{2q\cdot xP}{Q^2} = 1, \tag{5.254}$$

while

$$\frac{\partial^2}{Q^2} \to \frac{x^2P^2}{Q^2} \ll 1, \tag{5.255}$$

so we neglect the ∂^2/Q^2 term. In unpolarised DIS, $W^{\mu\nu}$ is symmetric under $\mu \leftrightarrow \nu$, so we can symmetrise the product of γ matrices in the numerator using

$$\frac{1}{2}\left(\gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu\right) = g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu - g^{\mu\nu}\gamma^\rho. \tag{5.256}$$

Then, in the numerator we obtain

$$\gamma^\mu(i\partial^\nu) + \gamma^\nu(i\partial^\mu) - g^{\mu\nu}(i\not{\partial}) + \gamma^\mu q^\nu + \gamma^\nu q^\mu - g^{\mu\nu}\not{q}. \tag{5.257}$$

$(i\not{\partial})$ on the quark field vanishes because of the Dirac equation; $\gamma^\mu q^\nu$, $\gamma^\nu q^\mu$ vanish because of current conservation. Thus

$$\int \mathrm{d}^4x\, e^{iq\cdot x}\bar{\psi}(x)\gamma^\mu\overbracket{\psi(x)\bar{\psi}}(0)\gamma^\nu\psi(0) = -i\bar{\psi}\left(\gamma^\mu(i\partial^\nu) + \gamma^\nu(i\partial^\mu) - g^{\mu\nu}\not{q}\right)\frac{1}{Q^2}\sum_{n=0}^{\infty}\left(\frac{2iq\cdot\partial}{Q^2}\right)^n\psi. \tag{5.258}$$

The diagram on the right-hand side of fig. 2.5 differs from the one on the left-hand side by swapping μ and ν and the points x and 0. Its Fourier transform is obtained from the one above by replacing $q \leftrightarrow -q$. So the sum of the contractions is

$$\begin{aligned}\int \mathrm{d}^4x\, e^{iq\cdot x}&\left[\bar{\psi}(x)\gamma^\mu\overbracket{\psi(x)\bar{\psi}}(0)\gamma^\nu\psi(0) + \overbracket{\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(0)\gamma^\nu\psi}(0)\right]\\ &= -i\bar{\psi}\left(2\gamma^{\{\mu}(i\partial^{\nu\}}) - g^{\mu\nu}\not{q}\right)\frac{1}{Q^2}\sum_{n=0}^{\infty}\left(\frac{2iq\cdot\partial}{Q^2}\right)^n\psi + (q^\mu \leftrightarrow -q^\mu),\end{aligned} \tag{5.259}$$

which has only an even number of terms in q^μ. $\{\mu...\nu\}$ labels symmetrised indices. All terms in the OPE contain the operator, $\bar{\psi}\gamma^{\mu_1}(i\partial^{\mu_2})...(i\partial^{\mu_n})\psi$, with an even number of indices. We can order the operators that contribute to the OPE of the two currents according to the irreducible representations of the Lorentz group to which they belong. When we take the matrix element $\langle P| ... |P\rangle$, we will get powers of the proton momentum P^μ, and the only Lorentz-covariant functions are the totally symmetric tensors, $P^{\mu_1}...P^{\mu_n}$. But these do not constitute irreducible representations of the Lorentz group, since they may be decomposed into symmetric traceless tensors. For example, from the symmetric tensors $X^{\mu\nu}$ and $X^{\mu\nu\rho}$, we can construct the tensors,

$$\hat{X}^{\mu\nu} = X^{\mu\nu} - \frac{1}{d}g^{\mu\nu}X_\rho^{\ \rho}, \qquad \text{with} \qquad g^\mu_{\ \mu} = d, \tag{5.260}$$

$$\hat{X}^{\mu\nu\rho} = X^{\mu\nu\rho} - \frac{1}{d+2}\left(g^{\mu\nu}X_\sigma^{\ \sigma\rho} + g^{\mu\rho}X_\sigma^{\ \sigma\nu} + g^{\nu\rho}X_\sigma^{\ \sigma\mu}\right), \tag{5.261}$$

whose traces vanish, $\hat{X}_\mu{}^\mu = 0$ and $\hat{X}_\mu{}^{\mu\rho} = 0$. Solving eqs. (5.260) and (5.261) for $X^{\mu\nu}$ and $X^{\mu\nu\rho}$, we decompose the latter into symmetric traceless tensors. So, the only operators that can contribute to the spin-averaged proton expectation value, $\langle P| \, T\big\{J^\mu(x)J^\nu(0)\big\} \, |P\rangle$, are the totally symmetric traceless tensors, $O_f^{(n)\mu_1 \dots \mu_n}$, which are obtained by symmetrising the operator $\bar{\psi}\gamma^{\mu_1}(i\partial^{\mu_2})...(i\partial^{\mu_n})\psi$ over $\mu_1....\mu_n$ and then subtracting terms proportional to $g^{\mu_i\mu_j}$ for each (μ_i, μ_j) pair so that the operator above is traceless over all pairs of indices.

Further, the currents $J^\mu(x)J^\nu(0)$ are invariant under colour gauge transformations, while the operator $\bar{\psi}\gamma^{\mu_1}(i\partial^{\mu_2})...(i\partial^{\mu_n})\psi$ is not. We can make it gauge invariant by replacing each factor $(i\partial^\mu)$ with the covariant derivative (iD^μ). This introduces terms proportional to the coupling g, which do not modify the OPE coefficients.

Finally, the symmetric traceless spin-n operator for quarks of flavour f is

$$O_f^{(n)\mu_1 \dots \mu_n} = \bar{\psi}_f \gamma^{\{\mu_1}(iD^{\mu_2}) \dots (iD^{\mu_n\}})\psi_f - \text{traces} \, . \tag{5.262}$$

For example, for $n = 2$ we obtain

$$O_f^{(2)\mu\nu} = \bar{\psi}_f \left[\frac{1}{2} \left(\gamma^\mu iD^\nu + \gamma^\nu iD^\mu \right) - \frac{1}{d} g^{\mu\nu} i\not{D} \right] \psi_f \, . \tag{5.263}$$

where the last term vanishes because of the Dirac equation.

Using eqs. (5.261) and (5.263), the OPE of the product of the two currents $J^\mu(x)J^\nu(0)$ begins as

$$\begin{aligned} i \int \mathrm{d}^4x \, e^{iq\cdot x} J^\mu(x) J^\nu(0) &= \sum_f Q_f^2 \bar{\psi}_f \frac{1}{Q^2} \left(2\gamma^\mu iD^\nu + 2\gamma^\nu iD^\mu - g^{\mu\nu} \frac{2\not{q} \, 2iq\cdot D}{Q^2} \right) \psi_f + \dots \\ &= \sum_f Q_f^2 \left[\frac{4}{Q^2} O_f^{(2)\mu\nu} - g^{\mu\nu} \frac{(2q_{\mu_1})(2q_{\mu_2})}{(Q^2)^2} O_f^{(2)\mu_1\mu_2} \right] + \dots \end{aligned} \tag{5.264}$$

where the ellipsis indicates terms with $n \geq 4$. In fact, one can work out the whole OPE,

$$\begin{aligned} & i \int \mathrm{d}^4x \, e^{iq\cdot x} J^\mu(x) J^\nu(0) \\ & = \sum_f Q_f^2 \sum_{n=2}^{\infty} \left[4 \frac{(2q^{\mu_1}) \dots (2q^{\mu_{n-2}})}{(Q^2)^{n-1}} O_f^{(n)\mu\nu\mu_1 \dots \mu_{n-2}} - g^{\mu\nu} \frac{(2q^{\mu_1}) \dots (2q^{\mu_n})}{(Q^2)^n} O_f^{(n)\mu_1 \dots \mu_n} \right] , \end{aligned} \tag{5.265}$$

where the sum runs over the even values of n.

5.8.3 Light-cone Expansion and Twist

In order to determine the hadronic tensor $\tilde{W}^{\mu\nu}(q, P)$ through the OPE, we need to evaluate the operators of the spin-averaged proton expectation value, $\langle P| \, T\{J^\mu(x)J^\nu(0)\} \, |P\rangle$, and so the spin-averaged matrix element of the operators $O_f^{(n)}$,

$$\langle P| \, O_f^{(n)\mu_1 \dots \mu_n} \, |P\rangle = 2A_f^n \, P^{\mu_1} ... P^{\mu_n} - \text{traces} \, , \tag{5.266}$$

where A^n_f are dimensionless coefficients. The factor 2 comes from spin averaging. In order to see it, take $n = 1$. Then,

$$\langle P| O^{(1)\mu}_f |P\rangle = \langle P| \bar{\psi}_f \gamma^\mu \psi_f |P\rangle \,. \tag{5.267}$$

Then average over the proton spin,

$$\langle P| O^{(1)\mu}_f |P\rangle = \frac{1}{2} \operatorname{Tr}\left(\gamma^\mu \gamma^\nu\right) P_\nu \cdot N_f = 2P^\mu N_f, \tag{5.268}$$

where N_f is the charge of the conserved flavour current, which is the net number of quarks of flavour f,

$$A^1_f = N_f = \begin{cases} 2 & f = u \\ 1 & f = d \end{cases} \,. \tag{5.269}$$

The canonical energy-momentum tensor, which is obtained by Noether's theorem, is neither symmetric nor traceless, but it can be cast in a symmetric and traceless form, which for a massless quark is[1]

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\left(\gamma^\mu iD^\nu + \gamma^\nu iD^\mu\right)\psi - g^{\mu\nu}\bar{\psi}i\not{D}\psi - F^{\mu\sigma a}F^{\nu}{}_{\sigma}{}^{a} + \frac{1}{4}g^{\mu\nu}\left(F^a_{\lambda\sigma}\right)^2 \,. \tag{5.270}$$

Thus, if we ignore the pure gauge part, terms of $\mathcal{O}(g)$ and use Dirac equation in the energy-momentum tensor above, $O^{(2)\mu\nu}_f$ in eq.(5.263) equals the contribution of the quark of flavour f to the energy-momentum tensor,

$$O^{(2)\mu\nu}_f = \frac{1}{2}\bar{\psi}_f\left(\gamma^\mu iD^\nu + \gamma^\nu iD^\mu\right)\psi_f = T^{\mu\nu}_f \,, \tag{5.271}$$

and then A^2_f is the fraction of the proton momentum carried by the quark of flavour f.

Finally, taking the matrix element of eq. (5.265) between proton states, we obtain

$$\begin{aligned} \tilde{W}^{\mu\nu}(q,P) &= i\int \mathrm{d}^4x\, e^{iq\cdot x}\, \langle P| T\big(J^\mu(x)J^\nu(0)\big) |P\rangle \\ &= \sum_f Q^2_f \left[8P^\mu P^\nu \sum_{n=2}^{\infty} \frac{(2q\cdot P)^{n-2}}{(Q^2)^{n-1}} A^n_f - 2g^{\mu\nu} \sum_{n=2}^{\infty} \frac{(2q\cdot P)^n}{(Q^2)^n} A^n_f\right], \end{aligned} \tag{5.272}$$

where the sum runs over even n, and all the trace terms in the operators $O^{(n)\mu_1\dots\mu_n}_f$ (5.262) have been neglected, since they replace terms like $P^\mu P^\nu$ by $g^{\mu\nu}m^2$. In fact, contracting with $q^\mu q^\nu$, we get m^2Q^2, to be compared with $(2q\cdot P)^2$. Since $x = Q^2/(2q\cdot P)$, then $m^2Q^2 \ll (2q\cdot P)^2$.

In the OPE of the product of two currents $J_\mu(x)J_\nu(0)$, if the mass dimension of an operator is $\left[O^{(n)}_f\right] = d$, its coefficient C^n_{12} has dimension $[C^n_{12}] = 6 - \left[O^{(n)}_f\right]$. Since in the Fourier transform we have the volume d^4x, the coefficient C^n_{12} behaves like $(1/Q)^{d-2}$. This is the dimensional analysis in a truly short-distance expansion where $x^\mu \to 0$. However, in DIS we are actually taking $Q^2 \to \infty$. Fourier transforming

$$\int \mathrm{d}^4x\, e^{iq\cdot x}\frac{1}{x^2} \quad \to \quad \frac{1}{Q^2}, \tag{5.273}$$

[1] e.g. see the analogous QED energy-momentum tensor in sec. 19.5 of Peskin-Schroeder's book.

we realise that this is a **light-cone expansion** for which $x^2 \to 0$, without necessarily having $x^\mu \to 0$. Since the matrix element of an operator of spin n carries n factors of momentum P^μ which appear in the combination $\left(2q \cdot P/Q^2\right)^n$, the coefficient C_{12}^n behaves like $(1/Q)^{d-n-2}$, and it has dimension,

$$[C_{12}^n] = 6 - t\,, \tag{5.274}$$

where

$$t = \left[O_f^{(n)}\right] - n, \tag{5.275}$$

is called the **twist**.

Note that $O_f^{(2)\mu\nu} = \bar{\psi}_f \gamma^{\{\mu}(iD^{\nu\}})\psi_f = T_f^{\mu\nu}$ has $t = 2$, and in fact all $O_f^{(2)\mu_1 \dots \mu_n}$ have twist $t = 2$. Accordingly, their coefficients C_{12}^n are not suppressed in $1/Q$. They are called **leading-twist** terms. In the operator product of two currents of different flavour, $\bar{u}\gamma^\mu u \bar{d}\gamma^\nu d$, the OPE starts with operators of dimension 6 and spin 2, so they have twist $t = 4$, and their coefficients behave like $1/Q^2$. They are higher-twist terms and are suppressed in DIS with respect to the leading-twist terms. As we shall see, the OPE implies the factorisation picture of the Parton Model, which is valid at leading twist.

Comparing the OPE of the hadronic tensor $\tilde{W}^{\mu\nu}$ above to the expression of $\tilde{W}^{\mu\nu} = 2\pi m_p W^{\mu\nu}$ (2.18) in terms of structure functions, we obtain

$$W_1 = \frac{1}{\pi m_p} \sum_f Q_f^2 \sum_{n=2}^{\infty} \frac{(2q \cdot P)^n}{(Q^2)^n} A_f^n, \tag{5.276}$$

$$W_2 = \frac{4 m_p}{\pi} \sum_f Q_f^2 \sum_{n=2}^{\infty} \frac{(2q \cdot P)^{n-2}}{(Q^2)^{n-1}} A_f^n, \tag{5.277}$$

which fulfils the Callan-Gross relation (2.93). It is not surprising that we obtain the Callan-Gross relation since we deal with OPE of fermionic currents, however, it is remarkable that we obtain it without any input from the Parton Model.

Using instead the Parton Model relation (2.91) between the structure function W_2 and the PDFs, $\sum_f x f_f(x) Q_f^2 = \frac{P \cdot q}{m_p} \operatorname{Im} W_2(P, q)$, we can write the PDFs $f_f(x)$ in terms of the OPE of the hadronic tensor $\tilde{W}^{\mu\nu}$.

5.8.4 Analytic Structure of W_2 and Coefficient Functions as Moments of PDFs

We have derived the OPE of $\tilde{W}^{\mu\nu}$ supposing that $Q^2 \to \infty$, however, as we know $\omega = 2q \cdot P \geq Q^2$, such that $x_{bj} = \dfrac{Q^2}{2q \cdot P} \leq 1$, in the DIS physical region. We can view W_2 as an analytic function of ω at fixed Q^2, in the ω complex plane (fig. 5.3). We know that the DIS cross section is proportional to the discontinuity of $\tilde{W}^{\mu\nu}$, and thus of W_2, which must be computed along the branch cut for $\omega \geq Q^2$. This corresponds to the s-channel process. Further, we know that $\tilde{W}^{\mu\nu}$ is invariant under $q \leftrightarrow -q$, thus $W_2(-\omega, Q^2) = W_2(\omega, Q^2)$, which implies that we must have also a branch cut for $\omega \leq -Q^2$. This corresponds to a u-channel process. Since $-Q^2 < 0$, there are no singularities

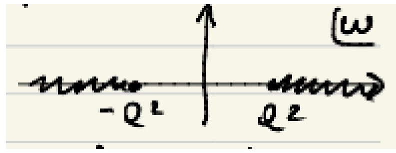
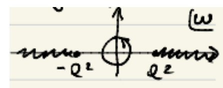


Figure 5.3: Representation of ω in the complex plane with branch cuts extending from Q^2 to ∞ and from $-Q^2$ to $-\infty$.

associated to the t channel. We may then consider the contour integral,

$$I_n = \oint \frac{\mathrm{d}\omega}{2\pi i} \frac{1}{\omega^{n-1}} W_2(\omega, Q^2) \tag{5.278}$$



,

on a contour around the origin. The residue of I_n is given by the ω^{n-2} term in $W_2(\omega, Q^2)$ (5.277). Then

$$I_n = \frac{4m_p}{\pi} \frac{1}{(Q^2)^{n-1}} \sum_f Q_f^2 A_f^n. \tag{5.279}$$

However, we can also deform the contour, so that it becomes an integral over the discontinuities (see fig. 5.4). Since $\mathrm{Disc} W_2 = 2i \,\mathrm{Im}\, W_2$ (2.84), and the two branch cuts yield equal contributions,

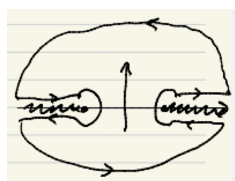


Figure 5.4: Representation of ω in the complex plane with branch cuts extending from Q^2 to ∞ and from $-Q^2$ to $-\infty$ and a specific contour.

we can also write

$$I_n = 2 \int_{Q^2}^{\infty} \frac{\mathrm{d}\omega}{2\pi i} \frac{1}{\omega^{n-1}} 2i \,\mathrm{Im}\, W_2(\omega, Q^2). \tag{5.280}$$

Changing variables to $x = Q^2/\omega$,

$$\begin{aligned} I_n &= 2 \int_0^1 \frac{\mathrm{d}x}{2\pi i} \frac{\omega}{x} \left(\frac{x}{Q^2}\right)^{n-1} 2i \,\mathrm{Im}\, W_2(\omega, Q^2) \\ &= \frac{2}{\pi} \frac{1}{(Q^2)^{n-1}} \int_0^1 \mathrm{d}x \, \omega x^{n-2} \,\mathrm{Im}\, W_2. \end{aligned} \tag{5.281}$$

Now, we use the Parton Model relation between W_2 and the PDFs $f_f(x)$ (2.91), rewritten in terms of ω,

$$\mathrm{Im}\, W_2 = \frac{2m_p}{\omega} \sum_f x f_f(x) Q_f^2. \tag{5.282}$$

So,

$$I_n = \frac{4m_p}{\pi} \frac{1}{(Q^2)^{n-1}} \int_0^1 \mathrm{d}x \, x^{n-1} \sum_f f_f(x) Q_f^2. \tag{5.283}$$

If we equate it to the expression of I_n obtained on a contour around the origin (5.279), we obtain that

$$A_f^n = \int_0^1 \mathrm{d}x\, x^{n-1} f_f(x), \tag{5.284}$$

i.e., *the coefficients of the proton matrix elements of the leading-twist operators of the OPE of* $\tilde{W}^{\mu\nu}$ *are the moments of the parton distributions.* More precisely, since the DIS cross section is

$$\frac{d\sigma}{dxdy} = \frac{2\pi\alpha^2 s}{(Q^2)^2}\left[1+(1-y)^2\right]\sum_i x f_i(x) Q_i^2, \tag{5.285}$$

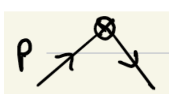
where the quark and antiquark of flavour f contribute equally, by $f_f(x)$ we mean $f_f^+(x) = f_f + \bar{f}_f$.

5.8.5 One-loop renormalisation of quark leading-twist operators

As we saw in eqs. (5.243) and (5.250), in the case of a product of conserved currents, $J^\mu(x)J^\nu(0)$, for which the anomalous dimensions vanish, $\gamma_1 = \gamma_2 = 0$, the anomalous dimension of the coefficient function C_{12}^n, or A_f^n above, is $-\gamma_n$, where γ_n is the anomalous dimension of the operator $O_f^{(n)}$ (5.240). So, the anomalous dimension of an operator is equal and opposite to the one of its coefficient function, $\gamma_{A_f^n} = -\gamma_{O_f^{(n)}} = -\gamma_n$. Then, with one-loop running of α_s, and fixing $\gamma_{A_f^n} = \frac{\alpha_s}{2\pi}\gamma^{(0)}_{A_f^n}$, the CS equation of A_f^n (5.250) implies that A_f^n scales like

$$A_f^n(Q^2) = \left(\frac{\log\left(Q^2/\Lambda^2\right)}{\log\left(\mu^2/\Lambda^2\right)}\right)^{\frac{\gamma^{(0)}_{A_f^n}}{\beta_0}} = \left(\frac{\log\left(Q^2/\Lambda^2\right)}{\log\left(\mu^2/\Lambda^2\right)}\right)^{-\frac{\gamma_n^{(0)}}{\beta_0}}. \tag{5.286}$$

Let us define the quark leading-twist operators to be



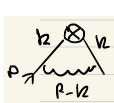
$$= \gamma^{\mu_1} P^{\mu_2} \ldots P^{\mu_n} - \text{traces} \qquad \text{at} \qquad p^2 = -M^2\,, \tag{5.287}$$

which is understood to have zero momentum inserted into the operator, to be symmetrised over the n indices, and to have all the trace terms subtracted.

Then, in order to compute the anomalous dimensions, one needs the counterterms for operators rescaling. At one loop, we need to compute terms like,

(5.288)

We consider first the diagram on the left-hand side, which we evaluate in Feynman gauge,



$$= I_1 = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, igt^a\gamma^\nu\, \frac{i\not{k}}{k^2}\, \gamma^{\mu_1} k^{\mu_2} \ldots k^{\mu_n}\, \frac{i\not{k}}{k^2}\, igt^a\gamma_\nu\, \frac{-i}{(p-k)^2}\,. \tag{5.289}$$

In order to get the correct scaling dimensions, we will evaluate it in $d = 4 - 2\epsilon$ dimensions, but at every step we will neglect the $\mathcal{O}(\epsilon)$ corrections, since we are only interested in the $1/\epsilon$ pole. We use the identity (easy to check)

$$\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\lambda \gamma_\nu = -2\gamma^\lambda \gamma^\mu \gamma^\rho + \mathcal{O}(\epsilon)\,, \tag{5.290}$$

so

$$I_1 = 2i\, g^2\, C_F \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{\not{k}\, \gamma^{\mu_1} k^{\mu_2} \dots k^{\mu_n}\, \not{k}}{(k^2)^2 (p-k)^2}\,. \tag{5.291}$$

Then we use Feynman parametrisation,

$$\frac{1}{D_1 D_2^n} = \int_0^1 \mathrm{d}x\, \mathrm{d}y \delta(1-x-y) \frac{n y^{n-1}}{(xD_1 + yD_2)^{n+1}}\,, \tag{5.292}$$

which in our case is

$$\begin{aligned} \frac{1}{(k^2)^2 (p-k)^2} &= \int_0^1 \mathrm{d}x \frac{2(1-x)}{[x(p-k)^2 + (1-x)(k^2)^2]^3} \\ &= \int_0^1 \mathrm{d}x \frac{2(1-x)}{[(k-xp)^2 + x(1-x)p^2]^3}\,. \end{aligned} \tag{5.293}$$

As usual, in integrating over $\mathrm{d}^d k$, we shift $k = k' + xp$ such that the denominator has no linear dependence on k'. Then $(k'+xp)^{\mu_2} \dots (k'+xp)^{\mu_n}$ will appear in the numerator. Upon integrating over $\mathrm{d}^d k'$, because of the integration range, only even powers of k' in the numerator will contribute, and these will yield $g^{\mu_i \mu_j}$ terms, which will be removed when subtracting traces, unless the indices μ_i or μ_j are contracted. Let us rename k' as k. The only left-over terms in the numerator are $\not{k}\gamma^{\mu_1} k^{\mu_j}$ and $\not{k}\gamma^{\mu_1}\not{k}$. Upon integration, the former yields $\gamma^{\mu_j}\gamma^{\mu_1}$, which upon symmetrising over μ_1 and μ_j yields $g^{\mu_1 \mu_j}$, which can be traced away. So we are left with

$$I_1 = 2i\, g^2\, C_F \int_0^1 \mathrm{d}x\, 2(1-x) x^{n-1}\, p^{\mu_2} \dots p^{\mu_n} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{\not{k}\, \gamma^{\mu_1} \not{k}}{(k^2 - \Delta)^3}\,, \tag{5.294}$$

with $\Delta = -x(1-x)p^2$. Now

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{k^\nu k^\sigma}{(k^2-\Delta)^3} = \frac{i}{(4\pi)^{2-\epsilon}}\, \frac{g^{\nu\sigma}}{2}\, \frac{\Gamma(\epsilon)}{\Gamma(3)}\, \frac{1}{\Delta^\epsilon}\,, \tag{5.295}$$

and

$$\gamma^\nu \gamma^{\mu_1} \gamma_\nu = -(d-2)\gamma^{\mu_1}\,, \tag{5.296}$$

so

$$I_1 = 2C_F \frac{g^2}{(4\pi)^{2-\epsilon}} \int_0^1 \mathrm{d}x\, (1-x) x^{n-1}\, \gamma^{\mu_1} p^{\mu_2} \dots p^{\mu_n}\, \frac{\Gamma(\epsilon)}{[-x(1-x)p^2]^\epsilon}\,, \tag{5.297}$$

where as usual we neglect $\mathcal{O}(\epsilon)$ corrections. Then we use

$$\int_0^1 \mathrm{d}x\, (1-x)^{1-\epsilon} x^{n-1-\epsilon} = \frac{\Gamma(2-\epsilon)\Gamma(n-\epsilon)}{\Gamma(n+2-2\epsilon)} = \frac{1}{n(n+1)} + \mathcal{O}(\epsilon) \tag{5.298}$$

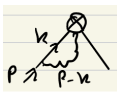
so

$$I_1 = \frac{\alpha_s}{4\pi}\, \frac{2C_F}{n(n+1)}\, \frac{\Gamma(\epsilon)}{(4\pi)^{-\epsilon}(-p^2)^\epsilon}\, \gamma^{\mu_1} p^{\mu_2} \dots p^{\mu_n}\,. \tag{5.299}$$

Using $S_\epsilon = (4\pi)^\epsilon \Gamma(1+\epsilon)$ (4.51), this can also be written in the $\overline{\mathrm{MS}}$ scheme as

$$I_1 = \frac{\alpha_s}{4\pi}\,\frac{2C_F}{n(n+1)}\,\frac{S_\epsilon}{\epsilon}(-p^2)^{-\epsilon}\,\gamma^{\mu_1}p^{\mu_2}\dots p^{\mu_n}\,. \tag{5.300}$$

The additional contributions (centre and right in eq. (5.288)) are due to the gluon field in any covariant derivative $iD^\mu = i\partial^\mu - gA^{\mu a}t^a$ of the operator (5.262) being contracted with the gluon on the external legs. This yields the contributions in the centre and right of eq. (5.288) with an index j ranging from 2 to n. We compute the first contribution,



$$\begin{aligned} = I_2 &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, igt^a\gamma^\nu\,\frac{i\not{k}}{k^2}\,\gamma^{\mu_1}k^{\mu_2}\dots k^{\mu_{j-1}}\,(-gt^a g^{\mu_j}{}_\nu)\,p^{\mu_{j+1}}\dots p^{\mu_n}\,\frac{-i}{(p-k)^2} \\ &= i\,g^2\,C_F\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{\gamma^{\mu_j}\not{k}\gamma^{\mu_1}k^{\mu_2}\dots k^{\mu_{j-1}}\,p^{\mu_{j+1}}\dots p^{\mu_n}}{(k^2)(p-k)^2}\,. \end{aligned} \tag{5.301}$$

We may symmetrise over μ_i and μ_j using eq. (5.256) as

$$\frac{1}{2}\left(\gamma^{\mu_1}\gamma^\rho\gamma^{\mu_j} + \gamma^{\mu_j}\gamma^\rho\gamma^{\mu_1}\right)k_\rho = \left(g^{\mu_1\rho}\gamma^{\mu_j} + g^{\mu_j\rho}\gamma^{\mu_1} - g^{\mu_1\mu_j}\gamma^\rho\right)k_\rho\,. \tag{5.302}$$

Then the last term can be traced away, and we are left with

$$I_2 = 2i\,g^2\,C_F\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{\gamma^{\mu_1}k^{\mu_2}\dots k^{\mu_j}\,p^{\mu_{j+1}}\dots p^{\mu_n}}{(k^2)(p-k)^2}\,. \tag{5.303}$$

where the symmetry over μ_1 and μ_j, as well as any pair of indices, is understood. We use Feynman parametrisation (5.292) with $n=1$. As above, we shift $k = k' + xp$ the integral over k, and we get $(k'+xp)^{\mu_2}\dots(k'+xp)^{\mu_{j-1}}$ in the numerator. Upon integrating over $\mathrm{d}^d k'$, we retain only the even powers of k', which yield $g^{\mu_i\mu_j}$ terms, which are removed when subtracting traces. So the only left-over term is

$$\begin{aligned} I_2 &= 2i\,g^2\,C_F\,\gamma^{\mu_1}\,p^{\mu_2}\dots p^{\mu_n}\int_0^1\mathrm{d}x\,x^{j-1}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{(k^2-\Delta)^2} \\ &= -2g^2\,C_F\,\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\,\gamma^{\mu_1}\,p^{\mu_2}\dots p^{\mu_n}\int_0^1\mathrm{d}x\,x^{j-1}\Delta^{-\epsilon}\,. \end{aligned} \tag{5.304}$$

Then we use

$$\int_0^1\mathrm{d}x\,(1-x)^{-\epsilon}x^{j-1-\epsilon} = \frac{\Gamma(j-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+j-2\epsilon)} = \frac{1}{j} + \mathcal{O}(\epsilon)\,, \tag{5.305}$$

so

$$I_2 = -\frac{\alpha_s}{4\pi}\,\frac{2C_F}{j}\,\frac{\Gamma(\epsilon)}{(4\pi)^{-\epsilon}(-p^2)^\epsilon}\,\gamma^{\mu_1}\,p^{\mu_2}\dots p^{\mu_n}\,, \tag{5.306}$$

and finally we sum over the insertion j from 2 to n.

The right-most contribution in eq. (5.288) yields the same result as in eq. (5.306), which together with eq. (5.299) allows us to write the sum of the three terms of eq. (5.288) as

$$I = \frac{\alpha_s}{4\pi}\,2C_F\left[\frac{1}{n(n+1)} - 2\sum_{j=2}^n\frac{1}{j}\right]\frac{\Gamma(\epsilon)}{(4\pi)^{-\epsilon}(-p^2)^\epsilon}\,\gamma^{\mu_1}\,p^{\mu_2}\dots p^{\mu_n}\,. \tag{5.307}$$

Thus the counterterm is

$$\delta_{O_f} = -\frac{\alpha_s}{4\pi}\, 2C_F \left[\frac{1}{n(n+1)} - 2\sum_{j=2}^{n} \frac{1}{j} \right] \frac{\Gamma(\epsilon)}{(4\pi)^{-\epsilon}(M^2)^{\epsilon}}\,,. \tag{5.308}$$

In sec. 4.7.1, we saw that the anomalous dimension of a local operator at one loop is given by eq. (4.158), in our case

$$\gamma_{O_f} = M\frac{\partial}{\partial M}\left(\delta_2 - \delta_{O_f}\right)\,. \tag{5.309}$$

where the counterterm δ_2 of the quark field is given e.g. in eq. (4.104), with $\zeta = 1$ in Feynman gauge. Then

$$\begin{aligned} \gamma_{O_f} &= \frac{\alpha_s}{4\pi}\, 2C_F \left[1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^{n} \frac{1}{j} \right] \\ &= -\frac{\alpha_s}{2\pi}\, 2C_F \left[\frac{3}{2} + \frac{1}{n(n+1)} - 2\sum_{j=1}^{n} \frac{1}{j} \right]\,. \end{aligned} \tag{5.310}$$

where in the second line we wrote the anomalous dimension in such a way as to make more readily contact with sec. 5.7.

As we stated above eq. (5.286), the anomalous dimension of the coefficient function A_f^n is $\gamma_{A_f^n} = -\gamma_{O_f^{(n)}}$. Thus,

$$\gamma_{A_f^n} = \frac{\alpha_s}{2\pi}\,\gamma^{(0)}_{A_f^n}\,, \qquad \text{with} \qquad \gamma^{(0)}_{A_f^n} = 2C_F \left[\frac{3}{2} + \frac{1}{n(n+1)} - 2\sum_{j=1}^{n} \frac{1}{j} \right]\,, \tag{5.311}$$

such that with one-loop running of α_s, A_f^n scales like in eq. (5.286). Note that up to a factor 2 due to the different normalisation, this agrees with the anomalous dimension of the non-singlet splitting function (5.203) we computed from the evolution equations, such that the scaling of A_f^n (5.286) is in agreement with the scaling of the moment of the non-singlet parton distribution (5.200).

That is not the end of the story. There are also gluon twist-2 operators,

$$O_g^{(n)\mu_1\ldots\mu_n} = F^{\mu_1\nu}(iD^{\mu_2})\ldots(iD^{\mu_{n-1}})F_{\nu}^{\mu_n} - \text{traces}\,, \tag{5.312}$$

where $F^{\mu_1\nu}$ is the field-strength tensor, and which is understood to be symmetric over the indices $\mu_1\ldots\mu_n$. Upon rescaling, the gluon twist-2 operators mix with the quark twist-2 operators. In fact, terms like,

,

will yield the $\gamma_{qg}(n)$ quark-gluon mixing. Terms like,

,

will yield the $\gamma_{gq}(n)$ gluon-quark mixing. Finally, terms like,

,

will yield the $\gamma_{gg}(n)$ anomalous dimension.

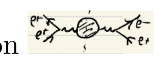
The outcome for the leading-order anomalous dimensions $\gamma_{ij}^{(0)}(n)$, and the way they mix, is the same as what we obtained in eqs. (5.209)-(5.211) from the DGLAP evolution equations, with anomalous dimension matrix given by eq. (5.215), so we will not re-compute it through the OPE. However, while the computation of the anomalous dimensions $\gamma_{ij}^{(0)}(n)$ through the parton evolution picture of sec. 5.7 is valid only at leading order in α_s, the OPE is more general and allows one to extend the computation beyond the leading order. In this way, the higher-order terms in

$$\gamma_{ij}(n) = \frac{\alpha_s}{2\pi}\gamma_{ij}^{(0)}(n) + \left(\frac{\alpha_s}{2\pi}\right)^2 \gamma_{ij}^{(1)}(n) + \left(\frac{\alpha_s}{2\pi}\right)^3 \gamma_{ij}^{(2)}(n) + \dots\,, \tag{5.313}$$

have been computed through $\gamma_{ij}^{(2)}(n)$. Further, it was the OPE that showed first that an asymptotically free field theory was needed in order to explain the scaling behaviour of DIS.

Finally, note that also the total cross section for $e^+e^- \to$ hadrons can be related to the forward scattering for $e^+e^- \to e^+e^-$,

$$\sigma_{tot}(e^+e^- \to \text{had}) = \frac{1}{2s}\,\text{Im}\,\mathcal{M}(e^+e^- \to e^+e^-), \tag{5.314}$$

whose amplitude can be expressed through the hadronic part of the vacuum polarisation , which can be written as the matrix element between vacuum states of the product $< 0|J^\mu(x)J^\nu(0)|0 >$ of two currents $J^\mu = \sum_f Q_f \bar{\psi}_f \gamma^\mu \psi_f$ of quarks, leading to an OPE.

5.9 Coherent Branching

So far, we have provided a somewhat general and systematic discussion of the collinear divergences, and how they arise in parton evolution. The soft divergences have been discussed specifically in sec. 5.1.1, in the context of a virtual photon decaying as $\gamma^* \to q\bar{q}g$. In this section, we will extend that discussion, leading to the concepts of **angular ordering** and **colour coherence**, although we will fall short of a general and systematic discussion of the soft divergences, which is beyond the scope of the present course.


