Lecture 1

03.04.2024

Plan

- description of a subtraction method at NLO in α_s

Single unresolved emissions

- scattering of two partons of flavour a and b, and momenta p_1 and p_2 within hadrons A and B with momenta P_A and P_B, with production of m jets plus any number of non-QCD particles (Higgs, W, Z, ...) denoted by X. The cross section is

$$\sigma(P_A, P_B) = \sum_{a,b} \int_0^1 d\eta_1\, f_{a/A}(\eta_1, \mu_F^2) \int_0^1 d\eta_2\, f_{b/B}(\eta_2, \mu_F^2) \left[\sigma_{ab}^{LO} + \sigma_{ab}^{NLO} + \sigma_{ab}^{NNLO} + \dots \right]$$

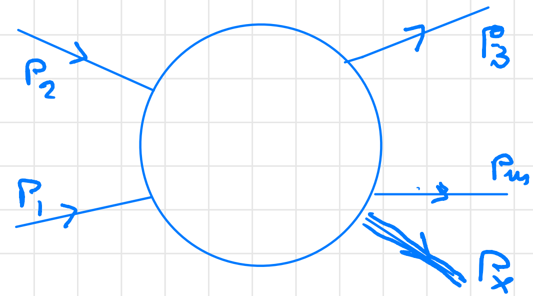
with $p_1 = \eta_1 P_A$ and $p_2 = \eta_2 P_B$ and $\sum_{a,b}$ sum over parton flavours

and $\sigma_{ab}^{N^kLO} \equiv \sigma_{ab}^{N^kLO}(p_1, p_2; \mu_R^2, \mu_F^2)$ $\quad k = 0, 1, 2, \dots$

is the parton cross section

leading order

$$\sigma_{ab}^{LO}(p_1,p_2) \sim \int_m d\sigma_m^{LO}(p_1,p_2)\, J_m$$



where $\int_m$ denotes briefly the integration over the m-parton phase space and J_m is a jet function

J_m Properties:

- J_m vanishes if one parton is soft or collinear to another
  ↳ $d\sigma^{LO}$ is integrable over the 1-parton IR phase space

next-to-leading order

$$\sigma_{ab}^{NLO}(p_1, p_2; \mu_R^2, \mu_F^2) = \int_{m+1} d\sigma^R_{m+1,ab} J_{m+1} + \int_m d\sigma^V_{m,ab} J_m + \int_m d\sigma^C_{m,ab} J_m$$

J_m Properties:

soft:

- $J_{m+1}(p_1, \ldots, p_j = \lambda q, \ldots, p_{m+1}) \to J_m(p_1, \ldots, \cancel{p_j}, \ldots, p_{m+1})$ if $\lambda \to 0$

collinear:

- $J_{m+1}(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_{m+1}) \to J_m(p_1, \ldots, \underline{P}, \ldots, p_{m+1})$

  if $p_i \to z\underline{P}$ $\quad p_j \to (1-z)\underline{P}$

- J_{m+1} vanishes if two partons are unresolved (both soft or collinear to another)

  ↳ $d\sigma^{LO}$ is integrable over the 2-parton IR phase space

Collinear subtraction term:

$$d\sigma^{c}_{m,ab}(\mu_F^2) = -\frac{\alpha_s}{2\pi} S_\epsilon \sum_{c,d} \int_0^1 dz_1 \int_0^1 dz_2 \, d\sigma^{LO}_{cd}(z_1 P_1, z_2 P_2)$$

$$\cdot \delta_{bd}\, \delta(1-z_2) \left(-\frac{1}{\epsilon} \left(\frac{\mu_R^2}{\mu_F^2}\right)^\epsilon P^{ac}(z_1) + K^{ac}_{FS}(z_1) \right)$$

$$\cdot \delta_{ac}\, \delta(1-z_1) \left(-\frac{1}{\epsilon} \left(\frac{\mu_R^2}{\mu_F^2}\right)^\epsilon P^{bd}(z_2) + K^{bd}_{FS}(z_2) \right)$$

