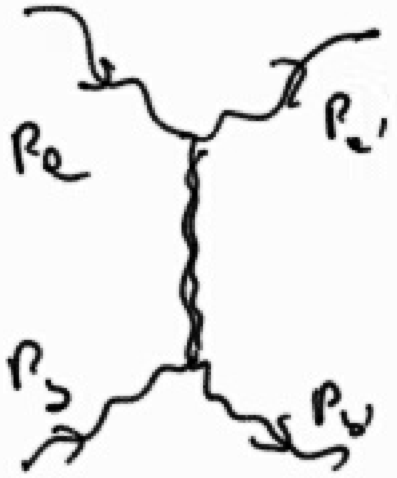
How do we go beyond LL ?

Let us consider firstly the virtual radiative corrections

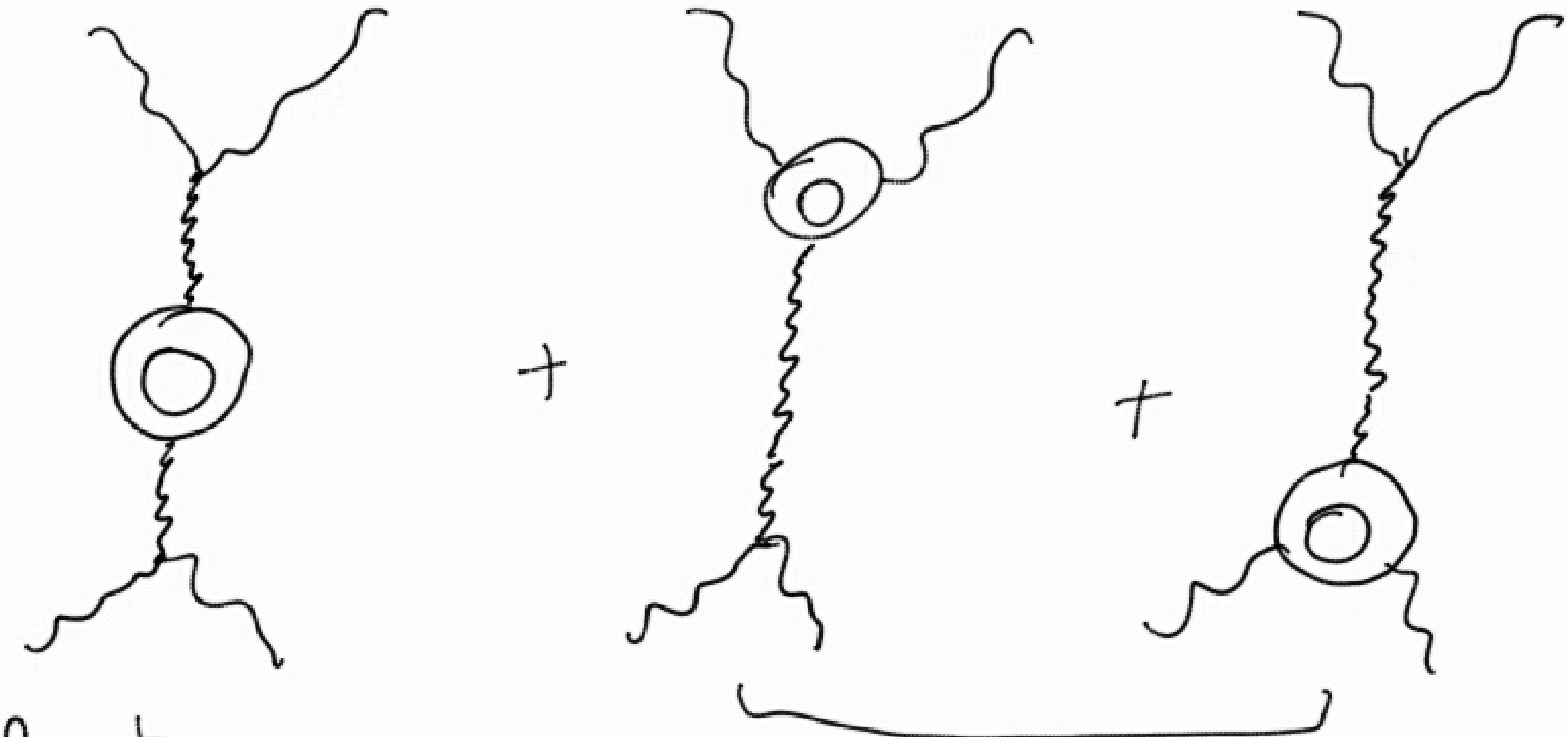


$$M^{a\,b\,a'\,b'}_{\lambda_a\lambda_b\lambda_{a'}\lambda_{b'}} = 2\hat{s}\left(i g f^{a a' c} C_{gg}^{\lambda_a\lambda_{a'}}(p_a, p_{a'})\right)$$
$$\frac{1}{\hat{t}}\left(\frac{s}{-t}\right)^{\alpha(t)}\left(i g f^{b b' c} C_{gg}^{\lambda_b\lambda_{b'}}(p_b, p_{b'})\right)$$

where $\alpha(\hat{t}) = \alpha_s N_c \hat{t} \int \frac{d^2k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 (q-k)_\perp^2} = 2g^2 N_c c_\Gamma \frac{1}{\epsilon}\left(\frac{\mu^2}{-t}\right)^\epsilon$

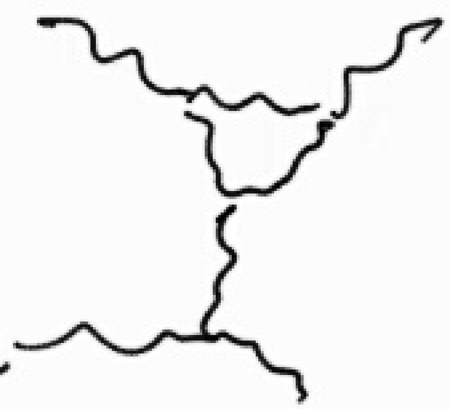
and at fixed helicities $C_{gg}^{-+}(p_a, p_{a'}) = 1$ $\quad C_{gg}^{-+}(p_b, p_{b'}) = \frac{p^*_{b'_\perp}}{p_{b'_\perp}}$

If we go beyond LL, there will be corrections that go with the ladder and corrections that go with the



log term

non-log terms

helicity-conserving vertices, like 

We need a prescription to disentangle them.

Firstly, let us make the amplitude explicitly $s \leftrightarrow u$ invariant beyond LL.

We know that for $\underline{8}_a$, $P^{ee'}_{b'b}(8_a) = - P^{ee'}_{bb'}(8_a)$

so also $A^8(s,t)$ must be antisymmetric under $\hat{s} \leftrightarrow \hat{u}$. *