,

In Sect. 5.5 and 5.6, we have shown that in the limit of a collinear emission from an incoming parton, we can write the scattering for the production of, say, $(n+1)$ partons as,

$$\mathrm{d}\sigma_{n+1} = \mathrm{d}\sigma_n \frac{\alpha_s}{2\pi}\frac{\mathrm{d}p_\perp^2}{p_\perp^2}\,\mathrm{d}z P_{ba}(z), \tag{5.315}$$

where the collinear emissions are ordered by the (space-like) virtuality of the propagators $\sim p_\perp^2$. Likewise, in a time-like branching, in the limit of a collinear emission of two final-state partons,

,

we can write the cross section for the production of $(n+1)$ partons as,

$$d\sigma_{n+1} = d\sigma_n \frac{\alpha_s}{2\pi} \frac{dp_\perp^2}{p_\perp^2} dz P_{ab}(z), \tag{5.316}$$

where the time-like splitting functions are, at leading order, the same as in the space-like case (only the indices are chosen differently, e.g.

,

so the space-like P_{qg} equals the time-like $P_{q\bar{q}}$). The only important difference is in the evolution, which in the space-like case evolves by increasing virtualities, while in the time-like case by decreasing virtualities.

5.9.1 Eikonal Current

In sec. 5.1.1, in the context of a virtual photon decaying as $\gamma^* \to q\bar{q}g$, we showed that in $q\bar{q}$ production, soft-gluon emission factorises (*eikonal approximation*), eq. (5.7),

$$i\mathcal{M}_{soft} = -g \left(\frac{p \cdot \epsilon(k)}{p \cdot k} - \frac{p' \cdot \epsilon(k)}{p' \cdot k} \right) \bar{u}(p) t^a (-ie\Gamma^\mu) u(p').$$

In fact, this can be generalised to an **eikonal current** which represents the emission of a soft gluon out of n partons,

$$J^\mu(k) = \sum_{i=1}^{n} T_i \frac{p_i \cdot \epsilon(k)}{p_i \cdot k}, \tag{5.317}$$

where we associate a colour charge T_i with the emission of a soft gluon from parton i, such that if the emitted soft gluon has colour c, then

$$T_i \equiv T_i^c \left| c \right\rangle, \tag{5.318}$$

and we define the operation of T_i on an n-parton colour space as,

$$< a_i, ..., a_n, c | T_i | b_1, ..., b_n >= \delta_{a_1 b_1} ... T^c_{a_i b_i} ... \delta_{a_n b_n}, \tag{5.319}$$

with

$$T^c_{ab} = \left\{ \begin{array}{l} t^c_{ab} \\ \bar{t}^c_{ab} = -t^c_{ba} \\ i f_{acb} \end{array} \right\} \quad \text{if } i \text{ is a} \left\{ \begin{array}{l} \text{quark} \\ \text{antiquark} \\ \text{gluon} \end{array} \right\}. \tag{5.320}$$

Then $T_i^2 = C_i$, with

$$C_i = \left\{ \begin{array}{c} C_F \\ C_A \end{array} \right\} \quad \text{for a} \left\{ \begin{array}{l} \text{(anti)quark} \\ \text{gluon} \end{array} \right\}. \tag{5.321}$$

For $\gamma^* \to q\bar{q}g$, or more in general $\gamma^* \to n$ partons plus a soft gluon, the sum of the colour charges of the n partons vanishes, so $|1, ..., n>$ is a colour-singlet state and colour conservation becomes

$$\sum_{i=1}^{n} T_i |1, ..., n >= 0, \tag{5.322}$$

or for short

$$\sum_{i=1}^{n} T_i = 0. \tag{5.323}$$

By squaring the eikonal current (5.7), in sec. 5.1.1, we computed the rate for soft-gluon emission (5.11). Neglecting the fermion mass, it reduces to

$$\mathrm{d}\sigma_{soft}(\gamma^* \to q\bar{q}g) = \mathrm{d}\sigma(\gamma^* \to \bar{q}q) g^2 C_F \frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0} \frac{2p \cdot p'}{p \cdot k p' \cdot k} \,. \tag{5.324}$$

Using the phase-space measure (5.12), with solid angle, $\mathrm{d}\Omega = \mathrm{d}\varphi \, \mathrm{d}\cos\theta$, eq. (5.324) becomes

$$\mathrm{d}\sigma_{soft}(\gamma^* \to q\bar{q}g) = \mathrm{d}\sigma(\gamma^* \to \bar{q}q) \, \frac{\alpha_s}{2\pi} C_F \, \frac{\mathrm{d}k^0}{k^0} \, \frac{\mathrm{d}\Omega}{2\pi} \, \frac{2p \cdot p'(k^0)^2}{p \cdot k p' \cdot k} \,, \tag{5.325}$$

whose integral over k^0 yields an IR logarithm.

Let us generalise eq. (5.325) to the emission of a soft gluon out of the production of n partons,

$$\mathrm{d}\sigma_{n+1} = \mathrm{d}\sigma_n \frac{\alpha_s}{2\pi} \frac{\mathrm{d}k^0}{k^0} \frac{\mathrm{d}\Omega}{2\pi} \sum_{i,j=1}^{n} C_{ij} W_{ij} \,, \tag{5.326}$$

with C_{ij} a colour factor and W_{ij} a radiation function,

$$W_{ij} = \frac{(k^0)^2 p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)} = \frac{1 - \cos\theta_{ij}}{(1 - \cos\theta_{ik})(1 - \cos\theta_{jk})} \,, \tag{5.327}$$

and the sum $\sum_{i,j=1}^{n}$ is over all pairs of partons (i, j). In the soft-gluon emission (5.326), we choose the colour factor as $C_{ij} = -T_i \cdot T_j$, such that in the 2-parton state, $\gamma^* \to q\bar{q}$,

$$T_q^2 = T_{\bar{q}}^2 = C_F, \tag{5.328}$$

and

$$T_q + T_{\bar{q}} = 0 \quad \Rightarrow \quad -T_q \cdot T_{\bar{q}} = \frac{T_q^2 + T_{\bar{q}}^2}{2} = C_F, \tag{5.329}$$

in agreement with eq. (5.325). In this case, there is only one radiation function $W_{q\bar{q}}$ to be computed. This occurs twice in the sum $\sum_{ij}$ and that explains the factor 2 in the radiation function of eq. (5.325).

5.9.2 Angular Ordering

Let us examine the generic radiation function (5.327), It has collinear singularities as i is collinear to k ($i||k$) and $j||k$. Since they are localised in space, it is better to disentangle them in W_{ij}, so that we can deal with them one at a time. This can be done in many different ways. We set,

$$W_{ij} = W_{ij}^{[i]} + W_{ij}^{[j]}, \tag{5.330}$$

with

$$W_{ij}^{[i]} = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos\theta_{ik}} - \frac{1}{1-\cos\theta_{jk}}\right). \tag{5.331}$$

Note that $W_{ij}^{[i]}$ has a collinear singularity only as $i||k$. To see it, we make $W_{ij}^{[i]}$ explicit,

$$\begin{aligned} W_{ij}^{[i]} &= \frac{1}{2}\left[\left(\frac{1-\cos\theta_{ij}}{1-\cos\theta_{ik}} - 1\right)\frac{1}{1-\cos\theta_{jk}} + \frac{1}{1-\cos\theta_{ik}}\right] \\ &= \frac{1}{2}\frac{1}{1-\cos\theta_{ik}}\left(1 + \frac{\cos\theta_{ik} - \cos\theta_{ij}}{1-\cos\theta_{jk}}\right). \end{aligned} \tag{5.332}$$

As $j||k$, $W_{ij}^{[i]}$ has no collinear singularity. Further, $W_{ij}^{[i]}$ has **angular ordering**. In order to see it, we may take parton i as reference and write the orientation of the other momenta with respect to parton i using Euler angles. If the angular integral of the soft gluon is $\mathrm{d}\Omega = \mathrm{d}\cos\theta_{ik}\,\mathrm{d}\varphi_{ik}$, we may average over the azimuthal angle of the soft gluon with respect to parton i. We obtain (please see Exercises),

$$\int_0^{2\pi}\frac{\mathrm{d}\varphi_{ik}}{2\pi}W_{ij}^{[i]} = \begin{cases} \dfrac{1}{1-\cos\theta_{ik}} & \text{if } \theta_{ik} < \theta_{ij} \\ 0 & \text{if } \theta_{ik} > \theta_{ij}. \end{cases} \tag{5.333}$$

So the integral contributes only if the soft gluon k is emitted in a cone centred on line i and with opening angle θ_{ij}.

In displaying angular ordering, we did not use colour and in fact this feature of soft emission is common to all gauge theories, and so both to QED and QCD. In QED, it implies the suppression of soft-photon emission from e^+e^- pairs when the emission angle θ is larger than the opening angle of the pair, $\theta_{e^+e^-}$, and it is called Chudakov effect. It means that wide-angle soft photons cannot appreciate the detailed structure of the e^+e^- pair. They only see the net charge of the e^+e^- pair, which is zero.

5.9.3 Colour Coherence

In order to see how angular ordering works in QCD, let us consider soft-gluon emission out of a 3-parton system. We can write the sum over radiation functions as,

$$W = -T_1\cdot T_2 W_{12} - T_1\cdot T_3 W_{13} - T_2\cdot T_3 W_{23}. \tag{5.334}$$

Colour conservation implies that

$$T_1 + T_2 + T_3 = 0. \tag{5.335}$$

Then,

$$-2T_1 \cdot T_2 = T_1^2 + T_2^2 - T_3^2, \tag{5.336}$$

and likewise for (1,3) and (2,3). So

$$W = \frac{1}{2}\left[T_1^2(W_{12} + W_{13} - W_{23}) + T_2^2(W_{12} + W_{23} - W_{13}) + T_3^2(W_{13} + W_{23} - W_{12})\right]. \tag{5.337}$$

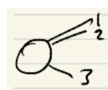
Now, use $W_{12} = W_{12}^{[1]} + W_{12}^{[2]}$ and cyclic permutations. Further, introduce the functions,

$$\tilde{W}_{23}^{[1]} = \frac{1}{2}\left(W_{13}^{[1]} - W_{12}^{[1]}\right), \qquad \text{and} \qquad \tilde{W}_{13}^{[2]} = \frac{1}{2}\left(W_{23}^{[2]} - W_{12}^{[2]}\right). \tag{5.338}$$

Note that $\tilde{W}_{23}^{[1]}$ has no collinear singularity in direction 1, and likewise for $\tilde{W}_{13}^{[2]}$ in direction 2. Then, with a bit of algebra we can decompose W as,

$$\begin{aligned} W = {} & T_1^2\left[W_{12}^{[1]} + \tilde{W}_{23}^{[1]} - \tilde{W}_{13}^{[2]} + \frac{1}{2}\left(W_{13}^{[3]} - W_{23}^{[3]}\right)\right] \\ & + T_2^2\left[W_{12}^{[2]} + \tilde{W}_{13}^{[2]} - \tilde{W}_{23}^{[1]} - \frac{1}{2}\left(W_{13}^{[3]} - W_{23}^{[3]}\right)\right] \\ & + T_3^2\left[\tilde{W}_{23}^{[1]} + \tilde{W}_{13}^{[2]} + \frac{1}{2}\left(W_{13}^{[3]} + W_{23}^{[3]}\right)\right]. \end{aligned} \tag{5.339}$$

Now, we study the configuration where $\theta_{12} \ll \theta_{13}, \theta_{23}$.



We can approximate the direction of 1 or 2 by the common direction (12) in any function which is not singular in the direction of 1 or 2. That is the case for $W_{13}^{[3]}$, $W_{23}^{[3]}$, $\tilde{W}_{23}^{[1]}$ and $\tilde{W}_{13}^{[2]}$. So we can write,

$$W_{13}^{[3]} \simeq W_{23}^{[3]} \simeq W_{(12)3}^{[3]} \qquad \text{and} \qquad \tilde{W}_{23}^{[1]} \simeq \tilde{W}_{13}^{[2]}, \tag{5.340}$$

and the radiation function (5.339) is reduced to

$$W \cong T_1^2 W_{12}^{[1]} + T_2^2 W_{12}^{[2]} + T_3^2\left[W_{(12)3}^{[3]} + \tilde{W}_{23}^{[1]} + \tilde{W}_{13}^{[2]}\right]. \tag{5.341}$$

Now, define

$$\tilde{W}_{(12)3}^{[12]} = \tilde{W}_{23}^{[1]} + \tilde{W}_{13}^{[2]} \simeq 2\tilde{W}_{23}^{[1]} \simeq 2\tilde{W}_{13}^{[2]}, \tag{5.342}$$

which has no collinear singularities by construction. After azimuthal averaging,

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi_{2k}}{2\pi} \tilde{W}_{13}^{[2]} = \left\{ \begin{array}{c} 0 \\ \dfrac{1}{1-\cos\theta_{2k}} \end{array} \right\} \text{if} \left\{ \begin{array}{l} \theta_{2k} < \theta_{12} \\ \theta_{12} < \theta_{2k} < \theta_{23} \end{array} \right\}, \tag{5.343}$$

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi_{1k}}{2\pi} \tilde{W}_{23}^{[1]} = \left\{ \begin{array}{c} 0 \\ \dfrac{1}{1-\cos\theta_{1k}} \end{array} \right\} \text{if} \left\{ \begin{array}{l} \theta_{1k} < \theta_{12} \\ \theta_{12} < \theta_{1k} < \theta_{13} \end{array} \right\}. \tag{5.344}$$

For $\theta_{12} < \theta_{2k} < \theta_{23}$ and $\theta_{12} < \theta_{1k} < \theta_{13}$, we have

$$\theta_{1k} \simeq \theta_{2k} \equiv \theta_{(12)k}. \tag{5.345}$$

Then, we can say

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi_{(12)k}}{2\pi} \tilde{W}^{[12]}_{(12)3} = \begin{cases} \dfrac{2}{1-\cos\theta_{(12)k}} & \text{for } \theta_{(12)k} > \theta_{12} \\ 0 & \text{for } \theta_{(12)k} < \theta_{12}\,, \end{cases} \tag{5.346}$$

Further, the system (12) has colour charge,

$$T_{(12)} = T_1 + T_2 = -T_3. \tag{5.347}$$

So we can finally write

$$W \cong T_1^2 W^{[1]}_{12} + T_2^2 W^{[2]}_{12} + T_3^2 W^{[3]}_{(12)3} + T^2_{(12)} W^{[12]}_{(12)3}, \tag{5.348}$$

i.e. the partons radiate proportionally to their colour charge squared. For emissions within the cone (12), $\theta_{ik} < \theta_{12}$, $i = 1, 2$, they sum incoherently. For $\theta_{ik} > \theta_{12}$, partons 1 and 2 yield a coherent contribution, proportional to their combined charge squared, as if it came from an internal line of momentum $p_{(12)} = p_1 + p_2$.

These features extend to more emissions, leading to a coherent parton branching, which is a tenet of any modern parton shower Monte Carlo. In collinear emissions, the evolution variable is the virtuality, in coherent branching the evolution variable is the opening angle, $1 - \cos\theta$. Then the basic formula for coherent branching is

$$\mathrm{d}\sigma_{n+1} = \mathrm{d}\sigma_n \frac{\alpha_s}{2\pi} \frac{\mathrm{d}\zeta}{\zeta} \,\mathrm{d}z P_{ba}(z), \tag{5.349}$$

with $\zeta = 1 - \cos\theta$.

Chapter 6

Epilogue

In sec. 5.4, we introduced inclusive cross sections as an example of infrared-safe quantities, i.e. quantities for which the soft and collinear divergences cancel order by order in perturbation theory, and we made the specific example of the production rate for $e^+e^- \to 3$ jets, up to the $\mathcal{O}(\alpha_s^2)$ corrections.

It is good then to have a formalism that deals with soft and collinear divergences to all orders of perturbation theory, and that enables us to tell that for a given infrared-safe quantity, the infrared divergences will cancel at any order in perturbation theory, although in practice we may be able to compute explicitly only a few lowest orders. Such a formalism is founded on the analytic structure of the Feynman integrals, which occur in the scattering amplitudes. Due to lack of time, we will not discuss it in the necessary detail, so we will just outline its most important tenets.

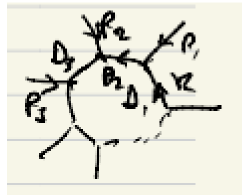
6.1 Analytic Structure of Feynman Integrals

In Euclidean space and for non-zero masses, Feynman integrals are analytic functions of the external momenta. When continuing to Minkowski, branch points occur, which are often associated with particle thresholds. Of course, any on-shell momentum is at threshold for the emission of (zero-momentum) soft partons, and any on-shell massless line is at threshold for the production of collinear particles. So we may see soft and collinear singularities as special cases of singularities of Feynman integrals in the complex plane.

Let us consider then an n-point function. It is a function of the Mandelstam invariants $\{Q_i^2\}$ formed by the n external legs, and by their masses $\{m_i\}$. In $d = 4-2\epsilon$ dimensions, we can write a one-loop Feynman integral of the n-point function as

$$I_n^{(1)}\left(\{\nu_i\}, \{Q_i^2\}, \{m_i\}\right) = e^{\gamma_E \epsilon} \int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \prod_{i=1}^{n} \frac{1}{D_i^{\nu_i}}, \tag{6.1}$$

and diagrammatically



,

with

$$\begin{cases} D_1 & = k^2 - m_1^2 + i\epsilon \\ D_i & = \ell_i^2(k, \{p_i\}) - m_i^2 + i\epsilon \qquad i = 2, ..., n, \end{cases} \tag{6.2}$$

and

$$\ell_i^\mu(k, \{p_i\}) = k^\mu + \sum_{j=1}^{i-1} \eta_j p_j^\mu, \qquad\qquad \eta_j = \begin{cases} -1 & \text{for outgoing momenta} \\ +1 & \text{for incoming momenta.} \end{cases} \tag{6.3}$$

In general, we also have a numerator $F\left(\{Q_i^2\}, \{m_i\}\right)$, but for the sake of simplicity we set $F = 1$, i.e. we take a scalar integral. We introduce the Feynman parametrisation,

$$\prod_{i=1}^n \frac{1}{D_i^{\nu_i}} = \frac{\Gamma(\sum_{i=1}^n \nu_i)}{\Gamma(\nu_1)...\Gamma(\nu_n)} \prod_{i=1}^n \int \mathrm{d}x_i\, x_i^{\nu_i - 1} \frac{\delta(1 - \sum_{i=1}^n x_i)}{D(\{x_i\}, k, \{p_r\})^{\sum_i \nu_i}}, \tag{6.4}$$

with

$$D(\{x_i\}, k, \{p_r\}) = \sum_{i=1}^n x_i D_i. \tag{6.5}$$

So, basically the one-loop Feynman integral behaves as

$$I_n^{(1)} \sim \prod_{\text{lines } i} \int \mathrm{d}x_i \delta(1 - \sum_{i=1}^n x_i) \int \mathrm{d}^d k D(\{x_i\}, k, \{p_r\})^{-\sum_i \nu_i}. \tag{6.6}$$

It is easy to extend the treatment to an ℓ-loop Feynman integral, and say that it behaves like

$$I_n^{(\ell)} \sim \prod_{\text{lines } i} \int \mathrm{d}x_i \delta(1 - \sum_{i=1}^n x_i) \prod_{\text{loops } j} \int \mathrm{d}^d k_j D(\{x_i\}, \{k_j\}, \{p_r\})^{-\sum_i \nu_i}, \tag{6.7}$$

with $D(\{x_i\}, \{k_j\}, \{p_r\})$ as in eq. (6.5), but with

$$D_i = \ell_i^2(\{k_j\}, \{p_r\}) - m_i^2 + i\epsilon, \qquad \text{with} \qquad \ell_i^\mu(\{k_j\}, \{p_r\}) = \sum_j \eta_{ij} k_j^\mu + \sum_r \eta_{ir} p_r^\mu, \tag{6.8}$$

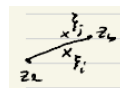
and matrices η_{ij} and η_{ir}, which are specific to the multiloop configuration.

Now, we must find the position of poles and branch points of $I_n^{(\ell)}$ as a function of the external momenta $\{p_i\}$. Take I as a function on $(n + \ell)$ complex coordinates $(\{x_i\}, \{k_j\})$. Singularities arise from zeros of $D(\{x_i\}, \{k_j\}, \{p_r\})$, but not all zeros yield a singularity (remember that we can contour deform the integral in the complex plane, and in particular isolated poles can always be avoided by contour deformation). E.g. take,

$$I(w) = \int_{z_a}^{z_b} \mathrm{d}z F(z, w), \tag{6.9}$$

with $z \in (\{x_i\}, \{k_j\})$. w are the remaining variables: $w \in (\{x_i\}, \{k_j\})/z$. Then no poles on the contour means that $I(w)$ is an analytic function of w. Even with a pole $\xi_i(w)$ on the contour, there is no singularity in the w plane if the contour can be deformed away. Singularities occur when contour deformation cannot avoid a pole. This can happen in two instances:

1. Suppose that the pole migrates to z_a or z_b, say that $\xi(w_0) = z_a$. Then the pole is at an **end-point** of the integral, and $I(w)$ is undefined.
2. Two (or more) poles $\xi_i(w)$ and $\xi_j(w)$ merge at a point $\xi_i(w_0) = \xi_j(w_0)$ on either side of the contour in the z plane.

   This yields a **"pinch" singularity** at $w = w_0$. In the w plane, that can be shown to be a branch point.

6.2 Landau Equations and Pinch Surfaces

In general, we can say that $D(\{x_i\}, \{k_j\}, \{p_r\})$ vanishes on sets of points $\{\bar{x}, \bar{k}\}$ which define **pinch surfaces** in $(\{x_i\}, \{k_j\})$ space. Now, suppose that $z = k_j$, the j-th loop momentum. D is independent of k_j, only if $x_i = 0$ for every line i in loop j. Otherwise, D is quadratic in k_j. D is analytic in the k_j plane, except at

$$D = \sum_{\text{lines i}} x_i(\ell_i^2(\{k_j\}, \{p_r\}) - m_i^2) = 0. \tag{6.10}$$

The k_j contour is trapped if the two k_j roots of $D = 0$ merge. This happens at

$$\left.\frac{\partial D}{\partial k_j}\right|_{D=0} = 0. \tag{6.11}$$

Using eq. (6.8),

$$\begin{aligned}\frac{\partial D}{\partial k_s} &= \frac{\partial}{\partial k_s} \sum_{\text{lines i}} \left[x_i \left(\left(\sum_j \eta_{ij} k_j + \sum_r \eta_{ir} p_r \right)^2 - m_i^2 \right) \right] \\ &= 2 \sum_{\text{lines i}} (x_i \ell_i^\mu) \eta_{is}.\end{aligned} \tag{6.12}$$

So, for $\ell_i^2 = m_i^2$, for every loop j that includes line i we must have that

$$\sum_{\text{lines i}} (x_i \ell_i^\mu) \eta_{ij} = 0. \tag{6.13}$$

For $\ell_i^2 \neq m_i^2$ we must have that

$$x_i = 0. \tag{6.14}$$

The two conditions above are known as **Landau equations**.

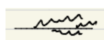
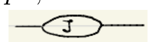
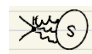
Suppose that line i goes on-shell, $\ell_i^2 = m_i^2$. Take $\Delta x_i^\mu \equiv y_i \ell_i^\mu$ as a space distance. Take two points in this space, and suppose that their distance is $x_b^\mu - x_a^\mu = y_1 \ell_1^\mu$. Take the set of momenta $\ell_2^\mu, ..., \ell_n^\mu$ and use them to connect the points a, b,

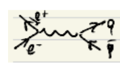
$$x_b^\mu - x_a^\mu = \sum_{i=2}^{n} y_i \ell_i^\mu. \tag{6.15}$$

For the distance $x_b^\mu - x_a^\mu$ to be well defined, we must have

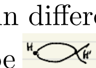
$$y_1 \ell_1^\mu + \sum_{i=2}^{n} (-y_i) \ell_i^\mu = 0. \tag{6.16}$$

This is fulfilled if we choose the y_i's to fulfil the Landau equation (6.13). If the line i is off-shell, $\ell_i^2 \neq m_i^2$, Landau equation (6.14) requires that $x_i = 0$. Then we have the following picture (known as **Coleman-Norton picture**): draw a reduced diagram of a pinch surface by contracting all off-shell lines to points. The other lines are on-shell and correspond to soft or collinear singularities.

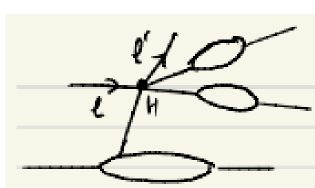
For example, a set of on-shell massless lines with finite energy and with momenta q_i^μ all proportional to a light-like momentum p^μ, , $q_i^\mu = \alpha_i p^\mu$, with $p^2 = 0$ form a jet of collinear particles, whose reduced diagram is . A set of on-shell massless lines with vanishing momenta, $q_i^\mu = 0$, form a soft subdiagram, whose reduced diagram is .

The most general pinch surface for $e^+e^- \to q\bar{q}$  generates the reduced diagram,


.

Note that there must be at least two jets, since the invariant mass of a jet is zero while the invariant mass of the final state is positive. To produce another set of jets, one would need another hard vertex at a later time. However, the two jets move in different directions, and can never meet at a later time (a reduced diagram for such a case would be , with Landau equation $x_1 \ell_1^\mu - x_2 \ell_2^\mu = 0$, with no solution unless $\ell_1^\mu = k \ell_2^\mu$, with k a constant, which is not an allowed singular point). Jets only interact through zero-momentum particles. So there can only be one hard vertex, and to every final-state particle of momentum p_i^μ, it corresponds a jet of total momentum p_i^μ. Thus, for massless particles in $e^+e^- \to$ hadrons, the reduced diagrams is described by collinear and soft divergences, as in the picture above.

In DIS, , the reduced diagram is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where there is at least the incoming jet, made of the proton constituents, and possible one or more outgoing jets out of the hard interaction with the lepton. However, through an infrared power counting (that we will not display here: you can find it in section 13.4 of Sterman's *An Introduction to Quantum Field Theory*),

,

it can be shown that the leading pinch surface is provided by the reduced diagram of forward Compton scattering.

From the infrared power counting, it can also be shown that for massless theories without vector fields, pinch surfaces with soft lines are power finite, thus scalar theories like φ^4 in 4 dimensions or φ^3 in 6 dimensions, have no soft divergences.

6.3 KLN Theorem

For a given theory, let $D(E_0)$ be a set of states with energy, $E_0 - \epsilon < E < E_0 + \epsilon$, and let P_{ij} be the probability density for a transition from state i to state j. Then, at any order of perturbation theory,

$$P(E_0, \epsilon) = \sum_{i,j \in D(E_0)} P_{ij}, \tag{6.17}$$

is free of singularities in the massless limit. This quantum mechanical statement can be proven and it is called Kinoshita-Lee-Nauenberg theorem, or **KLN theorem**.

So, if one sums over a degenerate set of particles within an energy window, e.g. over the collinear particles within a jet or over the set of all soft particles, KLN theorem ensures that the cross section will be finite[1].

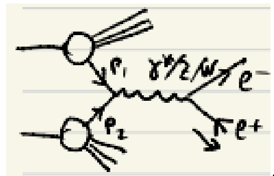
If we apply KLN theorem to cross sections with initial-state divergences, like DIS, the lack of an averaging over the initial state results in a residual dependence on ϵ, and so in left-over initial-state collinear divergences, which we take care of, through the factorisation procedure as we have seen in Sect. 5.6.3, by absorbing them in the parton distributions.

6.4 Drell-Yan Scattering

How does factorisation work for LHC processes? i.e. on scattering processes with two protons in the initial state and two leptons, jets or heavy objects (like vector bosons or Higgs bosons or top quarks) in the final state?

[1]From a KLN perspective, it is somewhat a coincidence that in QED, soft-photon divergences cancel by summing only over final states (cf. Bloch-Nordsieck dealing of soft photons in Sect. 5.2).

As an example, let us take **Drell-Yan scattering**,



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i.e. let us consider the production of a lepton pair in a $q\bar{q}$ annihilation in the collision of two protons, A and B. Let us set the momenta of incoming quark and antiquark,

$$p_1 = x_1 P_1, \qquad p_2 = x_2 P_2, \tag{6.18}$$

with x_1, x_2 the fractions of the proton momenta. The (time-like) photon momentum is

$$q = x_1 P_1 + x_2 P_2. \tag{6.19}$$

In the lab frame,

$$P_1 = (E, 0, 0, E), \qquad P_2 = (E, 0, 0 - E), \tag{6.20}$$

then

$$q = ((x_1 + x_2)E, 0, 0, (x_1 - x_2)E)\,. \tag{6.21}$$

In light-cone coordinates, $q^{\pm} = q^0 \pm q^z$, we rewrite it as

$$q = (2x_1 E, 2x_2 E; 0_{\perp}). \tag{6.22}$$

Let us set,

$$\begin{cases} Qe^{Y} & = 2x_1 E \\ Qe^{-Y} & = 2x_2 E \end{cases} \quad \Rightarrow \quad \begin{cases} Q^2 & = x_1 x_2 s = \hat{s} \\ e^{Y} & = \sqrt{\dfrac{x_1}{x_2}}\,, \end{cases} \tag{6.23}$$

where Q^2 is the squared invariant mass and Y is the rapidity of the photon. Then, the parton momentum fractions can be found,

$$x_1 = \frac{Q}{\sqrt{s}} e^{Y}, \qquad x_2 = \frac{Q}{\sqrt{s}} e^{-Y}. \tag{6.24}$$

We may reconstruct Q^2 and Y from the final state lepton momenta,

$$\ell_i = (p_{\perp_i} e^{y_i}, p_{\perp_i} e^{-y_i}; \vec{p}_{\perp_i}), \tag{6.25}$$

with $\vec{p}_{\perp_2} = -\vec{p}_{\perp_1} \equiv -\vec{p}_{\perp}$. Momentum conservation implies that

$$Qe^{Y} = p_{\perp}(e^{y_1} + e^{y_2}). \tag{6.26}$$

But Y is the rapidity of the lepton pair centre of mass,

$$Y = \frac{y_1 + y_2}{2}, \tag{6.27}$$

and y^* is the rapidity of the ℓ^- or ℓ^+ in the centre of mass,

$$y^* = \frac{y_1 - y_2}{2}, \tag{6.28}$$

so that

$$y_1 = Y + y^*, \qquad y_2 = Y - y^*. \tag{6.29}$$

Thus,

$$Qe^Y = p_\perp(e^{y^*} + e^{-y^*})e^Y \Rightarrow Q = 2p_\perp \cosh\frac{y_1 - y_2}{2}, \tag{6.30}$$

so $p_\perp, y_1, y_2$ let us reconstruct Q^2 and Y.

The parton model factorisation (2.63) implies that

$$\mathrm{d}\sigma(A(P_1)+B(P_2) \to \ell^+\ell^- + X) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \sum_f f_{f/A}(x_1) f_{\bar{f}/B}(x_2)\, \mathrm{d}\hat{\sigma}(q_f(p_1)+q_{\bar{f}}(p_2) \to \ell^+\ell^-), \tag{6.31}$$

where X accounts for any hadronic final state, and the sum is over the f quark flavours. Through the Jacobian,

$$\left|\frac{\partial(x_1, x_2)}{\partial(Q^2, Y)}\right| = \begin{vmatrix} \frac{x_1}{2Q^2} & \frac{x_2}{2Q^2} \\ x_1 & -x_2 \end{vmatrix} = \frac{x_1 x_2}{Q^2}, \tag{6.32}$$

we can write the factorisation formula as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}Y} = \sum_f x_1 f_{f/A}(x_1) x_2 f_{\bar{f}/B}(x_2) \frac{1}{Q^2} \hat{\sigma}(q\bar{q} \to \ell^+\ell^-), \tag{6.33}$$

where $\hat{\sigma}$ is the partonic cross section for $q\bar{q} \to \ell^+\ell^-$. This may be obtained from

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi}{3}\frac{\alpha^2}{\hat{s}}, \tag{6.34}$$

(that we computed in the Exercises), by averaging over the initial colour,

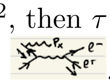
$$\hat{\sigma}(q\bar{q} \to \ell^+\ell^-) = \frac{4\pi}{3}\frac{Q_f^2}{N_c}\frac{\alpha^2}{\hat{s}}, \tag{6.35}$$

with $\hat{s} = Q^2$. Then, at lowest order, the Drell-Yan (DY) production rate is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}Y} = \sum_f x_1 f_{f/A}(x_1) x_2 f_{\bar{f}/B}(x_2) \frac{4\pi}{3}\frac{Q_f^2}{N_c}\frac{\alpha^2}{(Q^2)^2}. \tag{6.36}$$

We define the variable,

$$\tau = \frac{Q^2}{s}\,. \tag{6.37}$$

Since $\hat{s} = x_1 x_2 s$ and $\hat{s} = Q^2$, then $\tau = x_1 x_2$. However, if initial-state collinear radiation is emitted (before the $q\bar{q}$ annihilation) 

, then $q = x_1 P_1 + x_2 P_2 - p_x$, and $Q^2 \leq \hat{s}$. Using eq. (6.37) and $x_1 x_2 = \hat{s}/s$, we obtain that $\tau \leq x_1 x_2$. Setting,

$$\sigma_0 = \frac{4\pi}{3N_c}\frac{\alpha^2}{Q^2 s}, \tag{6.38}$$

so that $x_1 x_2 \hat{\sigma}(q\bar{q} \to \ell^+\ell^-) = Q_f^2 Q^2 \sigma_0$, and $z = \dfrac{Q^2}{\hat{s}}$, we can write the DY cross section as,

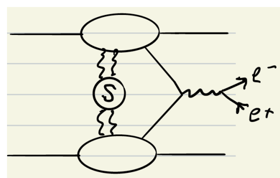
$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}Y} = \sigma_0 \sum_{i,j} Q_i^2 \int_\tau^1 \mathrm{d}z \int_0^1 \frac{\mathrm{d}x_1}{x_1} \int_0^1 \frac{\mathrm{d}x_2}{x_2} f_{i/A}(x_i, \mu_F^2) f_{j/B}(x_j, \mu_F^2) \\ \cdot\, \delta\left(z - \frac{Q^2}{x_1 x_2 s}\right) \delta\left(Y - \frac{1}{2}\log\frac{x_1}{x_2}\right) \cdot \omega_{ij\to\gamma*}\,, \quad (6.39)$$

with

$$\omega_{ij\to\gamma*} = \delta(1-z)\delta_{ij} + \mathcal{O}(\alpha_s). \quad (6.40)$$

In fact, replacing $\omega_{ij\to\gamma*} = \delta(1-z)\delta_{ij}$ in eq. (6.39), we get back the parton model formula (6.33) (please check!).

The most general pinch surface for DY yields the reduced diagram,



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However, the proof of factorisation for the DY process is complicated by the soft interactions between the two incoming protons. The proof goes through a Feynman diagrammatic analysis, using a good deal of the pinch-surface technology outlined in this lecture. A few more details are in chapters 13 and 14 of Sterman's *An Introduction to Quantum Field Theory*, a lot more in chapter 5 of Collins' *Foundations of Perturbative QCD*.

Appendix A

Exercises

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Contents

For any comment or correction feel free to write an email to armin.schweitzer@phys.ethz.ch in order to update the file to the best version for all of the students.

Exercise 1. ***Representation of the Lorentz group***

By definition the Lorentz group consist of all the transformations that leave the scalar product $x^\mu y_\mu := x^\mu g_{\mu\nu} y^\nu$ invariant, in particular we are going to look at the restricted Lorentz group, namely the group of proper orthochronous transformations:

$$SO^+(3,1) := \{\Lambda \in SL(4,\mathbb{R}) \,|\, \Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma},\, \Lambda^0{}_0 \geq 1\}. \tag{1.1}$$

a) The generators of the Lorentz transformations for 4-vectors are given by:

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right), \qquad \Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \tag{1.2}$$

where $\omega_{\mu\nu}$ is a totally anti-symmetric tensor.
Show that for a boost in the z-direction with rapidity η the transformation of (x^0, x^3) can be expressed as:

$$[\Lambda(\omega)]^\alpha{}_\beta x^\beta = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \cdot \begin{pmatrix} x^0 \\ x^3 \end{pmatrix}, \qquad \omega_{\mu\nu} = \begin{cases} \eta & \text{if } (\mu,\nu) = (0,3) \\ -\eta & \text{if } (\mu,\nu) = (3,0) \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

b) The generators of the Lorentz transformations for spinors are given by:

$$(\mathcal{S}^{\mu\nu})_{\alpha\beta} = \frac{i}{4}([\gamma^\mu, \gamma^\nu])_{\alpha\beta}, \qquad \Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{S}^{\mu\nu}\right) \tag{1.4}$$

Show that the Dirac equation for a $u(p)$ is invariant when

$$(\Lambda^{-1})_\nu{}^\mu \gamma_\mu = \Lambda_{\frac{1}{2}} \gamma_\nu \Lambda_{\frac{1}{2}}^{-1}. \tag{1.5}$$

Solution.

a) The transformation matrix $T^\alpha{}_\beta$ has as generator:

$$t^\alpha{}_\beta = -\frac{i}{2}(\omega_{\mu\nu}\mathcal{J}^{\mu\nu})^\alpha{}_\beta = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} \tag{S.1.1}$$

Then it follows:

$$T^\alpha{}_\beta = \exp(t^\alpha{}_\beta) = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}. \tag{S.1.2}$$

b) We start from the expression:

$$(\gamma_\mu p^\mu - m) u(p) = 0 \tag{S.1.3}$$

Then we apply a boost:

$$(\gamma_\mu \Lambda^\mu{}_\nu p^\nu - m)\, u'(\Lambda p) = \left((\Lambda^{-1})_\nu{}^\mu \gamma_\mu p^\nu - m\right) \Lambda_{\frac{1}{2}} u(p) \tag{S.1.4}$$

$$= \Lambda_{\frac{1}{2}} \left(\Lambda_{\frac{1}{2}}^{-1} \left[(\Lambda^{-1})_\nu{}^\mu \gamma_\mu\right] \Lambda_{\frac{1}{2}} p^\nu - m\right) u(p) \tag{S.1.5}$$

Where we used $u'(p) = \Lambda_{\frac{1}{2}} u(\Lambda^{-1}p)$ This become zero if

$$\Lambda_{\frac{1}{2}}^{-1} \left[(\Lambda^{-1})_\nu{}^\mu \gamma_\mu\right] \Lambda_{\frac{1}{2}} = \gamma_\nu, \tag{S.1.6}$$

which proves the relation.