P^{ab} : 4-dim DGLAP splitting function

K^{ab} : factorisation scheme dependent finite term ($=0$ in $\overline{MS}$)

$$S_\epsilon = \int \frac{d^{d-3}\Omega}{(2\pi)^{d-3}} = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

(other choices: $S_\epsilon = (4\pi)^\epsilon e^{-\gamma_E \epsilon}$)

Subtraction :

- define an approximate cross section $d\sigma^{R,A_1}_{ab}$ which matches the singularity structure of $d\sigma^R$ point-wise in the 1-parton IR phase space, and subtract it from $d\sigma^R_{ab}$

$$\sigma^{NLO}_{m+1,ab} = \int_{m+1} \left[d\sigma^R_{m+1,ab} J_{m+1} - d\sigma^{R,A_1}_{m+1,ab} J_m \right]_{d=4}$$

- it regularises the single unresolved emissions
- it is finite in $d=4$ by construction

Then

$$\int_m \left[\left(d\sigma^V_{m,ab} + d\sigma^C_{m,ab} + \int_1 d\sigma^{R,A_1}_{m+1,ab} \right) J_m \right]_{d=4}$$

is finite by KLN

Single unresolved emissions

write the approximate cross section symbolically as

$$d\sigma^{R,A_1}_{m+1,ab} = d\phi_{m+1}(\{p\}_{m+1+X}; p_1, p_2)\, \mathcal{A}_1 \left| \mathcal{M}^{(0)}_{m+1,ab}(\{p\}_{m+1+X}; p_1, p_2) \right|^2$$

with $d\phi_{m+1}$ parton phase space, $\mathcal{A}_1 \left| \mathcal{M}^{(0)}_{m+1,ab} \right|^2$ subtraction term

$$\mathcal{A}_1 \left| \mathcal{M}^{(0)}_{m+1,ab}(\{p\}_{m+1+X}; p_1, p_2) \right|^2 = \sum_{r\in F} \left[S^{(0)}_r + \sum_{\substack{i\in F \\ i\neq r}} \left(\frac{1}{2} C^{FF(0)}_{ir} - C^{FF}_{ir} S^{(0)}_r \right) + \sum_{c\in I} \left(C^{IF(0)}_{cr} - C^{IF}_{cr} S^{(0)}_r \right) \right]$$

$I = \{1,2\}$, $F = \{3, \ldots, m+3\}$ denote sets of initial and final state parton

$1, 2$: partons of flavour a, b, colour C_a, C_b, momentum p_1, p_2

i: parton of flavour f_i, colour C_i, momentum p_i $\quad i = 3, \ldots, m+3$

Subtraction terms

- $S_r^{(0)}$ regularises emission of single soft gluon
- $C_{ir}^{FF(0)}$, $C_{cr}^{IF(0)}$ regularise FF and IF collinear singularities
- $C_{ir}^{FF} S_r^{(0)}$, $C_{cr}^{IF} S_r^{(0)}$ account for double subtractions in overlapping soft-collinear regions

Final Final (FF) collinear subtraction

For final state collinear partons i and r, define counterterm

$$C_{ir}^{FF(0)}(\{p\}_{m+1+x}; P_1, P_2)$$

$$= 8\pi\alpha_s \mu^{2\epsilon} \frac{1}{s_{ir}} \hat{P}^{(0)}_{f_i f_r}(z_{i,r}, k_{\perp ir}; \epsilon) \otimes_s \left| \mathcal{M}^{(0)}_{m,ab}(\{\hat{p}\}_{m+x}; \hat{P}_1, \hat{P}_2) \right|^2$$

with spin product shorthand

$$\hat{P}_{f_i f_r} \otimes_s \left| \mathcal{M}_{m,ab}(\{\hat{p}\}_{m+x}; \hat{P}_1, \hat{P}_2) \right|^2 \equiv \sum_{ss'} \hat{P}^{ss'}_{f_i f_r} \, \mathcal{T}^{ss'}_{ab f f_3 \cdots f_{m+3}}$$

