High-Energy scattering and Wilson lines [Lecture 1]

Main ~~formula~~ :

$$\frac{\partial}{\partial \eta} U(x,y) = \frac{g^2 N_c}{8\pi^3} \int \frac{d^2 z\,(x-y)^2}{(x-z)_\perp^2 (z-y)_\perp^2} \left[U(x,z)U(z,y) - U(x,y) \right]$$

(Balitsky-Kovchegov eq.)

η: rapidity $\simeq \log(\text{Energy})$

$U(x,y)$: "dipole" Wilson loop exp. value

z: 2D transverse coordinates

Plan: - Why WL's? Why evolution eq.?

- Approximations:
  - eikonal
  - 1-loop ("$\frac{g^2 N_c}{8\pi^2}$")
  - Large-N_c

  } Relaxed on Friday

- Applications:
  - DIS
  - Parton scattering

  } Wednesday

- Systematic, gauge-invariant formulation of BFKL (WL $\leftrightarrow$ Reggeized gluon)

$\Rightarrow$ Evolution eq. resums large high-energy logs in Regge limit.

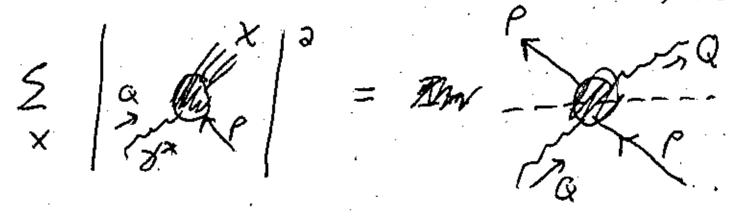
Related references:

- Iancu + Venugopalan ph/0303204
- SCH 1309.6521

Let's consider a specific process, $\gamma^* p \to \gamma^* p$, at high-energies.

This is related to DIS at ep machine:

$$\sigma[\gamma^* p \to X] = \mathrm{Im} \underbrace{\langle p | T J^\mu(Q) J^\nu(-Q) | p \rangle}_{A(p,Q)} \quad \text{proton}$$

pictorially, 

optical theorem

It will be useful to think about the full amplitude A, not only $\mathrm{Im}\, A$.

Parametrize momenta, in a frame where:

$$p^\mu = (0, p^-, 0) \qquad (\text{ignore } m_p \to 0)$$

$$Q^\mu = (Q^+, 0, Q_\perp)$$

$\left(\text{L.C. coords: } K^+ = \frac{K^0 + K^z}{2}, \quad K^- = K^0 - K^z, \quad K^2 = 2K^+K^- - K_\perp^2 \right)$

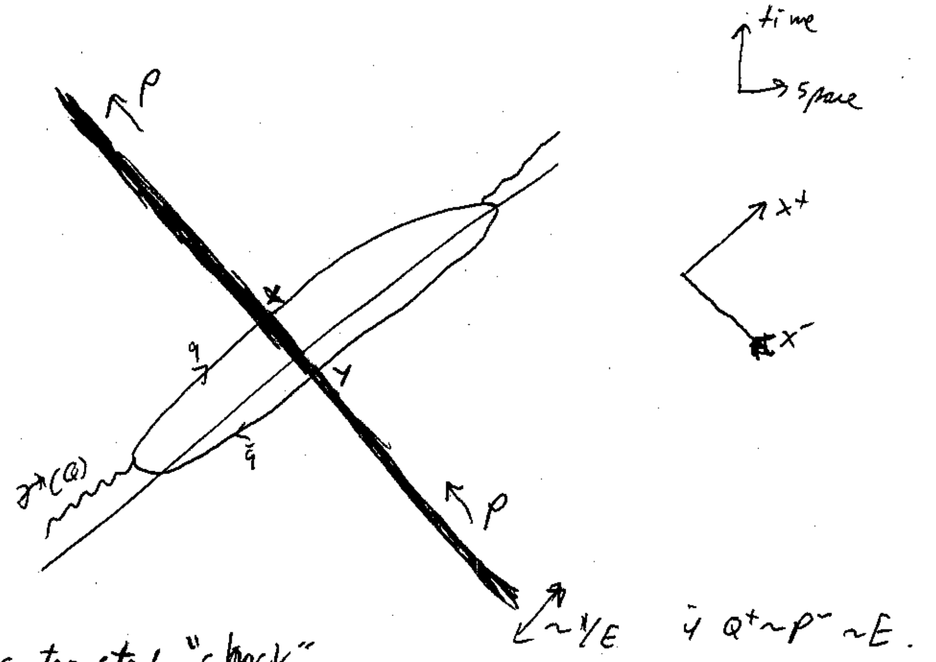
$\Rightarrow$ Bjorken variable $x_B = \dfrac{-Q^2}{2Q\cdot p} = \dfrac{Q_\perp^2}{2Q^+p^-}$.