So the u-channel contribution will be like

$$\hat{s}\left(ig f^{ae'c}\, C^{d_e d_{e'}}_{\delta\delta}\right)\frac{1}{t}\left[-\left(\frac{u}{-t}\right)^{\alpha(t)}\right]\left(ig f^{b'bc}\, C^{d_b d_{b'}}_{ff}\right)$$

* In principle there is also $\underline{10} + \underline{\overline{10}}$, but it does not contribute at NLL

so the whole amplitude may be written

$$M^{aa'bb'}_{\lambda_a\lambda_{a'}\lambda_b\lambda_{b'}} = s\left(ig f^{aa'c} C^{\lambda_a\lambda_{a'}}_{gg}\right) \frac{1}{t}\left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left(ig f^{bb'c} C^{\lambda_b\lambda_{b'}}\right)$$

where $u = -s-t \simeq -s$

Then, there may corrections to the vertex $C^{\lambda\lambda}_{gg}$ and to the trajectory,

$$C_{gg} = C^{(0)}_{gg}\left(1 + \tilde{g}_s^2 C^{(1)}_{gg} + \tilde{g}_s^4 C^{(2)}_{gg} + O(g_s^6)\right)$$

$$\alpha(t) = \tilde{g}_s^2 \alpha^{(1)}(t) + \tilde{g}_s^4 \alpha^{(2)}(t) + O(g_s^6)$$

$$\tilde{g}_s^2 = g_s^2 c_\Gamma \left(\frac{\mu^2}{-t}\right)^\epsilon$$

these can be either computed, or extracted from exact amplit

For example, we already said that the real part of the one-loop amplitude (unrenormalised) for $gg \to gg$ is

$$\mathrm{Re}\, M_4^{(1)}(p_a^-, p_{a'}^+, p_b^-, p_{b'}^+) = M_4^{(0)}(p_a^-, p_{a'}^+, p_b^-, p_{b'}^+)\, g^2 c_\Gamma$$

