Soft theorems and collinear divergences

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50 Years of Jets: Celebrating the Science of George Sterman Stony Brook 19 May 2026

in 1987-1990 I was a PhD student of George's:
… and by 1987 factorisation in hadron collisions was a *fait accompli*

Nuclear Physics B261 (1985) 104–142

FACTORIZATION FOR SHORT DISTANCE HADRON-HADRON SCATTERING

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Received 18 February 1985
(Revised 17 May 1985)

We show that factorization holds at leading twist in the Drell-Yan cross section $d\sigma/dQ^2\,dy$ and related inclusive hadron-hadron cross sections

We review the heuristic arguments for factorization, as well as the difficulties which must be overcome in a proof We go on to give detailed arguments for the all order cancellation of soft gluons, and to show how this leads to factorization

it was factorisation in hadron collisions at leading twist

J C Collins et al / Hadron-hadron scattering

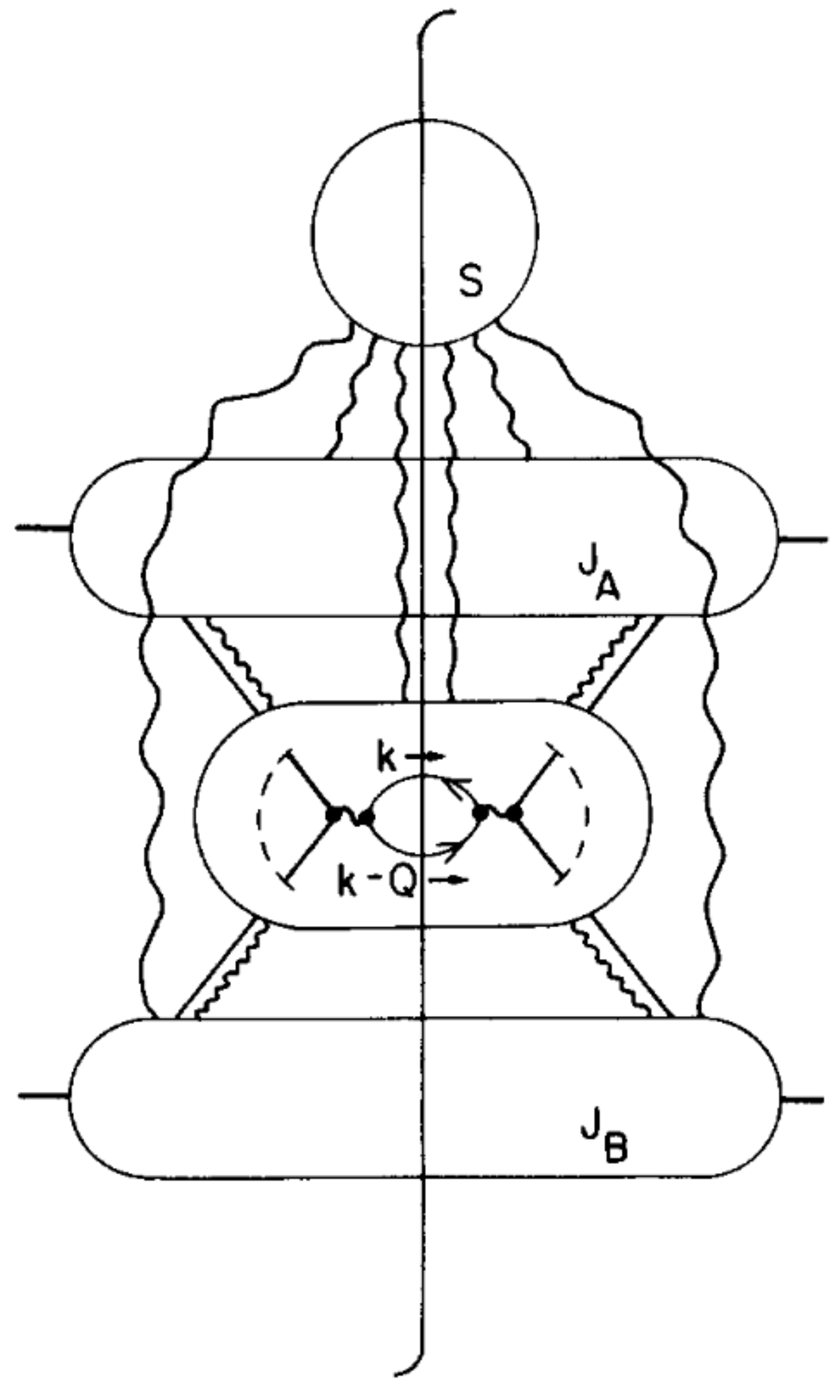

Fig 4 1 General leading surface with muon pair of momentum k^μ and $(Q-k)^\mu$ shown explicitly by 1989, George was already thinking about extending factorisation beyond the leading twist, and working with a postdoc, Janwei Qiu, on it