$\Rightarrow$ The limit $x_B \ll 1$ with $Q^2 = \text{fixed}$ represents <u>large relative boost</u> $Q^+p^- \to \infty$

$=$ <u>Regge limit</u>.

(For $2\to 2$ scattering, the "Regge limit" is often defined as $s \gg t$.
Here $s = 2Q^+p^-$
and $t = 0$ is fixed.)

For large $Q^2 \gg \Lambda^2_{QCD}$, the γ^*'s wavefunction will be <u>perturbatively calculable</u> (which is why we're starting with this process!)



Zeroth order:

$\gamma^* \to q\bar{q} \to \gamma^*$

Proton is Lorentz-contracted "shock"

$\Rightarrow$ No deflection of the quarks during the (short) crossing time. All they get is a <u>phase</u>.

~~Amplitude = just a phase~~

Amplitude factorizes:

$A(P,Q) \approx$ ~~[illegible]~~

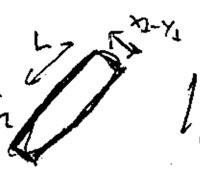
$_{x^+=0}\langle q\bar{q}|J|0\rangle$ — vacuum propagation of $(q,\bar{q})$ to $(x_\perp, y_\perp)$

$\cdot\ U(x_\perp, y_\perp)$ — phase for crossing the proton

$\cdot\ \langle 0|J|q\bar{q}\rangle_{x^+=0}$ — vacuum propagation from $(x_\perp, y_\perp)$ to second current.

For a charged particle, the "phase" is the exp. value of a Wilson line.
For quarks, it's really a matrix: $U(x_\perp) \sim P e^{+i\int_{-L}^{L} dx^+ A_+^a T^a}$.

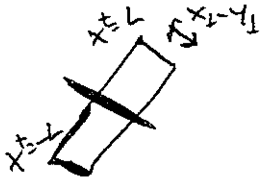
Multiplying the "phases" for the q and $\bar{q}$, we get the trace of a Wilson loop:



$U(x_\perp, y_\perp) = \frac{1}{N_c} \langle p|\ \mathrm{Tr}\ [\text{loop}]\ |p\rangle$

We've normalized it s.t. $U=1$ in vacuum.

Note that we've replaced the "roundish" contour by a rectangle



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Physically, this is because we expect the important interactions to occur near $x^+=0$, where the Lorentz-contracted proton sits.

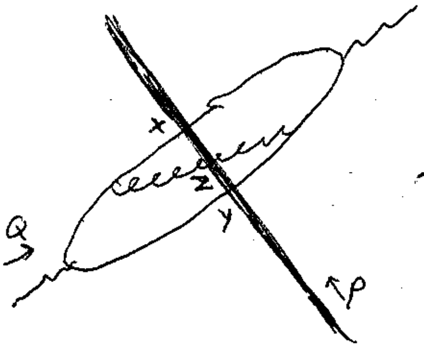
In fact, it is useful to imagine that $L \to \infty$, <u>however</u> we can't directly set $L=\infty$! This is because rapidity divergences appear, and some scale has to be retained to regulate them.

Why there is an evolution eq.?

The basic reason why the amplitude will depend on energy, is that <u>any charged particle comes with a cloud of soft radiation.</u>

To see this more explicitly, consider the next-order correction:

$\gamma^* \to q\bar{q}g \to \gamma^*$



$$\sim {}_{x^+=0}\langle q\bar{q}g|J|0\rangle \cdot U(x_\perp, y_\perp, z_\perp) \cdot \langle 0|J|q\bar{q}g\rangle_{x^+=0}$$

The phase now cares also about where the <u>gluon</u> crossed the shock.

Over a wide range of energies, the phase does not depend on the gluon energies, so we have to sum over them $\Rightarrow$ <u>large logs from soft gluons!</u>

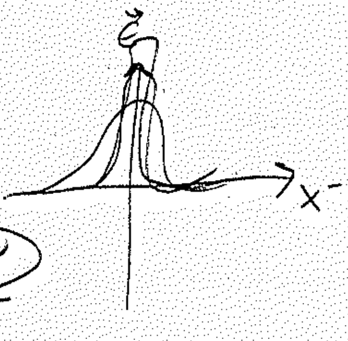
Weizsacker - William equivalent photon appror.