$$\left\{ \left(\frac{\mu^2}{-t}\right)^\epsilon \left[N_c \left(-\frac{4}{\epsilon^2} - \frac{11}{3\epsilon} + \frac{2}{\epsilon} \ln \frac{\hat{s}}{-t} - \frac{67}{9} + \pi^2 \right] + n_f \left(\frac{2}{3\epsilon} + \frac{10}{9} \right) - \frac{\beta_0}{\epsilon} \right\}$$

we know that the log term belongs to the Regge trajectory. Then we may assign the non-log terms to the one-loop correction to the C_{gg} vertex,

$$C_{gg}^{(1)} = \left[N_c \left(-\frac{2}{\epsilon^2} - \frac{11}{6\epsilon} - \frac{67}{18} + \frac{\pi^2}{2} \right) + n_f \left(\frac{1}{3\epsilon} + \frac{5}{9} \right) - \frac{\beta_0}{2\epsilon} \right]$$

$$\beta_0 = \frac{11 C_A - 2 n_f}{6}$$

(Schmidt, VDD 1998)

One more thing that can be checked is Regge factorisation:

For the $gg \to gg$ amplitude, as we saw,

$$M_{gg\to gg} = s\left(i g f^{aa'c}\, C_{gg}^{\lambda_a\lambda_{a'}}(p_a,p_{a'})\right)\frac{1}{t}\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left(i g f^{bb'c}\, C_{gg}^{\lambda_b\lambda_{b'}}\right)$$

For the $qq \to qq$ amplitude, one gets

$$M_{qq\to qq} = s\left(g\, T^{c}_{a\bar a}\, C_{\bar q q}^{\lambda_a\lambda_{a'}}(p_a,p_{a'})\right)\frac{1}{t}\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left(g\, T^{c}_{b\bar b}\, C_{\bar q q}^{\lambda_b\lambda_{b'}}\right)$$

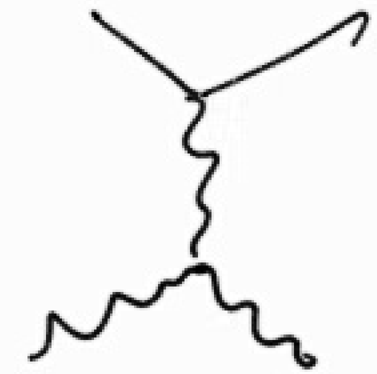
where we said that $C_{\bar q q}^{\lambda_a\lambda_{a'}(0)}(p_a,p_{a'}) = -i \qquad C_{\bar q q}^{\lambda_b\lambda_{b'}(0)}(p_b,p_{b'}) = i\sqrt{\frac{p_{b'_\perp}^{*}}{p_{b'_\perp}}}$

one gets that (unrenormalised)

$$C_{\bar q q}^{(1)} = N_c\left(-\frac{1}{\epsilon^2}+\frac{1}{3\epsilon}+\frac{13}{18}+\frac{\pi^2}{2}\right)+\frac{1}{N_c}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+4\right)+n_f\left(\frac{1}{3\epsilon}+\frac{5}{9}\right)-\frac{\beta_0}{2\epsilon}$$

(Schmidt VDD 1998)