Nuclear Physics B353 (1991) 105–136
North-Holland

POWER CORRECTIONS IN HADRONIC SCATTERING (I). LEADING $1/Q^2$ CORRECTIONS TO THE DRELL–YAN CROSS SECTION

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Received 2 July 1990

We review the leading $1/Q^2$ corrections to hadronic structure functions measured in deeply inelastic scattering, and we calculate the leading $1/Q^2$ corrections to Drell–Yan cross sections. We find that the leading $1/Q^2$ corrections to a Drell–Yan cross section is given by the convolution of a calculable short-distance hard part, a twist-2 matrix element and a twist-4 matrix element. At leading order in α_s, the normalization of the $1/Q^2$ longitudinal structure functions for Drell–Yan cross sections are determined by higher-twist longitudinal structure functions in deeply inelastic scattering. Other experimental tests of $1/Q^2$ corrections are also discussed.

sometime in 1989, George told me:

it would be good to try and extend factorisation beyond the leading twist …
Why don't you have a look at Low's theorem in the high-energy limit?

I assume this was just one of the different perspectives George had in mind
to extend factorisation beyond the leading twist,
but back then I was a bit concerned about the …

PHYSICAL REVIEW VOLUME 110, NUMBER 4 MAY 15, 1958

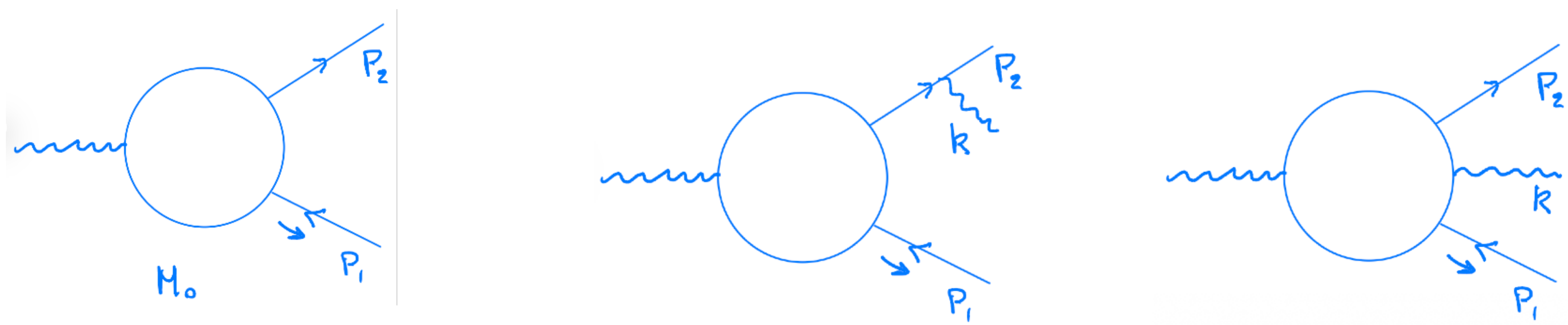
Bremsstrahlung of Very Low-Energy Quanta in Elementary Particle Collisions

F. E. Low
Department of Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts

(January 29, 1958)

It is shown that the first two terms in the series expansion of the differential bremsstrahlung cross section (in powers of the energy loss) may be calculated exactly in terms of the corresponding elastic amplitude and the electromagnetic constants of the participating particles.