Soft gluons can be understood classically. A good place to start is the electric field around a static electron at rest,

$$\vec{E}_\perp(x_\perp, z) = \frac{e}{4\pi} \frac{\vec{x}_\perp}{(\vec{x}_\perp^2 + z^2)^{3/2}} .$$

Boosting this gives the field around a fast electron ~~[illegible]~~

$$\vec{E}_\perp(x_\perp, x^-) \approx \frac{e}{4\pi} \frac{\vec{x}_\perp \, \gamma}{(x_\perp^2 + (\gamma x^-)^2)^{3/2}}$$ where γ is the boost factor.

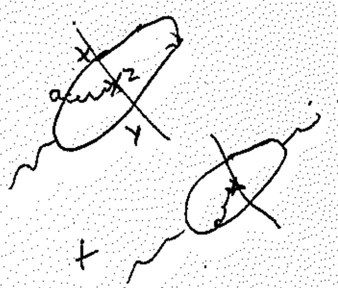


At $\gamma \to \infty$ this converges to a δ-fct, with area:

$$\int_{-\infty}^{\infty} dx^- \, \vec{E}_\perp(x_\perp, x^-) = \frac{e}{2\pi} \frac{\vec{x}_\perp}{x_\perp^2} .$$

$$\Rightarrow \vec{E}_\perp \approx \frac{e}{2\pi} \frac{\vec{x}_\perp}{x_\perp^2} \quad \text{WW}$$

For a $q\bar{q}$ dipole, the gluon field is a simple generalization:

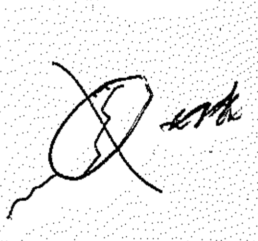


$$\vec{E}_\perp = \frac{g}{2\pi} T^a \left(\frac{(\vec{x}-\vec{z})_\perp}{(x-z)_\perp^2} - \frac{(\vec{y}-\vec{z})_\perp}{(y-z)_\perp^2} \right) \cdot \delta(x^-) \qquad (*)$$

When we multiply by the amplitude after the shock, we get a divergent energy

$$\vec{E}_\perp^2 \sim \delta(x^-)^2 \propto \int dk^+$$

The amplitude itself is only log-divergent,



$$A \sim g^2 \int \frac{dk^+}{k^+} \int d^2 z \, U(x_\perp, y_\perp, z_\perp) \left(\qquad \right)^2 .$$

The $\perp$-dependence is simple: $\left(\frac{\vec{a}}{a_\perp^2} - \frac{\vec{b}}{b_\perp^2} \right)^2 = \frac{(\vec{a}-\vec{b})_\perp^2}{a_\perp^2 \, b_\perp^2}$ for $\vec{a} = \vec{x} - \vec{z}$, $\vec{b} = \vec{y} - \vec{z}$.

Thus the factor $\frac{(x-y)^2}{(x-z)^2 (z-y)^2}$ in BK arises ~~[illegible]~~ from squaring $(*)$

What are the Cut-off?

$$\int_{k_{min}}^{k^+_{max}} \frac{dk^+}{k^+}$$

k^+_{max} ⟵ Max. energy of gluon $\sim Q^+$

k_{min} ⟵ gluons that are too soft don't actual do any scattering against target $\quad k_{min} \sim \frac{Q^2}{P^-} \sim$ "thickness of proton" in x^+ direction.

⇒ If we boost the virtual photon, the log range ~~...~~ increases ⇒ evolution eq.