Exercise 2. ***Solution of the Dirac equation***

Whenever an explicit representation is asked we consider the Chiral representation of the gamma matrices, which is given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \qquad P_{R,L} := \frac{1 \pm \gamma^5}{2}, \tag{1.6}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.7}$$

where $\sigma^\mu := (\mathbb{1}, \vec\sigma)$ and $\bar\sigma^\mu := (\mathbb{1}, -\vec\sigma)$.

a) Consider the Dirac equation for positive frequencies solution, $\psi(x) := u(p)e^{-ip\cdot x}$. Show that the solution for a particle at rest with normalization $\bar u\, u = 2m$, can be written as:

$$u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad (p_0)^\mu = (m, 0, 0, 0). \tag{1.8}$$

where ξ are two-component spinors satisfying $\xi^\dagger\xi = 1$

b) Split the component of the spinor into the the chiral components of $u(p)_{R,L} := P_{R,L}u(p)$ a s $u(p) = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix}$ and show that they do not mix under a boost (e.g in the z-direction).

c) Apply a boost in the z-direction the spinor for a particle at rest and express it in terms of the light-cone coordinates $p^\pm = E \pm p^3$.

d) Take the boosted solution and then send the mass to zero and give the expression for the spinor for a spin up and spin down particle, namely $\xi^t = (1, 0)$ and $\xi^t = (0, 1)$, respectively.

e) Follow the same procedure to find the expression for the negative frequencies solutions, $\psi(p) = v(p)e^{ip\cdot x}$ that represent a positron with positive energy.

Solution.

a) The equation that we need to solve is $(\gamma^\mu p_\mu - m)u(p) = 0$ for $p^\mu = (m, 0, 0, 0)$, then it reduces to

$$(\gamma^0 - 1)u(p) = \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p) = 0 \tag{S.1.7}$$

The only non trivial solution of this expression is given by $u(p) \propto \begin{pmatrix} \xi \\ \xi \end{pmatrix}$

b)

$$t_{\alpha_\beta} = -\frac{i}{2}(\omega_{\mu\nu}\mathcal{S}^{\mu\nu}) = \frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \tag{S.1.8}$$

Then,

$$T_{\alpha_\beta} = \cosh\frac{\eta}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \sinh\frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \tag{S.1.9}$$

This means that ξ_L and ξ_R do not mix under any Lorentz transformation.

c)

$$\sqrt{m}\begin{pmatrix} \cosh\left(\frac{\eta}{2}\right)\mathbb{1}+\sinh\left(\frac{\eta}{2}\right)\sigma^3 & 0 \\ 0 & \cosh\left(\frac{\eta}{2}\right)\mathbb{1}-\sinh\left(\frac{\eta}{2}\right)\sigma^3 \end{pmatrix}\cdot\begin{pmatrix}\xi\\ \xi\end{pmatrix} \tag{S.1.10}$$

we know that when a particle at rest of mass m is boosted in the z direction with rapidity η transforms:

$$\begin{pmatrix} m \\ 0 \end{pmatrix} \to \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix} \qquad \Rightarrow \qquad m\,e^{\pm\eta} = p^{\pm}. \tag{S.1.11}$$

where the light-cone coordinates are defined as $p^{\pm} = E \pm p^3$. Using the square root of this expression one can write:

$$u'(\Lambda^{\mu}{}_{\nu}p^{\nu}) = \begin{pmatrix} \left[\sqrt{p^+}\frac{1-\sigma^3}{2}+\sqrt{p^-}\frac{1+\sigma^3}{2}\right]\xi \\ \left[\sqrt{p^+}\frac{1+\sigma^3}{2}+\sqrt{p^-}\frac{1-\sigma^3}{2}\right]\xi \end{pmatrix} \qquad \Rightarrow \qquad m\,e^{\pm\eta} = p^{\pm}. \tag{S.1.12}$$

d) In the massless limit, for a particle moving in the positive z-direction, we have $p^- = 0$ and $p^+ = 2E$, giving us that for spin-up particles only the right-handed part survives while for spin down is the left-handed.

$$\xi = \begin{pmatrix}1\\0\end{pmatrix}, \quad m = 0 \qquad \Rightarrow \qquad u_R(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{S.1.13}$$

$$\xi = \begin{pmatrix}0\\1\end{pmatrix}, \quad m = 0 \qquad \Rightarrow \qquad u_L(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{S.1.14}$$

e) In the negative frequencies solution we have that $\psi(x) = v(p)e^{ip\cdot x}$. The Dirac equation new read:

$$(\gamma^{\mu}p_{\mu}+m)v(p) = 0.$$

Taking the massive momentum in its rest frame and following the same steps that we did for the positive energies solution one gets:

$$(\gamma^0+1)v(p) = \begin{pmatrix}\mathbb{1} & \mathbb{1}\\ \mathbb{1} & \mathbb{1}\end{pmatrix}v(p) = 0, \qquad v(p) \propto \begin{pmatrix}\xi\\ -\xi\end{pmatrix} \tag{S.1.15}$$

Exercise 3. *Helicity Operator*

The helicity operator h is defined as:

$$h := \frac{\vec{p}\cdot\vec{\Sigma}}{2\,|\vec{p}|}, \qquad \Sigma := \begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} \tag{1.9}$$

a) Show that the two chiral state $u_{R,L}$ are eigenstates of the helicity operator for the massless case for the particular case of a particle moving in the z-direction.

b) Argue that the up spin $\xi = (1,0)$ is oriented in the z-direction.

Hint. *You can use that for a massless spinor particle moving along the z-axis the chiral states are given by:*

$$u_R(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad u_L(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}. \tag{1.10}$$

Solution.

a) By acting with $h = \frac{1}{2}\gamma^3$ the proof is trivial.

b) We can perform a rotation around the three orthogonal directions x, y and z and see that only the latter leave the direction of ξ invariant.

$$R_{\frac{1}{2},z}(\theta) = \exp\left(-i\frac{\theta}{2}\begin{pmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}\right) = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma^3 & 0 \\ 0 & \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma^3\end{pmatrix} \tag{S.1.16}$$

This means that under such rotation the state pick up just a phase $u_{R,L} \to e^{\mp\frac{\theta}{2}} u_{R,L}$. In a similar way it is possible to show that under a rotation around any other axis will lead to a change of state since $\xi = (1,0$ is not an eigenstate of neither σ^1 nor σ^2.

Exercise 4. *General solution for massless spinors*

We want to derive the general expression for $u_{R,L}$ for the general momenta in the case of massless particles. We start by writing down the Dirac equaiton for the problem:

$$(\gamma^\mu p_\mu)u(p) = 0, \qquad u(p) = \begin{pmatrix}\xi_L \\ \xi_R\end{pmatrix} \tag{1.11}$$

a) Show that we can write the following two equations:

$$\begin{pmatrix}p^- & -p_\perp^* \\ -p_\perp & p^+\end{pmatrix}\xi_R = 0, \qquad \begin{pmatrix}p^+ & p_\perp^* \\ p_\perp & p^-\end{pmatrix}\xi_L = 0 \tag{1.12}$$

where,

$$p^\pm = p^0 \pm p^3, \qquad p_\perp = p^1 + i\,p^2. \tag{1.13}$$

Does these equations have non trivial solutions?

b) Find a solution for ξ_L imposing the normalization condition $\xi_L^\dagger \xi_L = 2E$, and show that it gives:

$$\xi_R = e^{i\alpha}\begin{pmatrix}\sqrt{p^+} \\ \sqrt{p^-}e^{i\phi_p}\end{pmatrix}, \qquad \xi_L = e^{i\beta}\begin{pmatrix}-\sqrt{p^-}e^{-i\phi_p} \\ \sqrt{p^+}\end{pmatrix}$$

where $e^{i\phi_p} = \frac{p_\perp}{\sqrt{p^+p^-}}$.

c) Show that in the corresponding limit we can recover (1.10).

Solution.

a) Using the explicit representation given in (1.6) it's possible to write:

$$(p^0 \cdot \mathbb{1} - p^i \cdot \sigma^i)\xi_R = 0, \tag{S.1.17}$$

$$(p^0 \cdot \mathbb{1} + p^i \cdot \sigma^i)\xi_L = 0. \tag{S.1.18}$$

This leads directly to the wanted expression by explicitly writing the expression for the Pauli matrices. To show that the space of solution does not have zero dimensions we need a zero determinant. This can be shown to be the case since we have the condition $p^2 = 0 = p^+p^- - |p_\perp|^2$.

b) The solution to (1.12) is easily found to be:

$$\xi_R = C_L \begin{pmatrix} p^+ \\ p_\perp \end{pmatrix}, \qquad \xi_L = C_L \begin{pmatrix} -p_\perp^* \\ p^+ \end{pmatrix}. \tag{S.1.19}$$

From the normalization condition, together with the on-shell condition, we find that $C_{R,L} = \frac{e^{i\alpha_{R,L}}}{\sqrt{p^+}}$ where $\alpha_{R,L}$ are some arbitrary phases.

The final expression is obtained by rewriting:

$$p_\perp = \sqrt{p^+ p^-} e^{i\phi_p}$$

Exercise 5. *Hadronic Tensor*

While computing the deep inelastic scattering the amplitude square can be factorized into a Lepronic and Hadronic tensor:

$$\overline{|\mathcal{M}|^2} = \frac{e^2}{(q^2)^2} L^{\mu\nu}(l,q) W_{\mu\nu}(p,q),$$

with q the exchanged momentum, l and p the initial lepton and parton momentum,respectively. Without any further information about the Hadronic vertex, we may write the gerneral form:

$$\begin{aligned} W_{\mu\nu} =& V_1 \cdot g_{\mu\nu} + V_2 \cdot p_\mu p_\nu + V_3 \cdot (p_\mu q_\nu + q_\mu p_\nu) \\ &+ V_4 \cdot (p_\mu q_\nu - q_\mu p_\nu) + V_5 \cdot q_\mu q_\nu + V_6 \cdot \epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta \end{aligned} \tag{1.14}$$

Use the fact that the leptonic tensor $L_{\mu\nu}$ is symmetric under interchange of the indices μ and ν, together with the requirement of current conservation that reads:

$$q^\mu W_{\mu\nu} = W_{\mu\nu} q^\nu = 0, \tag{1.15}$$

to cast the general expression for $W_{\mu\nu}$ into:

$$W_{\mu\nu} = -W_1 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{W_2}{p^2} \left[p_\mu - \frac{(q\cdot p) q_\mu}{q^2} \right] \left[p_\nu - \frac{(q\cdot p) q_\nu}{q^2} \right] \tag{1.16}$$

Give an expression for W_1 and W_2 in terms of the V_is.

Solution. From the symmetry condition we can set $V_4 = V_6 = 0$, from the ward identity we obtain:

$$\begin{aligned} 0 &= V_1 q_\nu + V_2 (q\cdot p) p_\nu + V_3((q\cdot p) q_\nu + q^2 p_\nu) + V_5 q^2 q_\nu & \text{(S.1.20)} \\ &= q_\nu (V_1 + V_3 (q\cdot p) + V_5 q^2) + p_\nu (V_2 (q\cdot p) + V_3 q^2) & \text{(S.1.21)} \end{aligned}$$

From this it follows that we can write everything in terms of V_1 and V_2:

$$V_3 = -V_2 \frac{(q\cdot p)}{q^2}, \tag{S.1.22}$$

$$V_5 = V_2 \frac{(q\cdot p)^2}{q^4} - \frac{V_1}{q^2}. \tag{S.1.23}$$

Then the correspondence between the W_i and the V_i factors becomes trivial:

$$W_{\mu\nu} = V_1 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + V_2 \left[p_\mu - \frac{(q\cdot p) q_\mu}{q^2} \right] \left[p_\nu - \frac{(q\cdot p) q_\nu}{q^2} \right]. \tag{S.1.24}$$

Exercise 1. ***Spinor Product***

Given the chiral spinors

$$u_R(p) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p^+} \\ \frac{p_\perp}{\sqrt{p^+}} \end{pmatrix}, \qquad u_L(p) = \begin{pmatrix} -\frac{p_\perp^*}{\sqrt{p^+}} \\ \sqrt{p^+} \\ 0 \\ 0 \end{pmatrix}, \tag{2.1}$$

where

$$p_\pm = p^0 \pm p^3, \qquad p_\perp = p^1 + ip^2. \tag{2.2}$$

We define the following spinor products:

$$\langle p\, k\rangle := \overline{u_L(p)}\, u_R(k), \tag{2.3}$$

$$[p\, k] := \overline{u_R(p)}\, u_L(k). \tag{2.4}$$

Recall that $\bar{\psi} = \psi^\dagger \gamma^0$. Check the following relations hold:

$$\langle k\, p\rangle = -\langle p\, k\rangle\,, \qquad [k\, p] = -[p\, k] \qquad \langle k\, p\rangle^* = [p\, k]\,, \tag{2.5}$$

$$\langle p\, k\rangle\ [k\, p] = 2p \cdot k. \tag{2.6}$$

Solution. We start by proving the first two relations:

$$\begin{aligned} \langle k\, p\rangle = u_L(k)^\dagger \gamma^0 u_R(p) &= -k_\perp \sqrt{\frac{p^+}{k^+}} + p_\perp \sqrt{\frac{k^+}{p^+}} \\ &= -\left(-p_\perp \sqrt{\frac{k^+}{p^+}} + k_\perp \sqrt{\frac{p^+}{k^+}}\right) \\ &= -\langle p\, k\rangle \end{aligned} \tag{S.2.1}$$

$$[k\, p] = u_L(k)^\dagger \gamma^0 u_R(p) = -k_\perp^* \sqrt{\frac{p^+}{k^+}} + p_\perp^* \sqrt{\frac{k^+}{p^+}} = -[p\, k] \tag{S.2.2}$$

By observing the explicit form of $\langle k\, p\rangle$ and $[k\, p]$ it is clear that $\langle k\, p\rangle^* = [k\, p]$. We are then left with showing the last relation:

$$\begin{aligned} \langle p\, k\rangle\ [k\, p] &= \left(-p_\perp \sqrt{\frac{k^+}{p^+}} + k_\perp \sqrt{\frac{p^+}{k^+}}\right)\left(-k_\perp^* \sqrt{\frac{p^+}{k^+}} + p_\perp^* \sqrt{\frac{k^+}{p^+}}\right) \\ &= |p_\perp|^2 \frac{k^+}{p^+} + |k_\perp|^2 \frac{p^+}{k^+} - 2\,\mathrm{Re}(p_\perp k_\perp^*) \\ &= p^- k^+ + k^- p^+ - 2\,\mathrm{Re}(p_\perp k_\perp^*) \\ &= -2p \cdot k \end{aligned} \tag{S.2.3}$$

Exercise 2. ***Spinor Convention***

We have seen that when a particle is massless, chirality and helicity overlap. In this case it is enough to know the momentum and helicity of the particle to define out states. This brings up the question of finding a more suitable basis for this kind of state.
In this exercise you are asked to prove some properties of a different representation of the massless spinors.

As already saw in an other exercise we can contract the the momentum with the Pauli matrices to obtain a 2×2 matrix:

$$p_{a\dot{a}} := (p_\mu \bar{\sigma}^\mu)_{a\dot{a}} = \begin{pmatrix} p^+ & p_\perp^* \\ p_\perp & p^- \end{pmatrix}_{a\dot{a}} \tag{2.7}$$

a) Show $\det(p_{a\dot{a}}) = p^2$.

b) Show that the above expression for massless particles can be written in terms of a two-dimensional vector λ_a as:

$$(p_\mu \bar{\sigma}^\mu)_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \tag{2.8}$$

We make the explicit choice for the form of the (massless) spinor λ and $\tilde{\lambda}$ associated with the momentum p:

$$\lambda_a(p) = \frac{1}{\sqrt{p^+}} \begin{pmatrix} p^+ \\ p_\perp \end{pmatrix}, \qquad \tilde{\lambda}_{\dot{a}}(p) = \frac{1}{\sqrt{p^+}} \begin{pmatrix} p^+ \\ p_\perp^* \end{pmatrix}. \tag{2.9}$$

c) Verify that indeed $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ for $p_{a\dot{a}}$ in (2.7) holds. Which additional freedom do we have in the definition of λ_a and $\tilde{\lambda}_{\dot{a}}$?

The spinor indices are raised and lowered by the totally anti-symmetric ϵ tensor with $\epsilon^{12} = 1$ and we use the shorthand notation $\lambda(p_i) := \lambda_i$.

d) Show that:

i) $\epsilon^{ab}(\lambda_1)_a(\lambda_2)_b = \langle p_1\, p_2 \rangle = -\langle p_2\, p_1 \rangle$,

ii) $\epsilon^{\dot{a}\dot{b}}(\tilde{\lambda}_1)_{\dot{a}}(\tilde{\lambda}_2)_{\dot{b}} = [p_1\, p_2] = -[p_2\, p_1]$,

iii) $2p_1^\mu = (\tilde{\lambda}_1)_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}a}(\lambda_1)_a$,
Hint. You can relate $\mathrm{Tr}\,(\sigma^\mu\sigma^\nu)$ *to* $\mathrm{Tr}\,(\gamma^\mu\gamma^\nu)$.

iv) $2p_1 \cdot p_2 = (\lambda_1)_a(\lambda_2)^a(\tilde{\lambda}_2)_{\dot{a}}(\tilde{\lambda}_1)^{\dot{a}}$.
Hint. You may use the Fierz identity for Pauli matrices: $(\sigma^\mu)_{a\dot{a}}(\sigma_\mu)^{\dot{b}b} = 2\delta_a^b\delta_{\dot{a}}^{\dot{b}}$.

Solution. Note: We are sticking to the conventions of **Peskin & Schröder**. In the literature you will mostly find different conventions used.

a) Follows directly from plugging in the definitions of $p^\pm$ and $p_\perp$ in (2.2).

b) $p_{a\dot{a}}$ is a two by two matrix with rank ≤ 2. Therefore we have the decomposition

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \mu_a \tilde{\mu}_{\dot{a}} \tag{S.2.4}$$

for some linearly indpendent vectors λ and μ. But for massless vectors $p^2 = \det(p_{a\dot{a}}) = 0$ and the matrix has not full rank and therefore

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

for some λ.

c) The only non-trivial part is:

$$p_{22} = \frac{1}{p^+} p_\perp p_\perp^* = \frac{1}{p^+}\left((p^1)^2 + (p^2)^2\right) = \frac{1}{p^+}\left((p^0)^2 - (p^3)^2\right) = p^- .$$

One can always redefine by multiplying λ with a phase $\exp(i\Theta)$ and $\tilde{\lambda}$ with its complex conjugate. This is known as *little group scaling*.

d) i) Note that $\lambda(p)_a$ is the positive chirality wave-function for a spin 1/2 particle of momentum p ($u_R(p)$). We furthermore have

$$\lambda_a(p)\epsilon^{ab} = (-\lambda_2(p), \lambda_1(p))^b = \frac{1}{\sqrt{p^+}}(-p_\perp, p^+)^b . \tag{S.2.5}$$

Comparison with Ex. 1 shows that it is $\bar{u_L}(p)$. The identity follows directly from Ex. 1.

ii) Comparing with Ex. 1 shows that $\tilde{\lambda}_{\dot{a}}\epsilon_{\dot{a}\dot{b}}$ corresponds to $u_L(p)$ and $\tilde{\lambda}_{\dot{a}}$ is $\bar{u_R}(p)$. The identities are the ones shown in Ex. 1.

iii) We have

$$\begin{aligned}
(\tilde{\lambda}_1)_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}a}(\lambda_1)_a &= (\bar{\sigma}^\mu)^{\dot{a}a}\left(p_\nu \bar{\sigma}^\nu\right)_{a\dot{a}} &\text{(S.2.6)}\\
&= \mathrm{Tr}\left(\bar{\sigma}^\mu\bar{\sigma}^\nu\right)p_\nu &\text{(S.2.7)}\\
&= 2p^\mu . &\text{(S.2.8)}
\end{aligned}$$

Where we have used from the hint that $2\,\mathrm{Tr}\left(\sigma^\mu\sigma^\nu\right) = \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right)$ and the γ-trace can be straightforwardly computed using the Clifford-algebra.

iv) We have

$$2p_1\cdot p_2 = \frac{1}{2}(\tilde{\lambda}_1)_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}a}(\lambda_1)_a(\tilde{\lambda}_2)_{\dot{b}}(\bar{\sigma}_\mu)^{\dot{b}b}(\lambda_2)_b = (\lambda_1)_a(\lambda_2)^a(\tilde{\lambda}_2)_{\dot{a}}(\tilde{\lambda}_1)^{\dot{a}} = \langle 1\,2\rangle\,[1\,2] \tag{S.2.9}$$

where me made use of the Fierz identity.

Exercise 3. *Optical Theorem*

Probability amplitudes for scattering of asymptotic states in QFT are described by the S-matrix. Since S connects to sets of orthonormal states, it must be unitary

$$S^\dagger S = \mathbb{1}.$$

We may write $S = \mathbb{1} + iT$, where the *transfer matrix* T describes the non-trivial scattering.

a) Show that the transfer matrix T fulfills:

$$i(T - T^\dagger) = -T^\dagger T. \tag{2.10}$$

The non-trivial scattering of initial-state $|i\rangle$ and final-state $|f\rangle$ can be written as:

$$\langle f|\, iT\, |i\rangle = (2\pi)^4\delta^{(4)}\left(p_i - p_f\right)\cdot iM_{fi},$$

where M_{fi} is called the *matrix element*.

b) Show that:

$$M_{fi} - M_{if}^* = (2\pi)^4 i\sum_n \int \mathrm{d}\Pi_n \delta^4\left(p_i - p_n\right) M_{\{n\}f}^* M_{\{n\}i}, \tag{2.11}$$

where the sum is over all possible intermediate states.

$\mathrm{d}\Pi_n = \prod_{i=1}^n \frac{\mathrm{d}^4 p_i}{(2\pi)^4}\delta_+^{(4)}\left(p_i^2 - m_i^2\right) = \prod_{i=1}^n \frac{\mathrm{d}^4 p_i}{(2\pi)^4}\delta^{(4)}\left(p_i^2 - m_i^2\right)\Theta(p_i^0) = \prod_{i=1}^n \frac{\mathrm{d}^3 p_i}{(2\pi)^4}\frac{1}{2E_i}$ denotes the Lorentz invariant phase-space measure for n particles of mass m_i.

Solution.

a) It follows directly from unitarity:

$$\mathbb{1} = S^\dagger S = \left(\mathbb{1} - iT^\dagger\right)(\mathbb{1} + iT) = \mathbb{1} + T^\dagger T + i(T - T^\dagger). \tag{S.2.10}$$

b) We start with

$$i\langle f|\left(T - T^\dagger\right)|i\rangle = i\,(2\pi)^4\,\delta^{(4)}\left(p_i - p_f\right)\left(M_{fi} - M_{if}\right). \tag{S.2.11}$$

Furthermore we have by inserting unity $\mathbb{1} = \sum_n \int \mathrm{d}\Pi_n\,|\{n\}\rangle\langle\{n\}|$

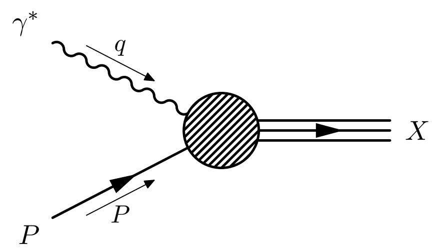
$$\begin{aligned} -\langle f|T^\dagger T|i\rangle &= -\sum_n \int \mathrm{d}\Pi_n \langle f|T^\dagger|\{n\}\rangle\langle\{n\}|T|i\rangle & \text{(S.2.12)}\\ &= -(2\pi)^8 \sum_n \int \mathrm{d}\Pi_n \delta^{(4)}\left(p_f - p_{\{n\}}\right)\delta^{(4)}\left(p_i - p_{\{n\}}\right) M^*_{\{n\}f} M_{\{n\}i} & \text{(S.2.13)}\\ &= -(2\pi)^8 \delta^{(4)}\left(p_f - p_i\right)\sum_n \int \mathrm{d}\Pi_n M^*_{\{n\}f} M_{\{n\}i}\delta^4\left(p_i - p_n\right), & \text{(S.2.14)} \end{aligned}$$

which yields the desired equation.

Exercise 4. *DIS Hard Cross Section*

We now want to compute the cross section between a virtual photon γ^* and a proton, $\gamma^* P \to X$. The final state denoted by X can be anything.

$i\mathcal{M}_{\gamma^* P \to X}(q, P) \quad =$



(2.12)

Show by means of the optical theorem that the corresponding cross section in the laboratory rest frame is given by:

$$\sigma_\lambda(\gamma^* P \to X) = \frac{4\pi^2\alpha}{|q|}\varepsilon^\mu_\lambda(q)\varepsilon^{\nu *}_\lambda(q)\,\mathrm{Im}(W_{\mu\nu}(q, P)) \tag{2.13}$$

with λ denoting the helicity of the incoming virtual photon.

Solution. We start by writing explicitly the cross section for a $2 \to n$ process:

$$\sigma_\lambda(\gamma^* P \to X) = \frac{1}{4E_P E_q|\vec{v}_P - \vec{v}_q|}\sum_n \int \mathrm{d}\phi_n |\mathcal{M}_{\gamma^* P \to X_n}|^2, \qquad \vec{v}_p = \frac{\vec{p}}{p^0}. \tag{S.2.15}$$

The numerator can be computed by noticing that $|\vec{v}_P - \vec{v}_q| = \frac{|q|}{E_q}$ and $E_P = m_P$, were we used the fact that in the lab frame the proton is at rest. The phase space for a n-particle final state is given by:

$$\mathrm{d}\phi_n = \prod_{i=1}^n \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^{(4)}\left(q + P - \sum_{j=1}^n p_j\right). \tag{S.2.16}$$

From the optical theorem one can use:

$$2e^2\varepsilon_\lambda^\mu(q)\varepsilon_\lambda^{\nu*}(q)\,\mathrm{Im}(\tilde{W}_{\mu\nu}) = \sum_n \int \prod_n \mathrm{d}\phi_n |\mathcal{M}_{\gamma^* P \to X_n}|^2 \tag{S.2.17}$$

with $\tilde{W}_{\mu\nu} = 2\pi m_P^2 q^2$.
Substituting the above expression into the definition of cross section gives us:

$$\sigma_\lambda(\gamma^* P \to X) = \frac{4\pi^2\alpha}{|q|}\varepsilon_\lambda^\mu(q)\varepsilon_\lambda^{\nu*}(q)\,\mathrm{Im}(W_{\mu\nu}(q,P)), \qquad \alpha = \frac{e^2}{4\pi}. \tag{S.2.18}$$

Exercise 1. ***SU(N)***

$SU(N)$ is the special group of unitary $N \times N$ matrices ($U^\dagger = U^{-1}$, $\det(U) = 1$). The group elements U_R in a given representation can be written in terms of generators t_R^a (also matrices) and real group parameters α_a:

$$U_R = \exp\left(i \sum_{a=1}^{N^2-1} \alpha_a t_R^a\right) = \mathbb{1} + i\alpha_a t^a + \mathcal{O}(\alpha^2). \tag{3.1}$$

(a) Consider small transformations ($\alpha_a \ll 1$) and show that the generators need to be traceless and hermitian using the fact the $SU(N)$ is the special unitary group:

$$U_R^\dagger U_R = \mathbb{1} \quad \Rightarrow \quad (t_R^a)^\dagger = t_R^a, \tag{3.2}$$
$$\det(U_R) = 1 \quad \Rightarrow \quad \mathrm{tr}(t_R^a) = 0. \tag{3.3}$$

Hint. $Det(e^X) = e^{\mathrm{tr}(X)}$ *if* $Det(e^X) \neq 0$.

The generators t_R^a satisfy the following commutation relations:

$$[t_R^a, t_R^b] = i f^{abc} t_R^c \tag{3.4}$$

where f^{abc} are the real and totally antisymmetric structure constants of $SU(N)$.

(b) What are the conditions on the structure constants f^{abc} in order to have an Abelian group?

Hint. An Abelian group is a group whose elements commute, i.e. $[A, B] = 0$

(c) The commutator satisfies the Jacobi identity,

$$[t^a, [t^b, t^c]] + [t^c, [t^a, t^b]] + [t^b, [t^c, t^a]] = 0. \tag{3.5}$$

Show that this enforces the following condition on the structure constants:

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0. \tag{3.6}$$

In any representation the generators can be chosen such that:

$$\mathrm{tr}\left(t_R^a t_R^b\right) = T_R \delta^{ab}, \qquad T_R \in \mathbb{R} \tag{3.7}$$

(d) The Casimir operator is an operator that commutes with all the generators and therefore all matrices in the group. Show that the quadratic Casimir operator $C_2(R) := \sum_a t_R^a t_R^a$ satisfies:

$$[C_2(R), t_R^b] = 0 \quad \forall b \qquad \Rightarrow \qquad C_2(R) = C_R \, \mathbb{1}, \tag{3.8}$$
$$C_R = T_R \frac{d(G)}{d(R)}. \tag{3.9}$$

where $d(R)$ and $d(G)$ are the dimensions of the representation and of the group, respecively.

(e) Show that the structure constant can be written as follows:

$$f^{abc} = \frac{-i}{T_R} \mathrm{tr}\left([t_R^a, t_R^b] t_R^c\right). \tag{3.10}$$

Solution.

(a) From unitarity we find

$$(\mathbb{1} + i\alpha_a t^a)(\mathbb{1} - i\alpha_a (t^a)^\dagger) = \mathbb{1} + i\alpha_a (t^a - (t^a)^\dagger) + \mathcal{O}(\alpha^2) = \mathbb{1}. \tag{S.3.1}$$

because α is arbitrary we have $(t^a)^\dagger = t^a$.
The determinant imposes a second condition:

$$\exp(i\alpha_a \operatorname{tr}(t^a)) = 1 + i\alpha_a \operatorname{tr}(t^a) + \mathcal{O}(\alpha^2) = 1. \tag{S.3.2}$$

meaning that $\operatorname{tr}(t^a) = 0$.

(b) If $f^{abc} = 0$ then the group is abelian. For the group to be non-abelian a non vanishing structure constant is needed.

$$\begin{aligned} 0 = AB - BA = &\left(\mathbb{1} + i\alpha_a t^a - \frac{(\alpha_a t^a)^2}{2}\right)\left(\mathbb{1} + i\beta_a t^a - \frac{(\beta_a t^a)^2}{2}\right) - \left(\mathbb{1} + i\beta_a t^a - \frac{(\beta_a t^a)^2}{2}\right)\left(\mathbb{1} + i\alpha_a t^a - \frac{(\alpha_a t^a)^2}{2}\right) \\ &+ \mathcal{O}(\alpha^3, \beta\alpha^2, .., \beta^3) \\ = &-\alpha\beta(t^a t^b - t^b t^a) \end{aligned}$$

(c) trival.

(d) The commutation relation can be proven as follows:

$$[t_R^a t_R^a, t_R^b] = if^{abc}(t_R^a t_R^c + t_R^c t_R^a) = 0. \tag{S.3.3}$$

We use that f^{abc} is totally anti-symmetric and that the expression inside the parentheses is symmetric under $a \leftrightarrow b$.
Because C_2 commutes with all the generators, it must be proportional to the identity, $C_2(R) = C_R \mathbb{1}$. To fix the constant we can perform the trace:

$$\left.\begin{aligned} &\operatorname{tr}(C_2(R)) = \operatorname{tr}\left(\textstyle\sum_a t_R^a t_R^a\right) = T_R d(G), \\ &\operatorname{tr}(C_2(R)) = C_R d(R). \end{aligned}\right\} \Rightarrow C_R = \frac{T_R d(G)}{d(R)}. \tag{S.3.4}$$

(e)

$$\operatorname{tr}\left([t_R^a, t_R^b] t_R^c\right) = if^{abd} \operatorname{tr}\left(t_R^d t_R^c\right) = if^{abc} T_R \tag{S.3.5}$$

Exercise 2. *SU(N) Representations*

A representation t_R^a of the generators is defined as a matrix that acts on some vector space V. We want now to discuss two representations that have a major relevance in QCD.

The **fundamental representation** t_F^a for $SU(N)$ is denoted by the complex $N \times N$-matrices that are traceless and hermitian. In this course we adopt the normalization $T_F = \frac{1}{2}$ see (3.7).

(a) Show that the Casimir operator for the fundamental representation is:

$$C_2(F) := C_F \, \mathbb{1}, \qquad C_F = \frac{N^2 - 1}{2N}. \tag{3.11}$$

The **adjoint representation** is defined as $(t_A^a)_{bc} := -if^{abc}$. The value of T_A is fixed by imposing the same structure constants for all representations and is $T_A = N$.

(b) Show that t_A^a is a representation of the generators of $SU(N)$.

Hint. Prove that it satisfies (3.4) *and that its elements are traceless.*

(c) The Adjoint representation can also be written explicitly in terms of the fundamental one.

$$t_A^a = t_F^a \otimes \mathbb{1} + \mathbb{1} \otimes t_F^a. \tag{3.12}$$

Use this result to show that $C_A = T_A = N$.

Hint. Useful relations involving the tensor product $\otimes$ are:

$$(r_1 \otimes r_2)(s_1 \otimes s_2) = r_1 s_1 \otimes r_2 s_2, \tag{3.13}$$

$$\mathrm{tr}(r_1 \otimes r_2) = \mathrm{tr}(r_1)\,\mathrm{tr}(r_2). \tag{3.14}$$

(d) Show that the Casimir operator for the adjoint representation is:

$$C_2(A) := C_A\,\mathbb{1}, \qquad C_A = N. \tag{3.15}$$

Solution.

(a) trivial, $d(F) = N$.

(b) For the trace we have,

$$tr(t_R^a) = -if^{abb} = 0. \tag{S.3.6}$$

For the commutation relation we just need to use the Jacobi identity:

$$\begin{aligned}
([t_A^a, t_A^b])_{cd} &= (t_A^a)_{ce}(t_A^b)_{ed} - (t_A^b)_{ce}(t_A^a)_{ed} \\
&= -f^{ace}f^{bed} + f^{bce}f^{aed} \\
&= (f^{cde}f^{eab} + f^{dae}f^{ecb}) + f^{bce}f^{aed} \\
&= f^{cde}f^{eab} \\
&= if^{abe}(-if^{eab}) = if^{abe}t_R^e.
\end{aligned} \tag{S.3.7}$$

(c) We start from the definition:

$$t_A^a = t_F^a \otimes \mathbb{1} + \mathbb{1} \otimes t_F^a \tag{S.3.8}$$

The we build the Casimir operator

$$\sum_a (t_A^a)^2 = \sum_a \left[(t_F^a)^2 \otimes \mathbb{1} + 2(t_F^a \otimes t_F^a) + \mathbb{1} \otimes (t_F^a))^2\right] \tag{S.3.9}$$

Finally we take the traces, (note that the $\mathrm{tr}(t^a) = 0$:

$$\mathrm{tr}(t_A^a)^2 = T_F, \qquad \mathrm{tr}(\mathbb{1}) = d(F) \tag{S.3.10}$$

$$C_A d(A) = 2T_F d(F) d(G) \to C_A = 2T_F d(F) = N \tag{S.3.11}$$

since $d(A) = d(G)$

(d) The dimension of the adjoint representation is $N^2 - 1$, then $C_A = T_A$.