- in a QED process with emission of a photon, Low's theorem shows that the first two orders of an expansion of the amplitude in the soft photon momentum can be expressed in terms of the associated non-radiative (i.e. without photon emission) amplitude



$$M_\mu \epsilon^\mu(k) = \sum_i Q_i \left[\frac{(2p_i + k)\cdot\epsilon}{2p_i \cdot k + k^2} \left(1 + k^\mu \frac{\partial}{\partial p_i^\mu}\right) - \epsilon^\mu \frac{\partial}{\partial p_i^\mu} + \bar{j}(p_i, k)\cdot\epsilon \right] M_0 + O(k)$$

- based on Ward identity $M_\mu k^\mu = 0$
- assumes a mass cut-off $k^0 < m$ $s \simeq m^2$
- dependence on spin of charged legs only in IR finite term

Low 1958
Weinberg 1965
Gribov 1967
Burnett Kroll 1968

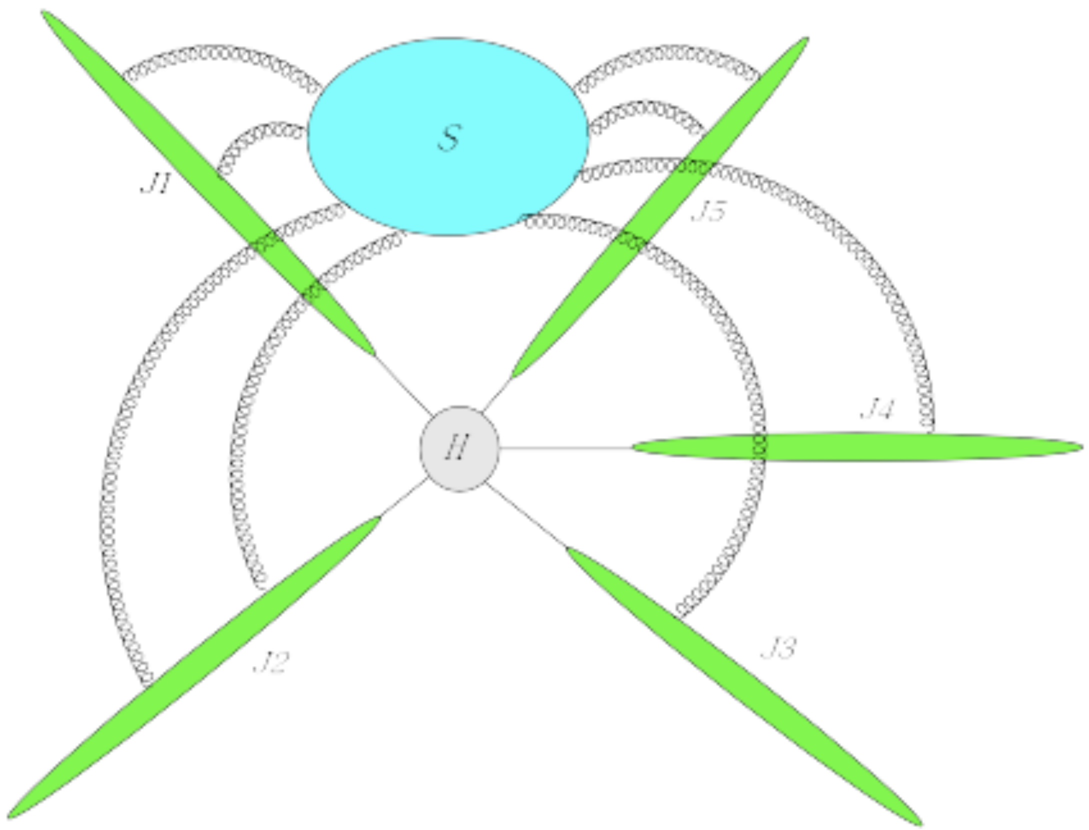
- Note that using $$\left(g^{\nu}_{\mu} - k^{\nu}\frac{(2p_i+k)_{\mu}}{2p_i\cdot k+k^2}\right)\epsilon^{\mu}(k) = \frac{(2p_i+k)_{\mu}}{2p_i\cdot k+k^2}F^{\mu\nu}(k,\epsilon)$$

Grammer-Yennie *K* + *G* split

field strength tensor $$F^{\mu\nu}(k,\epsilon) = k^{\mu}\epsilon^{\nu}(k) - k^{\nu}\epsilon^{\mu}(k)$$

$$M_{\mu}\epsilon^{\mu}(k) = \sum_i Q_i\frac{(2p_i+k)_{\mu}}{2p_i\cdot k+k^2}\left[\epsilon^{\mu}(k) - F^{\mu\nu}(k;\epsilon)\frac{\partial}{\partial p_i^{\nu}} + F^{\mu\nu}(k;\epsilon)j_{\nu}(p_i,k)\right]M_0 + O(k)$$

