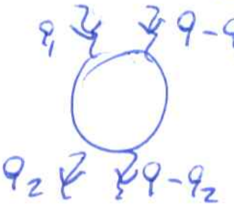
Singlet BFKL at $t \neq 0$ $\quad q^2 = t$



$$\left[\omega - \alpha(\hat{t}_1) - \alpha(\hat{t}_1')\right] f_\omega(q_1, q_2, q) =$$

$$= (2\pi)^2 \delta^2(q_1 - q_2) - 2\alpha_s N_c \int \frac{d^2k}{(2\pi)^2} \frac{K(q_1, k)}{k_\perp^2 (q-k)_\perp^2} f_\omega(k, q_2, q)$$

$$K(q_1, q_2) = q_\perp^2 - \frac{(q-q_1)_\perp^2 q_{2\perp}^2 + (q-q_2)_\perp^2 q_{1\perp}^2}{(q_1 - q_2)_\perp^2}$$

$$\hat{t}_1 = q_1^2 \quad \hat{t}_1' = (q-q_1)^2$$

Trans. coords:
$$\vec{\rho}_1 \Leftrightarrow q_1$$
$$\vec{\rho}_2 \Leftrightarrow q_2$$
$$\rho_1' \Leftrightarrow q - q_1$$
$$\rho_2' \Leftrightarrow q - q_2$$

Fourier transfor to coord. space:

$$f_\omega(\rho_1, \rho_2, \rho_1', \rho_2') = \int d^2q\, d^2q_2\, d^2q\; e^{i[q_1\cdot\rho_1 + (q-q_1)\cdot\rho_1' - q_2\cdot\rho_2 - (q-q_2)\cdot\rho_2']} \frac{f_\omega(q_1, q_2, q)}{q_2^2 (q_1 - q)^2}$$

$$\omega\, \partial_{\rho_1}^2 \partial_{\rho_1'}^2 f_\omega(\rho_1, \rho_2, \rho_1', \rho_2') = (2\pi)^4 \delta^2(\rho_1 - \rho_2) \delta^2(\rho_1' - \rho_2')$$

$$+ \frac{N_c \alpha_s}{2\pi^2} \Big\{ (2\pi)^2 \delta^2(\rho_1 - \rho_1') (\partial_{\rho_1} + \partial_{\rho_1'})^2 f_\omega(\rho_1, \rho_2, \rho_1', \rho_2')$$

$$+ \partial_{\rho_1}^2 \int \frac{d^2\rho_0}{(\rho_1 - \rho_0)^2} \left[\partial_{\rho_1'}^2 f_\omega(\rho_0, \rho_1', \rho_2, \rho_2') - \frac{(\rho_1 - \rho_1')^2}{(\rho_1 - \rho_0)^2 + (\rho_1' - \rho_0)^2} \partial_{\rho_1'}^2 f_\omega(\rho_1, \rho_1', \rho_2, \rho_2') \right]$$

$$+ \partial_{\rho_1'}^2 \int \frac{d^2\rho_0}{(\rho_1' - \rho_0)^2} \left[\partial_{\rho_1}^2 f_\omega(\rho_1, \rho_0, \rho_2, \rho_2') - \frac{(\rho_1 - \rho_1')^2}{(\rho_1 - \rho_0)^2 + (\rho_1' - \rho_0)^2} \partial_{\rho_1}^2 f_\omega(\rho_1, \rho_1', \rho_2, \rho_2') \right] \Big\}$$

if $\rho_i = x_i + i y_i \quad \rho_i' = x_i' + i y_i'$ then $\partial_{\rho_1}^2 \to 4 \frac{\partial^2}{\partial \rho_1 \partial \rho_1^*}$

$\rho_i \to \frac{a\rho_i + b}{c\rho_i + d} \quad ad - bc = 1$ conf. transf.

homog. eq. :

$$K \otimes \varphi(\rho, \rho') = \frac{N_c \alpha_s}{2\pi^2} \left\{ \partial_\rho^2 \int \frac{d^2\rho_0}{(\rho - \rho_0)^2} \left[\partial_{\rho'}^2 \varphi(\rho_0, \rho') - \frac{(\rho-\rho')^2}{(\rho-\rho_0)^2 + (\rho'-\rho_0)^2} \partial_{\rho'}^2 \varphi(\rho, \rho') \right] \right.$$

$$\left. + \partial_{\rho'}^2 \int \frac{d^2\rho_0}{(\rho'-\rho_0)^2} \left[\partial_\rho^2 \varphi(\rho, \rho_0) - \frac{(\rho-\rho')^2}{(\rho-\rho_0)^2 + (\rho'-\rho_0)^2} \partial_\rho^2 \varphi(\rho, \rho') \right] \right\}$$