Exercise 3. *SU(3): Gell-Mann matrices*

The aim is now to construct the Gell-Mann matrices. They are a possible manifestation of the fundamental representation.

(a) Starting from the fact the $SU(2)$ is a subspace of $SU(3)$ we define the first three matrices to be:

$$t^1 = k_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t^2 = k_2 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t^3 = k_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

Prove that $k_{1,2,3} = 1/2$, due to the normalization $T_F = \frac{1}{2}$.

(b) Using $tr(t^a t^b) = \frac{\delta^{ab}}{2}$ for $a = 1, 2, 3$ and $b = 4, ..., 8$, show that

$$(t^b)_{12} = (t^b)_{21} = 0, \qquad (t^b)_{11} = (t^b)_{22} = -\frac{(t^b)_{33}}{2}. \tag{3.17}$$

(c) Given that $SU(3)$ can only have two generators with only diagonal elements, and that one is already given by t^3, define t^8 as the second one and show that:

$$t^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.18}$$

(d) Prove that $t^{4,5,6}$ cannot have any diagonal element and that they are given by:

$$t^4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t^5 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t^6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t^7 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \tag{3.19}$$

Solution.

(a) Imposing $\operatorname{tr}\big(t^a t^b\big) = \frac{1}{2}\delta^{ab}$ together with $\operatorname{tr}\big(\sigma^a \sigma^b\big) = \delta^{ab}$ with σ^a the Pauli matrices, we find that $k_{1,2,3} = \frac{1}{2}$

(b)

$$t^b = \begin{pmatrix} t^b_{11} & t^b_{12} & t^b_{13} \\ t^b_{21} & t^b_{22} & t^b_{23} \\ t^b_{31} & t^b_{32} & t^b_{33} \end{pmatrix}, \tag{S.3.12}$$

$$\left.\begin{array}{rcl} 2\operatorname{tr}\big(t^b\big) & = & t^b_{11} + t^b_{22} + t^b_{33} = 0 \\ 2\operatorname{tr}\big(t^b t^3\big) & = & t^b_{11} - t^b_{22} = 0 \end{array}\right\} \quad \Rightarrow \quad t^b_{11} = t^b_{22} = -\frac{t^b_{33}}{2} \tag{S.3.13}$$

$$\left.\begin{array}{rcl} 2\operatorname{tr}\big(t^b t^2\big) & = & i(t^b_{12} - t^b_{21}) = 0 \\ 2\operatorname{tr}\big(t^b t^1\big) & = & t^b_{12} + t^b_{21} = 0 \end{array}\right\} \quad \Rightarrow \quad t^b_{12} = t^b_{21} = 0 \tag{S.3.14}$$

(c) Using the result from the previous point we have:

$$t^8 = k_8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{S.3.15}$$

k_8 is fixed by $\operatorname{tr}\big(t^8 t^8\big) = 1/2$ to be $k_8 = \frac{1}{2\sqrt{3}}$.

(d) Note that $\operatorname{tr}\big(t^b t^8\big) = 0$ for $b = 4, .., 7$.

$$2\sqrt{3}\operatorname{tr}\Big(t^b t^8\Big) = t^b_{11} + t^b_{22} - 2t^b_{33} = 6t^b_{11} = 0 \quad \Rightarrow \quad t^b_{11} = t^b_{22} = t^b_{33} = 0 \tag{S.3.16}$$

Again, we only need to compute the trace to fix the constant in front.

Note that this representation is not unique. This can also be seen in the simpler case of the $\mathfrak{su}(2)$ Lie algebra. This algebra can be represented by the Pauli matrices σ, with $t^a := \frac{\sigma_a}{2}$ then:

$$[t^i, t^j] = i\epsilon^{ijk} t^k \tag{S.3.17}$$

However the following generators will satisfy the same commutation relations:

$$\begin{array}{l} t'^1 = \cos(\theta) t^1 - \sin(\theta) t^2 \\ t'^2 = \sin(\theta) t^1 + \cos(\theta) t^2 \end{array}, \qquad t'^3 = t^3$$

and they also fulfil $\operatorname{tr}\big(t'^i t'^j\big) = \frac{\delta^{ij}}{2}$.

Exercise 1. ***Wilson Lines***

In the non-abelian case, as shown in the class, the Wilson line reads:

$$U(y,x) = \exp\left(-ie\int_{x,y} \mathrm{d}z^\mu A_\mu\right) \tag{4.1}$$

In the non abelian case, the matrices $A^a_\mu t^a$ do not commute at different space-time points. It's then convenient to have a parametric representation of the integral:

$$z(\tau) \qquad \text{with} \qquad z(0) = x \quad \text{and} \quad z(1) = y \tag{4.2}$$

Together with the path ordering of the matrices P defines as:

$$P\left(A^a_\mu(z(\tau_1)t^a A^b_\nu(z(\tau_2))t^b\right) = \\ A^a_\mu(z(\tau_1))t^a A^b_\nu(z(\tau_2))t^b\,\theta(\tau_1-\tau_2) + A^b_\nu(z(\tau_2))t^b A^a_\mu(z(\tau_1))t^a\,\theta(\tau_2-\tau_1). \tag{4.3}$$

The Wilson Line can be defined a s a path-ordered expansion of the exponential:

$$U_P(y,x) := P\left\{\exp\left(ig\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}z^\mu}{\mathrm{d}\tau} A^a_\mu(z(\tau))t^a\right)\right\}, \tag{4.4}$$

fulfilling,

$$\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau} D_\mu(A(z))U_P(z,x) = 0, \qquad D_\mu(A(z)) = \frac{\partial}{\partial z^\mu} - igA^a_\mu(z)t^a. \tag{4.5}$$

a) We would like to show that the Wilson line $U_P(y,x)$ transforms in the same way as the comparator, namely:
$$U_P(x,y,A') = V(y)U_P(y,x,A)V^\dagger(x). \tag{4.6}$$
Where A'^a_μ is the gauge transformed field of A^a_μ corresponding to $\psi(x) \to V(x)\psi(x)$.

b) Consider now an infinitesimal Wilson loop, and show that The resulting quantity is not an invariant under gauge transformation. Compute $U_P(z,z)$ dropping everything that is of $\mathcal{O}(\epsilon^3)$.

c) Show that the trace of the Wilson loop, $U_P(x,x)$, is invariant.

Solution.

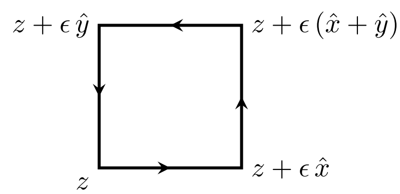


Figure 1: Infinitesimal Wilson loop.

1 Additional Material: Preface on Wilson Lines

When we consider Wilson lines, we are concerned with transporting the transformation behavior of a field at a point x to a point y. In the following we are going to derive/motivate the differential equation of the transport function $U(y,x)$, which we made first contact with already in the last exercise class and in the lectures. In particular, we saw that for an infinitesimal transport $U(x+\mathrm{d}x, x)$:

$$U(x+\mathrm{d}x, x) = \mathbb{1} + ig\partial_\mu\left(\alpha_a(x)t^a\right)\mathrm{d}x^\mu = \mathbb{1} + igA_\mu(x)\mathrm{d}x^\mu \ . \tag{S.4.1}$$

But now we are interested in the finite case. For that, we consider transport along a curve $\gamma : [0,1] \to \mathbb{R}^{1,3}$ with the curve parameter $t \mapsto x^\mu(t)$. We will furthermore use the sensible assumption, that the transport along a given curve is transitive, as depicted in fig. 2:

$$U_\gamma(\gamma(1),\gamma(0)) = U_\gamma(\gamma(1),\gamma(s))U_\gamma(\gamma(s),\gamma(0)) \ . \tag{S.4.2}$$

Let us now consider the transporter as a function of the endpoint $x^\mu(t)$ and the curve γ and consider a small variation $\Delta s > 0$ around a point $t = s$ on the path γ. The left-hand-side of (S.4.2) expanded for small Δs reads

$$U_\gamma(x^\mu(s+\Delta s), x^\mu(0)) \approxeq U_\gamma(x^\mu(s), x^\mu(0)) + \partial_\mu U_\gamma(x^\mu(s), x^\mu(0))\frac{\mathrm{d}x^\mu(s)}{\mathrm{d}s}\Delta s \tag{S.4.3}$$

while the right-hand-side

$$U_\gamma(x^\mu(s+\Delta s), x^\mu(s))U_\gamma(x^\mu(s), x^\mu(0)) \approxeq \left(\mathbb{1} + igA_\mu(x(s))\frac{\mathrm{d}x^\mu(s)}{\mathrm{d}s}\Delta s\right)U_\gamma(x^\mu(s), x^\mu(0)), \tag{S.4.4}$$

where we used (S.4.1). As we can see, this can be written as the first order differential equation in the curve parameter as:

$$\frac{\mathrm{d}}{\mathrm{d}s}U_\gamma(x(s),y) = \left(igA_\mu(x(s))\frac{\mathrm{d}x^\mu(s)}{\mathrm{d}s}\right)U_\gamma(x(s),y) \tag{S.4.5}$$

which is e.g. given in Peskin & Schröder, where the base-point $x(0) = y$ is fixed, but the endpoint varies. Alternatively we may write it as

$$\frac{\mathrm{d}x^\mu(s)}{\mathrm{d}s}\left(D_\mu U_\gamma(x(s),y)\right) = 0 \ . \tag{S.4.6}$$

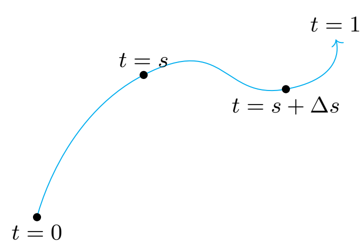


Figure 2: Sketch of a transport along a given curve $\gamma : \mathbb{R} \to \mathbb{R}^{1,3}$, $t \mapsto x^\mu(t)$, where we "split" the curve at $t = s$.

1.1 Iterated integrals and Path-ordering

In that case we have to solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \epsilon A(t)U(t) \ , \tag{S.4.7}$$

which is equivalent to the integral equation

$$U(t) - U_0 = \epsilon \int_{t=0}^{t} \mathrm{d}s A(s) U(s) \tag{S.4.8}$$

with $U_0 = U(x(t=0), x) = U(x, x) = \mathbb{1}$ and $\epsilon = g$ is some parameter. The approximative solution of either the differential or the integral equation is well known to physicists, since it is usually encountered by considering the time dependent Schrödinger equation

$$\frac{\mathrm{d}}{dt}\Psi(t, t_0) = \frac{i}{\hbar} H(t)\Psi(t, t_0) \tag{S.4.9}$$

with an explicit time dependent Hamiltonian. This problem is well formulated in terms of the Dyson series and the differential equation (S.4.7) admits the same method of solution which reads

$$\begin{aligned} U(t, t_0) &= T(t, t_0) U_0 && \text{(S.4.10)} \\ &= \left(\mathbb{1} + \epsilon \int_{t_0}^{t} A(s)\mathrm{d}s + \epsilon^2 \int_{t_0}^{t} A(s) \int_{t_0}^{s} A(s')\mathrm{d}s'\mathrm{d}s \ldots \right) U_0 && \text{(S.4.11)} \\ &= \left(\mathbb{1} + \sum_{n>1} \epsilon^n \int_{t_0 \leq s_1 \cdots \leq s_n \leq x} A(s_n) A(s_{n-1}) \ldots A(s_1) \mathrm{d}s_1 \ldots \mathrm{d}s_n \right) U_0 && \text{(S.4.12)} \\ &= \mathbb{P} e^{\epsilon \int_{t_0}^{t} A(s)\mathrm{d}s} U_0 && \text{(S.4.13)} \end{aligned}$$

where $\mathbb{P}$ denotes the *path ordering* [1] operator and the matrix $T(t_0, t)$ the *transport* of (S.4.7). For a discussion of these types of integrals, which arise also in a different context in perturbative QFT see e.g. chapter 2.1 in the lecture notes by Francis Brown `http://www.ihes.fr/~brown/ColombiaNotes7.pdf`.

a) We are looking at gauge transformation and we already know that:

$$D_\mu \to V(x) D_\mu V(x)^\dagger \tag{S.4.14}$$

We can start by assuming that $U_P(y, x, A')$ satisfy the DE (4.5), and by showing that it must follows that also $U_P(y, x, A)$ fulfills that same DE for any gauge transformation, together with the uniqueness of the solution for 1st order DE with fixed boundary condition, one can assert that the given transformation is indeed the right one. In order to prove it, we can act with the covariant derivative on both sides of the (4.6), showing that:

$$D_\mu(A') U_P(y, x, A') = V(y) D_\mu(A) U_P(y, x, A) V(x)^\dagger \tag{S.4.15}$$

$$\Rightarrow \quad D_\mu(A) U_P(y, x, A) = 0. \tag{S.4.16}$$

b) In the limit for $\epsilon \to 0$ we can write:

$$U_P(x + \epsilon\, n, x) = \exp\Big[i g\, \epsilon\, n^\mu A^a_\mu \left(x + \frac{\epsilon}{2} n\right) t^a + \mathcal{O}(\epsilon^3) \Big] \tag{S.4.17}$$

[1] Known as time ordering operator in QM.

$$U_P(z,z) = U_P(z, z+\epsilon\,\hat{y})U_P(z+\epsilon\,\hat{y}, z+\epsilon(\hat{x}+\hat{y}))U_P(z+\epsilon(\hat{x}+\hat{y}), z+\epsilon\,\hat{x})U_P(z+\epsilon\,\hat{x}, z) \tag{S.4.18}$$

$$\begin{aligned} &= \exp\Big[-ig\,\epsilon\,A^a_y\left(z+\frac{\epsilon}{2}\hat{y}\right)t^a - ig\,\epsilon\,A^a_x\left(z+\epsilon\hat{y}+\frac{\epsilon}{2}\hat{x}\right)t^a \\ &\qquad +ig\,\epsilon\,A^a_y\left(z+\epsilon\hat{x}+\frac{\epsilon}{2}\hat{y}\right)t^a + ig\,\epsilon\,A^a_x\left(z+\frac{\epsilon}{2}\hat{x}\right)t^a + \mathcal{O}(\epsilon^3)\Big] \end{aligned} \tag{S.4.19}$$

$$\begin{aligned} &= 1 + ig\epsilon\Bigg(\cancel{-A^a_y t^a} + \cancel{\frac{ig\epsilon}{2}A^a_y A^b_y t^a t^b} - \cancel{\frac{\epsilon}{2}\partial_y A^a_y t^a} \\ &\qquad \cancel{-A^a_x t^a} + \cancel{\frac{ig\epsilon}{2}A^a_x A^b_x t^a t^b} - \cancel{\frac{\epsilon}{2}\partial_x A^a_x t^a} - \epsilon\partial_y A^a_x t^a \\ &\qquad \cancel{+A^a_y t^a} + \cancel{\frac{ig\epsilon}{2}A^a_y A^b_y t^a t^b} + \cancel{\frac{\epsilon}{2}\partial_y A^a_y t^a} + \epsilon\partial_x A^a_y t^a \\ &\qquad \cancel{+A^a_x t^a} + \cancel{\frac{ig\epsilon}{2}A^a_x A^b_x t^a t^b} + \cancel{\frac{\epsilon}{2}\partial_x A^a_x t^a} \\ &\qquad \cancel{+ig\epsilon A^a_y A^b_x t^a t^b} - \cancel{ig\epsilon A^a_y A^b_y t^a t^b} - \cancel{ig\epsilon A^a_y A^b_x t^a t^b} \\ &\qquad -ig\epsilon A^a_x A^b_y t^a t^b - \cancel{ig\epsilon A^a_x A^b_x t^a t^b} + ig\epsilon A^a_y A^b_x t^a t^b\Bigg) + \mathcal{O}(\epsilon^3) \end{aligned} \tag{S.4.20}$$

$$= 1 + ig\epsilon^2\left\{\left(\partial_x A^a_y - \partial_y A^a_x\right)t^a - igA^a_x A^c_y[t^a, t^c]\right\} + \mathcal{O}(\epsilon^3) \tag{S.4.21}$$

That can be rewritten as:

$$U_P(z,z) = 1 + ig\epsilon^2 F^a_{\mu\nu}t^a + \mathcal{O}(\epsilon^3), \qquad F^a_{\mu\nu} = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ig\,f^{abc}A^b_\mu A^c_\nu\right)t^a \tag{S.4.22}$$

c) We have that the Wilson loop transforms as:

$$U_P(x,x) \to V(x)U_P(x,x)V(x)^\dagger \tag{S.4.23}$$

Then the trace is invariant, since:

$$\mathrm{tr}\Big(V(x)U_P(x,x)V(x)^\dagger\Big) = \mathrm{tr}\Big(V(x)^\dagger U_P(x,x)V(x)\Big) = \mathrm{tr}(U_P(x,x)). \tag{S.4.24}$$

Exercise 2. ***Dimensional Regularization: Minkowski Integral***

We want to compute the following integral:

$$\begin{aligned} \int d^dl\,\frac{A + B\,l^\mu + C\,l^\mu l^\nu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}} &= \int dx\,x^{\nu_1-1}(1-x)^{\nu_2-1} \\ &\qquad \frac{i(-1)^\nu \pi^{\frac{d}{2}}}{\Delta^{\nu-\frac{d}{2}}}\frac{\Gamma\left(-\frac{d}{2}+\nu\right)}{\Gamma(\nu_1)\Gamma(\nu_2)}\left(A' - C'\frac{g^{\mu\nu}}{2}\frac{1}{\left(\nu-\frac{d}{2}-1\right)\Delta}\right), \end{aligned} \tag{4.7}$$

where we introduce $\Delta = -x(1-x)p^2$ with $p^2 < 0$, while the A and B are generic expression independent of of the loop momentum l. Their expressions are mapped to:

$$A' = A - B\,(1-x)\,p^\mu + C\,(1-x)^2\,p^\mu p^\nu, \qquad C' = C. \tag{4.8}$$

(a) By means of the Feynman parameters show that:

$$\int d^dl\,\frac{A + B\,l^\mu + C\,l^\mu l^\nu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_1}} = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\int dx\,x^{\nu_1-1}(1-x)^{\nu_2-1}\int d^dl\frac{A' + C'\,l^2\frac{g^{\mu\nu}}{d}}{[l^2-\Delta]^{\nu_1+\nu_2}}, \tag{4.9}$$

Hint. *Recall that the Feynman parameters are defined as:*

$$\frac{1}{A^{\nu_1}A^{\nu_2}\ldots A^{\nu_n}} = \frac{\Gamma(\nu)}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int\left(\prod_{i=1}^n dx_i\,x^{\nu_i-1}\right)\frac{\delta\left(\Sigma_i x_i - 1\right)}{\left[\Sigma_i x_i A_i\right]^\nu}, \qquad \nu = \sum_{i=1}^n \nu_i. \tag{4.10}$$

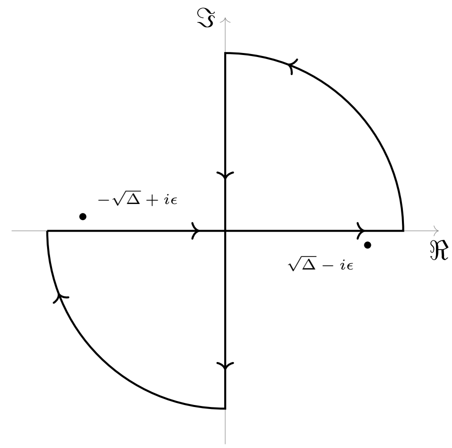


Figure 3: Wick rotation.

(b) We need to compute an integral of the form

$$\int d^d l \frac{(l^2)^a}{[l^2-\Delta]^\nu}, \tag{4.11}$$

Because of the particular form of the Minkowski metric, we cannot switch to d-dimensional polar coordinates. One possibility is to perform the integration over l^0 as a contour integral and then perform the remaining $(d-1)$-dimensional spherical integral. Another trick consists in performing a Wick rotation, rotating the integration of l^0 from the real to the imaginary axis. Then one defines the new euclidean coordinates:

$$l^0 = i\, l_E^0, \qquad \vec{l} = \vec{l}_E. \tag{4.12}$$

Using the results from the Euclidean integral, show that:

$$\int d^d l \frac{(l^2)^a}{[l^2-\Delta]^\nu} = \frac{i(-1)^{\nu+a}\pi^{\frac{d}{2}}}{\Delta^{\nu-a-\frac{d}{2}}} \frac{\Gamma\left(a+\frac{d}{2}\right)\Gamma\left(-a-\frac{d}{2}+\nu\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\nu)} \tag{4.13}$$

(c) Put everything together to recover (4.7).

(d) Set $A'=1$ and $C'=0$ and perform the integration over the Feynman parameters:

$$\int d^d l \, \frac{1}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_1}} = (i\pi^{\frac{d}{2}})(-1)^\nu(-p^2)^{\frac{d}{2}-\nu} \frac{\Gamma\left(-\frac{d}{2}+\nu\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(\frac{d}{2}-\nu_1)\Gamma(\frac{d}{2}-\nu_2)}{\Gamma(d-\nu)}. \tag{4.14}$$

Solution.

2 Additional Material: A brief outline on the relevance of this exercise

It is worth pointing out, that the particular approach covered in this exercise will almost never be used in praxis apart from these bubble type integrals, where it is possible to bring by a smart shift the denominator into the form $\left(l^2-\Delta^2\right)$. However, some of the techniques sketched in it, are important.

The first important point is, that in order to perform tensor integrals one either needs to project onto a Lorentz basis or perform a dimensional shift. Once the problem is reduced to scalar integrals, powerful techniques can be employed to compute them. In the following, we will outline the approach of projecting onto external momenta. We take the tensor Feynman integral an separate it by rank of the tensors in the numerator:

$$\int d^d l \, \frac{A + B\, l^\mu + C\, l^\mu l^\nu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}} = \int d^d l \, \frac{A}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}} + \int d^d l \, \frac{B\, l^\mu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}} + \int d^d l \, \frac{C\, l^\mu l^\nu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}} \ . \tag{S.4.25}$$

After integration, there is of course no loop-momentum dependence left and the integral will be of the form:

$$\int d^d l \, \frac{A + B\, l^\mu + C\, l^\mu l^\nu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}} = A\, \mathcal{I}_1 + BT_1^\mu(\{p\}) \cdot \mathcal{I}_2 + CT_2^{\mu\nu}(g, \{p\}) \cdot \mathcal{I}_3 \ , \tag{S.4.26}$$

where the T_i are Lorentz tensors dependent only on the external momenta, the metric and in some cases the Levi-Cevita tensor, while the $\mathcal{I}_k$ are the results of the integration of certain scalar integrals. In the following we want to see, which integrals $\mathcal{I}_k$ we need an how the tensors T_k are found. Tensor integrals will be denoted by its numerators but it is understood that all these identities hold only **after** integration. Let us first consider the rank one integral l^μ. Since we have one external momentum p, this integral will evaluate to something proportional to p^μ. We therefore make the ansatz:

$$\alpha p^\mu = l^\mu \ . \tag{S.4.27}$$

To find the coefficient α we contract both sides with p_μ

$$\alpha p^2 = (lp) \Leftrightarrow \alpha = \frac{(lp)}{p^2} \tag{S.4.28}$$

which leads to the identity

$$\int d^d l \, \frac{B\, l^\mu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}} = \frac{p^\mu}{p^2} \int d^d l \, \frac{B\,(lp)}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}} \ . \tag{S.4.29}$$

In general we need the most general rank one tensor we can build. For example if we would deal with a triangle with independent external momenta p_1 and p_2 our ansatz for the rank one tensor is $\alpha p_1^\mu + \beta p_2^\mu$.
Let us now consider the rank 2 case. We make the most general ansatz which can be build by the metric and one external momentum p:

$$\alpha_1 g^{\mu\nu} + \alpha_2 p_1^\mu p_2^\nu = l^\mu l^\nu \ . \tag{S.4.30}$$

Contracting with $g_{\mu\nu}$ and with $p_\mu p_\nu$ yields

$$l^2 = \alpha_1 d + \alpha_2 p^2 \qquad \text{and} \qquad (lp)^2 = \alpha_1 p^2 + \alpha_2 (p^2)^2 \ , \tag{S.4.31}$$

which we can easily solve for α_1 and α_2. For a triangle with independent external momenta p_1 and p_2 our ansatz for the rank two tensor is $\alpha p_1^\mu p_1^\nu + \alpha_2 p_2^\mu p_2^\nu + \alpha_3 p_1^\mu p_2^\nu + \alpha_4 p_2^\mu p_1^\nu + \alpha_5 g^{\mu\nu}$ and we see that we would need to invert a 5×5 matrix.
For high rank and multi-leg processes the inversion of these systems becomes a obstacle quickly. It is therefore often desirable, to use all symmetries and additional features of the underlying process. In particular, if the open Lorentz indices are related to contractions with external bosons, one can additionally impose Ward-identities or gauges to minimize the tensor-structures in analogy to the last exercise of the first exercise sheet.
With respect to the exercise discussed in the following: By performing the smart manipulations we arrived at the denominator $(l^2 - \Delta^2)$, which is the denominator of a tadpole diagram without inflowing external momentum but of mass Δ. The ansatz for the rank 2 tensor is therefore simply $\alpha_1 g^{\mu\nu}$, which yields the formula used in the solution.

(a) We start by using the Feynman parameters for the current case:

$$\int d^d l \, \frac{A + B\, l^\mu + C\, l^\mu l^\nu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_1}} = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int dx\, x^{\nu_1 - 1}(1-x)^{\nu_2 - 1} \int d^d l \frac{A + B\, l^\mu + C\, l^\mu l^\nu}{[(l+(1-x)p)^2 - x(1-x)p^2]^{\nu_1+\nu_2}}, \tag{S.4.32}$$

We can now perform a shift in the loop momentum $l \to l - (1-x)p$. This translation will cast the denominator in the form $[l^2-\Delta]^{\nu_1+\nu_2}$ with $\Delta=-p^2x(1-x)$. Note that the denominator is now an even function in l and this can be used to remove all the odd pieces from the numerator, in particular:

$$\int d^d l \frac{A'+B'l^\mu+C'l^\mu l^\nu}{[l^2-\Delta]^{\nu_1+\nu_2}}=\int d^d l \frac{A'+C'\,l^2\,\frac{g^{\mu\nu}}{d}}{[l^2-\Delta]^{\nu_1+\nu_2}}. \tag{S.4.33}$$

This will then give us exactly the expression that we were looking for.

(b) Note that the singularities are located at:

$$l^2-\Delta+i\epsilon=\left[l^0-\left(\sqrt{\vec{l}^2+\Delta}-i\epsilon\right)\right]\left[l^0+\left(\sqrt{\vec{l}^2+\Delta}-i\epsilon\right)\right]=0$$

The poles are then:

$$l^0=\pm\left(\sqrt{\vec{l}^2+\Delta}-i\epsilon\right) \tag{S.4.34}$$

In Fig.(3) we can see that the Wick rotation doesn't cross any of this poles, allowing us to exchange an integration along the real axis for one along the imaginary axis. Doing the rotation gives us:

$$\begin{aligned}\int d^d l \frac{(l^2)^a}{[l^2-\Delta]^\nu}&=i(-1)^{\nu+a}\int d^d l_E \frac{(l_E^2)^a}{[l_E^2+\Delta]^\nu}\\&=\frac{i(-1)^{\nu+a}\pi^{\frac{d}{2}}}{\Delta^{\nu-a-\frac{d}{2}}}\frac{\Gamma\left(a+\frac{d}{2}\right)\Gamma\left(-a-\frac{d}{2}+\nu\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\nu)}\end{aligned}$$

(c) We can now use the general solution from equation (4.13) for the two special cases of $a=0,1$:

$$\int d^d l \frac{1}{[l^2-\Delta]^\nu}=\frac{i(-1)^\nu\pi^{\frac{d}{2}}}{\Delta^{\nu-\frac{d}{2}}}\frac{\Gamma\left(-\frac{d}{2}+\nu\right)}{\Gamma(\nu)} \tag{S.4.35}$$

$$\int d^d l \frac{l^2}{[l^2-\Delta]^\nu}=\frac{i(-1)^{\nu+1}\pi^{\frac{d}{2}}}{\Delta^{\nu-1-\frac{d}{2}}}\frac{d}{2}\frac{\Gamma\left(-1-\frac{d}{2}+\nu\right)}{\Gamma(\nu)} \tag{S.4.36}$$

Then we can apply this two results to the expression that we compute at the beginning of the exercise:

$$\begin{aligned}\int d^d l\,\frac{A+B\,l^\mu+C\,l^\mu l^\nu}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_2}}=&\int dx\,x^{\nu_1-1}(1-x)^{\nu_2-1}\\&\frac{i(-1)^\nu\pi^{\frac{d}{2}}}{\Delta^{\nu-\frac{d}{2}}}\frac{\Gamma\left(-\frac{d}{2}+\nu\right)}{\Gamma(\nu_1)\Gamma(\nu_2)}\left(A'-C'\frac{g^{\mu\nu}}{2}\frac{1}{\left(\nu-\frac{d}{2}-1\right)\Delta}\right).\end{aligned} \tag{S.4.37}$$

(d) Let's start form (4.7) and set $C'=0$ and $A'=0$.

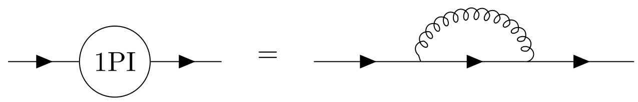
$$\begin{aligned}\int d^d l\,\frac{1}{(l^2)^{\nu_1}[(l+p)^2]^{\nu_1}}&=i(-1)^\nu\pi^{\frac{d}{2}}\frac{\Gamma\left(-\frac{d}{2}+\nu\right)}{\Gamma(\nu_1)\Gamma(\nu_2)}\int dx\,x^{\nu_1-1}(1-x)^{\nu_2-1}\Delta^{-\nu+\frac{d}{2}}\\&=(i\pi^{\frac{d}{2}})(-1)^\nu(-p^2)^{\frac{d}{2}-\nu}\frac{\Gamma\left(-\frac{d}{2}+\nu\right)}{\Gamma(\nu_1)\Gamma(\nu_2)}\int dx\,x^{-\nu_2-1+\frac{d}{2}}(1-x)^{-\nu_1-1+\frac{d}{2}}\\&=(i\pi^{\frac{d}{2}})(-1)^\nu(-p^2)^{\frac{d}{2}-\nu}\frac{\Gamma\left(-\frac{d}{2}+\nu\right)}{\Gamma(\nu_1)\Gamma(\nu_2)}\frac{\Gamma\left(\frac{d}{2}-\nu_2\right)\Gamma\left(\frac{d}{2}-\nu_1\right)}{\Gamma(d-\nu)}.\end{aligned} \tag{S.4.38}$$

Where in the last step we just used the definition of the β-function.

Exercise 1. ***Mass Renormalization***

We want to compute the renormalized mass for the fermions in QCD. The corrections at LO are given by a virtual gluon:

$i\Sigma_\psi =$



$$+ \quad \mathcal{O}(\alpha_s^2)$$
$$= i\frac{\alpha_s\,\mu^{2\epsilon}}{(4\pi)\epsilon}C_F\left(\not{p} - 4m\right) + \mathcal{O}(\alpha_s^2, \epsilon^0), \tag{5.1}$$

where $\alpha_s = \frac{g_s^2}{4\pi}$. We recall from the lecture that the renormalized propagator is defined as:

$$\begin{aligned} iG^{(R)}(\not{p}) &= Z_\psi^{-1} iG^{(0)}(\not{p}) \\ &= Z_\psi^{-1}\frac{i}{\not{p} - m^{(0)} + \Sigma_\psi(\not{p})}. \end{aligned} \tag{5.2}$$

With the upper index 0 to denote the bare parameters. Then the renormalized mass is defined as:

$$m^{(R)} = Z_m^{-1}\, m^{(0)}. \tag{5.3}$$

(a) Consider the renormalized propagator and write an expression for $\Sigma_R(\not{p}, m^{(R)})$ as a function of the renormalization constants δ_m and δ_ψ. The function $\Sigma_R(\not{p}, m^{(R)})$ is defined as:

$$iG^{(R)} = \frac{i}{\not{p} - m^{(R)} + \Sigma_R(\not{p}, m^{(R)})} \tag{5.4}$$

Show that the renormalization constant in the $\overline{MS}$ scheme are given by:

$$\begin{aligned} Z_\psi &= 1 + \delta_\psi & \qquad Z_m &= 1 + \delta_m \\ &= 1 - \tfrac{g_s^2}{(4\pi)^2\epsilon}C_F + \mathcal{O}(\alpha_s^2), & &= 1 - \tfrac{3g_s^2}{(4\pi)^2\epsilon}C_F + \mathcal{O}(\alpha_s^2). \end{aligned} \tag{5.5}$$

(b) The running of the mass is given by the renormalization group equation

$$\mu^2\frac{dm^{(R)}}{d\mu^2} = -\gamma_m\left(\alpha_s^{(R)}\right)m^{(R)}. \tag{5.6}$$

Use the fact that the bare mass $m^{(0)}$ is independent of μ in order to show that the *anomalous dimension* γ_m is given to order α_s by:

$$\gamma_m\left(\alpha_s^{(R)}\right) = 4\left(\frac{\alpha_s^{(R)}}{4\pi}\right) + \mathcal{O}\left(\left(\alpha_s^{(R)}\right)^2\right). \tag{5.7}$$

(c) Solve the differential equation for the renormalized mass $m^{(R)}$. The boundary condition for the integration are the two scales μ_0 and μ_1. Then the solution can be written as:

$$m^{(R)}(\mu_1) = m(\mu_0)\left(\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_0)}\right)^{\frac{4}{\beta_0}}. \tag{5.8}$$

Hint. Recall that the coupling constant g_s satisfies:

$$\frac{dg_s(\mu)}{d\log\mu} = -\frac{\beta_0\, g_s(\mu)^3}{(4\pi)^2}. \tag{5.9}$$

Solution.

(a)

$$iG^{(R)}(\not{p}) = \frac{1}{1+\delta_\psi}\frac{i}{\not{p} - m^{(R)} - \delta_m m^{(R)} + \Sigma_\psi(\not{p})} \tag{S.5.1}$$

$$= \frac{i}{\not{p} - m^{(R)} + \delta_\psi \not{p} - (\delta_\psi + \delta_m)m^{(R)} + \Sigma_\psi(\not{p})} \tag{S.5.2}$$

With this we can conclude that

$$\Sigma_R(\not{p}) = \delta_\psi \not{p} - (\delta_\psi + \delta_m)m^{(R)} + \Sigma_\psi(\not{p}) + \mathcal{O}(\alpha_s^2) \tag{S.5.3}$$

The bare 1-PI quark self-energy can be obtained from the electron one by including the suitable color factor. In Feynman gauge one gets::

$$-i\Sigma_0(\not{p}) = -ig^2 C_F \delta_{ij}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int dx \frac{d\,m - (d-2)x\,\not{p}}{[-p^2\,x(1-x) + m^2(1-x)]^{2-\frac{d}{2}}} \tag{S.5.4}$$

Keeping only the divergent term (i.e. by taking the limit $d = 0$ for the integrand), and truncating the perturbative expansion at $\mathcal{O}(\alpha_s)$, the first 1PI correction is:

$$\frac{g_s^2}{(4\pi)^2\epsilon}\left(\not{p} - 4\,m^{(R)}\right) + \mathcal{O}(\alpha_s^2, \epsilon^0) \tag{S.5.5}$$

Putting everything together, gives:

$$Z_\psi = 1 - \frac{g_s^2}{(4\pi)^2\epsilon}C_F + \mathcal{O}(\alpha_s^2, \epsilon^0), \qquad Z_m = 1 - \frac{3g_s^2}{(4\pi)^2\epsilon}C_F + \mathcal{O}(\alpha_s^2, \epsilon^0). \tag{S.5.6}$$

We can also extract the finite part form the integral over the Feynman parameter x. This can be done by noticing that the renormalization conditions are:

$$\begin{aligned} \Sigma_R(\not{p})\big|_{\not{p}=m^{(R)}} = 0 \quad &\Rightarrow \quad \Sigma_\psi(\not{p})\big|_{\not{p}=m^{(R)}} = -\delta_m, \\ \left.\frac{\mathrm{d}\Sigma_R(\not{p})}{\mathrm{d}\not{p}}\right|_{\not{p}=m^{(R)}} = 0 \quad &\Rightarrow \quad \left.\frac{\mathrm{d}\Sigma_\psi(\not{p})}{\mathrm{d}\not{p}}\right|_{\not{p}=m^{(R)}} = \delta_\psi. \end{aligned} \tag{S.5.7}$$

$$\delta_m = -g^2 C_F \delta_{ij}(m^{(R)})^{d-3}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int dx \frac{d-(d-2)x}{(1-x)^{4-d}}. \tag{S.5.8}$$

$$\delta_\psi = g^2 C_F \delta_{ij}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(m^{(R)})^{d-4}\int dx\left[\frac{-(d-2)x}{(1-x)^{4-d}} + \left(2-\frac{d}{2}\right)\frac{2x(1-x)(d-(d-2)x)}{(1-x)^{6-d}}\right]. \tag{S.5.9}$$

This integrals can then be easily solved by using the definition of the β-function:

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x \tag{S.5.10}$$

Then, taking $d = 4 - 2\epsilon$ and performing an expansion around $\epsilon = 0$:

$$\delta_\psi = -\frac{g_s}{(4\pi)^2}C_F\left[\frac{1}{\epsilon} - \left(\log\left(\frac{m^2}{4\pi}\right) + \gamma_E - 1\right) + \mathcal{O}(\epsilon^1)\right] \tag{S.5.11}$$

$$\delta_m = -3\frac{g_s}{(4\pi)^2}C_F\left[\frac{1}{\epsilon} - \left(\log\left(\frac{m^2}{4\pi}\right) + \gamma_E - \frac{4}{3}\right) + \mathcal{O}(\epsilon^1)\right] \tag{S.5.12}$$

Showing that the finite term contains a terms that is divergent as the renormalized mass goes to zero.