Factorisation of a multi-leg amplitude



Mueller 1979
Collins 1980
Sen 1981
Korchemsky 1989

$$M_0 \sim H(p_1, p_2) S(u_1, u_2) \prod_{i=1}^{2} J(p_i; u_i)$$

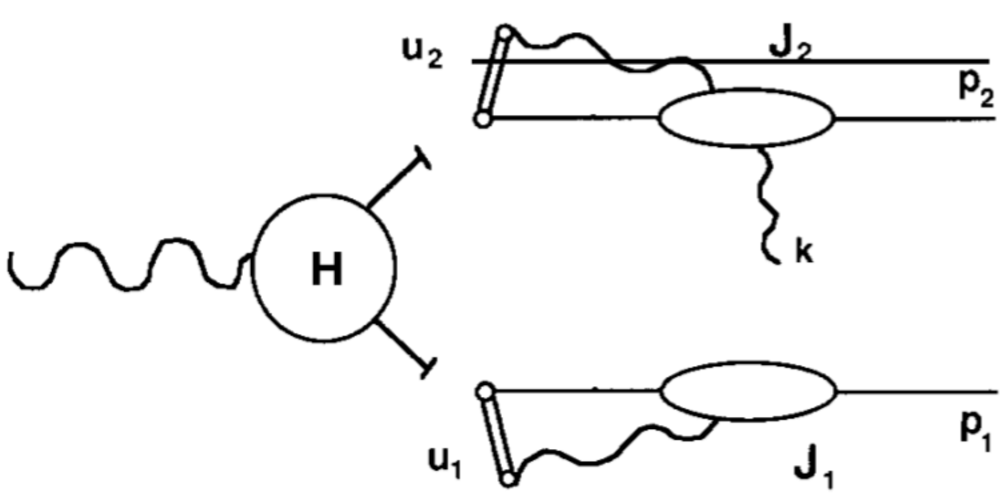
Mass divergences in the high energy limit

- George said: as it is, Low's theorem is valid only in the region because of collinear contributions to the jet functions $k^0 < \frac{m^2}{\sqrt{s}}$ $\quad s \gg m^2$

- I extended Low's theorem to the emission of a soft photon including the collinear contributions $\frac{m^2}{\sqrt{s}} < k^0 < m$

$$M_\mu \epsilon^\mu(k) = \sum_{i=1}^{2} Q_i \frac{(2p_i + k)_\mu}{2p_i \cdot k + k^2} \left\{ \left[\left(\epsilon^\mu(k) - F^{\mu\nu}(k;\epsilon) \frac{\partial}{\partial p_i^\nu} \right) H(p_1, p_2) \right] S(u_1, u_2) \prod_{k=1}^{2} J(p_k; u_k) \right.$$
$$\left. + F^{\mu\nu}(k;\epsilon) H(p_1,p_2) S(u_1,u_2) J_\nu(p_i, k; u_i) J(p_i'; u_i') \right\} + O(k)$$

VDD 1990



radiative jet function

$J_\nu(p_2, k; u_2)$

see Eric's talk

$J(p_1; u_1)$

Mass divergences?

- no mass divergences, in the case of soft gravitons

PHYSICAL REVIEW VOLUME 140, NUMBER 2B 25 OCTOBER 1965

Infrared Photons and Gravitons*

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Department of Physics, University of California, Berkeley, California
(Received 1 June 1965)

It is shown that the infrared divergences arising in the quantum theory of gravitation can be removed by the familiar methods used in quantum electrodynamics. An additional divergence appears when infrared photons or gravitons are emitted from noninfrared external lines of zero mass, but it is proved that for infrared gravitons this divergence cancels in the sum of all such diagrams. (The cancellation does not occur in massless electrodynamics.) The formula derived for graviton bremsstrahlung is then used to estimate the gravitational radiation emitted during thermal collisions in the sun, and we find this to be a stronger source of gravitational radiation (though still very weak) than classical sources such as planetary motion. We also verify the conjecture of Dalitz that divergences in the Coulomb-scattering Born series may be summed to an innocuous phase factor, and we show how this result may be extended to processes involving arbitrary numbers of relativistic or nonrelativistic particles with arbitrary spin.

I. INTRODUCTION

THE chief purpose of this article is to show that the infrared divergences in the quantum theory of gravitation can be treated in the same manner as in quantum electrodynamics. However, this treatment apparently does not work in other non-Abelian gauge theories, like that of Yang and Mills. The divergent phases encountered in Coulomb scattering will incidentally be explained and generalized.