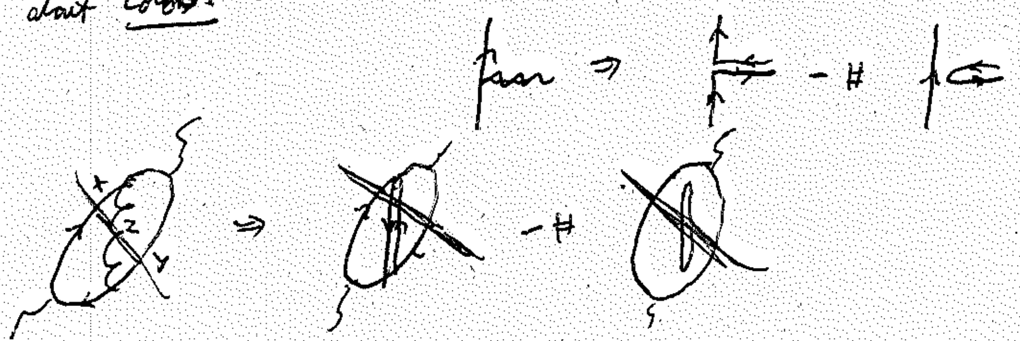
⇒ Meaning of BK: $\left[\text{Dipole with regulator } k^+_{max}(1+\delta n)\right] = \left[\text{Dipole with reg. } k^+_{max}\right] + \delta n \times$ extra gluon

(Note: possible rapidity regulators:
- Cutoff on k^+_{max}
- Finite length L^+ (⇒ $k^+_{max} \lesssim L^+ Q^2$, see supplement)
- Non-null WLs

⇒ all lead to the same evolution eq)

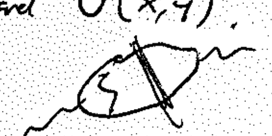
Finally, about colors: use double-line notation: adjoint charge $\simeq q \otimes \bar{q}$ (-trace .

f^{abc} ⇒



First graph has phase $U(x,y,z) = U(x,z)\,U(z,y)$, second $U(x,y)$.

The color structure $U(x,y)$ also arises from "virtual" corrections



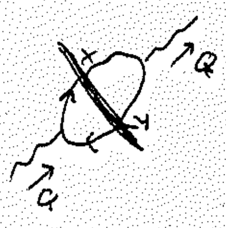
The coefficient is fixed because in vac. $\langle 0| U(x,y) |0\rangle = 1$ should be a fixed point. ⇒ This explains why BK has the combination $\left[U(x,z)U(z,y) - U(x,y)\right]$,

(Why $\langle U \rangle = 1$ is natural normalization?
For a time-like rectangle, $\langle U \rangle = e^{-i\tau V_{q\bar{q}}(x_\perp)}$ measures $q\bar{q}$ pot. (e.g. lattice).
For a null rectangle, proper time $\tau \to 0$ ⇒ U doesn't "measure" anything!)

Exercise.

Today's problem is a single problem related to the $\gamma^* \to q\bar{q} \to \gamma^*$ amplitude discussed in the lecture. The problem is easy to state:

Q.: What is the coefficient of $U(x_\perp, y_\perp)$ in the product of two currents (at zeroth order); e.g. how are x, y distributed?



$$\langle p | J^\mu(Q)\, J^\nu(-Q) | p \rangle = \int d^2x\, d^2y\, C^{\mu\nu}(x,y)\, U(x,y)$$

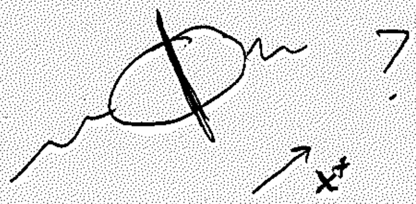
$$\Rightarrow \underline{\text{What's } C^{\mu\nu}(x,y)?} \qquad \left(\text{use } Q^\mu = (Q^+, 0, q_\perp)\right).$$

To simplify, you can consider a (fundamental) scalar loop instead of a fermion, and replace J^μ by $\phi^+\phi \equiv \mathcal{O}$.

Although easy to state, to solve this problem you'll probably need to use lightcone quantization, which I'll now describe. The goal of this problem is to help you familiarise with this tool. (See supplement)

Supplement: ~~lightfront~~ lightfront quantization

How does one calculate the "vacuum propagation" amplitudes in the diagram



?

The best tool is to work in a mixed rep. where x^+ is "time", p^+ = energy, and $p_\perp$ = transverse mom:

$$\phi = \phi(x^+, p^+, p_\perp).$$

It may seem surprising to speak of "time" and "energy" at the same time, but on the lightcone these actually commute.

Fourier phase: $$e^{ip\cdot x} = e^{i(p^+x^- + p^-x^+ - p_\perp \cdot x_\perp)}$$

$\Rightarrow$ "energy" p^+ is conjugate to x^-

"time" x^+ is conjugate to p^-.