(b) We use

$$\mu^2\frac{dm^{(0)}}{d\mu^2} = 0. \tag{S.5.13}$$

This equation can be written in terms of the renormalized mass and the Z_m parameter. Taking the derivative leads us to:

$$\begin{aligned}\mu^2\frac{dZ_m m^{(R)}}{d\mu^2} &= \mu^2\frac{dZ_m}{d\mu^2}m^{(R)} + \mu^2\frac{dm^{(R)}}{d\mu^2}Z_m \\ &= \left(\mu^2\frac{dZ_m}{d\mu^2}\right) + \mu^2\frac{dm^{(R)}}{d\mu^2} + \mathcal{O}(\alpha_s^2,\epsilon^0)\end{aligned} \tag{S.5.14}$$

Then the first contribution to the anomalous dimension is given by:

$$\mu^2\frac{dZ_m}{d\mu^2} = -\frac{3C_F}{(2\epsilon)(4\pi)}\mu\frac{d\alpha_s^{(R)}}{d\mu} \tag{S.5.15}$$

For simplicity let's use from now on $\alpha_s = \alpha_s^{(R)}$.
The derivative above is nothing but the β-function $\beta(\alpha_s)$, given by:

$$\beta(\alpha_s) := \mu\frac{d\alpha_s^{(R)}}{d\mu} = -(2\epsilon)\alpha_s - 2\alpha_s\sum_i\left(\frac{\alpha_s}{4\pi}\right)\beta_i, \tag{S.5.16}$$

with β_i some coefficients that don't contribute at the current order. Putting everything together and using the definition for the casimir operator $C_F = \frac{8}{6}$, we find:

$$\begin{aligned}\gamma_m(\alpha_s) &= \mu^2\frac{dZ_m}{d\mu^2} + \mathcal{O}\left(\alpha_s^2,\epsilon^0\right) \\ &= 4\left(\frac{\alpha_s}{4\pi}\right) + \mathcal{O}\left(\alpha_s^2,\epsilon^0\right).\end{aligned} \tag{S.5.17}$$

(c) The differential equation leads to the integral expression:

$$\int_{m^{(R)}(\mu_0)}^{m^{(R)}(\mu_1)}\frac{dm^{(R)}}{m^{(R)}} = -\frac{2}{\pi}\int_{\mu_0}^{\mu_1}\alpha_s(\mu)\,d\log(\mu) \tag{S.5.18}$$

The differential equation for α_s give us a change of variable to be able to integrate the right-hand side of the expression, in particular:

$$\frac{d\alpha_s}{d\log(\mu)} = -2\beta_0\frac{\alpha_s^2}{4\pi} \qquad \Rightarrow \qquad \alpha_s\,d\log(\mu) = -\frac{2\pi}{\beta_0}\frac{d\alpha_s}{\alpha_s}. \tag{S.5.19}$$

Plugging in this substitution and carrying out the integration over α_s and m one finds:

$$\log\left(\frac{m^{(R)}(\mu_1)}{m^{(R)}(\mu_0)}\right) = \log\left(\left[\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_0)}\right]^{\frac{4}{\beta_0}}\right). \tag{S.5.20}$$

By exponentiating the above expression one can recover the expression for $m^{(R)}(\mu_1)$.

Exercise 2. *Dimensional Regularization: Euclidean Integral*

Compute the superficial degree of divergence D of loop diagrams in d-dimensional QCD, defined as:

$$D := (\text{power of loop momenta in numerator}) - (\text{power of loop momenta in denominator}). \tag{5.10}$$

You may start by the explicit expression:

$$D = d\cdot L - P_q - 2P_g - 2P_c + V_{3g} + V_{gc}, \tag{5.11}$$

Then express it in terms of the number of external particles and number of vertices.

Use the following notation to express your results:

$N_{q,g,c}$	:	number of external gluon or ghost, respectively.
$P_{q,g,c}$	:	number of quark, gluon or ghost propagators, respectively.
V_{gq}	:	number of quark-gluon vertices,
V_{gc}	:	number of ghost-gluon vertices,
V_{3g}	:	number of three gluon vertices,
V_{4g}	:	number of four gluon vertices.
V	:	total number of vertices.

Hint. What is the expression for L? Can you get rewrite all the propagators in terms of externals and vertices?

Solution. First we notice that the number of loop is given by,

$$L = P_q + P_g + P_c - (V - 1). \tag{S.5.21}$$

Moreover, the propagators can be expressed as,

$$2P_q + N_q = 2V_{qg}, \tag{S.5.22}$$

$$2P_c + N_c = 2V_{cg}, \tag{S.5.23}$$

$$2P_g + N_g = 4V_{4g} + 3V_{4g} + V_{qg} + V_{cg}, \tag{S.5.24}$$

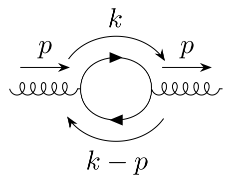
Plugging this expression into eq. (5.11) and after some algebra:

$$D = d + \frac{d-4}{2}V + \frac{d-4}{2}V_{4g} - \frac{d-1}{2}N_q - \frac{d-2}{2}N_g - \frac{d-2}{2}N_c \tag{S.5.25}$$

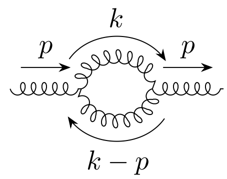
Exercise 1. ***Vacuum polarization***

We want to compute the 1PI contributions to the gluon self-energy at order g_s^2,

$$i\Sigma^{ab,\nu\mu} =$$



$$+$$



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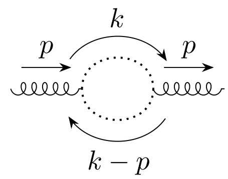
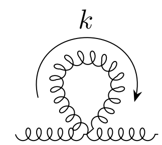


$$+$$



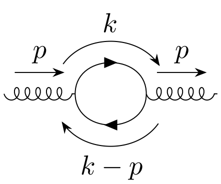
$$+ \quad \dots \tag{6.1}$$

In equation (6.1) we can see the corrections due to a fermion- gluon- and ghost-loop, there are also higher order corrections that are $\mathcal{O}(g_s^4)$ that we won't consider. From the Ward identity we expect the result to have the form:

$$\Sigma^{ab,\nu\mu} = (p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(-p^2) \tag{6.2}$$

(a) We start by computing the correction coming from the fermionic loop. Apply the Feynman rules in order to obtain:

$$i\Sigma^{ab,\nu\mu}_{\text{fermion}} :=$$



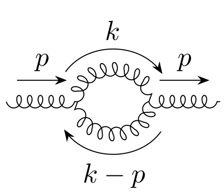
$$= -\, g_s^2 \,\mathrm{tr}\Big(t^a t^b\Big)\mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\mathrm{tr}\big(\gamma^\mu (\not{k} - \not{p})\gamma^\nu \not{k}\big)}{k^2 (k-p)^2} \tag{6.3}$$

Perform the integration and show that the final solution fulfil the ward identity and reads:

$$i\left[g^{\mu\nu}p^2 - p^\mu p^\nu\right]\delta^{ab}\left(-\alpha_s \left(\frac{4\pi\mu^2}{-p^2}\right)^{\epsilon} 8\, n_f\, T_F \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\right) \tag{6.4}$$

(b) Compute the correction coming from the gluon loops. In the lecture you saw that the contribution coming from the tadpole is zero, therefore we only need to consider the gluon-loop with the 3-gluon vertices:

$$i\Sigma^{ab,\nu\mu}_{\text{gluon}} :=$$



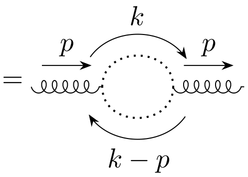
$$= -\,\frac{g_s^2}{2}\mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{f^{acd} f^{bdc} N^{\mu\nu}}{k^2 (k-p)^2}, \tag{6.5}$$

$$\begin{aligned} N^{\mu\nu} =& \left(g^{\rho\sigma}(p-2k)^\mu + g^{\mu\sigma}(k-2p)^\rho + g^{\mu\rho}(k+p)^\sigma\right) \\ & \left(g_\rho{}^\sigma (2k-p)^\nu + g^\nu{}_\sigma (2p-k)_\rho + g^\nu{}_\rho(-k-p)_\sigma\right) \end{aligned} \tag{6.6}$$

Use the Feynman rules from the lecture in Feynman gauge ($\xi = 1$) to derive the expression above. Carry out the integration over the loop momentum neglecting the integration over the Feynman parameters, show that the result is:

$$i\Sigma^{ab,\nu\mu}_{\text{gluon}} = C_A\delta^{ab}g_s^2\mu^{4-d}\int dx \frac{i\Delta^{\frac{d}{2}-2}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left[-g^{\mu\nu}p^2\left(\left(3\frac{(d-1)}{(d-2)}+1\right)x(1-x)-\frac{5}{2}\right)\right.$$
$$\left.+p^\mu p^\nu\left(\frac{(d-6)}{2}+(3-2d)x(1-x)\right)\right]. \quad (6.7)$$

(c) The result coming from the gluon loop is giving us a result that does not have a projection over the transverse polarizations. In order to fix this we need to include the correction coming from the ghost loop:

$$i\Sigma^{ab,\nu\mu}_{\text{ghost}} :=$$


$$= g_s^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{f^{acd}f^{bdc}k^\mu(k-p)^\nu}{k^2(k-p)^2} \quad (6.8)$$

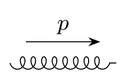
Integrating as in the gluon correction, show that the solution can be cast in the following form:

$$i\Sigma^{ab,\nu\mu}_{\text{ghost}} = C_A\delta^{ab}g_s^2\mu^{4-d}\int dx\frac{i\Delta^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{p^2g^{\mu\nu}}{d-2}+p^\mu p^\nu\right)x(1-x) \quad (6.9)$$

(d) Sum the contributions from the gluon-loop (6.7) and the ghost-loop (6.10), and carry out the remaining integration over the Feynman parameters.

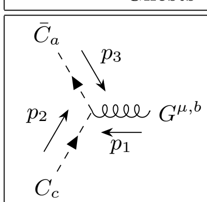
$$i\left(\Sigma^{ab,\nu\mu}_{\text{gluon}}+\Sigma^{ab,\nu\mu}_{\text{ghost}}\right) = i\left[g^{\mu\nu}p^2-p^\mu p^\nu\right]\delta^{ab}\left(\frac{\alpha_s}{4\pi}\left(\frac{4\pi\mu^2}{-p^2}\right)^\epsilon C_A\frac{2(3d-2)}{d-2}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\right) \quad (6.10)$$

Solution. The gluon propagator is given in R_ξ gauge:

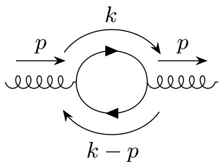


$$= \frac{-i\delta^{ab}}{p^2+i\epsilon}\left(g^{\mu\nu}-(1-\xi)\frac{p^\mu p^\nu}{p^2+i\epsilon}\right), \quad (S.6.1)$$

while the ghosts:

Ghosts Vertex	Ghost Propagator
 $= g_s f^{abc}p_3^\mu$	 $= i\frac{\delta^{ab}}{p^2+i\epsilon}$

(a) We start by computing the correction coming from the fermionic loop:



$$= -\, g_s^2 \,\mathrm{tr}\Big(t^a t^b\Big) \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\mathrm{tr}\big(\gamma^\mu (\not{k} - \not{p}) \gamma^\nu \not{k}\big)}{k^2 (k-p)^2} \tag{S.6.2}$$

$$= -g_s^2 T_F \delta^{ab} \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{4(2k^\mu k^\nu - k^\mu p^\nu - k^\nu p^\mu - g^{\mu\nu}(k^2 - k\cdot p))}{k^2 (k-p)^2} \tag{S.6.3}$$

$$= -4\, g_s^2 T_F \delta^{ab} \mu^{4-d} \int dx \int \frac{d^d k}{(2\pi)^d} \frac{-2x(1-x) p^\mu p^\nu + g^{\mu\nu}\left[\left(\frac{2}{d} - 1\right) k^2 + x(1-x) p^2\right]}{(k^2 - \Delta)^2} \tag{S.6.4}$$

$$= -4\, g_s^2 T_F \delta^{ab} \mu^{4-d} \int dx \int \frac{d^d k}{(2\pi)^d} \frac{\Delta \cdot \left(g^{\mu\nu} - 2\frac{p^\mu p^\nu}{p^2}\right) - \frac{2}{d} g^{\mu\nu} \left(1 - \frac{d}{2}\right) k^2}{(k^2 - \Delta)^2} \tag{S.6.5}$$

We use

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta)^2} = \frac{-i}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \Gamma\left(1 - \frac{d}{2}\right) \Delta^{\frac{d}{2}-1}$$
$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^2} = \frac{i}{(4\pi)^{\frac{d}{2}}} \Gamma\left(2 - \frac{d}{2}\right) \Delta^{\frac{d}{2}-2}$$

The result of the integral over the loop momentum is:

$$\frac{-2i}{(4\pi)^{\frac{d}{2}}} \Gamma(\epsilon)\, \Delta^{\frac{d}{2}-1} \left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right] \tag{S.6.6}$$

By performing also the integral over the Feynman parameter one gets:

$$i\Sigma^{ab,\nu\mu}_{\text{fermion}} = -g_s^2 T_F\, d\, \mu^{4-d} \delta^{ab} (-p^2)^{\frac{d}{2}-1} \frac{\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} \frac{-8i}{(4\pi)^{\frac{d}{2}}} \Gamma(\epsilon) \left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right] \tag{S.6.7}$$

This can be already used as final result by extracting explicitly the pole in ϵ coming from $\Gamma(\epsilon)$.
Let's see what an expansion around $\epsilon = 0$ looks like in this case. First we redefine $\tilde{\mu}^2 := 4\pi\, e^{-\gamma_E} \mu^2$ as in the $\overline{MS}$ scheme, then

$$8i\, T_F \delta^{ab} \frac{g_s^2(-p^2)\left(\frac{\mu^2}{-p^2}\right)^\epsilon e^{\epsilon\,\gamma_E}}{(4\pi)^2} \frac{\Gamma\left(\frac{d}{2}\right)^2 \Gamma(\epsilon)}{\Gamma(d)} \left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right] =$$
$$= -8i\, T_F \delta^{ab} \frac{g_s^2}{16\pi^2} \left[p^2 g^{\mu\nu} - p^\mu p^\nu\right] \underbrace{\left(\frac{\tilde{\mu}^2}{-p^2}\right)^\epsilon e^{\epsilon\,\gamma_E} \frac{\Gamma\left(\frac{d}{2}\right)^2 \Gamma(\epsilon)}{\Gamma(d)}}_{\frac{1}{8}\left(\frac{4}{3\epsilon} + \left(\frac{4}{3}\log\left(\frac{\tilde{\mu}^2}{-p^2}\right) + \frac{20}{9}\right) + O(\epsilon^1)\right)} \tag{S.6.8}$$

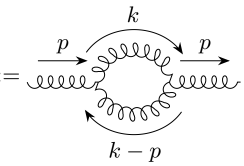
Using

$$e^{\epsilon\,\gamma_E} \Gamma(\epsilon) = \frac{1}{\epsilon} - \frac{\zeta(2)}{2}\epsilon + \mathcal{O}(\epsilon^2)$$
$$\frac{\Gamma(2-\epsilon)^2}{\Gamma(4-2\epsilon)} = \frac{1}{6} + \frac{5}{18}\epsilon + \mathcal{O}(\epsilon^2)$$

$$i\Sigma^{ab,\nu\mu}_{\text{fermion}} = -i\, T_F \delta^{ab} \frac{g_s^2}{16\pi^2} \left[p^2 g^{\mu\nu} - p^\mu p^\nu\right] \left[\frac{4}{3\epsilon} + \left(\frac{4}{3}\log\left(\frac{\tilde{\mu}^2}{-p^2}\right) + \frac{20}{9}\right) + O\left(\epsilon^1\right)\right] \tag{S.6.9}$$

(b) We start by computing the correction coming from the gluonic loop:

$$i\Sigma^{ab,\nu\mu}_{\text{gluon}} :=$$



$$= -\frac{g_s^2}{2}\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{f^{acd}f^{bdc}N^{\mu\nu}}{k^2(k-p)^2}, \tag{S.6.10}$$

note the symmetry factor $\frac{1}{2}$.
We simplify the structure constants:

$$-\frac{g_s^2}{2}\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{f^{acd}f^{bdc}N^{\mu\nu}}{k^2(k-p)^2} = C_A\delta^{ab}\frac{g_s^2}{2}\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{N^{\mu\nu}}{k^2(k-p)^2} \tag{S.6.11}$$

The tensor part reads:

$$\begin{aligned} N^{\mu\nu} =& \left(g^{\rho\sigma}(p-2k)^\mu + g^{\mu\sigma}(k-2p)^\rho + g^{\mu\rho}(k+p)^\sigma\right) \\ & \left(g_\rho{}^\sigma(2k-p)^\nu + g^\nu{}_\sigma(2p-k)_\rho + g^\nu{}_\rho(-k-p)_\sigma\right) \end{aligned} \tag{S.6.12}$$

Now we can use the Feynman parameters for the denominator and then we apply the shift $k \to k + xp$. Once we apply this shift we can drop all the linear terms in the loop momentum:

$$N^{\mu\nu} = -g^{\mu\nu}\left(k^2 6\left(1-\frac{1}{d}\right) + p^2\left(2x^2-2x+5\right)\right) - p^\mu p^\nu\left(d(1-2x)^2 - 6\left(x^2-x+1\right)\right)$$

Alternatively one can notice that the structure of $N^{\mu\nu}$ can be reconstructed by making use of the fact the only available tensor structure are $g^{\mu\nu}$ and $p^\mu p^\nu$. This can be done by computing indices trough projectors. After integration and using,

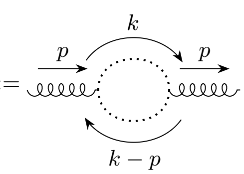
$$\Gamma\left(1-\frac{d}{2}\right) = -2\frac{\Gamma\left(2-\frac{d}{2}\right)}{d-2} \tag{S.6.13}$$

one obtain

$$\begin{aligned} C_A\delta^{ab}g_s^2\mu^{4-d}\int dx \frac{i\Delta^{\frac{d}{2}-2}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\Bigg[& -g^{\mu\nu}p^2\left(\left(3\frac{(d-1)}{(d-2)}+1\right)x(1-x)-\frac{5}{2}\right) \\ & + p^\mu p^\nu\left(\frac{(d-6)}{2} + (3-2d)x(1-x)\right)\Bigg] \end{aligned} \tag{S.6.14}$$

(c) Let's see what is the contribution coming from the ghosts:

$$i\Sigma^{ab,\nu\mu}_{\text{ghost}} :=$$



$$= g_s^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{f^{acd}f^{bdc}k^\mu(k-p)^\nu}{k^2(k-p)^2} \tag{S.6.15}$$

$$\begin{aligned} -C_A\delta^{ab}g_s^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{k^\mu(k-p)^\nu}{k^2(k-p)^2} =& -C_A\delta^{ab}g_s^2\mu^{4-d}\int dx\int \frac{d^dk}{(2\pi)^d}\frac{\frac{g^{\mu\nu}}{d}k^2 + \Delta\frac{p^\mu p^\nu}{p^2}}{(k^2-\Delta)^2} \\ =& -C_A\delta^{ab}g_s^2\mu^{4-d}\int dx \frac{i}{(4\pi)^{\frac{d}{2}}}\left(-\frac{g^{\mu\nu}}{d}\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right) + \frac{p^\mu p^\nu}{p^2}\Gamma\left(2-\frac{d}{2}\right)\right)\Delta^{\frac{d}{2}-1} \\ =& -C_A\delta^{ab}g_s^2\mu^{4-d}\int dx \frac{i}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\left(-\frac{g^{\mu\nu}}{d}\frac{d}{2}\frac{1}{1-\frac{d}{2}} + \frac{p^\mu p^\nu}{p^2}\right)\Delta^{\frac{d}{2}-1} \\ =& -C_A\delta^{ab}g_s^2\mu^{4-d}\int dx \frac{i}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{g^{\mu\nu}}{d-2} + \frac{p^\mu p^\nu}{p^2}\right)\Delta^{\frac{d}{2}-1} \\ =& C_A\delta^{ab}g_s^2\mu^{4-d}\int dx \frac{i\Delta^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{p^2g^{\mu\nu}}{d-2} + p^\mu p^\nu\right)x(1-x) \end{aligned}$$

(d) The sum reads:

$$C_A \delta^{ab} g_s^2 \mu^{4-d} \int dx \frac{i\Delta^{\frac{d}{2}-2}}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \left[-g^{\mu\nu} p^2 \left(\frac{2(2d-3)}{d-2} x(1-x) - \frac{5}{2}\right)\right.$$
$$\left. + p^\mu p^\nu \left(\frac{(d-6)}{2} + (4-2d)x(1-x)\right)\right] \quad \text{(S.6.16)}$$

Then we can integrate over the Feynman parameters:

$$i\left[g^{\mu\nu}p^2 - p^\mu p^\nu\right] \delta^{ab} \left(\frac{\alpha_s}{4\pi} \left(\frac{4\pi\mu^2}{-p^2}\right)^\epsilon C_A \frac{2(3d-2)}{d-2} \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\right) \quad \text{(S.6.17)}$$

Exercise 1. ***Installation***

In this tutorial we would like to explore some tools to compute Feynman diagrams using Mathematica. In order run the auxiliary file you would need the following software:

- `Mathematica`
- `FeynCalc + FeynArt` for `Mathematica`

The `Mathematica` software can be freely downloaded from the ETH IT shop, you can access it with the nethz login. There are two versions available ”floating” and ”node”, you need to download the node version if you want to have the license on your computer. All the details for the installation can be found on the IT shop page.

A guide for the installation of `FeynCalc` can be found on their wiki (`https://github.com/FeynCalc/feyncalc/wiki`). You will need to install the development version. This can easily be done by running a Mathematica session and executing:

```
Import["https://raw.githubusercontent.com/FeynCalc/feyncalc/master/install.m"]
InstallFeynCalc[InstallFeynCalcDevelopmentVersion -> True]
```

This command will also install the necessary `FeynArt` dependencies.

Exercise 2. ***One-loop corrections in QCD***

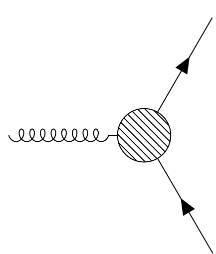
Your task is to compute all one-loop vertex corrections and self-energies in QCD.

i) Familiarize yourself to a certain degree with the code in **QCD_START.m**. Both FeynArts and FeynCalc have extensive documentations.

ii) Complete the computation of all one-loop counter-terms.

iii) Rewrite the final results in terms of Casimir’s operators for the fundamental and adjoint representation.

Solution. The final complete version of the package that is found on the moodle page of this course give you a routine to compute various one-loop corrections to several QCD vertices, together with many propagators.
The three point functions corrections include: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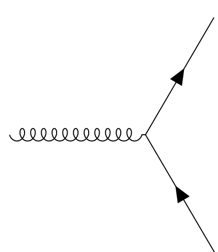
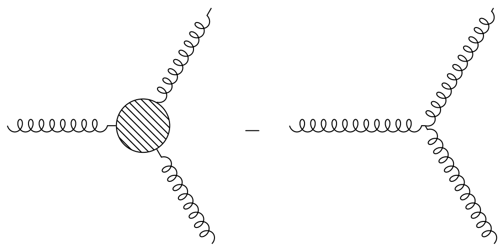

$$= \frac{g_s^3 (C_F - C_A) T^a \gamma^\mu}{\epsilon} \mathcal{O}\left(\alpha_s^2, \epsilon^0\right)$$

Where one has to rewrite the final output in terms of Casimir's operators for the fundamental and adjoint representation.
Another contribution consists in the 3-point gluon vertex:



$$= -\frac{g_s f^{abc}\left(9C_A\xi_g - 17C_A + 8n_f\right)}{12\epsilon}\left(g^{\nu\rho}\left(p_1^\mu + 2p_2^\mu\right) - g^{\mu\rho}\left(2p_1^\nu + p_2^\nu\right) + g^{\mu\nu}\left(p_1^\rho - p_2^\rho\right)\right) + \mathcal{O}\left(\alpha_s^2, \epsilon^0\right)$$

Exercise 1. ***Relating Subtraction Schemes***

We consider the renormalized electron propagator:

$$G_R = \frac{1}{\not{p} - m_R + \Sigma_R(\not{p})} \tag{8.1}$$

in QED. This propagator has the beginning of a branch-cut at the location of the physical electron-mass, which we will refer to as the **pole mass** m_P, which residue 1. We therefore have

$$G_R = \frac{1}{\not{p} - m_P} + \text{terms regular at } \not{p} = m_P \tag{8.2}$$

and the pole mass is independent of the subtraction-scheme.

i.) Derive the two relations between the m_P, m_R and Σ_R which follow from the above considerations and are independent of the subtraction scheme.

Solution. $G_R(\not{p})$ has to have a pole at $\not{p} = m_P$ and therefore the 1PI graphs must fulfill

$$\Sigma_R\big|_{\not{p}=m_P} = m_R - m_P. \tag{S.8.1}$$

The condition on the residue yields

$$\begin{aligned} 1 &= \lim_{\not{p}\to m_P} (\not{p} - m_P)\frac{1}{\not{p} - m_R + \Sigma_R(\not{p})} && \text{(S.8.2)} \\ &= \lim_{\not{p}\to m_P} \frac{1}{1 + \frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma_R(\not{p})} && \text{(S.8.3)} \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma_R(\not{p})\bigg|_{\not{p}=m_P} &= 0, && \text{(S.8.4)} \end{aligned}$$

where we used the rule of L'Hospital.

In exercise sheet 5 you derived the relation:

$$\Sigma_R(\not{p}) = \Sigma_2(\not{p}) + \delta_2\not{p} - (\delta_m + \delta_2)m_R, \tag{8.3}$$

where Σ_2 is the one-loop fermion self-energy. In the *on-shell* subtraction scheme, the renormalized mass m_R is identified with the pole mass m_P and one finds

$$\delta_2^{OS} = -\frac{\alpha}{2\pi}\left(\frac{1}{\epsilon} + \frac{1}{2}\log\left(\frac{\tilde{\mu}^2}{m_R^2}\right) + \frac{5}{2} + \log\left(\frac{m_\gamma^2}{m_P^2}\right)\right) \tag{8.4}$$

$$\delta_m^{OS} = \frac{\alpha}{2\pi}\left(-\frac{3}{\epsilon} - \frac{3}{2}\log\left(\frac{\tilde{\mu}^2}{m_R^2}\right) - \frac{5}{2}\right) \tag{8.5}$$

where $\tilde{\mu}^2 = (4\pi)^\epsilon \exp^{-\gamma_E \epsilon} \mu^2$.

i.) What is the renormalized mass in the $\overline{MS}$- and MS-subtraction scheme.

ii.) How are they related to the pole mass?

Solution. Both questions are answered together. If we ant our subtraction scheme to be MS ($\overline{MS}$), we simply take the (modified) pole-part of the renormalization functions:

$$\delta_m^{\mathrm{MS}} = -\frac{3\alpha}{4\pi}\frac{2}{\epsilon} \tag{S.8.5}$$

$$\delta_m^{\overline{MS}} = -\frac{3\alpha}{4\pi}\left(\frac{2}{\epsilon} + \log(4\pi \exp(-\gamma_E))\right). \tag{S.8.6}$$

From the pole condition of the renormalized propagator and using $m_P = m_R$ at leading order, we get in $\overline{MS}$

$$m_R = m_P\left[1 - \frac{\alpha}{4\pi}\left(5 + 3\log\left(\frac{\mu^2}{m_P^2}\right)\right) + \mathcal{O}\left(\alpha^3\right)\right], \tag{S.8.7}$$

whereas in MS, we have $\tilde{\mu}$ instead of μ.

iii.) How can you relate parameters renormalized in different schemes?

Solution. We can always use $\kappa^0 = Z_\kappa^X \kappa_R^X$ to relate renormalization schemes X.

iv.) Can you think of a reason, why an *on-shell* scheme may be problematic for interpreting quark-masses in QCD?

Solution. In QED, the electron pole mass and the on-shell coupling can be determined in a low energy regime, which is accessible in experiments. In QCD however, free partons do not exist and the mass has to be extracted by fitting experimental data to theoretical predictions. The prediction however is not scheme independent in fixed order computations and in particular depends on the simulation of parton showers, which have to be set up in a particular scheme. The relevant mass in QCD scattering experiments will never be the pole mass, since the scattered partons are never free.

Exercise 2. *The vertex counterterm* $\delta_{q\bar{q}}$

In the lecture the computation of the counterterm

$$\delta_{q\bar{q}} = -\left[d - (1-\xi)\right] C_F \frac{\alpha_s}{4\pi}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}} \tag{8.6}$$

was outlined.

Compute this result by performing explicitly all necessary steps.

Solution. The integral under consideration is

$$I_1 = \int \frac{\mathrm{d}^d l}{(2\pi)^d}\left(igt^a\gamma^\mu i\frac{\not{p}+\not{l}}{(p+l)^2}\right)\mathbb{1}\left(\frac{\not{k}+\not{l}}{(k+l)^2}igt^a\gamma^\nu\right)\frac{-i}{l^2}\left(g_{\mu\nu} - (1-\xi)\frac{l^\mu l^\nu}{l^2}\right) =: \int \mathrm{d}^d l\frac{N(p,k,l)}{D(p,k,l)}\ . \tag{S.8.8}$$

We can now perform power counting a see that only the parts of the numerator containing the loop-momentum l have a non-vanishing degree of divergence. The UV-approximation is therefore the same as expanding the numerator in the limit $p, k \ll l$

$$I_1^{UV} = \mathrm{d}^d l\frac{N(0,0,l)}{D(p,k,l)}. \tag{S.8.9}$$

Performing the numerator algebra yields:

$$N(0,0,l) = ig^2 l^2 C_F(d-(1-\xi))\delta_{ij} \tag{S.8.10}$$

and we need to integrate a scalar bubble with incoming momentum q

$$I_1^{UV} = ig^2 C_F(d-(1-\xi))\delta_{ij}\int \mathrm{d}^d l \frac{1}{(l)^2(l-q)^2}. \tag{S.8.11}$$

This integral can easily be performed with the techniques discussed in the last exercise class:

$$\begin{aligned}\int \mathrm{d}^d l \frac{1}{(l)^2(l-q)^2} &\propto \Gamma(2-d/2)(-q^2)^{-\epsilon}\int_0^1 1/((1-x)x)^{\epsilon}\mathrm{d}x & \text{(S.8.12)}\\ &= (-q^2)^{-\epsilon}\Gamma(2-d/2)\frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} & \text{(S.8.13)}\\ &= (-q^2)^{-\epsilon}\Gamma(2-d/2)(1+\mathcal{O}(\epsilon)), & \text{(S.8.14)}\end{aligned}$$

yielding the result of the lecture.

Exercise 3. Λ_{QCD} *at two-loops and beyond*

You saw in the lecture that at leading-order:

$$\alpha_s(\mu^2) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} \; . \tag{8.7}$$

This means, there is a characteristic scale $\Lambda \equiv \Lambda_{QCD}$, in which the coupling becomes strong and a perturbative expansion will break down. In the following, we want to investigate this scale dependence beyond the leading behaviour.

The RGE for the coupling constant $\alpha_s(\mu^2)$ is given by

$$\frac{\mathrm{d}\alpha_s(\mu^2)}{\mathrm{d}\ln(\mu^2)} \stackrel{\text{def}}{=} \beta(\alpha_s(\mu^2)) = -\sum_{n=0}^{k}\beta_n \alpha_s(\mu^2)^{n+2} + \mathcal{O}\left(\alpha_s(\mu^2)^{k+2}\right) \; . \tag{8.8}$$

i) Show that

$$\ln\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{1}{\beta_0\alpha_s(\mu^2)} + \frac{\beta_1 \ln\left(\alpha_s(\mu^2)\right)}{\beta_0^2} + \frac{\left(\beta_0\beta_2-\beta_1^2\right)\alpha_s(\mu^2)}{\beta_0^3} + C + \mathcal{O}\left(\alpha_s(\mu^2)^2\right), \tag{8.9}$$

where C is a integration constant conventionally chosen to be $\beta_1/\beta_0^2\ln(\beta_0)$, and $\Lambda = \Lambda_{QCD}$ denotes the scale at which perturbation theory breaks down.

Solution. We have:

$$\mathrm{d}\ln(\mu^2) = \frac{\mathrm{d}\alpha_s(\mu^2)}{\beta(\alpha_s)} \approx \left(-\frac{1}{\beta_0\alpha_s^2} + \frac{\beta_1}{\beta_0^2\alpha_s} + \frac{\beta_0\beta_2-\beta_1^2}{\beta_0^3} + O\left(\alpha_s^1\right)\right)\mathrm{d}\alpha_s \; . \tag{S.8.15}$$

Integrating it yields the desired formula.

ii) Show that in principle Λ is determined by only knowing the first two coefficients of the β-function, β_0 and β_1.

Hint. Think about an appropriate limit and remember what you know about asymptotic freedom.

Solution. Λ is determined by:

$$\Lambda^2 = \mu^2 \exp\left(-\frac{1}{\beta_0\alpha_s(\mu^2)} - \frac{\beta_1\ln\left(\beta_0\alpha_s(\mu^2)\right)}{\beta_0^2} - \frac{\left(\beta_0\beta_2-\beta_1^2\right)\alpha_s(\mu^2)}{\beta_0^3} + \mathcal{O}\left(\alpha_s(\mu^2)^2\right)\right) \; . \tag{S.8.16}$$

The limit $\mu \to \infty$ will correspond to $\alpha_s \approx 0$. Therefore we have

$$\Lambda^2 = \lim_{\mu^2 \to \infty} \mu^2 \exp\left(-\frac{1}{\beta_0 \alpha_s(\mu^2)} - \frac{\beta_1 \ln\left(\beta_0 \alpha_s(\mu^2)\right)}{\beta_0^2} \right) . \tag{S.8.17}$$

Nonetheless, higher order terms improve the determination of Λ_{QCD}, since α_s is only known for finite μ^2.

iii) The running of α_s can be written in terms of a power series in $\ln\left(\Lambda^2/\mu^2\right)$ as

$$\alpha_s(\mu^2) = \frac{1}{\beta_0 \ln\left(\mu^2/\Lambda^2\right)} - \frac{\beta_1}{\beta_0^2} \frac{\ln\left(\ln\left(\mu^2/\Lambda^2\right)\right)}{\ln^2\left(\mu^2/\Lambda^2\right)} + \mathcal{O}\left(\frac{\ln\ln\left(\ln\left(\mu^2/\Lambda^2\right)\right)}{\ln^3\left(\mu^2/\Lambda^2\right)} \right) . \tag{8.10}$$

Derive the given terms in the series.