~~$\rho \neq \rho'$~~

~~eigenfn. $\varphi_{n,\nu}(\rho, \rho')$~~ in complex notation :

$$K \otimes \varphi(\rho, \rho') = \frac{4 N_c \alpha_s}{\pi^2} \left\{ \partial_\rho \partial_{\rho^*} \int \frac{d\rho_0 \, d\rho_0^*}{|\rho - \rho_0|^2} \left[\partial_{\rho'} \partial_{\rho'^*} \varphi(\rho_0, \rho') - \frac{|\rho-\rho'|^2}{|\rho-\rho_0|^2 + |\rho'-\rho_0|^2} \partial_{\rho'} \partial_{\rho'}^* \varphi(\rho, \rho') \right] \right.$$

$$\left. + \partial_{\rho'} \partial_{\rho'^*} \int \frac{d\rho_0 \, d\rho_0^*}{|\rho' - \rho_0|^2} \left[\partial_\rho \partial_{\rho^*} \varphi(\rho, \rho_0) - \frac{|\rho-\rho'|^2}{|\rho-\rho_0|^2 + |\rho'-\rho_0|^2} \partial_\rho \partial_{\rho^*} \varphi(\rho, \rho') \right] \right\}$$

eigenfn's : $\varphi_{n,\nu}(\rho, \rho', \rho_0) = \left(\frac{\rho - \rho'}{(\rho - \rho_0)(\rho' - \rho_0)} \right)^{\frac{1-n}{2} + i\nu} \left(\frac{\bar\rho - \bar\rho'}{(\bar\rho - \bar\rho_0)(\bar\rho' - \bar\rho_0)} \right)^{\frac{1+n}{2} + i\nu}$

generate irrep. of conformal group.

Casimir op's are: $(\rho - \rho')^2 \partial_\rho \partial_{\rho'}$

$(\bar\rho - \bar\rho')^2 \partial_{\bar\rho} \partial_{\bar\rho'}$

w/ eigenvalues: $(\rho - \rho')^2 \partial_\rho \partial_{\rho'} \varphi_{n,\nu} = h(h-1) \varphi_{n,\nu}$

$(\bar\rho - \bar\rho')^2 \partial_{\bar\rho} \partial_{\bar\rho'} \varphi_{n,\nu} = h'(h'-1) \varphi_{n,\nu}$

$$h = \frac{1+n}{2} - i\nu \qquad h' = \frac{1-n}{2} - i\nu$$

$$h(h-1) = \left(\frac{1+n}{2} - i\nu\right)\left(\frac{n-1}{2} - i\nu\right) = \left(\frac{n}{2} - i\nu\right)^2 - \frac{1}{4}$$

$$h'(h'-1) = \left(\frac{1-n}{2} - i\nu\right)\left(-\frac{1+n}{2} - i\nu\right) = \left(\frac{n}{2} + i\nu\right)^2 - \frac{1}{4}$$

$$\varphi_{n,\nu}(\rho, \rho', \rho_0) = \left(\frac{(\rho-\rho_0)(\rho'-\rho_0)}{\rho-\rho'}\right)^{\frac{n-1}{2} - i\nu} \left(\frac{(\bar\rho-\bar\rho_0)(\bar\rho'-\bar\rho_0)}{\bar\rho-\bar\rho'}\right)^{-\frac{n+1}{2} - i\nu}$$

$$\omega_{\nu,n} = -\psi\left(\frac{|n|+1}{2} + i\nu\right) - \psi\left(\frac{n+1}{2} - i\nu\right) - 2\gamma_E \qquad \text{as at } t=0$$

Completeness:

$$\sum_n \int d\nu \int d^2\rho_0 \, 16\left(\nu^2 + \frac{n^2}{4}\right) \frac{\varphi_{n,\nu}(\rho_1, \rho_1', \rho_0)\, \varphi^*_{n,\nu}(\rho_2, \rho_2', \rho_0)}{|\rho_1 - \rho_1'|^2 \, |\rho_2 - \rho_2'|^2}$$

$$= (2\pi)^4 \, \delta^2(\rho_1 - \rho_2)\, \delta^2(\rho_1' - \rho_2')$$

solution of $t \neq 0$ singlet BFKL eq: (Lipatov 1986)

$$f_\omega(\rho_1, \rho_2, \rho_1', \rho_2') =$$

$$= \sum_n \int d\nu \int d^2\rho_0 \frac{\left(\nu^2 + \frac{n^2}{4}\right)}{\left[\nu^2 + \left(\frac{n-1}{2}\right)^2\right]\left[\nu^2 + \left(\frac{n+1}{2}\right)^2\right]} \frac{\varphi_{n,\nu}(\rho_1, \rho_1', \rho_0)\, \varphi^*_{n,\nu}(\rho_2, \rho_2', \rho_0)}{\omega - \omega_{\nu,n}}$$