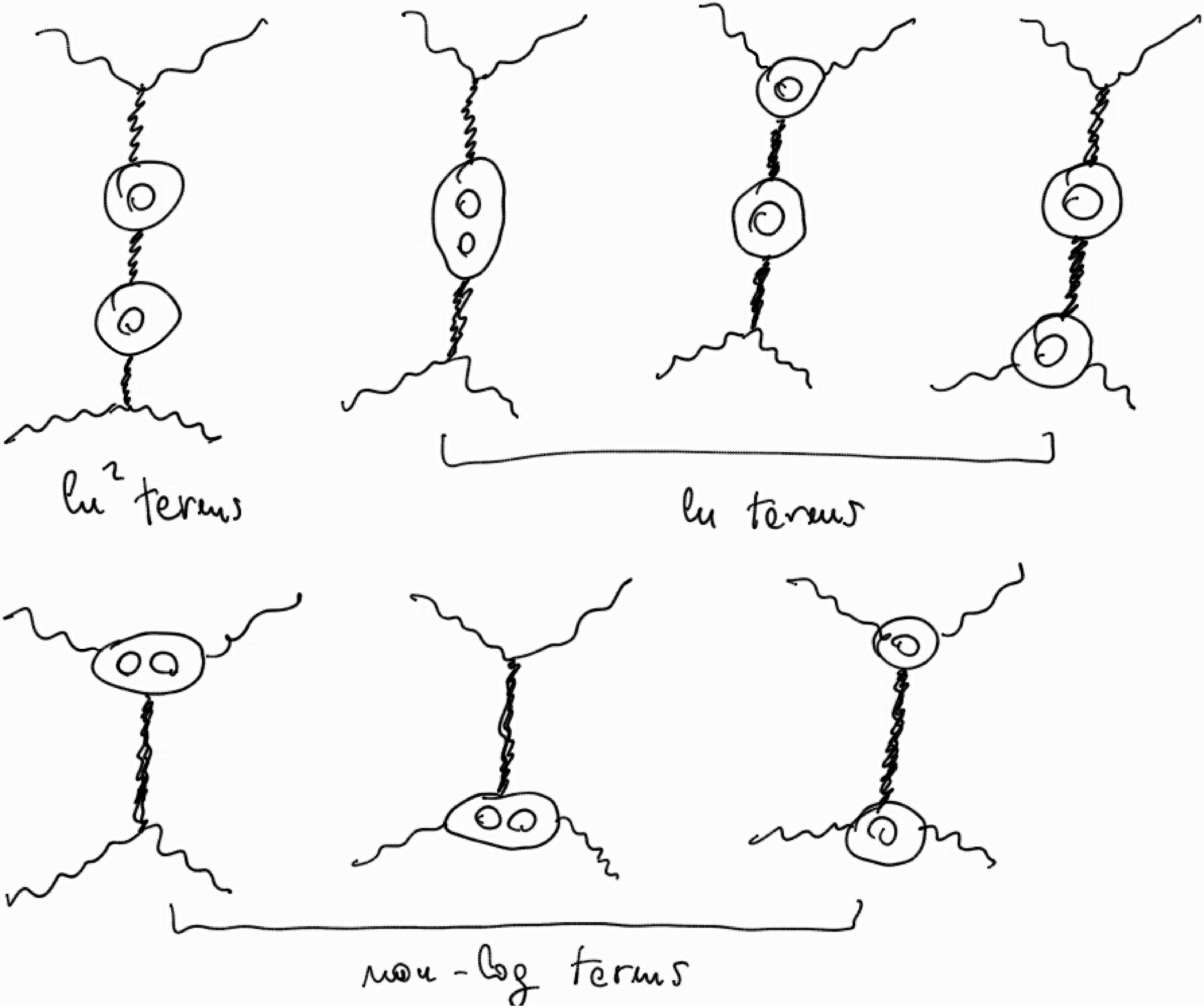
then one can check that the $qg \to qg$ amplitude can indeed be written as



$$M_{qg\to qg} = \hat{s}\left(g\, T^{c}_{a\bar a}\, C^{\lambda_a \lambda_{a'}}_{\bar q q}(p_a, p_{a'})\right)\frac{1}{t}\left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left(i g f^{b b' c}\, C^{\lambda_b \lambda_{b'}}_{gg}\right)$$

and this holds true including the non-log terms.

This is because at NLO only the octet is exchanged in the real part of the amplitude


ln² terms
ln terms
non-log terms

To obtain the NLL corrections to the trajectory, one needs to consider the two-loop amplitude. One finds that the (unrenormalised) trajectory is

$$\alpha^{(2)}(t) = C_A \beta_0 \frac{2}{\epsilon^2} + C_A \hat{\gamma}_K^{(2)} \frac{2}{\epsilon} + C_A^2 \left(\frac{404}{27} - 2\zeta_3 \right) - \frac{56}{27} C_A n_f$$

with $\hat{\gamma}_K^{(2)} = \left(\frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} n_f$ the 2-loop cusp anomalous dimension

(Fadin, Fiore, Kotsky 1995)