It would be difficult to pretend that the gravitational infrared divergence problem is very urgent. My reasons for now attacking this question are:

(1) Because I can. There still does not exist any satisfactory quantum theory of gravitation, and in lieu of such a theory it would seem well to gain what experience we can by solving any problems that can be solved with the limited formal apparatus already at our disposal. The infrared divergences are an ideal case of this sort, because we already know all about the coupling of a very soft graviton to any other particle,[1]

wrong. In Sec. II we see that the dependence on the infrared cutoffs of real and virtual gravitons cancels just as in electrodynamics.

However, there is a more subtle difficulty that might have been expected. Ordinary quantum electrodynamics would contain unremovable logarithmic divergences if the electron mass were zero, due to diagrams in which a soft photon is emitted from an external electron line with momentum parallel to the electron's.[3] There are no charged massless particles in the real world, but hard neutrinos, photons, and gravitons do carry a gravitational "charge," in that they can emit soft gravitons. In Sec. III we show that diagrams in which a soft graviton is emitted from some other hard massless particle line do contain divergences like the lnm_e terms in massless electrodynamics, but that these divergences cancel when we sum all such diagrams.[4] However, this cancellation is definitely due to the details of gravitational coupling, and does not save theories (like Yang and Mills's) in which massless particles can emit soft massless particles of spin one.

PHYSICAL REVIEW VOLUME 135, NUMBER 4B 24 AUGUST 1964

Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass*

STEVEN WEINBERG†
Physics Department, University of California, Berkeley, California
(Received 13 April 1964)

We give a purely S-matrix-theoretic proof of the conservation of charge (defined by the strength of soft photon interactions) and the equality of gravitational and inertial mass. Our only assumptions are the Lorentz invariance and pole structure of the S matrix, and the zero mass and spins 1 and 2 of the photon and graviton. We also prove that Lorentz invariance alone requires the S matrix for emission of a massless particle of arbitrary integer spin to satisfy a "mass-shell gauge invariance" condition, and we explain why there are no macroscopic fields corresponding to particles of spin 3 or higher.

I. INTRODUCTION

IT is not yet clear whether field theory will continue to play a role in particle physics, or whether it will ultimately be supplanted by a pure S-matrix theory. However, most physicists would probably agree that the place of local fields is nowhere so secure as in the theory of photons and gravitons, whose properties seem indissolubly linked with the space-time concepts of gauge invariance (of the second kind) and/or Einstein's equivalence principle.

The purpose of this article is to bring into question the need for field theory in understanding electromagnetism and gravitation. We shall show that there are no general properties of photons and gravitons, which *have* been explained by field theory, which cannot also be understood as consequences of the Lorentz invariance and pole structure of the S matrix for massless particles of spin 1 or 2.[1] We will also show why there can be no macroscopic fields whose quanta carry spin 3 or higher.

What are the special properties of the photon or graviton S matrix, which might be supposed to reflect specifically field-theoretic assumptions? Of course, the usual version of gauge invariance and the equivalence principle cannot even be stated, much less proved, in terms of the S matrix alone. (We decline to turn on external fields.) But there are two striking properties of the S matrix which *seem* to require the assumption of gauge invariance and the equivalence principle:

(1) The S matrix for emission of a photon or graviton can be written as the product of a polarization "vector" ϵ^μ or "tensor" $\epsilon^\mu\epsilon^\nu$ with a covariant vector or tensor amplitude, and it vanishes if any ϵ^μ is replaced by the photon or graviton momentum q^μ.

(2) Charge, defined dynamically by the strength of soft-photon interactions, is additively conserved in all reactions. Gravitational mass, defined by the strength of soft graviton interactions, is equal to inertial mass for all nonrelativistic particles (and is twice the total energy for relativistic or massless particles).

Property (1) is actually a straightforward consequence of the well-known[2,3] Lorentz transformation properties of massless particle states, and is proven in Sec. II for massless particles of arbitrary integer spin. (It has already been proven for photons by D. Zwanziger.[4])

Property (2) does not at first sight appear to be derivable from property (1). Even in field theory (1) does not prove that the photon and graviton "currents" $J_\mu(x)$ and $\theta_{\mu\nu}(x)$ are conserved, but only that their matrix elements are conserved for light-like momentum transfer, so we cannot use the usual argument that $\int d^3x J^0(x)$ and $\int d^3x\theta^{0\mu}(x)$ are time-independent. And in pure S-matrix theory it is not even possible to define what we mean by the operators $J^\mu(x)$ and $\theta^{\mu\nu}(x)$.

