Lecture 5

05.04.2024

Soft subtraction

For a soft gluon r, define counterterm

$$S_r^{(0)}(\{p\}_{m+1+x};p_1,p_2)$$

$$= -8\pi\alpha_s\mu^{2\epsilon}\sum_{\substack{j,k\in I\cup F\\ j,k\neq r}}\frac{1}{2}S_{\tilde{j}\tilde{k}}(r)\,T_j T_k\otimes_c\left|\mathcal{M}^{(0)}_{m,ab}(\{\tilde{p}\}_{m+x};\tilde{p}_1,\tilde{p}_2)\right|^2$$

with eikonal factor $\quad S_{\tilde{j}\tilde{k}}(r)=\dfrac{2s_{\tilde{j}\tilde{k}}}{s_{\tilde{j}r}\,s_{\tilde{k}r}}\qquad j,k\in I\cup F\qquad j,k\neq r$

with colour product shorthand

$$T_j T_k\otimes_c\left|\mathcal{M}^{(0)}_{m,ab}(\{p\}_{m+x};p_1,p_2)\right|^2$$

$$=\left[\mathcal{M}_{ab\,f\,f_3\cdots f_{m+3}}^{c_1\cdots c_j'\cdots c_k\cdots c_{m+3};s_1\cdots,s_{m+3}}\right]^* T^{c_r}_{c_j'c_j}\,T^{c_r}_{c_kc_k'}\,\mathcal{M}_{ab\,f\,f_3\cdots f_{m+3}}^{c_1\cdots c_j\cdots c_k'\cdots c_{m+3};s_1\cdots,s_{m+3}}$$

Colour matrices of different partons commute

$$T_j T_k = T_k T_j \quad \text{if } j \neq k \quad \text{and} \quad T^{c_r}_{c_j' c_j} T^{c_r}_{c_j c_j''} = (T_j)^2_{c_j' c_j''} = C_j \delta_{j'j''}$$

The sum over initial and final state partons in $S_r^{(0)}$ can be spelled out

$$S_r^{(0)}(\{p\}_{m+1+x}; P_1, P_2)$$
$$= -8\pi\alpha_s \mu^{2\epsilon} \Bigg[\sum_{\substack{j,k\in F \\ j,k\neq r}} \frac{1}{2} S_{\tilde{j}\tilde{k}}(r) T_j T_k + \sum_{\substack{j\in F \\ j\neq r}} S_{\tilde{1}\tilde{j}} T_a T_j + \sum_{\substack{j\in F \\ j\neq r}} S_{\tilde{2}\tilde{j}} T_b T_j + S_{\tilde{1}\tilde{2}} T_a T_b \Bigg]$$
$$\otimes_c \left| \mathcal{M}^{(0)}_{m,ab}(\{\tilde{p}\}_{m+x}; \tilde{P}_1, \tilde{P}_2) \right|^2$$

We anticipate that $S_{\tilde{1}\tilde{j}}$, $S_{\tilde{2}\tilde{j}}$, $S_{\tilde{1}\tilde{2}}$ are degree-zero homogeneous functions of λ_r, so we can substitute

$$S_{\tilde{1}\tilde{j}}(r) = S_{1\tilde{j}}(r), \qquad S_{\tilde{2}\tilde{j}}(r) = S_{2\tilde{j}}(r), \qquad S_{\tilde{1}\tilde{2}}(r) = S_{12}(r)$$

We define the set of shifted momenta $\{\tilde{p}_1, \tilde{p}_2, \{\tilde{p}\}_{m+X}\}$, with $\{\tilde{p}\}_{m+X} = \{\tilde{p}_3, \dots, \tilde{p}_{m+3}, \tilde{p}_X\}$ the set of $m+X$ shifted momenta obtained by removing p_r, through the shift equations

$$\begin{cases} \tilde{p}_1^\mu = \lambda_r p_1^\mu \\ \tilde{p}_2^\mu = \lambda_r p_2^\mu \\ \tilde{p}_k^\mu = \Lambda(P, \tilde{P})^\mu{}_\nu \, p_k^\nu & k = 3, \dots, m+3 \quad k \neq r \\ \tilde{p}_X^\mu = \Lambda(P, \tilde{P})^\mu{}_\nu \, p_X^\nu \end{cases}$$