for $n = \pm 1$, take principal value

Having introduced complex coord.'s

$$\rho_k = x_k + i y_k \qquad p_k = i \frac{\partial}{\partial \rho_k}$$

in 1993,
Lipatov could re-phrase the homogeneous part of the BFKL eq. for the singlet at generic $\hat{t}$ as an eigenvalue eq. for 2 Reggeised gluons $\quad H_{12} \psi(\vec{\rho}_1, \vec{\rho}_2) = E \psi(\vec{\rho}_1, \vec{\rho}_2)$

w/ $$H_{12} = \frac{1}{p_1 p_2^*} \ln|\rho_{12}|^2 p_1 p_2^* + \frac{1}{p_1^* p_2} \ln|\rho_{12}|^2 p_1^* p_2 + \ln|p_1 p_2|^2 - 4\psi(1)$$

the Hamiltonian for 2 Reggeised gluons, and
w/ $\rho_{12} = \rho_1 - \rho_2$.

H_{12} is invariant under conformal (Möbius) transf.s $\quad \rho_k \to \dfrac{a\rho_k + b}{c\rho_k + d}$

Eigenvalues of the Casimir op.'s are $\begin{cases} m(m-1) \\ \tilde{m}(\tilde{m}-1) \end{cases}$

w/ $m = \dfrac{1+n}{2} + i\nu \qquad \tilde{m} = \dfrac{1-n}{2} + i\nu$

in terms of the anomalous dim. $\gamma = 1 + 2i\nu$ and conf. spin n of the composite op.'s $O_{n\tilde{n}}(\rho)$

More precisely, $\quad M^2 f_{m\tilde{m}} = m(m-1) f_{m\tilde{m}}$;

$$M^{*2} f_{m\tilde{m}} = \tilde{m}(\tilde{m}-1) f_{m\tilde{m}}$$

w/ the Casimir op.: $M^2 = 2 M_1^a M_2^a = -\rho_{12}^2 \partial_1 \partial_2$

where we used the conformal (Möbius) group generators:

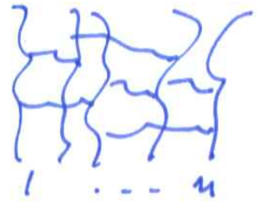
$$M_k^z = \rho_k \partial_k \qquad M_k^- = \partial_k \qquad M_k^+ = -\rho_k^2 \partial_k$$

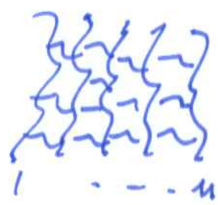
Then the eigenvalue of the BFKL eq. $H_{12}\psi = E\psi$ is

with $E_{m\tilde{m}} = \psi(m) + \psi(1-m) + \psi(\tilde{m}) + \psi(1-\tilde{m}) - 4\psi(1)$

$$\omega_{m\tilde{m}} = -\frac{\alpha_s N_c}{2\pi} E_{m\tilde{m}}$$

Then Lipatov considered the exchange of n Reggeized gluons in the $\hat{t}$ channel. In the large-N_c limit, the interaction among the n Reggeized gluons is reduced to pair-like interactions among nearest neighbours, and the Hamiltonian has holomorphic separability (the same happens in the linearised BK equation at large N_c - see SCH's lectures).





Then the Hamiltonian is written as

$$H^{(0)} = \sum_{k=1}^{n} H_{k,k+1} = h^{(0)} + \bar{h}^{(0)} \qquad [h^{(0)}, \bar{h}^{(0)}] = 0$$

w/ $h^{(0)} = \sum_{k=1}^{n} h^{(0)}_{k,k+1}$ w/ $h^{(0)}_{k,k+1} = \frac{1}{p_k} \ln(\rho_{k,k+1})\, p_k + \frac{1}{p_{k+1}} \ln(\rho_{k,k+1})\, p_{k+1} + \ln p_k p_{k+1} - 2\psi(1)$

the index (0) reminds us we are dealing w/ singlet exchange

The eigenvalue eq. is $H^{(0)}\psi = E\psi$

where the wave fn. has holomorphic factorisation

$$\Psi_{m,\bar{m}}(\vec{\rho}_1,\ldots,\vec{\rho}_n) = \sum_{r\ell} c_{r,\ell}\, f_m^r(\rho_1,\ldots,\rho_n)\, f_{\bar{m}}^\ell(\bar{\rho}_1,\ldots,\bar{\rho}_n)$$

where r, ℓ enumerate degenerate solutions in the subspaces:

$$h^{(0)} f_m = \epsilon_m f_m \qquad \bar{h}^{(0)} f_{\bar{m}} = \epsilon_{\bar{m}} f_{\bar{m}} \qquad \text{w/} \quad E_{m,\bar{m}} = \epsilon_m + \epsilon_{\bar{m}}$$

As in 2-dim. CFTs, the coefficients $c_{r,\ell}$ are fixed by the single-valuedness condition for $\Psi_{m,\bar{m}}$.