On the lightfront, Klein-Gordon eq. looks like Schrödinger:

$$(p^2 - m^2)\phi = 0 \Rightarrow \left(2ip^+\frac{\partial}{\partial x^+} - p_\perp^2 - m^2\right)\phi(x^+, p^+, p_\perp) = 0$$

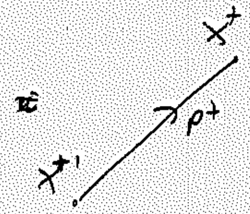
Divide by $2p^+$ $\Rightarrow$ $$\boxed{i\frac{\partial}{\partial x^+}\phi = \frac{p_\perp^2 + m^2}{2p^+}\phi}.$$

The propagator ~~also~~ looks Schrodinger-like: (say $p^+ > 0$).

$$\langle 0|T\,\phi(x^+, p^+, p_\perp)\,\phi(x^{+\prime}, p^{+\prime}, p_\perp')|0\rangle = (2\pi)^{3-2\varepsilon}\delta^{2-2\varepsilon}(p_\perp + p_\perp')\,\delta(p^+ + p^{+\prime}) \cdot \int_{-\infty}^{\infty}\frac{dp^-}{2\pi}\frac{i}{2p^+p^- - p_\perp^2 + i0 - m^2}e^{-ip^-(x^+ - x^{+\prime})}$$

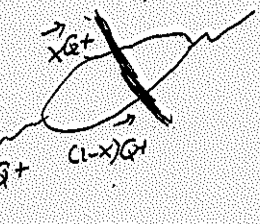
close contour above/below:

$$= \begin{cases} \dfrac{1}{2p^+}e^{-i(x^+ - x^{+\prime})(p_\perp^2 + m^2)/2p^+}, & x^+ > x^{+\prime} \\ 0, & x^+ < x^{+\prime} \end{cases}$$



$\Rightarrow$ modes with $p^+ > 0$ propagate strictly forward in time!

($\Rightarrow$ in the loop



, the energy fraction will obey $0 < x < 1$, so that both $q, \bar{q}$ move forward in time!)

When we turn on a gauge-field, KG becomes

$$\Big[2(i\partial_+ + gA_+)(p^+ + eA^+) - (p_\perp + eA_\perp)^2 - m^2\Big]\phi = 0$$

Dividing by p^+ is not so nice, <u>unless</u> we use gauge where $A^+ = 0$.

Then $$i\partial_+\phi = \left[gA_+ + \frac{(p_\perp + eA_\perp)^2 + m^2}{2p^+}\right]\phi \quad = \text{Schrödinger-like again.}$$

Note: "good" $(A^+ = A_- = 0)$ versus "bad" $(A_+ = 0)$ light-cone gauge.

$\Rightarrow$ what we set to 0 is $A_- = 0$ is the field dual to x^- motions, whereas our particle moves along x^+ and couples most strongly to A_+ !!

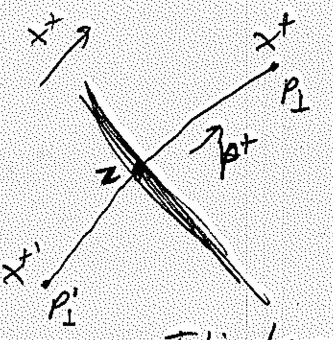
In that gauge, $p^+ = p^+ + gA^+$ = the physical, kinetic momentum of the particle.

(The "wrong" gauge $A_+ = 0$, for this problem, is like using $A_0 = 0$ gauge to solve the H-atom. It works, but why do that?)

As we cross the shock, only the gA_+ term matters as the kinetic term takes time to act.

$$\Rightarrow \phi^{\text{just after}} = \phi^{\text{just before}} \cdot P e^{i\int_{\text{before}}^{\text{after}} dx^+ A_+^a T^a}$$

Combining this with vacuum evolution on both sides of the shock, we get the propagator:



$$\boxed{\begin{aligned}&\langle \phi(x^+, p^+, p_\perp)\, \phi(x^{+\prime}, p^{+\prime}, p_\perp') \rangle_{\text{shock}} \\ &= \frac{1}{2p^+}\int d^{2-2\epsilon}z\, U(z)\, e^{i(p_\perp - p_\perp')\cdot z} \cdot e^{-i\left(\frac{p_\perp^2+m^2}{2p^+}x^+ + \frac{p_\perp'^2+m^2}{2p^+}(-x^{+\prime})\right)} \\ &\quad \cdot 2\pi\delta(p^+ + p^{+\prime})\end{aligned}}$$

This formula is useful to compute impact factors as in the exercise, or loop corrections to them.