with f the flavour of the parent parton ir, and spin tensor

$$\mathcal{T}^{ss'}_{ab f f_3 \cdots f_{m+3}} = \sum_{\text{spins}} \sum_{\text{colour}} \left[\mathcal{M}^{c, c_3 \cdots c_{m+3}; s, s_3 \cdots, s_{m+3}}_{ab f f_3 \cdots f_{m+3}} \right]^* \mathcal{M}^{c, c_3 \cdots c_{m+3}; s', s_3 \cdots, s_{m+3}}_{ab f f_3 \cdots f_{m+3}}$$

(flavour indices are not summed over)

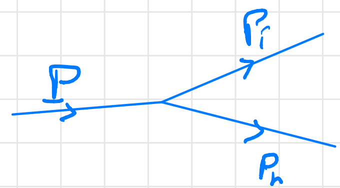
$\hat{P}^{(0)}_{f_i f_r}(z_{i,r}, k_{\perp ir}; \epsilon)$ are the tree DGLAP splitting functions for $F \to FF$

(e.g. for $g \to q\bar{q}$, $\hat{P}^{(0)\mu\nu}_{q\bar{q}}(z, k_\perp; \epsilon) = T_R\left(-g^{\mu\nu} + 4z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right)$)

of the variables $z_{i,r}, k_{\perp ir}$, which parametrise the collinear kinematics, which we analyse now in detail.

We consider the light-cone decomposition of p_i and p_r

$$\begin{cases} p_i^\mu = z_i p_{ir}^\mu + k_{\perp i}^\mu + a_i n^\mu \\ p_r^\mu = z_r p_{ir}^\mu + k_{\perp r}^\mu + a_r n^\mu \end{cases}$$



with $p_{ir}^\mu = \left(\sqrt{(\vec{p}_i + \vec{p}_r)^2}, \vec{p}_i + \vec{p}_r\right)$ and n^μ two light-like vectors, $p_{ir}^2 = n^2 = 0$ which specify the light-cone directions, $p_{ir} \cdot k_\perp = n \cdot k_\perp = 0$.

The massless on-shell conditions $p_i^2 = p_r^2 = 0$ allow us to fix a_i

$$a_j = -\frac{k_{\perp j}^2}{2 z_j \, p_{ir} \cdot n} \qquad j = i, r$$

The momentum fractions are $z_i = \dfrac{p_i \cdot n}{p_{ir} \cdot n}$, $z_r = \dfrac{p_r \cdot n}{p_{ir} \cdot n}$

with $z_i + z_r = \dfrac{P \cdot n}{p_{ir} \cdot n} \neq 1$, where $P^\mu = p_i^\mu + p_r^\mu$ is the parent momentum

However we can use the longitudinal-boost invariant variables

$$\bar{z}_i = \frac{z_i}{z_i + z_r} = \frac{p_i \cdot n}{P \cdot n} \qquad \bar{z}_r = \frac{z_r}{z_i + z_r} = \frac{p_r \cdot n}{P \cdot n}$$ with $\bar{z}_i + \bar{z}_r = 1$

so
$$\begin{cases} p_i^\mu = \bar{z}_i \, p_{ir}^\mu + k_\perp^\mu - \dfrac{k_\perp^2}{2 \bar{z}_i \, p_{ir} \cdot n} n^\mu \\[2ex] p_r^\mu = \bar{z}_r \, p_{ir}^\mu - k_\perp^\mu - \dfrac{k_\perp^2}{2 \bar{z}_r \, p_{ir} \cdot n} n^\mu \end{cases}$$ with $k_\perp^\mu = k_{\perp i}^\mu = -k_{\perp r}^\mu$