We overcome these obstacles by a trick, which replaces the operator calculus of field theory with a little simple polology. After defining charge and gravitational mass as soft photon and graviton coupling constants in Sec. III, we prove in Sec. IV that if a reaction violates charge conservation, then the same process with inner bremsstrahlung of a soft extra photon would have an S matrix which does not satisfy property (1), and hence would not be Lorentz invariant; similarly, the inner bremstrahlung of a soft graviton would violate Lorentz invariance if any particle taking part in the reaction has an anomalous ratio of gravitational to inertial mass.

Appendices A, B, and C are devoted to some technical problems: (A) the transformation properties of polarization vectors, (B) the construction of tensor amplitudes for massless particles of general integer spin, and (C) the presence of kinematic singularities in the conventional $(2j+1)$-component "M functions."

A word may be needed about our use of S-matrix theory for particles of zero mass. We do not know whether it will ever be possible to formulate S-matrix

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Weinberg's "tool"

We also note the very important transformation rule, proved in Appendix A,

$$(\Lambda_\nu{}^\mu - q^\mu\Lambda_\nu{}^0/|\mathbf{q}|)\epsilon_\pm{}^\nu(\Lambda\hat{q}) = \exp\{\pm i\Theta[\mathbf{q},\Lambda]\}\epsilon_\pm{}^\mu(\hat{q}), \quad (2.13)$$

with Θ the same angle as in (2.1).

If it were not for the q^μ term in (2.13), the polarization "tensor" $\epsilon_\pm{}^{\mu_1}\cdots\epsilon_\pm{}^{\mu_j}$ would be a true tensor, and the tensor transformation law (2.3) for $M_\pm{}^{\mu_1\cdots\mu_j}$ would be sufficient to ensure the correct behavior (2.1) of the S matrix. But $\epsilon_\pm{}^\mu$ is not a vector,[7] and (2.3) and (2.13)

- polarisation vector does not transform covariantly under little group
- field strength tensor does transform covariantly

Subsequent works

- next-to-leading-power threshold logarithms in QCD amplitudes, with an explicit NLO construction of the radiative jet function

Bonocore Laenen Magnea Melville Vernazza White 1503.05156

… and many more works on the radiative jet function, in QED or QCD

see Eric's talk

- … in QED, more on the interplay between photon energy and electron mass

van Bijleveld Laenen Marinissen Vernazza Wang 2503.10810

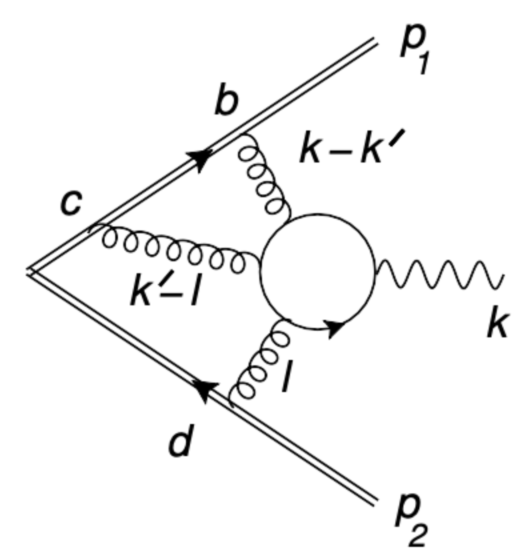
- … and practical aspects of soft gluon emission in massive QCD

Czakon Minguez Eschment 2510.21655

- … but there are $O(\alpha_s^3)$ corrections to leading power in $1/k$ in massless QCD

$$M_3^{(f)}(\{p_i\},k,\epsilon(k)) = \left[e_f + \Gamma_{\rm EM}^{(f)}(\alpha_s)\left(\sum_{n=1}^{n_0} e_n\right)\right]\left[\frac{p_1\cdot\epsilon(k)}{p_1\cdot k} - \frac{p_2\cdot\epsilon(k)}{p_2\cdot k}\right] M_2^{(f)}(\{p_i\})$$

Ma Sterman Venkata 2311.06912



What about gravity?