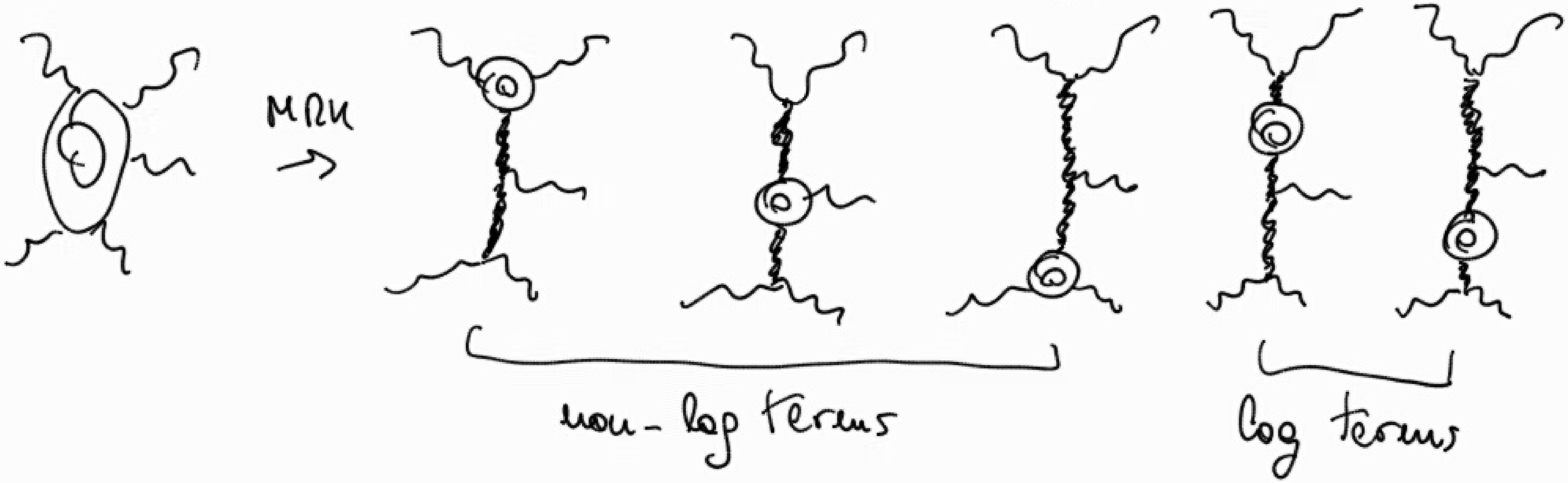
and this trajectory is indeed the same

for $gg \to gg$, $qq \to qq$ and $qg \to qg$ amplitudes, which means that Regge factorisation holds at NLL level, which is a consequence of the fact that only 8_A is exchanged in the real part of the amplitude at NLL.

However, Regge factorisation fails when we consider the non-log terms, i.e. at NNLO level, (Glover, VDD 2001) by terms of $\mathcal{O}\left(\frac{\pi^2}{\epsilon^2}\right)$. This is because the amplitude becomes non-diagonal in the t-channel basis at NNLO

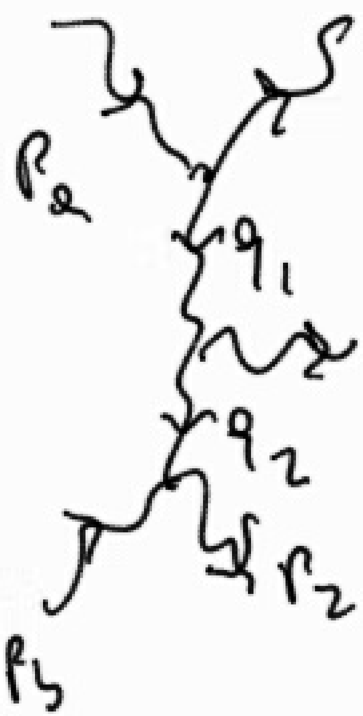
(Duhr, Gardi, Magnea, White, VDD 2011)

Going beyond 4-points, we can consider the virtual corrections to the central-emission vertex, in MRK



To compute these, we need the one-loop 5-pt amplitude in MRK

$$
\begin{aligned}
M^{a a' d b b'}_{\lambda_a \lambda_{a'} \lambda \lambda_b \lambda_{b'}} &= 2s \left(i g f^{a a' c} C^{\lambda_a \lambda_{a'}}_{gg} \right) \frac{1}{t_1} \left(\frac{s_1}{-t_1} \right)^{\alpha(t_1)} \left(i g f^{c d c'} C^{\lambda}_{g} \right) \\
&\quad \cdot \frac{1}{t_2} \left(\frac{s_2}{-t_2} \right)^{\alpha(t_2)} \left(i g f^{b b' c'} C^{\lambda_b \lambda_{b'}}_{gg} \right)
\end{aligned}
$$



where $s_1 \simeq p_0^+ p_1^-$ $\qquad s_2 \simeq p_1^+ p_2^-$

$t_1 \simeq -|p_{0_\perp}|^2$ $\qquad t_2 \simeq -|p_{2_\perp}|^2$

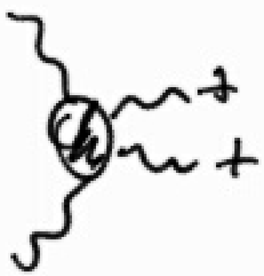
$$C_g^{+(0)}(q_1, q_2) = \sqrt{2}\, \frac{q_{1_\perp}^* q_{2_\perp}}{p_{1_\perp}} \qquad p_1 = q_1 - q_2$$