Deutschmann, Lionetti, VDD 2019

with $\Lambda(P, \tilde{P})^\mu{}_\nu$ the Lorentz transformation that takes the massive momentum $P^\mu = p_1^\mu + p_2^\mu - p_r^\mu$ into the momentum $\tilde{P}^\mu = \tilde{p}_1^\mu + \tilde{p}_2^\mu = \lambda_r (p_1^\mu + p_2^\mu) = \sum_{k=3, k\neq r}^{m+3} \tilde{p}_k^\mu + \tilde{p}_X^\mu$ of the same mass $P^2 = \tilde{P}^2$

with $P^2 = (p_1 + p_2 - p_r)^2 = s_{12} - s_{r(12)}$ and $\tilde{P}^2 = \lambda_r^2 s_{12}$

so we have the constraint $\lambda_r^2 = 1 - \frac{S_{r(12)}}{S_{12}}$

with $\tilde{p}^2 = M_\chi^2$ for $m = 0$ which entails the bounds

$\lambda^2_{min} = \frac{M_\chi^2}{S_{12}}$ and $\lambda^2_{max} = 1$

Phase space convolution

Inserting the identity,

$$1 = \frac{d^d P}{(2\pi)^d} (2\pi)^d \delta^d(P + p_r - p_1 - p_2) \, dP^2 \, \delta_+(P^2 - s_{12} + s_{r(12)})$$

with $\delta_+(P^2 - s_{12} + s_{r(12)}) = \delta_+(P^2 - \lambda_r s_{12})$

the $(m+1+X)$-particle phase space

$$d\phi_{m+1+X}(\{p\}_{m+1+X}; p_1, p_2)$$

$$= \frac{d^d p_r}{(2\pi)^{d-1}} \delta_+(p_r^2) \prod_{\substack{k=3\\k\neq r}}^{m+3} \frac{d^d p_k}{(2\pi)^{d-1}} \delta_+(p_k^2) \quad \frac{d^d p_X}{(2\pi)^{d-1}} \delta_+(p_X^2 - M_X^2) \quad (2\pi)^d \delta^d\Big(p_1 + p_2 - p_X - \sum_{i=3}^{m+3} p_i\Big)$$

can be written as the convolution

$$d\phi_{m+1+X}(\{p\}_{m+1+X}; p_1, p_2)$$

$$= \frac{dP^2}{2\pi} \, d\phi_{m+X}(p_3, \ldots, \cancel{p_r}, \ldots, p_{m+3}; p_X; P) \; d\phi_2(P, p_r; p_1, p_2)$$

We exchange the momenta $\{p_k\}$ and p_X with $\{\tilde{p}_k\}$ and $\tilde{p}_X$ through the shift equations. Just like in the IF collinear case, since p_k, p_X and P are related to $\tilde{p}_k$, $\tilde{p}_X$ and $\tilde{P}$ by Lorentz transformations, with $\det(\Lambda) = 1$, then

$$\delta^d\left(p_1 + p_2 - p_r - \sum_k p_k - p_X\right) = \delta^d\left(P - \sum_k p_k - p_X\right)$$
$$= \delta^d\left(\tilde{P} - \sum_k \tilde{p}_k - \tilde{p}_X\right)$$

Then, we insert another identity

$$1 = d\lambda^2\, \delta\left(\lambda^2 - 1 + \frac{s_{r(12)}}{s_{12}}\right) = s_{12}\, d\lambda^2\, \delta\left(s_{12}\lambda^2 - s_{12} + s_{r(12)}\right)$$
$$= s_{12}\, d\lambda^2\, \delta\left(P^2 - s_{12} + s_{r(12)}\right)$$

the $(m+1+X)$-particle phase space becomes

$$d\phi_{m+1+X}\left(\{p\}_{m+1+X}; p_1, p_2\right) = \frac{s_{12}}{2\pi} \int_{\lambda^2_{min}}^{\lambda^2_{max}} d\lambda^2\, d\phi_{m+X}\left(\{\tilde{p}\}_{m+X}; \lambda(p_1 + p_2)\right) d\phi_2\left(P, p_r; p_1, p_2\right)$$