The Hamiltonian $h^{(0)}$ can ~~also~~ be written as:

$$h^{(0)}_{k,k+1} = \rho_{k,k+1} \ln(p_k p_{k+1})\, \rho^{-1}_{k,k+1} + \ln \rho^2_{k,k+1} - 2\psi(1)$$

In fact, $h^{(0)}$ is invariant under the duality transformation

$p_i \to \rho^{-1}_{i,i+1}$ combined w/ the transposition, and in particular,

$$h^{(0)T} = \prod_{j=1}^{N} p_j\, h^{(0)} \prod_{j=1}^{N} p_j^{-1}$$

$$= \prod_{j=1}^{N} \rho^{-1}_{j,j+1}\, h^{(0)} \prod_{j=1}^{N} \rho_{j,j+1}$$

Thus $h^{(0)}$ commutes w/ the operator A, $[h^{(0)}, A] = 0$

w/ $A = \rho_{12}\rho_{23}\cdots\rho_{N1}\, p_1 \cdots p_N$

~~In fact~~ we may, introducing the Lax operator for the n^{th} site [Reggeised gluon]

$$L_{n,a}(u) = u\, 1_n \otimes 1_a + i \sum_{j=1}^{3} M_n^j \otimes \sigma^j$$

which in the auxiliary space may be viewed as a 2×2 matrix

$$L_{n,a}(u) = \begin{pmatrix} u + i\rho_n \partial_n & i\partial_n \\ -i\rho_n^2 \partial_n & u - i\rho_n \partial_n \end{pmatrix}$$

(the Möbius group generators coincide w/ the rotation group op's)

and the monodromy matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = L_1(u)\, L_2(u) \cdots L_N(u)$$

and the transfer matrix

$$\tau(u) = \mathrm{tr}_a\, T(u) = A(u) + D(u)$$

$$= 2u^N + S(S+1)\, u^{N-2} + Q_3\, u^{N-3} + \cdots + Q_N$$

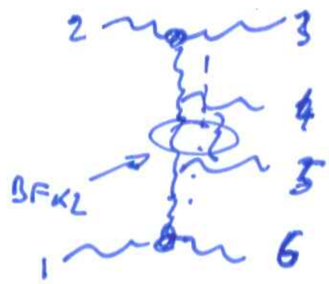
w/ $N-2$ commuting op's $Q_k = \sum_{N \geq i_1 > i_2 > \cdots > i_k \geq 1} i^k \rho_{i_1 i_2} \rho_{i_2 i_3} \cdots \rho_{i_k i_1} \partial_{i_1} \cdots \partial_{i_k}$

where in particular $Q_N = A$

The Hamiltonian is given by $H = \frac{1}{i} \frac{d}{du} \ln \tau(u) \Big|_{u = \frac{i}{2}}$

Faddeev-Korchemsky (1995) showed that H commutes w/ all Q_k, and that the Hamiltonian coincides w/ the one of a XXX Heisenberg chain

Also in planar $N=4$ SYM in the high-energy limit, the amplitudes feature the exchange of 2 Reggeised gluons, in the octet channel.

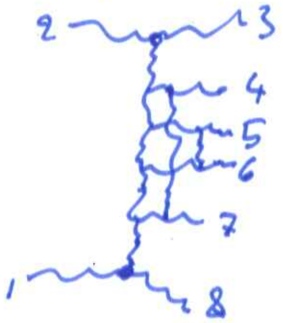


Take the 6-pt. amplitude in the region

$s>0\ ,\ s_{45}>0 \qquad s_{56}, s_{34}<0$

The imaginary part of the amplitude, given by the discontinuity of A, is computed by evaluating the (Mandelstam) cut through s_{45}, characterised by 2-Reggeised gluon exchange. (see Claude's lectures)

Likewise, it's easy to imagine that the 8-pt amplitude,



in the region $\quad s>0 \quad s_{34}, s_{78}<0$

$s_{4567}>0 \quad s_{45}, s_{67}>0$

$s_{56}<0$

has a double discontinuity with a cut characterised by 3-Reggeised gluon exchange,

and that in general a $(2+2N)$-pt amplitude has a RS region which has a $(N-1)^{\text{th}}$ discontinuity due to a cut characterised by N-Reggeised gluon exchange.

Thus, it is worth considering the ~~octet~~ octet exchange of N Reggeised gluons in the t channel in $N=4$ SYM.