Note that $s_{ir} = 2 p_i \cdot p_r = -\frac{\bar{z}_i}{\bar{z}_r} k_\perp^2 - \frac{\bar{z}_r}{\bar{z}_i} k_\perp^2 - 2 k_\perp^2$

$$= -\frac{(\bar{z}_i + \bar{z}_r)^2}{\bar{z}_r \, \bar{z}_i} k_\perp^2 = -\frac{k_\perp^2}{\bar{z}_i \, \bar{z}_r}$$

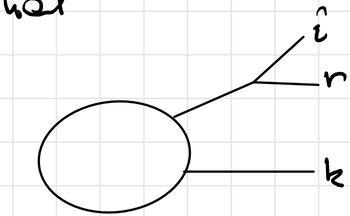
so $s_{ir} \to 0$ as $k_\perp^2 \to 0$, i.e. as p_i^μ and p_r^μ become collinear

so the collinear limit can also be defined through the rescaling

$k_\perp^\mu \to \lambda k_\perp^\mu$ with $\lambda \ll 1$

The parent parton with momentum $P^\mu = p_i^\mu + p_r^\mu$ is off shell.

$P^2 = s_{ir} = O(k_\perp^2)$. It goes on shell only in the strict collinear limit, where factorisation is exact. In order to achieve exact factorisation also elsewhere, a trick (Catani Seymour 1996) is to shift momenta and enforce the parent parton to be on shell. That requires to shift at least another parton, say p_k^μ, besides the parent parton $p_i^\mu + p_r^\mu$, in such a way that



$$\begin{cases} \hat{p}_{ir}^\mu = A\left(p_i^\mu + p_r^\mu\right) + B\, p_k^\mu \\ \hat{p}_k^\mu = C\left(p_i^\mu + p_r^\mu\right) + D\, p_k^\mu \end{cases}$$

with A, B, C, D, constants to be determined.

Momentum conservation requires that $\hat{p}_{ir}^\mu + \hat{p}_k^\mu = p_i^\mu + p_r^\mu + p_k^\mu \equiv Q^\mu$

which implies that $A + C = 1 \qquad B + D = 1$

On-shellness requires that $p_i^2 = p_r^2 = p_k^2 = \hat{p}_k^2 = \hat{p}_{ir}^2 = 0$

Squaring the shift equations above, we obtain that

$$A^{\cancel{2}}\, s_{ir} + \cancel{A}B\left(s_{ik} + s_{rk}\right) = 0$$

$$C^{\cancel{2}}\, s_{ir} + \cancel{C}D\left(s_{ik} + s_{rk}\right) = 0$$

summing the equations above, leads to an inconsistency: $s_{ik} + s_{rk} + s_{ir} = 0$

unless either $C=0$ or $D=0$.

$D=0$ is easily discarded, since $\hat{p}_k^2=0$ implies that also $C=0$.

We set then $C=0$, thus momentum $\hat{p}_k$ is simply rescaled.

Momentum conservation fixes $A=1$, $B+D=1$

Setting $Q^2 = s_{ik}+s_{rk}+s_{ir}$ and $y_{ir} = \frac{s_{ir}}{Q^2}$

$\hat{p}_{ir}^2 = 0$ implies that $s_{ir} + 2B(p_i+p_r)\cdot p_k = 0$

thus $B = -\frac{s_{ir}}{s_{ik}+s_{rk}} = -\frac{y_{ir}}{1-y_{ir}}$ and $D = \frac{1}{1-y_{ir}}$

so the shift is

$$\begin{cases} \hat{p}_{ir}^\mu = p_i^\mu + p_r^\mu - \frac{y_{ir}}{1-y_{ir}} p_k^\mu = \frac{p_i^\mu + p_r^\mu - y_{ir} Q^\mu}{1-y_{ir}} \\ \hat{p}_k^\mu = \frac{1}{1-y_{ir}} p_k^\mu \end{cases}$$