Hint. Solve the RGE iteratively. The ansatz for the unkown order δ_{α_s} in the log*-expansion will translate into a differential equation for δ_{α_s}.*

Solution. The first order can be easily obtained by solving

$$\mathrm{d} \ln\left(\mu^2\right) = -\frac{\mathrm{d}\alpha_s}{\beta_0 \alpha_s^2} + \mathcal{O}(\alpha_s^{-1}). \tag{S.8.18}$$

Comparing with our convention from (8.9) fixes the boundary condition and we get:

$$\alpha_s = \frac{1}{\beta_0 l} + \delta_{\alpha_s}, \tag{S.8.19}$$

where $\delta_{\alpha_s} = \mathcal{O}\left(l^{-2}\right)$ and $l = \ln\left(\mu^2/\Lambda^2\right)$. To get the higher order we use the RGE (8.8). Having $\mathrm{d}l = \mathrm{d}\ln\left(\mu^2\right)$ we get

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln(\mu^2)} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\beta_0 l} + \delta_{\alpha_s} \right) = -\frac{1}{\beta_0 l^2} + \frac{\mathrm{d}}{\mathrm{d}l}\delta_{\alpha_s} . \tag{S.8.20}$$

on the LHS of (8.8). The RHS reads

$$-\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 + \mathcal{O}(\alpha_s^4) = -\beta_0 \left(\frac{1}{\beta_0 l} + \delta_{\alpha_s} \right)^2 \left(1 + \frac{\beta_1 \left(\frac{1}{\beta_0 l} + \delta_{\alpha_s} \right)}{\beta_0} \right) + \mathcal{O}(\alpha_s^4) \tag{S.8.21}$$

$$= -\frac{1}{\beta_0 l^2} - \frac{2\delta_{\alpha_s}}{l} - \frac{\beta_1}{l^3 \beta_0^2} + \mathcal{O}(l^{-4}) . \tag{S.8.22}$$

from which we read off the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}l}\delta_{\alpha_s} = -\frac{2\delta_{\alpha_s}}{l} - \frac{\beta_1}{\beta_0^2 l^3} . \tag{S.8.23}$$

The solution to that differential equation is

$$\delta_{\alpha_s} = \frac{c_1}{l^2} - \frac{\beta_1 \log(l)}{\beta_0^2 l^2}, \tag{S.8.24}$$

where c_1 is some boundary constant. Fixing $c_1 = 0$ gives the desired result, which is currently standard. However, (8.9) would be basically obtained by $c = C$, which is reassuring.

Exercise 1. ***Plus Distribution***

In the lecture has been introduced the plus-distribution, defined as follows:

$$\int_0^1 dx\, F(x)_+\, g(x) = \int_0^1 dx\, (g(x) - g(1))F(x) \tag{9.1}$$

(a) This definition can be extended to include different integration boundaries. Show the following identity:

$$\int_z^1 dx\, F(x)_+\, g(x) = \int_z^1 dx\, (g(x) - g(1))F(x) - g(1)\int_0^z F(x) \tag{9.2}$$

(b) Take $F(x) = (1-x)^{-1+a\epsilon}$ and extract explicitly the divergence at $x = 0$ as a pole in ϵ by means of the plus-distribution and show the following identity:

$$(1-x)^{-1+a\epsilon} = \frac{\delta(1-x)}{a\epsilon} + \sum_{n=0}^{\infty} \frac{(a\epsilon)^n}{n!}\left[\frac{\log(1-x)^n}{1-x}\right]_+ \tag{9.3}$$

(c) The plus distribution can contain also regular parts that can be extracted.

$$\left[\frac{1+x^2}{1-x}\right]_+ = \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \tag{9.4}$$

Solution.

(a) In order to prove this identity we endow the regular function $g(x)$ with a theta function:

$$\begin{aligned}\int_0^1 (\theta(x-z)\, g(x))F(x)_+ &= \int_0^1 (\theta(x-z)\, g(x) - g(1))F(x)\\ &= \int_0^1 \theta(x-z)(g(x) - g(1))F(x) - g(1)\int_0^1 \theta(z-x)F(x)\\ &= \int_z^1 (g(x) - g(1))F(x) - g(1)\int_0^z \theta(z-x)F(x).\end{aligned} \tag{S.9.1}$$

(b) First we regulate the function around the limit $x \to 0$ because the two limits do not commute:

$$\int_0^1 \frac{g(x) - g(1)}{1-x}(1-x)^{a\epsilon} + \frac{g(1)}{a\epsilon}. \tag{S.9.2}$$

Now it is possible to expand around $\epsilon = 0$:

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(a\epsilon)^n \log(1-x)^n}{n!}\frac{g(x) - g(1)}{1-x} + \frac{g(1)}{a\epsilon}. \tag{S.9.3}$$

This can then be translated into

$$(1-x)^{-1+a\epsilon} = \frac{\delta(1-x)}{a\epsilon} + \sum_{n=0}^{\infty} \frac{(a\epsilon)^n}{n!}\left[\frac{\log(1-x)^n}{1-x}\right]_+. \tag{S.9.4}$$

(c)

$$\begin{aligned}
\int_0^1 dx \left[\frac{1+x^2}{1-x}\right]_+ g(x) &= \int_0^1 dx\, \frac{g(x)-g(1)}{1-x}(1+x^2) \\
&= \int_0^1 dx\, \frac{(1+x^2)g(x)-2g(1)}{1-x} + \int_0^1 dx\, \frac{g(1)}{1-x}(1-x^2) \\
&= \int_0^1 dx\, \frac{(1+x^2)g(x)-2g(1)}{1-x} + \frac{3}{2}g(1) \\
&= \int_0^1 dx \left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right) g(x)
\end{aligned} \tag{S.9.5}$$

Exercise 2. *Convolutions*

We define the following convolution:

$$(f \otimes g)(z) := \int_0^1 dx \int_0^1 dy\, f(x)\, g(y)\, \delta(xy - z). \tag{9.5}$$

(a) Show that the convolution can also be written as:

$$(f \otimes g)(z) = \int_z^1 \frac{dx}{x} f(x) g\left(\frac{z}{x}\right). \tag{9.6}$$

(b) Show that it has the following properties, keeping track of the integration boundaries:

$$\begin{aligned}
(f \otimes g)(z) &= (g \otimes f)(z) && \text{(commutativity)} \\
(f \otimes (g+h))(z) &= (f \otimes g)(z) + (f \otimes h)(z) && \text{(linearity)} \\
((f \otimes g) \otimes h)(z) &= (f \otimes (g \otimes h))(z) && \text{(associativity)}
\end{aligned}$$

Solution.

(a) From integrating the δ-function in y we have the constrain:

$$0 < \frac{z}{x} < 1 \qquad \Rightarrow \qquad x > z, \tag{S.9.6}$$

that reduce the integration volume as shown in the expression.

(b) Commutativity and linearity are trivial, one comes form the commutativity of multiplication and the other from the linearity of the integral. The last relation that one needs to show is associativity:

$$\begin{aligned}
((f \otimes g) \otimes h)(z) &= \int_0^1 dz_1 dx_3 (f \otimes g)(z_1) h(x_3)\, \delta(x_1 x_2 - z) \\
&= \int_0^1 dz_1 dx_1 dx_2 dx_3 f(x_1) g\,(x_2)\, h(x_3)\, \delta(z_1 x_3 - z)\, \delta(x_1 x_2 - z_1) \\
&= \int_0^1 dx_1 dx_2 dx_3 f(x_1) g\,(x_2)\, h(x_3)\, \delta(x_1 x_2 x_3 - z).
\end{aligned} \tag{S.9.7}$$

This shows that is associative. From integrating the δ-function we have the condition $0 < x_1 x_2 < 1$ which is always satisfied for $x_{1,2} \in (0,1)$, therefore the integration volume is unchanged.

Exercise 3. ***One-Loop Correction in QED***

Let's consider the QED case of the one loop correction to the $q\bar{q}\gamma$ vertex. In particular the one particle irreducible corrections Γ^μ, that can be written as $\Gamma^\mu = -e\gamma^\mu + \Lambda^\mu + \dots$.

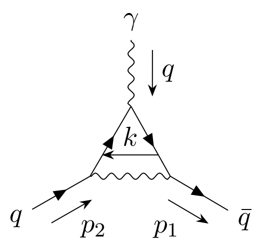
$\Lambda^\mu =$  (9.7)

Figure 4: One loop correction to the fermion-photon interaction

(a) Argue how this contribution is equivalent to the term proportion to C_F in the QCD correction to the $q\bar{q}\gamma$ vertex.

(b) Write down the integral Λ^μ corresponding to the diagram above. Consider the quarks having mass m and work in Feynman gauge.

(c) We now want to consider the infrared divergences of Λ^μ. In the soft region every component of the loop-momentum scales as $k_\mu \propto \delta Q \ll 1$. Perform a power counting in δQ. Is there a difference between off-shell and on-shell external quarks?

(d) Consider now on-shell external quarks $p_1 = p_2 = m^2$ and a virtual photon, perform the algebra for the leading IR divergent part and verify

$$\Lambda^\mu_{IR} = e\mu^\epsilon \gamma^\mu (e\mu^\epsilon)^2 \int \frac{d^D k}{(2\pi)^D} \frac{4 p_1 \cdot p_2}{((k+p_1)^2 - m^2)\,((k+p_2)^2 - m^2)\, k^2}. \tag{9.8}$$

Is Λ^μ_{IR} UV-finite or UV-divergent?

(e) We now want to extract the leading IR divergent part of the integral and for on-shell external quarks $p_1 = p_2 = m^2$ and a virtual photon in dimensional regularization. Therefore we have to compute the scalar triangle I in Λ_{IR}. Computing this integral is non-trivial and we are therefore performing it step by step.

i) We have

$$I \propto \tilde{I} = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathcal{U}^{3-d}}{\mathcal{F}^{3-\frac{d}{2}}} \delta\left(1 - \sum_{i=1}^{3} x_i\right) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3. \tag{9.9}$$

Determine the graph polynomials as discussed in the exercise class. The integral has a endpoint singularity. Determine its location.

ii) The extraction of the singular behavior is a two step process. At first, split the integration domain into the three regions

- region 1: $x_1 > x_2$ and $x_1 > x_3$,

- region 2: $x_2 > x_1$ and $x_2 > x_3$,
- region 3: $x_3 > x_1$ and $x_3 > x_2$.

Perform in every region $i = 1, \ldots, 3$ the rescaling

$$x_j = \begin{cases} t_{j-1} x_i & j > i \\ x_i & j = i \\ t_j x_i & j < i \end{cases} \tag{9.10}$$

and resolve the δ with respect to x_i. You will find

$$\tilde{I} = I_1 + 2 I_2 \tag{9.11}$$

where only I_1 exhibits a endpoint singularity. Argue that I_2 does not contribute to the pole part.

iii) In the previous step you found

$$I_1 = (s_E)^{-1-\epsilon} \int\limits_{[0,1]\times[0,1]} \mathrm{d}t_1 \mathrm{d}t_2 \, (t_1 + t_2 + 1)^{2\epsilon} \left(\frac{m^2 \, (t_1 + t_2)^2}{s_E} + t_1 t_2 \right)^{-1-\epsilon} \tag{9.12}$$

Perform the decomposition of the integration domain and the rescaling as before to extract the singular behavior $\propto x^{-1-2\epsilon}$ explicitly. Expand the integrand in ϵ (use plus distributions) and integrate the pole part.

Hint. Work in the Euclidean regime $q^2 = -s_E > 0$. The analytic continuation can always be performed at a later stage if needed.

Solution.

(a) Because the only difference between the $q\bar{q}\gamma$ and $q\bar{q}g$ is the presence of the color matrix T_a in the fundamental representation. The contribution proportion to C_F is the one that has the contraction of two such matrices which is precisely the same diagram as shown here with the photon replaced by gluons.

(b) The diagram can be written as an integral by applying the Feynman rules for QED, the result reads:

$$\int \frac{d^D k}{(2\pi)^D} (-i g^{\alpha\beta}) \frac{(-ie\mu^\epsilon \gamma_\alpha) i(\not{p}_1 + \not{k} + m)(-ie\mu^\epsilon \gamma^\mu) i(\not{p}_2 + \not{k} + m)(-ie\mu^\epsilon \gamma_\beta)}{[(p_1+k)^2 - m^2][(p_2+k)^2 - m^2][k^2]} \tag{S.9.8}$$

(c) We have the scaling

$$\mathrm{d}^d k \propto d\delta Q \tag{S.9.9}$$

$$\frac{1}{k^2} \propto (\delta Q)^{-2} \tag{S.9.10}$$

$$\frac{1}{k^2 + 2\,(p1k) + p_1^2 - m^2} \propto \frac{1}{2\delta Q\,(p1k) + p_1^2 - m^2} \propto \begin{cases} (\delta Q)^{-1} & \text{on-shell} \\ (\delta Q)^0 & \text{off-shell} \end{cases} . \tag{S.9.11}$$

From that we conclude that there is no soft-divergence for the case of off-shell quarks. The only soft divergence for the on-shell case arises if the numerator of the diagram scales as $(\delta Q)^0$, that means the part of the diagram without loop-momenta in the numerator.

(d) By performing the algebra we make use of the Dirac equation. The Λ_{IR} has a superfical UV degree of divergence of $d - 6$ and is therefore UV-finite in $d \to 4$.

(e) i) We construct the Feynman parameter integral as discussed in the exercises [2]. We label the massive lines by x_1, x_2 and the photon line by x_3. We have the trivial spanning trees obtained by deleting

[2] For a complete derivation of the procedure see e.g. https://arxiv.org/abs/1002.3458 section 4.

one edge. We therefore have

$$\mathcal{U} = x_1 + x_2 + x_3. \tag{S.9.12}$$

The spanning two-trees are obtained by deleting an additional edge from the spanning trees. We therefore have

$$\mathcal{F}_0 = -m^2 x_1 x_3 - m^2 x_2 x_3 + s_E x_1 x_2 \tag{S.9.13}$$

and

$$\mathcal{F} = \mathcal{U}\left(x_1 m^2 + x_2 m^2\right) + \mathcal{F}_0 = s_E x_1 x_2 + m^2 (x_1 + x_2)^2 \tag{S.9.14}$$

with $s_E = -q^2$. The relevant integral is therefore

$$\tilde{I} = (s_E)^{-1-\epsilon} \int_0^\infty \int_0^\infty \int_0^\infty (x_1 + x_2 + x_3)^{2\epsilon - 1} \left(\frac{m^2 \left(x_1 + x_2\right)^2}{s_E} + x_1 x_2\right)^{-\epsilon - 1} \delta\left(1 - \sum_{i=1}^3 x_i\right) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3. \tag{S.9.15}$$

The integrals has a endpoint singularity if x_1 and x_2 go to zero *simultaneously*.

ii) We will look at region I. The integral in that region reads

$$I_2 = (s_E)^{-1-\epsilon} \int_0^\infty \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_1} \mathrm{d}x_3 \, (x_1 + x_2 + x_3)^{2\epsilon - 1} \left(\frac{m^2 \left(x_1 + x_2\right)^2}{s_E} + x_1 x_2\right)^{-\epsilon - 1} \delta\left(1 - \sum_{i=1}^3 x_i\right) \tag{S.9.16}$$

$$= (s_E)^{-1-\epsilon} \int_0^\infty \mathrm{d}x_1 \int_0^1 x_1 \mathrm{d}t_1 \int_0^1 x_1 \mathrm{d}t_2 \left[x_1^{2\epsilon - 1} \left((1 + t_1 + t_2)\right)^{2\epsilon - 1}\right] \tag{S.9.17}$$

$$\times \left[x_1^{-2-2\epsilon} \left(\frac{m^2 \left(1 + t_1\right)^2}{s_E} + t_1\right)^{-\epsilon - 1}\right] \delta\left(1 - x_1(1 + t_1 + t_2)\right) \tag{S.9.18}$$

$$= (s_E)^{-1-\epsilon} \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 \, (t_1 + t_2 + 1)^{2\epsilon} \left(\frac{m^2 \left(t_1 + 1\right)^2}{s_E} + t_1\right)^{-\epsilon - 1}. \tag{S.9.19}$$

As we can see, this integral has no singularities in the integration domain. We can therefore expand it in ϵ and integrate it. However, the expansion starts at $\mathcal{O}(\epsilon^0)$ and does not contribute to the pol part. Region II is the same as region one, due to the $x_1 \leftrightarrow x_2$ symmetry of the original integral. Region III yields

$$I_1 = (s_E)^{-1-\epsilon} \int\limits_{[0,1]\times[0,1]} \mathrm{d}t_1 \mathrm{d}t_2 \, (t_1 + t_2 + 1)^{2\epsilon} \left(\frac{m^2 \left(t_1 + t_2\right)^2}{s_E} + t_1 t_2\right)^{-1-\epsilon} \tag{S.9.20}$$

and we see the endpoint singularity at $t_1 = t_2 = 0$. This integral will therefore contribute to the pol part. Furthermore we notice the $t_1 \leftrightarrow t_2$ symmetry.

iii) Splitting the integration domain again and performing the rescaling yields

$$I_1 = 2\left(s_E\right)^{-1-\epsilon} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}t_1 t_1^{-2\epsilon - 1} \left(t_1 x + t_1 + 1\right)^{2\epsilon} \left(\frac{m^2 (x+1)^2}{s_E} + x\right)^{-\epsilon - 1}, \tag{S.9.21}$$

where the factor of 2 is the symmetry factor. The divergence of this integral is explicit in the factor $t_1^{-2\epsilon - 1}$, which however we can expand in terms of plus distributions. The expansion yields

$$I_1 = 2\left(s_E\right)^{-1-\epsilon} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}t_1 \left[\frac{\delta(t_1)}{-2\epsilon} + \mathcal{O}(\epsilon^0)\right] \left[\frac{1}{\frac{m^2 (x+1)^2}{s_E} + x} + \mathcal{O}(\epsilon)\right] = -\left(s_E\right)^{-1-\epsilon} \left(\frac{1}{\epsilon}\frac{1}{\beta} \log\left(\frac{\beta + 1}{\beta - 1}\right)\right) \tag{S.9.22}$$

with $\beta = \sqrt{1 + \frac{4m^2}{s_E}}$. Including all prefactors will yield the pole part shown in the lecture.

Exercise 4. ***Splitting functions***

(a) Consider an electron of momentum p^μ that emits a photon momentum q^μ and has final momentum k^μ. This emitted photon is virtual but close to be on-shell. Parameterize the momenta as:

$$p^\mu = (p, 0, 0, p), \tag{9.13a}$$

$$k^\mu = \left((1-z)p, -\vec{p}_\perp, (1-z)p - \frac{\vec{p}_\perp^2}{2(1-z)p}\right), \tag{9.13b}$$

$$q^\mu = \left(z\,p, \vec{p}_\perp, z\,p + \frac{\vec{p}_\perp^2}{2(1-z)p}\right). \tag{9.13c}$$

Compute the amplitude,

$$i\mathcal{M} = \bar{u}_L(k)(-ie\gamma^\mu)u_L(p)\varepsilon^*_{\perp\mu}(q) \tag{9.14}$$

for the photon transverse polarization.

Hint. *Use the spinors in light-cone coordinates computed in a previous tutorial. Parameterize the transverse polarization as* $\varepsilon^\mu_{\perp R,L}(q) = \frac{1}{\sqrt{2}}(0, 1, \pm i, \varepsilon^3)$ *and use* $\varepsilon_\perp(q)\cdot q = 0$ *and* $\varepsilon^*_\perp \cdot \varepsilon_\perp = -1$ *to compute* ε^3 *as an expansion in* $p_\perp$*. Keep only the leading term of the expansion.*

(b) Take the photon on shell, and thus the outgoing electron as slightly off-shell. Parameterize their momenta. Check that, to lowest order in $p_\perp$, the spinors and the transverse polarization do no change.

(c) Consider the splitting of a photon with momentum p spitting into a e^+e^- pair with momenta q and k, respectively. Parameterize the momenta as,

$$q^\mu = \left(z\,p, \vec{p}_\perp, z\,p - \frac{\vec{p}_\perp^2}{2z\,p}\right), \tag{9.15a}$$

$$k^\mu = \left((1-z)p, -\vec{p}_\perp, (1-z)p - \frac{\vec{p}_\perp^2}{2(1-z)p}\right), \tag{9.15b}$$

$$p^\mu = q + k = \left(p, 0, 0, p - \frac{\vec{p}_\perp^2}{2z(1-z)p}\right)). \tag{9.15c}$$

For the photon transverse polarization, compute the amplitude,

$$i\mathcal{M} = \bar{u}_L(k)(-ie\gamma^\mu)v_L(p)\varepsilon_{\perp\mu}(q). \tag{9.16}$$

Hint. *Recall that here the transverse polarization are as usual,* $\varepsilon^\mu_{\perp R,L}(q) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$

Solution.

(a)-(b) As we have seen in Lecture 3 and in the tutorials, in a chiral basis $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$, we can write $u_L(k) = \sqrt{k^+}\begin{pmatrix} \xi(k) \\ 0 \end{pmatrix}$ with $\xi(k)$ a 2-spinor $\xi(k) = \begin{pmatrix} -\frac{k^*_\perp}{k^+} \\ 1 \end{pmatrix}$ so

$$\bar{u}_L(k)\gamma^\mu u_L(p) = \sqrt{k^+p^+}\xi^\dagger(k)\bar{\sigma}^\mu\xi(p). \tag{S.9.23}$$

The transverse polarization has one zero component, so

$$\xi^\dagger(k)\bar{\sigma}^\mu\xi(p)\varepsilon^*_{\perp,\mu}(q) = \xi^\dagger(k)\sigma^i\xi(p)\varepsilon^{i*}_\perp(q) \tag{S.9.24}$$

so the amplitude is

$$i\mathcal{M} = (-ie)\sqrt{2p}\sqrt{2(1-z)p}\xi^\dagger(k)\sigma^i\xi(p)\varepsilon_\perp^{i*}. \tag{S.9.25}$$

For our parameterization, $\xi(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\xi(k) = \begin{pmatrix} -\frac{p_\perp^*}{2(1-z)p} \\ 1 \end{pmatrix}$, the transverse polarization must be such that:

$$\varepsilon_\perp(q)\cdot q = 0, \qquad \varepsilon_\perp^*\cdot\varepsilon_\perp = -1, \tag{S.9.26}$$

so we can parameterize the polarization as:

$$\varepsilon^\mu_{\perp R,L}(q) = \frac{1}{\sqrt{2}}(0,1,\pm i,\varepsilon^3), \qquad \varepsilon^3 = -\frac{p_\perp^*}{z\,p} + \mathcal{O}(p_\perp^3) \tag{S.9.27}$$

thus we get:

$$\varepsilon^{\mu*}_{\perp L}(q) = \frac{1}{\sqrt{2}}(0,1,i,-\frac{p_\perp}{zp})\varepsilon^{\mu*}_{\perp R}(q) = \frac{1}{\sqrt{2}}(0,1,-i,-\frac{p_\perp^*}{zp}) \tag{S.9.28}$$

Consider the R-handed photon, and use the fact that in the chiral representation $\sigma^1 - i\sigma^2 = 2\bar{\sigma}_\perp$ with $\bar{\sigma}_\perp\xi(p) = 0$, so we can rewrite eq.(S.9.25) as:

$$i\mathcal{M}(e_L^- \to e_L^-\gamma_R) = (-ie)\sqrt{2p}\sqrt{2(1-z)p}\frac{p_\perp^*}{\sqrt{2}zp} = -ie\frac{\sqrt{2(1-z)}}{z}p_\perp^*. \tag{S.9.29}$$

Doing the same for the L-handed photon one obtains:

$$i\mathcal{M}(e_L^- \to e_L^-\gamma_L) = -ie\frac{\sqrt{2}}{z\sqrt{1-z}}p_\perp. \tag{S.9.30}$$

(c) Taking again the spinors from Lecture 3 as in part i), then:

$$\begin{aligned}\bar{u}_L(k)\gamma^\mu v_L(p)\varepsilon_{\perp,\mu}(q) &= \sqrt{2zp}\sqrt{2(1-z)p}\xi^\dagger(q)\bar{\sigma}^\mu\xi(k)\varepsilon_{\perp,\mu}(p)\\ &= \sqrt{2zp}\sqrt{2(1-z)p}\xi^\dagger(q)\sigma^i\xi(k)\varepsilon^i_\perp(p).\end{aligned}$$

For (R,L)-handed photon we have $\varepsilon^\mu_{\perp L,R}(p) = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$, then one gets:

$$\xi^\dagger(q)\sigma^i\xi(k)\varepsilon^i_{\perp,R}(p) = -\frac{1}{\sqrt{2}}\frac{p_\perp}{zp}, \qquad \xi^\dagger(q)\sigma^i\xi(k)\varepsilon^i_{\perp,L}(p) = \frac{1}{\sqrt{2}}\frac{p_\perp^*}{(1-z)p},$$

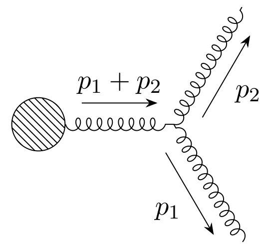
and we obtain the amplitudes:

$$i\mathcal{M}(\gamma_R \to e_L^- e_R^+) = ie\frac{\sqrt{2z(1-z)}}{z}p_\perp, \qquad i\mathcal{M}(\gamma_L \to e_L^- e_R^+) = -ie\frac{\sqrt{2z(1-z)}}{1-z}p_\perp^*.$$

Exercise 1. ***Splitting function*** $\hat{P}_{g\to gg}$

We continue our mission of computing splitting functions, this time we consider $\hat{P}_{g\to gg}$:

$$i\mathcal{M}_{n+1}(p_1, p_2, \dots) \quad = \tag{10.1}$$



(a) Show that the matrix element $i\mathcal{M}_{n+1}$ is:

$$T^{\mu,a}(p_1+p_2,\dots)\frac{d_{\mu\alpha}(p_1+p_2)}{s_{12}} g_s f^{ac_1c_2} \varepsilon^*_{(\lambda_1),\mu_1}(p_1)\varepsilon^*_{(\lambda_2),\mu_2}(p_2)$$
$$\times \left(g^{\mu_1\mu_2}\,(p_2-p_1)^{\,\alpha} - (p_1+2p_2)^{\,\mu_1} g^{\alpha\mu_2} + (2p_1+p_2)^{\,\mu_2} g^{\alpha\mu_1}\right),$$

where $d_{\mu\alpha}(p_1+p_2)$ is a consequence of using the transverse gauge for the gluon propagator:

$$d_{\mu\nu}(q) := -g_{\mu\nu} + \frac{q_\mu n_\nu + q_\nu n_\mu}{q\cdot n}, \qquad n^2 = 0. \tag{10.2}$$

We use the parametrization

$$p_1^\mu = z\,p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z\,2p\cdot n} n^\mu, \qquad p_2^\mu = (1-z)\,p^\mu - k_\perp^\mu - \frac{k_\perp^2}{(1-z)\,2p\cdot n} n^\mu, \tag{10.3}$$

decomposing the splitting momenta in terms of two light-like momenta n, p (i.e. $n^2 = p^2 = 0$) and a orthogonal part $k_\perp$ ($p\cdot k_\perp = n\cdot k_\perp = 0$). In what follows we neglect all higher orders in $k_\perp$ and when doing so we use $\approx$ instead of the equal sign.

(b) Write the expression above in terms of p and $k_\perp$ in order to obtain:

$$i\mathcal{M}_{n+1}(p_1,p_2,\dots) \approx 2\,\frac{T^{\mu,a}(p,\dots)}{s_{12}} g_s f^{ac_1c_2}\varepsilon^*_{(\lambda_1),\mu_1}(p)\varepsilon^*_{(\lambda_2),\mu_2}(p)$$
$$\times\left[g^{\mu_1\mu_2}k_{\perp,\mu} + d_{\mu\alpha}(p)\left(\frac{k_\perp^{\mu_1}}{z}g^{\alpha\mu_2} + \frac{k_\perp^{\mu_2}}{1-z}g^{\alpha\mu_1}\right)\right]. \tag{10.4}$$

Hint. It is convenient to choose the polarization vectors to satisfy, $\varepsilon_i(p_i)\cdot n = 0$ *together with the usual relation* $\varepsilon_i(p_i)\cdot p_i = 0$*. Using the reference vector* n *for the polarization yields the physical polarization sum* $\sum_\lambda \varepsilon_{(\lambda),\nu}(p)\varepsilon^*_{(\lambda),\mu}(p) = d_{\mu\nu}(p)$*. Express* $d_{\mu\nu}(p_1+p_2)$ *in terms of* $d_{\mu\nu}(p)$*,* $k_\perp$*,* n *and* p*. What is* $d_{\mu\nu}(p)\,p^\nu_{1,2}$*?*

(c) As usual compute $|\mathcal{M}_{n+1}|^2$ summing over the possible final state colors and polarizations to obtain:

$$\sum_{\substack{pol\\col}} |\mathcal{M}_{n+1}(p_1, p_2, ...)|^2 \approx \frac{2g_s^2}{s_{12}} T^{\mu,a}(p,...)(T^\dagger)^{\nu,a}(p,...)$$

$$\times \underbrace{2C_A \left[d_{\mu\nu}(p)\left(\frac{z}{1-z}+\frac{1-z}{z}\right) - 2(1-\epsilon)z(1-z)\frac{k_{\perp,\mu}k_{\perp\nu}}{k_\perp^2}\right]}_{\hat{P}_{g\to gg}}, \quad (10.5)$$

(d) We found an expression for the splitting function $\hat{P}^{\mu\nu}_{g\to g\bar{g}}$. Show that averaging over $k_\perp$, denoted by $\langle\cdot\rangle$, leads to the expression:

$$\left\langle \sum_{\substack{pol\\col}} |\mathcal{M}_{n+1}(p_1, p_2, ...)|^2 \right\rangle = \frac{2g_s^2}{s_{12}} \sum_{\substack{pol\\col}} |\mathcal{M}_n(p,...)|^2 \langle \hat{P}_{g\to g\bar{g}}\rangle, \quad (10.6)$$

where

$$\langle \hat{P}^{\mu\nu}_{g\to g\bar{g}}\rangle = d^{\mu\nu}\langle \hat{P}_{g\to g\bar{g}}\rangle, \qquad \langle \hat{P}_{g\to g\bar{g}}\rangle = 2C_A\left(\frac{z}{1-z}+\frac{1-z}{z}+z(1-z)\right), \quad (10.7)$$

and $\mathcal{M}_n$ defined as

$$i\mathcal{M}_n(p,...) = iT^{\mu,a}(p,...)\varepsilon^*_\mu(p). \quad (10.8)$$

Solution.

(a) Feynman rules

(b) We first notice that

$$s_{12} = (p_1+p_2)^2 = -\frac{k_\perp^2}{(1-z)z}\ . \quad \text{(S.10.1)}$$

Since we are only interested in the leading behavior of $\mathcal{M}_{n+1}$ in $k_\perp$ we can subsequently drop all terms which will give rise to order $\mathcal{O}(k_\perp^0)$ and higher.
We furthermore notice that

$$d_{\mu\nu}(p_1+p_2) = d_{\mu\nu}(p) - k_\perp^2 \frac{n_\mu n_\nu}{z(1-z)(n\cdot p)^2}. \quad \text{(S.10.2)}$$

Using that $d_{\mu\nu}(p)$ is transverse to both p^μ and n^μ and furthermore $k_\perp\cdot p = k_\perp \cdot n = 0$ one can easily verify that

$$d_{\mu\nu}(p)\, p^\nu_{1,2} = d_{\mu\nu}(p)\ (\pm k_\perp^\nu) = \mp\, k_{\perp\mu}\ , \quad \text{(S.10.3)}$$

leading to:

$$d_{\mu\alpha}(p)(p_2-p_1)^\alpha = 2k_{\perp\mu}. \quad \text{(S.10.4)}$$

For the other two contribution we can use $\varepsilon^*(p_i)\cdot p_i = \varepsilon^*(p_i)\cdot n = 0$ to rewrite everything in terms of $k_\perp$:

$$\varepsilon^*(p_1)\cdot(p_1+2p_2) = \varepsilon^*(p_1)\cdot\left(-2\frac{1-z}{z}p_1 + 2p_2\right) = -2\varepsilon^*(p_1)\cdot\left(\frac{k_\perp}{z}\right),$$
$$\varepsilon^*(p_2)\cdot(p_2+2p_1) = \varepsilon^*(p_2)\cdot\left(-2\frac{z}{1-z}p_2 + 2p_1\right) = 2\varepsilon^*(p_2)\cdot\left(\frac{k_\perp}{1-z}\right).$$

Notice that the rescaling of $\alpha\varepsilon(p_i)\cdot p_i = \varepsilon(p_i)\cdot p_i = 0$ is a useful consequence of the transversality.

Putting everything together gives:

$$2\,\frac{T^{\mu,a}(p_1+p_2,\dots)}{s_{12}}g_s f^{ac_1c_2}\varepsilon^*_{(\lambda_1),\mu_1}(p_1)\varepsilon^*_{(\lambda_2),\mu_2}(p_2)$$
$$\times\left[g^{\mu_1\mu_2}k_{\perp,\mu}+d_{\mu\alpha}(p)\left(\frac{k_\perp^{\mu_1}}{z}g^{\alpha\mu_2}+\frac{k_\perp^{\mu_2}}{1-z}g^{\alpha\mu_1}\right)+\mathcal{O}(k_\perp^2)\right].$$

Since the expression in the squared parenthesis starts linear $k_\perp$, the expression in front will contribute only with it's leading term. This is a consequence of the fact that the final expression - the squared matrix element - will be $\mathcal{O}(k_\perp^{-2})$ and that is already given by $s_{12}\propto k_\perp^2$. The factor in front can then be written as:

$$T^{\mu,a}(p_1+p_2,...)\varepsilon^*_{(\lambda_1),\mu_1}(p_1)\varepsilon^*_{(\lambda_2),\mu_2}(p_2)\approx T^{\mu,a}(p,...)\varepsilon^*_{(\lambda_1),\mu_1}(p)\varepsilon^*_{(\lambda_2),\mu_2}(p) \tag{S.10.5}$$

and we can finally reach the expression:

$$i\mathcal{M}_{n+1}(p_1,p_2,\dots)\approx 2\,\frac{T^{\mu,a}(p,\dots)}{s_{12}}g_s f^{ac_1c_2}\varepsilon^*_{(\lambda_1),\mu_1}(p)\varepsilon^*_{(\lambda_2),\mu_2}(p)$$
$$\underbrace{\times\left[g^{\mu_1\mu_2}k_{\perp,\mu}+d_{\mu\alpha}(p)\left(\frac{k_\perp^{\mu_1}}{z}g^{\alpha\mu_2}+\frac{k_\perp^{\mu_2}}{1-z}g^{\alpha\mu_1}\right)\right]}_{=:I_\mu^{\mu_1\mu_2}}.$$

(c) We now want to look at the squared matrix element:

$$\sum_{\substack{pol\\col}}|\mathcal{M}_{n+1}(p_1,p_2,...)|^2\approx\sum_{\substack{pol\\col}}4g_s^2\,\frac{T^{\mu,a}(p,\dots)}{s_{12}}\frac{\left(T^\dagger\right)^{\nu,b}(p,\dots)}{s_{12}}f^{ac_1c_2}f^{bd_1d_2} \tag{S.10.6}$$
$$\times\,\varepsilon^*_{(\lambda_1),\mu_1}(p)\varepsilon^*_{(\lambda_2),\mu_2}(p)\varepsilon^*_{(\lambda_1),\nu_1}(p)\varepsilon^*_{(\lambda_2),\nu_2}(p)I_\mu^{\mu_1\mu_2}I_\nu^{\nu_1\nu_2} \tag{S.10.7}$$

using the physical polarization sum

$$\sum_\lambda\varepsilon_{(\lambda),\nu}(p)\varepsilon^*_{(\lambda),\mu}(p)=d_{\mu\nu}(p). \tag{S.10.8}$$

The product of the matrix can be evaluated by using again that:

$$d_{\mu\nu}(p)k_\perp^\nu=-k_{\perp,\mu},\qquad d_\mu{}^\rho(p)d_{\rho\nu}(p)=-d_{\mu\nu}(p). \tag{S.10.9}$$

As first step one gets:

$$\sum_{\substack{pol\\col}}|\mathcal{M}_{n+1}(p_1,p_2,...)|^2\approx\frac{2g_s^2}{s_{12}}T^{\mu,a}(p,...)(T^\dagger)^{\nu,a}(p,...)\frac{2C_A}{s_{12}}\left[-d_{\nu_1\nu_2}(p)k_{\perp,\mu}+\frac{k_{\perp,\nu_1}}{z}d_{\mu\nu_2}(p)+\frac{k_{\perp,\nu_2}}{1-z}d_{\mu\nu_1}(p)\right]$$
$$\times\left[g^{\nu_1\nu_2}k_{\perp,\nu}+\frac{k_\perp^{\nu_1}}{z}d_\nu{}^{\nu_2}(p)+\frac{k_\perp^{\nu_2}}{1-z}d_\nu{}^{\nu_1}(p)\right]. \tag{S.10.10}$$

Leading to,

$$\sum_{\substack{pol\\col}}|\mathcal{M}_{n+1}(p_1,p_2,...)|^2\approx\frac{2g_s^2}{s_{12}}T^{\mu,a}(p,...)(T^\dagger)^{\nu,a}(p,...)$$
$$\times\underbrace{2C_A\left[d_{\mu\nu}(p)\left(\frac{z}{1-z}+\frac{1-z}{z}\right)-2(1-\epsilon)z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right]}_{\hat{P}^{\mu\nu}_{g\to gg}}, \tag{S.10.11}$$

using $s_{12}=-\frac{k_\perp^2}{z(1-z)}$ and $d=4-2\epsilon$.
One can use the fact that we are looking at the limit where the gluon with momentum p_1+p_2 is an external on-shell particle with momentum p. This allow us to take advantage of the Ward identity and write:

$$T^\mu(p,...)(T^\dagger)^\nu(p,...)d_{\mu\nu}(p)=T^\mu(p,...)(T^\dagger)^\nu(p,...)(-g_{\mu\nu}) \tag{S.10.12}$$

This is the way it is sometimes represented in the literature.