- Cachazo-Strominger soft theorem for tree graviton amplitudes

Cachazo Strominger 1404.4091

$$M_{n+1}(p_i,\ldots,p_n,q) = \epsilon_{\mu\nu}(k)\sum_{i=1}^{n}\left(\frac{p_i^\mu p_i^\nu}{p_i\cdot k} - i\frac{p_i^\mu k_\rho J_i^{\rho\nu}}{p_i\cdot k} - \frac{k_\rho J_i^{\rho\mu} k_\sigma J_i^{\sigma\nu}}{2p_i\cdot k}\right) M_n(p_i,\ldots,p_n) + O(k^2)$$

- leading (Weinberg) term gets no loop corrections (Weinberg's soft theorem)

Weinberg 1965

- next-to-leading-power term gets one-loop corrections
  next-to-next-to-leading-power term gets up to two-loop corrections

Bern Davies Nohle 1405.1015

- no mass divergences, in the case of soft-graviton (eikonal) limit

Weinberg 1965

- no mass divergences, in the case of wide-angle scattering

Akhoury Saotome Sterman 1109.0270

- at most a finite collinear contribution at $O(k^0)$

Bern Dixon Perelstein Rozowsky hep-th/9811140

PHYSICAL REVIEW D **84,** 104040 (2011)

Collinear and soft divergences in perturbative quantum gravity

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Collinear and soft divergences in perturbative quantum gravity are investigated to arbitrary orders in amplitudes for wide-angle scattering, using methods developed for gauge theories. We show that collinear singularities cancel when all such divergent diagrams are summed over, by using the gravitational Ward identity that decouples unphysical polarizations from the S-matrix. This analysis generalizes a result previously demonstrated in the eikonal approximation. We also confirm that the only virtual graviton corrections that give soft logarithmic divergences are of the ladder and crossed ladder types.

PHYSICAL REVIEW D **103,** 064036 (2021)

High energy scattering in perturbative quantum gravity at next-to-leading power

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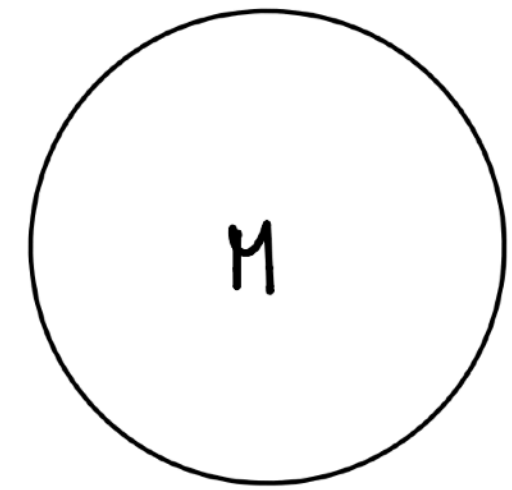
We consider the relativistic scattering of unequal-mass scalar particles through graviton exchange in the small-angle high-energy regime. We show the self-consistency of expansion around the eikonal limit and compute the scattering amplitude up to the next-to-leading power correction of the light particle energy, including gravitational effects of the same order. The first power correction is suppressed by a single power of the ratio of momentum transfer to the energy of the light particle in the rest frame of the heavy particle, independent of the heavy particle mass. We find that only gravitational corrections contribute to the exponentiated phase in impact parameter space in four dimensions. For large enough heavy-particle mass, the saddle point for the impact parameter is modified compared to the leading order by a multiple of the Schwarzschild radius determined by the mass of the heavy particle, independent of the energy of the light particle.

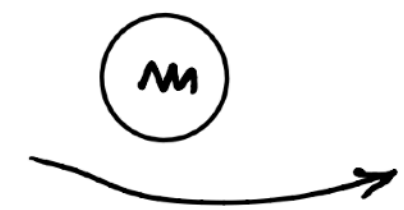
- analyse the case of light mass-*m* and heavy mass-*M* black holes
- expand to first order in $\sqrt{|t|}/E$

Self force



- Probe limit corresponds to Schwarzschild geometry: static (infinitely heavy) black hole

- in Self Force (SF) expansion,
  one expands Einstein's equation in powers of m/M
  and solves them (numerically)



0 SF = Probe limit = Schwarzschild
1 SF is known for generic orbits
2 SF is known for specific cases (e.g. quasi circular orbits)