Catani Seymour 1996

For a set of $m+1+X$ momenta, $\{p\}_{m+1+X} = \{p_3, \dots, p_{m+3}, p_X\}$, with $Q^\mu = p_1^\mu + p_2^\mu = \sum_{k=3}^{m+3} p_k^\mu + p_X^\mu$, rather than rescaling just one momentum p_k we can think of dealing in the same way with all the $m-1+X$ momenta recoiling against the collinear limit, by rescaling them all by the same factor (such that at NNLO, all the partons which were not involved in the NLO collinear limit are treated in the same way). Then $\{\hat{p}\}_{m+X} = \{\hat{p}_3, \dots, \hat{p}_{ir}, \dots, \hat{p}_{m+3}, \hat{p}_X\}$ is the set of $m+X$ shifted momenta obtained by removing p_i and p_r and replacing them with $\hat{p}_{ir}$ and the shift is

$$\begin{cases} \hat{p}_{ir}^\mu = A\left(p_i^\mu + p_r^\mu\right) + B\left(\sum_{k=3,\, k\neq i,j}^{m+3} p_k^\mu + p_X^\mu\right) \\ \hat{p}_k^\mu = D\, p_k^\mu \\ \hat{p}_X^\mu = D\, p_X^\mu \end{cases}$$

$k = 3, \dots, \cancel{i}, \dots, \cancel{j}, \dots, m+3$

Momentum conservation $\sum_{k=3}^{m+3} p_k^\mu + p_x^\mu = \hat{p}_{ir}^\mu + \sum_{k=3,\, k\neq ir}^{m+3} \hat{p}_k^\mu + \hat{p}_x^\mu = p_1^\mu + p_2^\mu = p_{(12)}^\mu$

fixes again $A = 1$, $B + D = 1$.

the shift equation $\hat{p}_{ir}^\mu = p_i^\mu + p_r^\mu + B\left[p_{(12)}^\mu - (p_i^\mu + p_r^\mu)\right]$

becomes $\hat{p}_{ir}^\mu = (1-B)(p_i^\mu + p_r^\mu) + B\, p_{(12)}^\mu$

The on-shell condition $\hat{p}_{ir}^2 = 0$ yields a quadratic equation in B, since the recoiler $p_{(12)}^\mu - (p_i^\mu + p_r^\mu)$ is timelike (to be contrasted with the linear equation in B in the Catani-Seymour shift, where p_k^μ is light-like).

We change variables, by setting $B = -\dfrac{\alpha_{ir}}{1-\alpha_{ir}} \rightarrow D = \dfrac{1}{1-\alpha_{ir}}$,

which lets us write the shift equations in a similar way as in Catani-Seymour

$$
\begin{cases}
\hat{P}_{ir}^{\mu} = \dfrac{P_i^{\mu} + P_r^{\mu} - \alpha_{ir} P_{(12)}^{\mu}}{1 - \alpha_{ir}} \\[2ex]
\hat{P}_{k}^{\mu} = \dfrac{1}{1 - \alpha_{ir}} P_k^{\mu} \qquad\qquad k = 3, \ldots, \cancel{r}, \ldots, \cancel{i}, \ldots, n+3 \\[2ex]
\hat{P}_{X}^{\mu} = \dfrac{1}{1 - \alpha_{ir}} P_X^{\mu}
\end{cases}
$$

with $\hat{P}_{ir}^2 = 0$ yielding a quadratic equation in α_{ir}

$$S_{ir} + \alpha_{ir}^2 S_{12} - 2\alpha_{ir} (P_i + P_r) \cdot P_{(12)} = 0$$

whose solutions are

$$\alpha_{ir,\pm} = \frac{(P_i + P_r) \cdot P_{(12)} \pm \sqrt{\left[(P_i + P_r) \cdot P_{(12)}\right]^2 - S_{12} S}}{S_{12}}$$

which, using $2(P_i + P_r) \cdot P_{(12)} = S_{(ir)(12)}$ can be written as

$$\alpha_{ir,\pm} = \frac{S_{(ir)(12)} \pm \sqrt{S_{(ir)(12)}^2 - 4 S_{12} S_{ir}}}{2 S_{12}}$$