(d) Note that we want to average over $k_\perp$. We can do it with an integral of the form:

$$\int d^d k_\perp \delta(k_\perp \cdot p)\delta(k_\perp \cdot n). \tag{S.10.13}$$

For the constants (in $k_\perp$) there is no contribution, while for $\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}$ we still need to compute it. First consider the general case

$$T^{\mu\nu} = \int d^d k_\perp \delta(k_\perp \cdot p)\delta(k_\perp \cdot n) k_\perp^\mu k_\perp^\nu F(k_\perp^2). \tag{S.10.14}$$

We can then perform a tensor decomposition in terms of the external momenta p^μ, n^μ and the tensor $g^{\mu\nu}$ making the ansatz:

$$T^{\mu\nu} = -g^{\mu\nu} T + p^\mu n^\nu T_{pn} + n^\mu p^\nu T_{np} + p^\mu p^\nu T_{pp} + n^\mu n^\nu T_{nn} \ . \tag{S.10.15}$$

But since we have $k_\perp \cdot n = k_\perp \cdot p = 0$ imposing

$$n_\mu T^{\mu\nu} = n_\nu T^{\mu\nu} = p_\mu T^{\mu\nu} = p_\nu T^{\mu\nu} = 0 \tag{S.10.16}$$

yields

$$T_{nn} = T_{pp} = 0 \ , \qquad\qquad T_{np} = T_{pn} = \frac{T}{n \cdot p} \tag{S.10.17}$$

and the ansatz reduces to

$$T^{\mu\nu} = T\left(-g^{\mu\nu} + \frac{n^\nu p^\mu + n^\mu p^\nu}{n \cdot p}\right) \ , \tag{S.10.18}$$

which squares to $d-2$. One then gets:

$$\int d^d k_\perp \delta(k_\perp \cdot p)\delta(k_\perp \cdot n) k_\perp^\mu k_\perp^\nu F(k_\perp^2) = \frac{\left(g^{\mu\nu} - \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n}\right)}{d-2} \int d^d k_\perp \delta(k_\perp \cdot p)\delta(k_\perp \cdot n) k_\perp^2 F(k_\perp^2), \tag{S.10.19}$$

which leads to

$$\left\langle \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right\rangle = -\frac{d_{\mu\nu}(p)}{2(1-\epsilon)}. \tag{S.10.20}$$

Exercise 2. *Evolution equations*

(a) Show that the difference of the evolution equations for two different quark flavors is:

$$\mu_F^2 \frac{\partial (q_i - q_j)}{\partial \mu_F^2} = P_{qq}^V \otimes (q_i - q_j) + P_{q\bar{q}}^V \otimes (\bar{q}_i - \bar{q}_j) \tag{10.9}$$

Hint. *use*

$$P_{q_i q_j} = \delta_{ij} P_{qq}^V + P_{qq}^S \tag{10.10}$$
$$P_{q_i \bar{q}_j} = \delta_{ij} P_{q\bar{q}}^V + P_{q\bar{q}}^S \tag{10.11}$$

(b) Show that the combination $q_i^+ - q_j^+$, where $q^+ = q + \bar{q}$, yields the following equation,

$$\mu_F^2 \frac{\partial (q_i^+ - q_j^+)}{\partial \mu_F^2} = (P_{qq}^V + P_{q\bar{q}}^V) \otimes (q_i^+ - q_j^+) \tag{10.12}$$

(c) Show that the distribution $q_i^- = q_i - \bar{q}_i$ yields the equation:

$$\mu_F^2 \frac{\partial q_i^-}{\partial \mu_F^2} = (P_{qq}^V + P_{q\bar{q}}^V) \otimes q_i^- + (P_{qq}^S + P_{q\bar{q}}^S) \otimes \sum_{j=1}^{n_f} q_j^- \tag{10.13}$$

(d) Show that the singlet combination

$$\Sigma(x, \mu_F^2) := \sum_{j=1}^{n_f} q_i^+ = \sum_i (q_i + \bar{q}_i), \tag{10.14}$$

yields an evolution equation which is coupled to that of a gluon distribution,

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \begin{pmatrix} \Sigma(\mu_F^2) \\ g(\mu_F^2) \end{pmatrix} = \begin{pmatrix} P_{qq}^\Sigma & 2n_f P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma(\mu_F^2) \\ g(\mu_F^2) \end{pmatrix}, \tag{10.15}$$

where $P_{qq}^\Sigma = P_{qq}^V + P_{q\bar{q}}^V + n_f(P_{qq}^S + P_{q\bar{q}}^S)$.

Solution.

(a)

$$\begin{aligned} \mu_F^2 \frac{\partial(q_i - q_j)}{\partial \mu_F^2} &= (P_{q_i q_k} - P_{q_j q_k}) \otimes q_k + (P_{q_i \bar{q}_k} - P_{q_j \bar{q}_k}) \otimes \bar{q}_k \\ &= \left(\delta_{ik} P_{qq}^V + P_{qq}^S - \delta_{jk} P_{qq}^V - P_{qq}^S\right) \otimes q_k + \left(\delta_{ik} P_{q\bar{q}}^V + P_{q\bar{q}}^S - \delta_{jk} P_{q\bar{q}}^V - P_{q\bar{q}}^S\right) \otimes \bar{q}_k \\ &= P_{qq}^V \otimes (q_i - q_j) + P_{q\bar{q}}^V \otimes (\bar{q}_i - \bar{q}_j) \end{aligned} \tag{S.10.21}$$

(b) It follows directly form the results of the previous point:

$$\mu_F^2 \frac{\partial(q_i - q_j)}{\partial \mu_F^2} = P_{qq}^V \otimes (q_i - q_j) + P_{q\bar{q}}^V \otimes (\bar{q}_i - \bar{q}_j) \tag{S.10.22}$$

$$\mu_F^2 \frac{\partial(\bar{q}_i - \bar{q}_j)}{\partial \mu_F^2} = P_{q\bar{q}}^V \otimes (q_i - q_j) + P_{qq}^V \otimes (\bar{q}_i - \bar{q}_j) \tag{S.10.23}$$

One only need to map the $q+$ expression to the q as,

$$q_i^+ - q_j^+ = (q_i - q_j) + (\bar{q}_i - \bar{q}_j), \tag{S.10.24}$$

in order to recover eq.(10.12).

(c)

$$\mu_F^2 \frac{\partial(q_i - \bar{q}_i)}{\partial \mu_F^2} = (P_{q_i q_j} - P_{\bar{q}_i q_j}) \otimes (q_j - \bar{q}_j) \tag{S.10.25}$$

$$= (\delta_{ij} P_{qq}^V + P_{qq}^S - \delta_{ij} P_{q\bar{q}}^V - P_{q\bar{q}}^S) \otimes (q_j - \bar{q}_j). \tag{S.10.26}$$

Because of the implicit summation over the index j:

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} q_i^- = (P_{qq}^V - P_{q\bar{q}}^V) \otimes q_i^- + (P_{qq}^S - P_{q\bar{q}}^S) \otimes \sum_{j=1}^{n_f} q_j^- \tag{S.10.27}$$

(d) The evolution equation for a quark and anti-quark are:

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} q_i = P_{q_i q_j} \otimes q_j + P_{q_i \bar{q}_j} \otimes \bar{q}_j + P_{qg} \otimes g \tag{S.10.28}$$

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \bar{q}_i = P_{\bar{q}_i q_j} \otimes q_j + P_{q_i q_j} \otimes \bar{q}_j + P_{qg} \otimes g. \tag{S.10.29}$$

Writing $\Sigma = \sum_i (q_i + \bar{q}_i)$ we obtain:

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \Sigma = \sum_i (P_{q_i q_j} + P_{q_i \bar{q}_j}) \otimes (q_j + \bar{q}_j) + 2n_f P_{qg} \otimes g. \tag{S.10.30}$$

Note that the second term correspond to the upper right term in the matrix. Let us focus on the first term,

$$\sum_i (\delta_{ij} P^V_{qq} + P^S_{qq} + \delta_{ij} P^V_{q\bar{q}} + P^S_{q\bar{q}}) \otimes (q_j + \bar{q}_j). \tag{S.10.31}$$

remember that the index j is summed too. We then get,

$$\begin{aligned} (P^V_{qq} + P^V_{q\bar{q}}) \otimes \sum_i (q_i + \bar{q}_i) + n_f (P^S_{qq} + P^S_{q\bar{q}}) \otimes \sum_j (q_j + \bar{q}_j) \\ = (P^V_{qq} + P^V_{q\bar{q}} + n_f (P^S_{qq} + P^S_{q\bar{q}})) \otimes \Sigma. \end{aligned} \tag{S.10.32}$$

i.e. the first entry of the matrix.

Exercise 1. ***Sum Rules and Evolution Equations***

In QED, in the splitting beyond the leading order, also the $P_{e\leftarrow\bar{e}}(z,\alpha)$ can occur.

i) Extend the Gribov-Lipatov equations for the electron and positron in order to include also the $P_{e\leftarrow\bar{e}}(z,\alpha)$ splitting.

Hint. Use the charige parity $P_{e\leftarrow\bar{e}}(z,\alpha) = P_{\bar{e}\leftarrow e}(z,\alpha)$.

ii) The net electron number

$$\int_0^1 \mathrm{d}x \left[f_e(x,Q^2) - f_{\bar{e}}(x,Q^2)\right] = \text{const} \tag{11.1}$$

and the total momentum is conserved

$$\int_0^1 \mathrm{d}x x \left[f_e(x,Q^2) + f_{\bar{e}}(x,Q^2) + f_\gamma(x,Q^2)\right] = \text{const} \tag{11.2}$$

to all orders in α. Find out what the conditions are for that to be true to all orders.

Hint. Extend the leading-order conditions $\int_0^1 \mathrm{d}z P^{(0)}_{e\leftarrow e}(z,\alpha) = 0$ *and* $\int_0^1 \mathrm{d}z z \left(P^{(0)}_{e\leftarrow e}(z,\alpha) + P^{(0)}_{\gamma\leftarrow e}(z,\alpha)\right) = 0$,$\int_0^1 \mathrm{d}z z \left(P^{(0)}_{\gamma\leftarrow\gamma}(z,\alpha) + 2P^{(0)}_{e\leftarrow\gamma}(z,\alpha)\right) = 0$.

iii) Likewise, in QCD the net quark number is conserved,

$$\int_0^1 \mathrm{d}x \left[q_i(x,Q^2) - \bar{q}_i(x,Q^2)\right] = \text{const} \tag{11.3}$$

to all order in α_s.

Hint. Use the $P_{q_iq_j}$ *splitting functions in terms of flavor singlet (S) and non-singlet (V) quantities,*

$$P_{q_iq_j} = \delta_{ij}P^V_{qq} + P^S_{qq} \tag{11.4}$$

$$P_{q_i\bar{q}_j} = \delta_{ij}P^V_{q\bar{q}} + P^S_{q\bar{q}}. \tag{11.5}$$

Use the evolution equation for

$$\mu_F^2 \frac{\partial}{\partial\mu_F^2} q_i^- = (P^V_{qq} - P^V_{q\bar{q}}) \otimes q_i^- + (P^S_{qq} - P^S_{q\bar{q}}) \otimes \sum_{j=1}^{n_f} q_j^- \tag{11.6}$$

$q_i^- = q_i - \bar{q}_i$ *and extend the leading order condition* $\int_0^1 \mathrm{d}z P^{(0)}_{qq}(z) = 0$. *Use the evolution equation*

$$\mu_F^2 \frac{\partial}{\partial\mu_F^2} \begin{pmatrix}\Sigma(\mu_F^2)\\ g(\mu_F^2)\end{pmatrix} = \begin{pmatrix}P^\Sigma_{qq} & 2n_f P_{qg}\\ P_{gq} & P_{gg}\end{pmatrix} \otimes \begin{pmatrix}\Sigma(\mu_F^2)\\ g(\mu_F^2)\end{pmatrix}, \tag{11.7}$$

where $P^\Sigma_{qq} = P^V_{qq} + P^V_{q\bar{q}} + n_f(P^S_{qq} + P^S_{q\bar{q}})$ *and extend the leading order conditions* $\int_0^1 \mathrm{d}z z \left(P^{(0)}_{qq}(z,\alpha) + P^{(0)}_{gq}(z,\alpha)\right) = 0$, $\int_0^1 \mathrm{d}z z \left(P^{(0)}_{gg}(z,\alpha) + 2n_f P^{(0)}_{qg}(z,\alpha)\right) = 0$

Solution.

i) The extended Gribov-Lipatov equations are

$$Q^2\partial_{Q^2}f_e(x,Q^2) = \int_0^1 \frac{\mathrm{d}z}{z}\left[P_{e\leftarrow\gamma}(z,\alpha)f_\gamma\left(\frac{x}{z},Q^2\right) + P_{e\leftarrow e}(z,\alpha)f_e\left(\frac{x}{z},Q^2\right) + P_{e\leftarrow\bar{e}}(z,\alpha)f_{\bar{e}}\left(\frac{x}{z},Q^2\right)\right] \tag{S.11.1}$$

$$,Q^2\partial_{Q^2}f_{\bar{e}}(x,Q^2) = \int_0^1 \frac{\mathrm{d}z}{z}\left[P_{e\leftarrow\gamma}(z,\alpha)f_\gamma\left(\frac{x}{z},Q^2\right) + P_{e\leftarrow e}(z,\alpha)f_{\bar{e}}\left(\frac{x}{z},Q^2\right) + P_{e\leftarrow\bar{e}}(z,\alpha)f_e\left(\frac{x}{z},Q^2\right)\right], \tag{S.11.2}$$

where we used the charge parity.

ii) Use the generalized Gribov-Lipatov equations above. Then we find that the net electron number is conserved:

$$\int_0^1 \mathrm{d}x\left[f_e(x,Q^2) - f_{\bar{e}}(x,Q^2)\right] = \text{const.} \tag{S.11.3}$$

if

$$\int_0^1 \mathrm{d}z\,[P_{e\leftarrow e} - P_{e\leftarrow\bar{e}}] \tag{S.11.4}$$

and the total momentum

$$\int_0^1 \mathrm{d}x x\left[f_e(x,Q^2) + f_{\bar{e}}(x,Q^2) + f_\gamma(x,Q^2)\right] = \text{const.} \tag{S.11.5}$$

is conserved if

$$\int_0^1 \mathrm{d}zz\,[P_{e\leftarrow e}(z,\alpha) + P_{\bar{e}\leftarrow e}(z,\alpha) + P_{\gamma\leftarrow e}(z,\alpha)] = 0 \tag{S.11.6}$$

and

$$\int_0^1 \mathrm{d}zz\,[P_{\gamma\leftarrow\gamma}(z,\alpha) + 2P_{e\leftarrow\gamma}(z,\alpha)] = 0, \tag{S.11.7}$$

so the second condition is the same as at leading order.

iii) Using the evolution equation for the distribution $q_i^- = q_i - \bar{q}_i$ we see that the net quark number is conserved,

$$\int_0^1 q_i^-\,\mathrm{d}x = \text{const.} \tag{S.11.8}$$

if

$$\int_0^1 \mathrm{d}z\left[P_{qq}^V - P_{q\bar{q}}^V\right] = 0 \tag{S.11.9}$$

and

$$\int_0^1 \mathrm{d}z\left[P_{qq}^S - P_{q\bar{q}}^S\right] = 0. \tag{S.11.10}$$

Using the evolution equation for the gluon and the singlet distributions, we see that the total momentum is conserved,

$$\int_0^1 \mathrm{d}x x \left[\sum_{i=1}^{n_f} \left(q_i(x,Q^2) + \bar{q}(x,Q^2) \right) + g(x,Q^2) \right] = \text{const.} \tag{S.11.11}$$

if

$$\int_0^1 \mathrm{d}z z \left[P_{qq}^V + P_{q\bar{q}}^V + n_f \left(P_{qq}^S + P_{q\bar{q}}^S \right) + P_{gq}^0 \right] = 0 \tag{S.11.12}$$

and

$$\int_0^1 \mathrm{d}z z \left[P_{gg} + 2n_f P_{qg} \right] = 0. \tag{S.11.13}$$

Note that the second condition is the same as for leading order.

Exercise 2. *Anomalous Dimensions*

Compute the anomalous dimensions,

$$\gamma_{ij}^{(0)}(N) = \int_0^1 \mathrm{d}z\, z^{N-1} P_{ij}^{(0)}(z), \tag{11.8}$$

for the following splitting functions:

$$P_{qg}^{(0)}(z) = T_F \left[z^2 + (1-z)^2 \right] \tag{11.9}$$

$$P_{gq}^{(0)}(z) = C_F \frac{1+(1-z)^2}{z} \tag{11.10}$$

$$P_{qq}^{(0)}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right] \tag{11.11}$$

$$P_{gg}^{(0)}(z) = 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \left[\frac{11}{6}C_A - \frac{2}{3}T_F n_f \right] \delta(1-z) \tag{11.12}$$

Hint. *Use,*

$$\int_0^1 \mathrm{d}z\, z^{N-1} = \frac{1}{N}, \qquad \int_0^1 \mathrm{d}z \frac{z^{N-1}}{(1-z)_+} = -\sum_{i=1}^{N-1} \frac{1}{i} \tag{11.13}$$

Solution.

$$\gamma_{qg}^{(0)}(N) = T_F \left(\frac{1}{N} - \frac{2}{N+1} + \frac{2}{N+2} \right) = T_F \frac{N^2+N+2}{N(N+1)(N+2)} \tag{S.11.14}$$

$$\gamma_{gq}^{(0)}(N) = C_F \left[\frac{2}{N-1} - \frac{2}{N} + \frac{1}{N+1} \right] = C_F \frac{N^2+N+2}{N(N^2-1)} \tag{S.11.15}$$

$$\gamma_{qq}^{(0)}(N) = C_F \left[-\sum_{i=1}^{N-1} \frac{1}{i} - \sum_{i=1}^{N+1} \frac{1}{i} + \frac{3}{2} \right] = C_F \left[\frac{3}{2} + \frac{1}{N(N+1)} - 2\sum_{i=1}^{N} \frac{1}{i} \right] \tag{S.11.16}$$

$$\gamma_{gg}^{(0)}(N) = 2C_A \left[\frac{1}{N-1} - \frac{1}{N} + \frac{1}{N+1} - \frac{1}{N+2} - \sum_{i=1}^{N} \frac{1}{i} + \frac{11}{12} \right] - \frac{2}{3} T_F n_f \tag{S.11.17}$$

$$= 2C_A \left[\frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} + \frac{11}{12} - \sum_{i=1}^{N} \frac{1}{i} \right] - \frac{2}{3} T_F n_f \tag{S.11.18}$$

Exercise 3. ***Eigenstates for Evolution Equations***

In momentum space the evolution equation for the singlet and the gluon distributions is, at leading order,

$$\frac{\partial}{\partial\log(\mu_F^2)}\begin{pmatrix}\Sigma(N,\mu_F^2)\\ g(N,\mu_F^2)\end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix}\gamma_{qq}^{(0)} & 2n_f\gamma_{qg}^{(0)}\\ \gamma_{gq}^{(0)} & \gamma_{gg}^{(0)}\end{pmatrix}\begin{pmatrix}\Sigma(N,\mu_F^2)\\ g(N,\mu_F^2)\end{pmatrix}. \tag{11.14}$$

The eigenvalue are

$$\gamma_\pm = \frac{1}{2}\left[\gamma_{gg}+\gamma qq \pm \sqrt{(\gamma_{gg}-\gamma_{qq})^2+8n_f\gamma_{qg}\gamma_{gq}}\right] \tag{11.15}$$

(a) Write the evolution equation for the second moment, N=2.

(b) Diagonalize and solve the evolution equation.

(c) Show that the contribution $O^+(2) = \Sigma(2,\mu_F^2)+g(2,\mu_F^2)$ is independent of μ_F^2.

Hint. Use the anomalous dimensions of the previous exercises with $T_F = \frac{1}{2}$*- Find the eigenvalues and eigenvectors.*

Solution.

(a) For the second moment, N=2,

$$\gamma_{qq}^{(0)}(2) = C_F\left(\frac{3}{2}+\frac{1}{6}-3\right) = -\frac{4}{3}C_F \tag{S.11.19}$$

$$\gamma_{qg}^{(0)}(2) = T_F\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{2}\right) = \frac{T_F}{3} = \frac{1}{6} \tag{S.11.20}$$

$$\gamma_{gq}^{(0)}(2) = C_F\left(2-1+\frac{4}{3}\right) = \frac{4}{3}C_F \tag{S.11.21}$$

$$\gamma_{gg}^{(0)}(2) = 2C_A\left(\frac{1}{2}+\frac{1}{12}+\frac{11}{12}-\frac{3}{2}\right) - \frac{2}{3}T_F n_f = -\frac{n_f}{3}. \tag{S.11.22}$$

Then the evolution equation becomes:

$$\frac{\partial}{\partial\log(\mu_F^2)}\begin{pmatrix}\Sigma(2,\mu_F^2)\\ g(2,\mu_F^2)\end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix}-\frac{4}{3}C_F & \frac{n_f}{3}\\ \frac{4}{3}C_F & -\frac{n_f}{3}\end{pmatrix}\begin{pmatrix}\Sigma(2,\mu_F^2)\\ g(2,\mu_F^2)\end{pmatrix}. \tag{S.11.23}$$

(b) The eigenvalues are

$$\gamma^+(2) = 0, \qquad \gamma^-(2) = -\left(\frac{4}{3}C_R+\frac{n_f}{3}\right). \tag{S.11.24}$$

The eigenvectors are given by:

$$v_+ = \begin{pmatrix}1\\ \frac{4C_F}{n_f}\end{pmatrix}, \qquad v_- = \begin{pmatrix}1\\ -1\end{pmatrix}. \tag{S.11.25}$$

In the eigenvector basis, i.e.

$$Q\cdot\begin{pmatrix}\Sigma(2,\mu_F^2)\\ g(2,\mu_F^2)\end{pmatrix} := \begin{pmatrix}1 & 1\\ \frac{4C_F}{n_f} & -1\end{pmatrix}\begin{pmatrix}\Sigma(2,\mu_F^2)\\ g(2,\mu_F^2)\end{pmatrix} = \begin{pmatrix}\Sigma(2,\mu_F^2)+g(2,\mu_F^2)\\ \frac{4C_F}{n_f}\Sigma(2,\mu_F^2)-g(2,\mu_F^2)\end{pmatrix}, \tag{S.11.26}$$

the anomalous dimension matrix M is diagonal, $\gamma\mathbb{1} = QMQ^{-1}$

$$\frac{\partial}{\partial\log(\mu_F^2)}\begin{pmatrix}\Sigma+g\\ \frac{4C_F}{n_f}\Sigma-g\end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix}0 & 0\\ 0 & -\left(\frac{4}{3}C_F-\frac{n_f}{3}\right)\end{pmatrix}\begin{pmatrix}\Sigma+g\\ \frac{4C_F}{n_f}\Sigma-g\end{pmatrix} \tag{S.11.27}$$

(c) As a consequence of the previous expression the combination $O^{+}(2) = \Sigma(2, \mu_F^2) + g(2, \mu_F^2)$ is independent of μ_F^2. The evolution equation for the other combination, $O^{-}(2) = \frac{4C_F}{n_f}\Sigma(2, \mu_F^2) - g(2, \mu_F^2)$, has solution,

$$O^{-}(2, \mu_F^2) = O^{-}(2, \mu^2) \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_F^2)} \right)^{\frac{2\gamma^{-}(2)}{\beta_0}} \tag{S.11.28}$$

with exponent

$$\frac{2\gamma^{-}(2)}{\beta_0} = -2 \frac{\frac{4}{3}C_R + \frac{n_f}{3}}{\frac{11}{3}N_c - \frac{2}{3}n_f} \tag{S.11.29}$$

Exercise 1. ***Three particle phase space***

We now consider the phase-space for three particles, namely:

$$d\phi_{p_1+p_2\to p_3+p_4+k} = \left(\prod_{j=3}^{5} \frac{d^d p_j}{(2\pi)^d}(2\pi)\delta_+(p_j^2)\right)(2\pi)^d\delta^{(d)}\left(p_1+p_2-p_3-p_4-k\right), \tag{12.1}$$

where $p_{1,2}$ are the incoming particles momenta and $p_{3,4}$, k the outgoing one. The aim is to massage this expression and make the divergent region explicit in the collinear and soft limit of the momentum k.

(a) We want to split the phase space in two where one part will depend only on the momentum k. Insert in the phase space integral the following identities:

$$1 = \int \frac{d^dQ}{(2\pi)^d}(2\pi)^d\delta^{(d)}(Q-p_3-p_4), \tag{12.2}$$

$$1 = \int \frac{dx}{2\pi}(2\pi)\delta(Q^2-x), \tag{12.3}$$

and show that the phase space now reads:

$$\int d\phi_{p_1+p_2\to p_3+p_4+k} = \int \frac{dx}{2\pi}\int d\phi_{p_1+p_2\to k+Q}\int d\phi_{Q\to p_3+p_4} \tag{12.4}$$

(b) Show that for a d-dimensional Euclidean space spanned by $\vec{e}_1, ..., \vec{e}_d$ one can write:

$$d\Omega_d = d\Omega_{d-1}d\theta(\sin\theta)^{d-2}, \qquad d>2 \tag{12.5}$$

where θ is the angle with $\vec{e}_d$ and goes from 0 to π. The solid angle in two dimensions is given by $d\Omega_2 = d\phi$, with $\phi\in[0,2\pi]$.

Hint. Consider the measure $\int d^d\vec{r}$.

(c) Show that,

$$\int d^dk\,\delta_+(k^2)\Theta(\sqrt{s}-k^0) = \frac{s^{1-\epsilon}\Omega_{d-2}}{2^{2\epsilon}}\int_0^1 dz\int_0^1 dy\, z^{1-2\epsilon}\,\left[y(1-y)\right]^{-\epsilon}, \tag{12.6}$$

where z is the momentum fraction of the gluon $z := \frac{\left|\vec{k}\right|^2}{\sqrt{s}}$ and y the parametrization of the angle as $1-2y=\cos\theta$.

Solution.

(a) After inserting the two identities for Q and x we can arrange them as follows:

$$\int d\phi_{p_1+p_2\to p_3+p_4+k} = \int dx\left(\int\frac{d^dk\,d^dQ}{(2\pi)^{d-1}}\delta(Q^2-x)\delta_+(k^2)\delta^{(d)}\left(p_1+p_2-Q-k\right)\right)$$
$$\left(\int\frac{d^dp_3\,d^dp_4}{(2\pi)^{d-2}}\delta_+(p_3^2)\delta_+(p_3^2)\delta^{(d)}(Q-p_3-p_4)\right) \tag{S.12.1}$$

Thus,

$$\int d\phi_{p_1+p_2\to p_3+p_4+k} = \int dx\int d\phi_{p_1+p_2\to k+Q}\int d\phi_{Q\to p_3+p_4} \tag{S.12.2}$$

(b) The measure can be written in two ways, one in terms of Ω_d and the other Ω_{d-1}.

$$\int d^d\vec{r} = \int \Omega_d \int_0^\infty dr\, r^{d-1} = \int \Omega_{d-1} \int_{-\infty}^\infty dr_0 \int_0^\infty d\rho\, \rho^{d-2} \tag{S.12.3}$$

We can use polar coordinates to write:

$$\int_{-\infty}^\infty dr_0 \int_0^\infty d\rho\, \rho^{d-2} = \int_0^\infty dr \int_0^\pi d\theta\, r \cdot (r\sin\theta)^{d-2} \tag{S.12.4}$$

Thus,

$$\int_0^\pi d\theta \int \Omega_{d-1} (\sin\theta)^{d-2} = \int \Omega_d. \tag{S.12.5}$$

(c)

$$\int d^dk\, \delta_+(k^2)\Theta(k^0)\Theta(\sqrt{s}-k^0) = \int \frac{d^{d-1}\vec{k}}{2|\vec{k}|} = \int d\Omega_{d-1} \int_0^{\sqrt{s}} \frac{d|\vec{k}|}{2} |\vec{k}|^{d-3} \tag{S.12.6}$$

$$= \Omega_{d-2} \int \frac{d|\vec{k}|}{2} \int_{-1}^1 dcos\theta\, (\sin\theta)^{d-4} |\vec{k}|^{d-3} \tag{S.12.7}$$

$$= \Omega_{d-2} \int \frac{d|\vec{k}|}{2} \int_{-1}^1 dy\, 2^{d-3}\, [y(1-y)]^{\frac{d-4}{2}}\, |\vec{k}|^{d-3} \tag{S.12.8}$$

$$= s^{\frac{d-2}{2}} \Omega_{d-2} \int \frac{d\Omega_{d-2}}{2} \int_0^1 dz \int_0^1 dy\, 2^{d-3}\, [y(1-y)]^{\frac{d-4}{2}}\, z^{d-3} \tag{S.12.9}$$

$$= \frac{s^{1-\epsilon}\Omega_{d-2}}{2^{2\epsilon}} \int_0^1 dz \int_0^1 dy\, [y(1-y)]^{-\epsilon}\, z^{1-2\epsilon} \tag{S.12.10}$$

In the third line we use $\cos\theta = 1-2y$ then $\sin\theta = 2\sqrt{y(1-y)}$ and the jacobian is just a factor of 2. In the next step we rescale $z = \frac{|\vec{k}|}{\sqrt{s}}$.

Exercise 2. *Real correction to $e^+e^- \to q\bar{q}$ at NLO*

We want now consider the QCD correction at NLO for the process $e^+e^- \to q\bar{q}$. Because of the nature of this process we need to consider the simpler process $\gamma^* \to q\bar{q}$, where a virtual photon decays into a quark anti-quark pair. At LO this process contains only the QED interactions, at NLO we have two types of corrections, we can either have a virtual loop from the exchange of a gluon between the two fermions, or the emission of a real radiation from either one of them. Our aim in this exercise is to compute the real correction.

(a) Consider the three body differential decay:

$$\frac{\overline{|\mathcal{M}|}^2}{E_{cm}} d\phi_3, \tag{12.7}$$

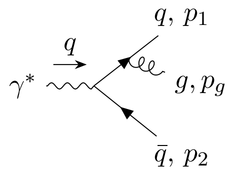
where $d\phi_3$ is the three particle phase space.

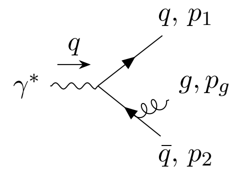
$$d\phi_3 = \frac{Q^2}{16(2\pi)^4} \left(\frac{Q^2}{4\pi}\right)^{-2\epsilon} \frac{1}{\Gamma(2-2\epsilon)} \left(\frac{1-z^2}{4}\right)^{-\epsilon} x_1^{-2\epsilon}\mathrm{d}x_1\, x_2^{-2\epsilon}\mathrm{d}x_2 \tag{12.8}$$

with Q^2 is the energy of the virtual photon in its rest frame, $x_i = \frac{2E_i}{Q}$, and $z = \cos\theta_{12}$ the angle between the two fermions. Show that the angle can be written as:

$$z = 1 + 2(1-x_1-x_2)/(x_1x_2). \tag{12.9}$$

(b) The diagrams that contribute to this correction are:

$\mathcal{A}_1 =$ 

, $\mathcal{A}_2 =$ 

. (12.10)

Show that, when considering all the final state particles to be massless the matrix element squared reads:

$$\overline{\left|\mathcal{M}_{\gamma^*\to e^+e^-g}\right|}^2 = 32e_q^2e^2g_s^2\mu^{2\epsilon}F(x_1,x_2), \tag{12.11}$$

$$F(x_1,x_2) = (1-\epsilon)^2\left[\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}\right] - 2\epsilon(1-\epsilon)\left[\frac{2-2x_1-2x_2+x_1x_2}{(1-x_1)(1-x_2)}\right] \tag{12.12}$$

Hint. It's convenient to split the computation in three pieces: $S_{11} := \overline{|\mathcal{A}_1|}^2$, $S_{22} := \overline{|\mathcal{A}_2|}^2$ *and* $S_{12} := \overline{\mathcal{A}_1\mathcal{A}_2^\dagger + \mathcal{A}_2\mathcal{A}_1^\dagger}$

(c) After combining the matrix elements squared together with the phase space integral, one can write:

$$\sigma^R = \frac{2\alpha_s}{3\pi}\sigma_0\left(\frac{Q^2}{4\pi\mu^2}^{-\epsilon}\right)\frac{1}{\Gamma(2-\epsilon)}\int_0^1 \mathrm{d}x_1\, x_1^{-2\epsilon}\int_0^1 \mathrm{d}x_2\, x_2^{-2\epsilon}\left(\frac{1-z^2}{4}\right)^{-\epsilon}F(x_1,x_2). \tag{12.13}$$

Show that you can decouple the integration by the change of variable $x_2 = 1 - vx_1$.

$$\int_0^1 \mathrm{d}v(1-v)^{-\epsilon}v^{-\epsilon}\int_0^1 \mathrm{d}x_1\,(1-x_1)^{-\epsilon}\,x_1^{-2\epsilon}x_1F(x_1,1-vx_1) \tag{12.14}$$

Argue how this integral can then be easily evaluated and explain at which order in ϵ can the regular pieces be truncated (i.e that are not distributions).

Solution.

(a) We can use momentum conservation to write:

$$0 = (p_g)^2 = (q-p_1-p_2)^2 = Q^2\left(1-x_1-x_2+\frac{1}{2}x_1x_2(1-z)\right) \tag{S.12.11}$$

where we use that in the virtual photon rest frame:

$$2p_1\cdot q = 2E_1Q = Q^2x_1, \tag{S.12.12}$$

$$2p_2\cdot q = 2E_2Q = Q^2x_2, \tag{S.12.13}$$

$$2p_1\cdot p_2 = 2E_1E_2(1-\cos\theta_{12}) = \frac{Q^2}{2}x_1x_2(1-z), \tag{S.12.14}$$

. Solving the expression for z one obtains:

$$z = 1 + 2\frac{1-x_1-x_2}{x_1x_2}. \tag{S.12.15}$$

(b) 1. We start by computing $S_{11} := \overline{|\mathcal{A}_1|}^2$. From applying the Feynman, squaring, and summing over all possible final states (polarizations and colors):

$$S_{11} = 64(1-\epsilon)^2e_q^2e^2g_s^2\mu^{2\epsilon}\frac{2(p_1\cdot q)(p_2\cdot q) - q^2(p1\cdot p2)}{(q-p_1)^4} \tag{S.12.16}$$

$$= 32e_q^2e^2g_s^2\mu^{2\epsilon}(1-\epsilon)^2\frac{1-x_2}{1-x_1} \tag{S.12.17}$$

where we used $C_F = \frac{N^2-1}{2N}$ and $C_A = N$, together with the fact that the final states are considered to be massless and that e_q is the fractional charge of the corresponding quark.