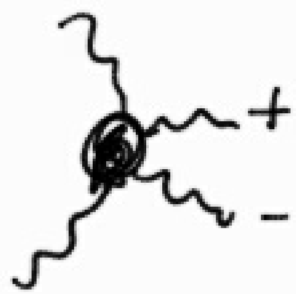
and $$C_g^{\lambda}(q_i, q_{i+1}) = C_g^{\lambda(0)}\left[1 + g_s^2 C_g^{\lambda(1)} + O(g_s^4)\right]$$

One needs then the real corrections to the central-emission vertex. Because they must generate the NLL corrections to the kernel of BFKL equation (together with the virtual corrections just considered), one must obtain one large rapidity less out of the

integration. So one must consider the amplitude in quasi-multi-Regge kinematics (QMRK), i.e. to relax the strong rapidity ordering and require, say for the 6-pt. amplitude, that

$$y_0 \gg y_1 \simeq y_2 \gg y_3$$

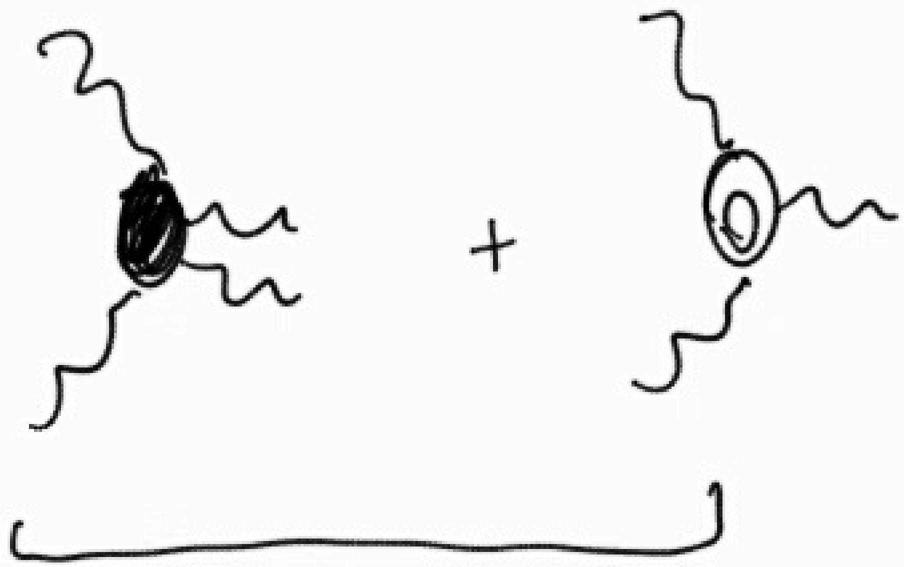
Then one needs to compute the 2-gluon central-emission vertex 



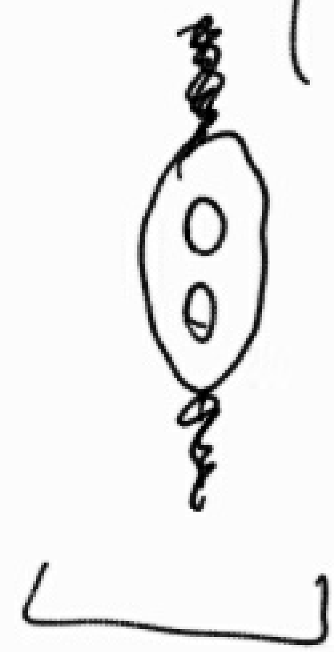
for the two independent helicity configurations.

At this level, one sees even more how powerful the helicity amplitude formulation is, as compared to the Lorentz covariant formulation.

Then the building blocks of the BFKL equation at NLL level are (Fadin Lipatov 1998)



in the kernel



in the homogeneous part of the eq.