Note that in the colour singlet case, $m=0$, the phase space $d\phi_{m+x}$ becomes

$$d\phi_x(\tilde{p}_x; \lambda(p_1+p_2)) = \frac{d^d\tilde{p}_x}{(2\pi)^{d-1}} \delta_+(\tilde{p}_x^2 - M_x^2) \, (2\pi)^d \delta^d(\lambda(p_1+p_2) - \tilde{p}_x)$$

$$= \frac{2\pi}{s_{12}} \delta\left(\lambda^2 - \frac{M_x^2}{s_{12}}\right)$$

Integrated counterterm

$$\int_1 \frac{1}{\Phi(p_1\cdot p_2)}\, d\phi_{m+1+x}(\{p\}_{m+1+x}; p_1, p_2)\, S_r^{(0)}(\{p\}_{m+1+x}; p_1, p_2)\, J_m(\{\hat{p}\}_{m+x})$$

$$= \int d\lambda^2 \frac{1}{\Phi(\tilde{p}_1\cdot\tilde{p}_2)}\, d\phi_{m+x}(\{\tilde{p}\}_{m+x}; \lambda(p_1+p_2))$$

$$\cdot \frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{s_{12}}\right)^\epsilon \sum_{\substack{j,k\in I\cup F\\ j,k\neq r}} \left[S_r^{(0)}\right]^{(j,k)} T^j T^k \otimes_c \left| \mathcal{M}^{(0)}_{m,\hat{a}b}(\{\tilde{p}\}_{m+x}; \tilde{p}_1, \tilde{p}_2)\right|^2 J_m(\{\tilde{p}\}_{m+x})$$

where the flux factor is, in terms of mapped momenta, $\Phi(p_1\cdot p_2) = \dfrac{\Phi(\tilde{p}_1\cdot\tilde{p}_2)}{\lambda^2}$
and we use the shorthand

$$s_{12}^{-\epsilon} S_\epsilon \left[S_r^{(0)}\right]^{(j,k)} = -8\pi s_{12} \int_1 d\phi_2(P, p_r; p_1, p_2)\, \lambda^2\, \frac{1}{2}\, S_{\tilde{j}\tilde{k}}(r)$$

FF collinear-soft subtraction

For a final-state parton i and a soft gluon r, which are collinear, since we can use the light-cone decomposition,

$$p_i \to z_{i,r}\, p_{ir} + O(k_\perp)$$

$$p_r \to z_{r,i}\, p_{ir} + O(k_\perp)$$

$$s_{ik} \to 2 z_{i,r}\, p_{ir} \cdot p_k$$

$$s_{rk} \to 2 z_{r,i}\, p_{ir} \cdot p_k \quad \text{with} \quad z_{r,i} = 1 - z_{i,r}$$

and so the eikonal factor becomes $\quad S_{ik}(r) = \dfrac{2 z_{ir}}{(1 - z_{\tilde{ir}})\, s_{ir}}$

we define the counterterm,

$$C_{ir}^{FF} S_r^{(0)}\left(\{p\}_{m+1+X}; p_1, p_2\right) = 8\pi\alpha_s \mu^{2\epsilon} \frac{2 z_{ir}}{(1 - z_{\tilde{ir}})\, s_{\tilde{ir}}} T_i^2 \left| \mathcal{M}_{m,ab}^{(0)}\left(\{\tilde{p}\}_{m+X}; \tilde{p}_1, \tilde{p}_2\right)\right|^2$$

with $z_{\tilde{i},r} = \dfrac{s_{\tilde{i}(12)}}{s_{(\tilde{i}r)(12)}}$ and $z_{r,\tilde{i}} = 1 - z_{\tilde{i}r}$

where $s_{(\tilde{i}r)(12)} = 2(\tilde{p}_i + p_r)\cdot(p_1+p_2)$

The counterterm is defined through the soft mapping, so shift equations and phase-space convolution are the same as in the soft case

Remember that

$$\mathcal{A}_1 \left| \mathcal{M}^{(0)}_{m+1,ab} \right|^2 = \sum_{r\in F} \left[S_r^{(0)} + \sum_{\substack{i\in F\\ i\neq r}} \left(\frac{1}{2} C_{ir}^{FF(0)} - C_{ir}^{FF} S_r^{(0)} \right) + \dots \right.$$

so $s_{\tilde{i}r}$ and $z_{\tilde{i},r}$ must use $\tilde{p}_i$, because in the $r \to 0$ soft limit $\tilde{p}_i \to p_i$, so $C_{\tilde{i}r}^{FF} S_r^{(0)}$ cancels $C_{\tilde{i}r}^{FF}$ in the soft limit, but in the $i // r$ collinear limit, $\tilde{p}_i \neq p_i$, so one needs $\tilde{p}_i$ for $C_{\tilde{i}r}^{FF} S_r^{(0)}$ to cancel $S_r^{(0)}$.