2. In a similar way one can compute $S_{22} := \overline{|\mathcal{A}_2|}^2$.

$$S_{22} = 64(1-\epsilon)^2 e_q^2 e^2 g_s^2 \mu^{2\epsilon} \frac{2(p_1 \cdot q)(p_2 \cdot q) - q^2(p1 \cdot p2)}{(q-p_2)^4} \tag{S.12.18}$$

$$= 32 e_q^2 e^2 g_s^2 \mu^{2\epsilon}(1-\epsilon)^2 \frac{1-x_1}{1-x_2} \tag{S.12.19}$$

3. We start by computing $S_{12} := \overline{\mathcal{A}_1\mathcal{A}_2^\dagger + \mathcal{A}_2\mathcal{A}_1^\dagger}$. In the same way as before we perform the trace over the spinor indices and obtain:

$$S_{12} = 64(1-\epsilon)e_q^2 e^2 g_s^2 \mu^{2\epsilon} \frac{2(p_1 \cdot p_2)^2 + (p_1 \cdot p_2)(q^2(2+\epsilon) - 2(p_1 \cdot q + p_2 \cdot q)) - 2\epsilon(p_1 \cdot q)(p_2 \cdot q)}{(q-p_1)^2(q-p_2)^2} \tag{S.12.20}$$

$$= -64 e^2 g_s^2 \mu^{2\epsilon}(1-\epsilon)\frac{(1-x_1)(1-x_2)\epsilon + 1 - x_1 - x_2}{(1-x_1)(1-x_2)} \tag{S.12.21}$$

A convenient way to bring the sum of these three contribution in the form of $F(x_1, x_2)$ is to notice that when $\epsilon = 0$ the only surviving term of $F(x_1, x_2)$ is the first one. This provide an efficient way on how to split S_{12}.

$$S_{12} = (1-\epsilon)^2 \left[S_{12}\right]_{\epsilon=0} + \left(S_{12} - (1-\epsilon)^2\left[S_{12}\right]_{\epsilon=0}\right) \tag{S.12.22}$$

$$= -64 e_q^2 e^2 g_s^2 \mu^{2\epsilon}(1-\epsilon)^2 \frac{1-x_1-x_2}{(1-x_1)(1-x_2)} - 64 e_q^2 e^2 g_s^2 \mu^{2\epsilon}\epsilon(1-\epsilon)\frac{(2-2x_1-2x_2+x_1x_2}{(1-x_1)(1-x_2)} \tag{S.12.23}$$

Then we can finally write:

$$\begin{aligned} F(x_1, x_2) &= \frac{S_{11}+S_{22}+S_{12}}{32 e_q^2 e^2 g_s^2 \mu^{2\epsilon}} \\ &= (1-\epsilon)^2\left[\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}\right] - 2\epsilon(1-\epsilon)\left[\frac{2-2x_1-2x_2+x_1x_2}{(1-x_1)(1-x_2)}\right] \end{aligned} \tag{S.12.24}$$

(c) By applying the change of variable on top of $z = 1 + 2(1-x_1-x_2)/(x_1x_2)$, together with the measure change $\mathrm{d}x_2 = x_1 \mathrm{d}v$, the given expression follows. Once the integral is cast into this new form, with the new variables, the expression for $F(x_1, x_2)$ reads:

$$\frac{\left(v^2+1\right)x_1^2 - 2vx_1 + 1}{v(1-x_1)} - 2\epsilon\frac{\left(v^2-v+1\right)x_1^2 - x_1 + 1}{v(1-x_1)} + \epsilon^2\left((v-1)x_1+1\right)^2 + \mathcal{O}(\epsilon^3) \tag{S.12.25}$$

This means that can be easily solved in term of simple β-function.

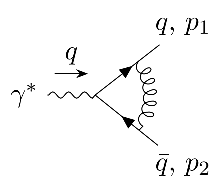
$$\beta(a,b) := \int_0^1 \mathrm{d}x(1-x)^{a-1}x^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{S.12.26}$$

The reason to keep up to $\mathcal{O}(\epsilon^2)$ comes form the fact the the integration of the complete result is expected to produce poles of the form $1/\epsilon$ and $1/\epsilon^2$ coming form the soft and collinear regions of the radiation.

Exercise 3. *Virtual correction to $e^+e^- \to q\bar{q}$ at NLO*

In the previous exercise we computed the real correction to $e^+e^- \to q\bar{q}$. In order to have a complete correction we need to add the virtual correction:

$\mathcal{A} =$



. (12.15)

After performing the color and Lorentz algebra one can solve the integral in dimensional regularization by means of Feynman parameters. This correction reads:

$$\sigma^V = \frac{2\alpha_s}{3\pi}\sigma_0\left[-\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(-2\log\left(\frac{\bar{Q}^2}{4\pi\mu^2}\right) - 2\gamma_E + 3\right) - \log^2\left(\frac{\bar{Q}^2}{4\pi\mu^2}\right)\right.$$
$$\left. -(2\gamma_E - 3)\log\left(\frac{\bar{Q}^2}{4\pi\mu^2}\right) - \gamma_E^2 + 3\gamma_E + \frac{\pi^2}{6} - 8 + \mathcal{O}(\epsilon)\right] \quad (12.16)$$

for $\bar{Q}^2 = -q^2 > 0$.
Argue why you will get an expression in terms of $\bar{Q}$ and find the expression for the case of a timelike virtual photon, i.e. $q^2 > 0$, by using analytic continuation and keeping only the real part.

Hint. *Assume all the positive kinematic invariants to carry the $+i0$ prescription.*

Solution. We can use

$$\log(-(x+i0)) = \log\left(x \cdot e^{-i\pi + i0}\right) = \log(x) - i\pi$$

Then

$$\sigma^V = \frac{2\alpha_s}{3\pi}\sigma_0\left[-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(2\log\left(\frac{Q^2}{4\pi\mu^2}\right) + 2\gamma_E - 3\right) - \log^2\left(\frac{Q^2}{4\pi\mu^2}\right)\right.$$
$$\left. -(2\gamma_E - 3)\log\left(\frac{Q^2}{4\pi\mu^2}\right) - \gamma_E^2 + 3\gamma_E + \frac{\pi^2}{6} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right] \quad (S.12.27)$$

Exercise 4. *NLO finite IR correction*

In the previous exercises we discussed the computation of the real contribution to the NLO corrections, after performing all the trivial integrations over x_1 and v and expanding in terms of the dimensional regulator one gets:

$$\sigma^R = \frac{2\alpha_s}{3\pi}\sigma_0\left[\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(2\log\left(\frac{Q^2}{4\pi\mu^2}\right) + 2\gamma_E - 3\right) + \log^2\left(\frac{Q^2}{4\pi\mu^2}\right)\right.$$
$$\left. +(2\gamma_E - 3)\log\left(\frac{Q^2}{4\pi\mu^2}\right) + \gamma_E^2 - 3\gamma_E - \frac{7\pi^2}{6} + \frac{57}{6} + \mathcal{O}(\epsilon)\right] \quad (12.17)$$

Combine the real and virtual correction to show that the sum of the two results is a finite cross section.

Solution.

$$\begin{aligned}\sigma^R + \sigma^V &= \frac{2\alpha_s}{3\pi}\sigma_0\left(-\frac{7\pi^2}{6} + \frac{57}{6} + \frac{\pi^2}{6} - 8 + \pi^2\right)\\ &= \frac{2\alpha_s}{3\pi}\sigma_0\left(\frac{3}{2}\right)\\ &= \frac{\alpha_s}{\pi}\sigma_0\end{aligned} \quad (S.12.28)$$

Exercise 1. ***DIS at NLO***

We want to show that compared with the results from $e^+e^- \to q\bar{q}$ we have the following expressions:

$$\sigma^R_{e^+e^-} + \sigma^V_{e^+e^-} = \sigma_0 \left(1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right), \tag{13.1}$$

$$\sigma^R_{DIS} + \sigma^V_{DIS} = \sigma_0 \left(1 - \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right). \tag{13.2}$$

where the virtual photon is now in the t channel instead of the s channel, and the LO process is $\gamma^* q \to q$.
Consider the virtual correction to the process, $\gamma^* q \to qg$:

$$\sigma^R(z, Q^2) = \frac{z}{32\pi Q^2}\left(\frac{(1-z)Q^2}{4\pi z}\right)^{-\epsilon} \frac{I}{2^{-\epsilon}\Gamma(1-\epsilon)}, \tag{13.3}$$

where $Q^2 = -q^2 > 0$, $z = \frac{2Q^2}{s+2Q^2}$ with $s = 2p_1 \cdot q$, and the integral I reads,

$$I = \int_0^1 \mathrm{d}y (1-y^2)^{-\epsilon} \left|\overline{\mathcal{M}}\right|^2, \tag{13.4}$$

$$\begin{aligned} \left|\overline{\mathcal{M}}(\gamma^* q \to qg)\right|^2 = 16\pi^2 \alpha\alpha_s \mu^{2\epsilon} e_q^2 \frac{4}{3}\frac{1}{2} \bigg(& \frac{4(4z^2 + 4yz - 4z + y^2 - 2y + 5)}{(1-y)(1-z)} \\ & -2\epsilon \frac{4(4z^2 - 4z + y^2 + 3)}{(1-y)(1-z)} + 4\epsilon^2 \frac{4z^2 - (4y+4)z + y^2 + 2y + 1}{(1-y)(1-z)}\bigg) \end{aligned} \tag{13.5}$$

where $\frac{4}{3}$ is the color factor and $\frac{1}{2}$ comes from averaging over the initial state quark spin.

(a) Show that the integrated cross section is:

$$\sigma^R(z, Q^2) = \frac{16\pi\alpha\alpha_s e_q^2 z}{3Q^2}\left(\frac{(1-z)Q^2}{\mu^2 4\pi z}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(-\frac{1+z^2}{1-z}\frac{1}{\epsilon} - \frac{3}{2}\frac{1}{1-z} - z + 3 + \mathcal{O}(\epsilon)\right) \tag{13.6}$$

where e_q represents the fractional charge of the quark.

(b) Applying the following relation:

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma^R(z, Q^2)}{\mathrm{d}z} = \frac{Q^2}{8\pi^2 \alpha e_q^2 z(1-\epsilon)}\sigma^R(z, Q^2). \tag{13.7}$$

Show that integrating over $z \in (0, 1)$ and expanding in the dimensional regulator ϵ yields:

$$\sigma^R(Q^2) = \frac{2\alpha_s}{3\pi}\sigma_0 \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\left(-\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{13}{2}\right) \tag{13.8}$$

(c) You can now use the results for the virtual correction computed in the previous exercise sheet and combine it with the real correction to obtain the total correction at NLO for the deep inelastic scattering of a quark.

Solution.

(a) As for the case of the e^+e^- we see that the integration over y translates into a sum of β functions. Once all the contribution coming form the numerator are combined together one obtains precisely the given expression

(b) the integration over z lead again a sum over β functions, for simplicity we factor out the term $\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}$ which is finite in epsilon.

(c) After expanding in ϵ one gets:

$$\sigma^R = \frac{2\alpha_s}{\pi}\sigma_0\left[\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(-2\log\left(\frac{Q^2}{4\pi\mu^2}\right)-2\gamma+3\right)+\log^2\left(\frac{Q^2}{4\pi\mu^2}\right)\right.$$
$$\left.+(2\gamma_E-3)\log\left(\frac{Q^2}{4\pi\mu^2}\right)-\frac{\pi^2}{6}+\gamma_E^2-3\gamma_E+\frac{13}{2}+\mathcal{O}(\epsilon)\right] \quad \text{(S.13.1)}$$

and from last time we have:

$$\sigma^V = \frac{2\alpha_s}{3\pi}\sigma_0\left[-\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\left(-2\log\left(\frac{Q^2}{4\pi\mu^2}\right)-2\gamma_E+3\right)-\log^2\left(\frac{Q^2}{4\pi\mu^2}\right)\right.$$
$$\left.-(2\gamma_E-3)\log\left(\frac{Q^2}{4\pi\mu^2}\right)-\gamma_E^2+3\gamma_E+\frac{\pi^2}{6}-8+\mathcal{O}(\epsilon)\right] \quad \text{(S.13.2)}$$

By combining this two results together one gets:

$$\sigma = \sigma^V + \sigma^R = \frac{2\alpha_s}{3\pi}\sigma_0\left(-\frac{3}{2}\right) = -\frac{\alpha_s}{\pi}\sigma_0 \quad \text{(S.13.3)}$$

which is the opposite sign w.r.t. the result obtain for $e^+e^- \to q\bar{q}$. The difference in the computation of the real correction arises from the fact that in one case we are looking at a $1 \to 3$ process while in the second to a $2 \to 2$. We can also notice how the color singlet state of the $\gamma^* \to q\bar{q}$ leads to a QCD correction that enhances the cross section (given by $+\frac{\alpha_s}{\pi}\sigma_0$) while in deep inelastic scattering this is not the case anymore and the correction leads to a reduction of the total cross. section
The above results can written as a single integral as follows:

$$\int_0^1 \mathrm{d}x\left[\frac{\mathrm{d}\sigma^R}{\mathrm{d}x}+\left(\sigma^V+\frac{\alpha_s}{\pi}\sigma_0\right)\delta(1-x)\right]=0. \quad \text{(S.13.4)}$$

Let's introduce the plus distribution for the differential cross section:

$$\int_0^1 \mathrm{d}x\left[\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right]_+ = 0, \qquad \frac{1}{\sigma_0}\frac{\mathrm{d}d\sigma}{\mathrm{d}z} = \frac{1}{\sigma_0}\left[\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right]_+ + \alpha_s\, I_{DIS}\delta(1-z) \quad \text{(S.13.5)}$$

with,

$$\sigma_{DIS} = \sigma_0(1+\alpha_s\, I_{DIS}+\dots), \qquad \alpha_s\, I_{DIS} = -\frac{\alpha_s}{\pi}. \quad \text{(S.13.6)}$$

In this case we have,

$$\frac{1}{\sigma_0}\left[\frac{\mathrm{d}\sigma^R}{\mathrm{d}x}\right]_+ = \frac{\alpha_s}{2\pi}P_{q\to qg}(z)\log\left(\frac{Q^2}{\mu^2}\right)+\alpha_s f^R_{DIS}(z), \quad \text{(S.13.7)}$$

where

$$P_{q\to qg}(z) = \frac{4}{3}\left[\frac{1+z^2}{1-z}\right]_+, \quad \text{(S.13.8)}$$

$$\alpha_s f^R_{DIS}(z) = \frac{2\alpha_s}{2\pi}\left((1+z^2)\left[\frac{\log(1-z)}{1-z}\right]_+ - \frac{1+z^2}{1-z}\log(z) - \frac{3}{2}\frac{1}{[1-z]_+} - z + 3 - \left(\frac{\pi^2}{3}+3\right)\delta(1-z)\right)$$
$$+\frac{\alpha_s}{2\pi}P_{q\to qg}(z)\log\left(\frac{Q^2}{\mu^2}\right)\left(\frac{2}{\epsilon}+\gamma_E-\log(4\pi)\right). \quad \text{(S.13.9)}$$

More details can be found in Field's book *Applications of Perturbative QCD.*

Exercise 2. ***Towards the Symmetric Energy-Momentum tensor in QED***

The *canonical* energy-momentum tensor is defined as

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\ell)} \partial^\nu \Phi^\ell - \eta^{\mu\nu}\mathcal{L} \tag{13.9}$$

$$= (\Pi^\mu)_\ell \Phi^\ell - \eta^{\mu\nu}\mathcal{L}. \tag{13.10}$$

In the exercise class you saw that the anti-symmetric part of the energy-momentum tensor for Poincare invariant field theories can be written as a total derivative. Furthermore the symmetric, gauge-invariant Belifante energy-momentum tensor can be obtained by adding a total derivative:

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\kappa B^{\kappa\mu\nu}, \tag{13.11}$$

with

$$\partial_\kappa B^{\kappa\mu\nu} = -\frac{i}{2}\partial_\kappa \left[(\Pi^\kappa)_l \left(\Sigma^{\mu\nu}\right)^l{}_m - (\Pi^\mu)_l \left(\Sigma^{\kappa\nu}\right)^l{}_m - (\Pi^\nu)_l \left(\Sigma^{\kappa\mu}\right)^l{}_m \right] \Phi^m. \tag{13.12}$$

The field Φ^m is a representation of the Lorentz group and $\left(\Sigma^{\mu\nu}\right)^l{}_m$ are the generators in this representation. The QED Lagrangian is

$$\mathcal{L} = \frac{i}{2}\bar{\Psi}\gamma^\mu D_\mu \Psi - \frac{i}{2}\bar{D}_\mu \bar{\Psi}\gamma^\mu \Psi - m\bar{\Psi}\Psi + \mathcal{L}_{\text{gauge}} \tag{13.13}$$

with

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\lambda A^\lambda\right)^2. \tag{13.14}$$

i) Restrict yourself to the case of the Lagrangian of free spinor fields and compute the *canonical* energy-momentum tensor $\Theta^{\mu\nu}$, the Belifante tensor $B^{\mu\nu\rho}$ and the Belifante energy-momentum tensor $T^{\mu\nu}$.

ii) Compute the trace $T^\alpha{}_\alpha$.

iii) (Optional) Compute the Belifante energy-momentum tensor for full QED.

Solution. The Lagrangian for the free Spinor-field reads

$$\mathcal{L} = \frac{i}{2}\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \left(\frac{i}{2}\partial_\mu\bar{\Psi}\right)\gamma^\mu\Psi - m\bar{\Psi}\Psi \tag{S.13.10}$$

and the canonical energy stress tensor is

$$\Theta^{\mu\nu} = \underbrace{\left(\frac{i}{2}\bar{\Psi}\gamma^\mu\right)}_{:=\Pi^\mu_\Psi}\partial^\nu\Psi + (\partial^\nu\bar{\Psi})\underbrace{\left(\frac{i}{2}\gamma^\mu\Psi\right)}_{:=\Pi^\mu_{\bar{\Psi}}} - \eta^{\mu\nu}\mathcal{L}. \tag{S.13.11}$$

Note however, that the last term vanishes if we impose the equation of motions. Let us look at the first term B_1 of the Belifante tenor only, since the others are simply obtained by permuting the symmetries. We have

$$B_1^{\kappa\mu\nu} = -\frac{i}{2}\Pi^\kappa_\Psi \Sigma^{\mu\nu}\Psi - \frac{i}{2}\bar{\Psi}\bar{\Sigma}^{\mu\nu}\Pi^\kappa_{\bar{\Psi}}, \tag{S.13.12}$$

since $\bar{\Psi}$ transforms under Lorentz boosts with the inverse transformation. For the spinor representation, the generators of the Lorentz group are

$$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{S.13.13}$$

and we get

$$B_1^{\kappa\mu\nu} = \frac{i}{16}\bar{\Psi}\left(\gamma^\kappa\left[\gamma^\mu,\gamma^\nu\right] + \left[\gamma^\mu,\gamma^\nu\right]\gamma^\kappa\right)\Psi \tag{S.13.14}$$

$$= \frac{i}{16}\bar{\Psi}\left(\{\gamma^\kappa,[\gamma^\mu,\gamma^\nu]\}\right)\Psi \; . \tag{S.13.15}$$

So the Belifante tensor is

$$B^{\kappa\mu\nu} = \frac{i}{16}\bar{\Psi}\left(\{\gamma^\kappa,[\gamma^\mu,\gamma^\nu]\} - \{\gamma^\mu,[\gamma^\kappa,\gamma^\nu]\} - \{\gamma^\nu,[\gamma^\kappa,\gamma^\mu]\}\right)\Psi \; . \tag{S.13.16}$$

For the last two terms of the above equation we can use the commutator relation

$$\{B,[A,C]\} + \{C,[A,B]\} = [A,\{B,C\}] \, . \tag{S.13.17}$$

For γ-matrices however, the anti-commutator is proportional to the identity because of the Clifford algebra. Therefore $[A,\{B,C\}]$ vanishes identically and the Belifante tensor takes the simple form

$$B^{\kappa\mu\nu} = \frac{i}{16}\bar{\Psi}\left(\{\gamma^\kappa,[\gamma^\mu,\gamma^\nu]\}\right)\Psi \; . \tag{S.13.18}$$

For the derivative we get two types of terms

$$\partial_\kappa B^{\kappa\mu\nu} = \frac{i}{16}\big((\partial_\kappa\bar{\Psi})\gamma^\kappa\left[\gamma^\mu,\gamma^\nu\right]\Psi + \bar{\Psi}\left[\gamma^\mu,\gamma^\nu\right]\gamma^\kappa\partial_\kappa\Psi\big) + \frac{i}{16}\big((\partial_\kappa\bar{\Psi})\left[\gamma^\mu,\gamma^\nu\right]\gamma^\kappa\Psi + \bar{\Psi}\gamma^\kappa\left[\gamma^\mu,\gamma^\nu\right]\partial_\kappa\Psi\big) \; . \tag{S.13.19}$$

The first one vanishes as soon as we impose the equation of motion. The second one needs a bit of tedious algebra. To facilitate the computation one can use

$$(\partial_\kappa\bar{\Psi})\left[\gamma^\mu,\gamma^\nu\right]\gamma^\kappa\Psi = (\partial_\kappa\bar{\Psi})\left[\left[\gamma^\mu,\gamma^\nu\right],\gamma^\kappa\right]\Psi + (\partial_\kappa\bar{\Psi})\gamma^\kappa\left[\gamma^\mu,\gamma^\nu\right]\Psi. \tag{S.13.20}$$

Now one can impose the equation of motion and use the identity $[\gamma^\alpha,[\gamma^\beta,\gamma^\delta]] = 4\left(\eta^{\alpha\beta}\gamma^\delta - \eta^{\alpha\delta}\gamma^\beta\right)$. When all the dust settles we arrive at the final Belifante energy stress tensor which is

$$T^{\mu\nu} = \frac{i}{4}\bar{\Psi}\left(\gamma^\mu\overrightarrow{\partial^\nu} + \gamma^\nu\overrightarrow{\partial^\mu} - \gamma^\mu\overleftarrow{\partial^\nu} - \gamma^\nu\overleftarrow{\partial^\mu}\right)\Psi, \tag{S.13.21}$$

where the arrow indicates on which field we are acting. The trace ${T^\alpha}_\alpha$ can be trivially computed by using the equations of motion and one gets

$${T^\alpha}_\alpha = m\bar{\Psi}\Psi. \tag{S.13.22}$$

In particular, for massless quarks one will have a vanishing trace in all dimensions.

Exercise 1. *On Angular Ordering*

In a scattering with production of $(n+1)$ massless partons, the rate for soft-gluon emission is

$$\mathrm{d}\sigma_{n+1} = \mathrm{d}\sigma_n \frac{\alpha_s}{2\pi}\frac{\mathrm{d}k^0}{k^0}\frac{\mathrm{d}\Omega}{2\pi}\sum_{i,j} C_{ij}W_{ij}, \tag{14.1}$$

where k^0 is the gluon energy, $\mathrm{d}\Omega$ is the solid angle for the emitted gluon, C_{ij} is a colour factor and

$$W_{ij} = \frac{\left(k^0\right)^2 (p_i\cdot p_j)}{(p_i\cdot k)(p_j\cdot k)} = \frac{1-\cos(\vartheta_{ij})}{(1-\cos(\vartheta_{ik}))(1-\cos(\vartheta_{jk}))} \tag{14.2}$$

is the radiation function. The sum $\sum_{i,j}$ is over all pairs of partons (i,j). Set $W_{ij} = W_{ij}^{[i]} + W_{ij}^{[j]}$ with

$$W_{ij}^{[i]} = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos(\vartheta_{ik})} - \frac{1}{1-\cos(\vartheta_{jk})}\right). \tag{14.3}$$

If $\mathrm{d}\Omega = \mathrm{d}\cos(\vartheta_{ik})\mathrm{d}\varphi_{ik}$, show that

$$\int_0^{2\pi}\frac{\mathrm{d}\varphi_{ik}}{2\pi}W_{ij}^{[i]} = \begin{cases}\frac{1}{1-\cos(\vartheta_{ik})} & \text{if } \vartheta_{ik}<\vartheta_{ij}\\ 0 & \text{if } \vartheta_{ik}>\vartheta_{ij}.\end{cases} \tag{14.4}$$

Hint. Parametrise the momenta using Euler angles:

$$p_i = p_i^0(1,0,0,1) \tag{14.5}$$
$$p_j = p_j^0(1,0,\sin(\vartheta_{ij}),\cos(\vartheta_{ij})) \tag{14.6}$$
$$k = k^0(1,\sin(\vartheta_{ik})\sin(\varphi_{ik}),\sin(\vartheta_{ik})\cos(\varphi_{ik}),\cos(\vartheta_{ik}))\ . \tag{14.7}$$

Solution. We have with the parametrization from the hint

$$(p_j\cdot k) = p_j^0 k^0\left(1-\cos\vartheta_{jk}\right) \tag{S.14.1}$$
$$= p_j^0 k^0\left(1-\sin(\vartheta_{ij})\sin(\vartheta_{ik})\cos(\varphi_{ik})-\cos(\vartheta_{ij})\cos(\vartheta_{ik})\right) \tag{S.14.2}$$

and we can set

$$1-\cos(\vartheta_{jk}) = a - b\cos(\varphi_{ik}) \tag{S.14.3}$$

with $a = 1-\cos(\vartheta_{ij})\cos(\vartheta_{ik})$ and $b = \sin(\vartheta_{ij})\sin(\vartheta_{ik})$. We have

$$W_{ij}^{[i]} = \frac{1}{2}\left[\left(\frac{1-\cos(\vartheta_{ij})}{1-\cos(\vartheta_{ik})}-1\right)\frac{1}{1-\cos(\vartheta_{jk})}+\frac{1}{1-\cos(\vartheta_{ik})}\right] \tag{S.14.4}$$
$$= \frac{1}{2}\frac{1}{1-\cos(\vartheta_{ik})}\left(1+\frac{\cos(\vartheta_{ik})-\cos(\vartheta_{ij})}{1-\cos(\vartheta_{jk})}\right) \tag{S.14.5}$$

and we need to compute the integral

$$I_{ij}^{[i]} = \int_0^{2\pi}\frac{\mathrm{d}\varphi_{ik}}{2\pi}\frac{1}{1-\cos(\vartheta_{jk})} = \int_0^{2\pi}\frac{\mathrm{d}\varphi_{ik}}{2\pi}\frac{1}{a-b\cos(\varphi_{ik})}. \tag{S.14.6}$$

The change of variables $z = \exp(i\varphi_{ik})$ yields

$$I_{ij}^{[i]} = \oint_\gamma \frac{\mathrm{d}z}{2\pi i b} \frac{1}{a - b\frac{z^2+1}{2z}} \tag{S.14.7}$$

$$= -\oint_\gamma \frac{\mathrm{d}z}{\pi i b} \frac{1}{(z - z_+)(z - z_-)}, \tag{S.14.8}$$

where the contour γ is a counter-clockwise oriented circle of radius 1 centered at the origin and

$$z_\pm = \frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1} \tag{S.14.9}$$

denote the poles. To determine the location of the poles we compute

$$a^2 - b^2 = 1 + \cos^2(\vartheta_{ij})\cos^2(\vartheta_{ik}) - 2\cos(\vartheta_{ij})\cos(\vartheta_{ik}) - \sin^2(\vartheta_{ij})\sin(\vartheta_{ik}) \tag{S.14.10}$$

$$= (\cos(\vartheta_{ij}) - \cos(\vartheta_{ik}))^2 \geq 0, \tag{S.14.11}$$

where we used $1 = \cos^2(\vartheta_{ij}) + \sin^2(\vartheta_{ij})$. We can therefore conclude $z_+ > 1$ and $z_- < 1$ and only the pole z_- is inside the contour γ. Thus, by applying the residue theorem,

$$I_{ij}^{[i]} = \frac{2}{b} \frac{1}{z_- - z_+} \tag{S.14.12}$$

$$= \frac{1}{\sqrt{a^2 - b^2}} \tag{S.14.13}$$

$$= \frac{1}{|\cos(\vartheta_{ij} - \cos(\vartheta_{ik}))|} \tag{S.14.14}$$

and we obtain

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi_{ik}}{2\pi} W_{ij}^{[i]} = \frac{1}{2} \frac{1}{1 - \cos(\vartheta_{ik})} \left(1 + \frac{\cos(\vartheta_{ik}) - \cos(\vartheta_{ij})}{|\cos(\vartheta_{ij}) - \cos(\vartheta_{ik})|}\right) \tag{S.14.15}$$

$$= \begin{cases} \frac{1}{1-\cos(\vartheta_{ik})} & \text{if } \vartheta_{ik} < \vartheta_{ij} \\ 0 & \text{if } \vartheta_{ik} > \vartheta_{ij}. \end{cases} \tag{S.14.16}$$

Exercise 2. ***Towards HEFT***

A major break-through for particle physics was the discovery of the Higgs-boson at the LHC. In the following we want to investigate its dominant production channel at the LHC, called gluon-fusion. The coupling of the Higgs boson is proportional to the mass of the particles it couples to. Gluon fusion is therefore a loop-induced process with the LO-contribution shown in fig. 5.

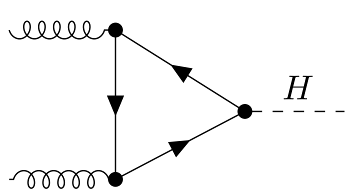


Figure 5: LO Higgs production in gluon fusion. The process where the gluon-legs are crossed does contribute the same.

The contribution from the depicted diagram reads:

$$i\mathcal{A} = -(i)^3(-ig_s)^2\epsilon_\mu(p)\epsilon_\nu(q)\,\mathrm{Tr}\Big(t^a t^b\Big)\left(\frac{-im_Q}{v}\right)\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{t^{\mu\nu}}{\Big(l^2-m_Q^2\Big)\,((l+p)^2-m_Q^2)((l-q)^2-m_Q^2)} \tag{14.8}$$

where

$$t^{\mu\nu} = \mathrm{Tr}\left[(\not{l}+m_Q)\gamma^\mu(\not{l}+\not{p}+m_Q)(\not{l}-\not{q}+m_Q)\gamma^\nu\right], \tag{14.9}$$

m_Q denotes the mass of the quark running in the loop and v the so-called vacuum expectation value.
Before starting any computations answer the following question

a) In LHC collisions the typical momentum fraction x is small $x \ll 1$. Why do we expect gluon induced processes to be dominant?

Solution. To answer that question, we remember the x-dependence of the PDF (see e.g. fig. 2 in https://arxiv.org/pdf/1709.04922.pdf). The gluon PDF is by far dominant in the low x-regime. Processes involving initial state gluons are therefore PDF-enhanced for typical LHC processes.

b) The top quark is roughly 43 times heavier than the bottom quark, which is the second heaviest quark in the SM. Why would we naively expect the gluon fusion process to be dominated by virtual top-quarks? What do we have to verify in order to support or contradict our naive expectations?

Solution. Since the Yukawa coupling is proportional to the mass, we would expect a big enhancement due to the extreme quark mass gap and the top-contribution to be the by far dominant one. However, for that expectation to hold, we have to study the mass dependence of the loop-diagram as well.

We now want to set out to get an idea on the size of the contributions of different quark flavours for Higgs production. To do that, proceed as followed:

i.) Perform the numerator algebra, got to Feynman parameters and complete the square.

Solution. After performing the trace, the numerator reads:

$$t^{\mu\nu} = 4m_Q\left[g^{\mu\nu}(m_Q^2 - l^2 - \frac{m_H^2}{2}) + 4l^\mu l^\nu + p^\nu p^\mu\right]. \tag{S.14.17}$$

Going to Feynman parameters yields

$$\frac{1}{\Big(l^2-m_Q^2\Big)\,((l+p)^2-m_Q^2)((l-q)^2-m_Q^2)} = 2\int \mathrm{d}x\mathrm{d}y\frac{1}{\left[l^2-m_Q^2+2(l\cdot(px-qy))\right]^3} \tag{S.14.18}$$

$$= 2\int \mathrm{d}x\mathrm{d}y\frac{1}{[l'^2-m_Q^2+m_H^2 xy]^3}, \tag{S.14.19}$$

where we completed the square with $l' = l + px - qy$. Completing the square allows the terms linear in l' in the shifted numerator to be dropped due to symmetry.

ii.) Verify, that even though power-counting suggests it, the integral is free of ϵ-poles due to UV-divergences.

Solution. We can use the projection already derived in a previous exercise sheet

$$\int \mathrm{d}^d \frac{k^\mu k^\nu}{(k^2 - C)^m} = \frac{g^{\mu\nu}}{d} \int d^d k \frac{k^2}{(k^2 - C)^m} \tag{S.14.20}$$

to write

$$i\mathcal{A} = -\frac{4g_s^2 m_Q^2}{v} \delta^{ab} \epsilon_\mu(p)\epsilon_\nu(q) \tag{S.14.21}$$

$$\times \int \frac{\mathrm{d}^d l}{(2\pi)^d} \int \frac{\mathrm{d}x\mathrm{d}y}{\left(l^2 - m_Q^2 + m_H^2 xy\right)^3} \left[g^{\mu\nu} \left[m_Q^2 + \frac{4-d}{d} l^2 + m_H^2 (xy - \frac{1}{2}) \right] + p^\nu p^\mu (1 - 4xy) \right]. \tag{S.14.22}$$

Notice that the UV-divergent piece $\propto l^2$ is multiplied by $d - 4 = \epsilon$, which cancels the pole, but will give a non-vanishing finite piece.

iii.) Show that the complete amplitude for LO Higgs production in gluon-fusion can be written as

$$\mathcal{A}(gg \to H) = -\frac{\alpha_s}{\pi v} \delta^{ab} \left(g^{\mu\nu} \frac{m_H^2}{2} - p^\nu q^\mu \right) \epsilon_\mu(p)\epsilon_\nu(q) \int \mathrm{d}x\mathrm{d}y \left(\frac{1 - 4xy}{1 - \frac{m_H^2}{m_Q^2} xy} \right) . \tag{14.10}$$

Solution. Performing the loop-momentum integration by inserting the relevant tadpole integrals and expanding in ϵ will give the result stated in *iii.)* up to a factor of two, which we included to account for the diagram where the gluon legs are crossed.

iv.) Study the amplitude for the two diametric limits $m_Q \ll m_H$ and $m_H \ll m_Q$. Relate your findings to question *b*) asked above.

Solution. The only mass dependence left in the amplitude is in the integral

$$I(\frac{m_H^2}{m_Q^2}) = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}x\mathrm{d}y \frac{1 - 4xy}{1 - \frac{m_H^2}{m_Q^2} xy}. \tag{S.14.23}$$

Expanding for $m_Q \ll m_H$ yields

$$I(\frac{m_H^2}{m_Q^2}) \xrightarrow{\frac{m_H^2}{m_Q^2} \to \infty} -\frac{m_Q^2}{2m_H^2} \log^2 \frac{m_Q^2}{m_H^2} . \tag{S.14.24}$$

This result shows us, that the light quark contribution are even stronger suppressed than the naive coupling analysis in part *b*.) suggested. For a heavy quark approximation $m_Q^2 \gg m_H^2$ we find

$$I(\frac{m_H^2}{m_Q^2}) \xrightarrow{\frac{m_H^2}{m_Q^2} \to 0} \frac{1}{3}. \tag{S.14.25}$$

Higher order terms will be suppressed by $(\frac{m_H^2}{m_Q^2})^n \log^n(\frac{m_H^2}{m_Q^2})$. This suppression allows for an effective description of Higgs-physics called Higgs-Effective-Field-Theory (HEFT) in which the top-quark is assumed to be much heavier than the Higgs and is integrated out from the Lagrangian. All light-quarks will be replaced by massless quarks, which will not couple to the Higgs due to the Yukawa interaction which is proportional to the mass. The relevant effective Lagrangian will then involve effective interactions between the gluons and the Higgs, where the mediating top-quark loop is shrunk to a point. The processes we studied above will e.g. translate to an effective vertex as shown in fig. 6, where a matching of the C_1-coefficient to the SM will require to compute the large mass expansion of the full SM-process. For single Higgs production this effective field theory approach is known to work extremely well and as a matter of fact, even the NNLO

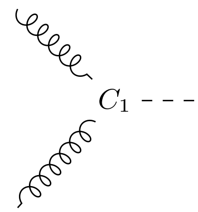


Figure 6: Effective Higgs Coupling for LO Higgs production in gluon-fusion

fully inclusive cross-section with full mass-dependence is as of yet not available in analytical form. This is owed to the extreme mathematical complications which arise in the computation of multi-loop multi-scale Feynman integrals.