We consider unitarity and the optical theorem to relate the forward amplitude with the exchange of a two-gluon ladder to the total cross section for gluon-gluon scattering wit exchange of a one-gluon ladder. Now,

$$SS^{\dagger} = 1 = (1 + iM)(1 - iM^{\dagger}) = 1 + i(M - M^{\dagger}) + MM^{\dagger}$$

so $\quad MM^{\dagger} = -i \,\mathrm{Disc}(M)$

$$= -\,\mathrm{Disc}(iM)$$

so

$$\hat{\sigma}_{tot} = \frac{1}{2\hat{s}} \int dP \, MM^{\dagger} = -\frac{1}{2\hat{s}} \,\mathrm{Disc}\left(i\bar{M}(s, t=0)\right)$$

where $\overline{M}(s,t=0)$ is the forward amplitude summed over final colors and helicities, and averaged over the initial ones.

The colour-averaged forward scattering is

$$\overline{M} = \frac{\delta^{aa'}}{N_c^2-1}\frac{\delta^{bb'}}{N_c^2-1}\sum_T P^{aa'}_{bb'}(T)\,A^T(s,t)$$

$$= \frac{1}{N_c^2-1} P^{bb'}_{aa'}(\mathbb{1}) \sum_T P^{aa'}_{bb'}(T)\,A^T(s,t)$$

$$= \frac{1}{N_c^2-1} P^{aa'}_{aa'}(\mathbb{1})\,A^{(1)}(s,t)$$

$$= \frac{1}{N_c^2-1}\frac{\delta^{aa'}\delta_{aa'}}{N_c^2-1} A^{(1)}(s,t) = \frac{1}{N_c^2-1} A^{(1)}(s,t)$$

So the total cross section is

$$\hat{\sigma}_{tot} = -\frac{1}{2\hat{s}} \,\mathrm{Disc}\left(i\bar{M}(s, t=0)\right)$$

$$= -\frac{1}{2\hat{s}} \frac{i}{N_c^2-1} \,\mathrm{Disc}\, A^{(1)}$$

$$= -\frac{1}{2\hat{s}} \frac{i}{N_c^2-1} \, 16 i \hat{s} \alpha_s^2 N_c^2 \int d^2 q_a \, d^2 q_b \, \frac{1}{q_a^2 q_b^2} f(q_a, q_b, y)$$

$$= \frac{8 N_c^2}{N_c^2-1} \alpha_s^2 \int d^2 q_a \, d^2 q_b \, \frac{1}{q_a^2 q_b^2} f(q_a, q_b, y)$$

As we noted, the BFKL eigenvalue, and thus the solution of the BFKL equation for the singlet at $t=0$, have no IR or UV divergences.

Accordingly, the solution for the singlet at $t=0$ is finite order by order in α_s. This is consistent with the relation through the optical theorem between the singlet solution at $t=0$ and the total cross section, which must be finite order by order in α_s, because of the KLN theorem.

Finally, we note that because of the leading singularity at $A = 4\ln 2 \frac{\alpha_s N_c}{\pi}$, the total cross section grows like s^A, thus violating the Froissart bound $s \sim \ln^2 s$

From the parton total cross section,

$$\hat{\sigma}_{tot} = C_A^2 \alpha_s^2 \int d^2k_a \, d^2k_b \, \frac{1}{k_a^2 k_b^2} f(k_a, k_b, y)$$

we can extract the parton cross section for the production of two gluons

$$\frac{d\hat{\sigma}_{gg}}{d^2k_a \, d^2k_b} = \left(\frac{C_A \alpha_s}{k_{a_\perp}^2}\right) f(k_a, k_b, y) \left(\frac{C_A \alpha_s}{k_{b_\perp}^2}\right)$$

where $y = |y_a - y_b|$ and $f(k_a, k_b, y)$ is the inverse Laplace transform of the singlet solution

$$f(k_a, k_b, y) = \int \frac{d\omega}{2\pi i} e^{\omega y} f_\omega(k_a, k_b) \qquad \text{with } \omega = \ell - 1$$

$$f(k_a, k_b, y) = \frac{1}{(2\pi)^2} \frac{1}{\sqrt{k_a^2 k_b^2}} \sum_n \int d\nu \, e^{\omega(\nu,n) y} \, e^{i\nu \ln k_a^2/k_b^2} \, e^{in(\varphi_a - \varphi_b)}$$

$e^{\omega(\nu,n) y}$ resums $\alpha_s y$ to all orders

Besides $e^{\omega(\nu,n) y} = 1 + O(\alpha_s)$

In addition, we know that

$$\delta^2(k_a - k_b) = \frac{1}{2\pi^2} \frac{1}{\sqrt{k_a^2 k_b^2}} \sum_n \int d\nu \, e^{i\nu \ln k_a^2/k_b^2} \, e^{in(\varphi_a - \varphi_b)}$$

so $f(k_a, k_b, y) = \frac{1}{2} \delta^2(k_a - k_b) + O(\alpha_s)$

then the two-gluon cross section is

$$\frac{d\sigma}{d^2k_2\, d^2k_3} = \frac{C_A\alpha_S}{k_2^2}\left[\frac{1}{2}\delta^2(k_2-k_3) + O(\alpha_S)\right]\frac{C_A\alpha_S}{k_3^2}$$

Integrating out a transverse momentum,

$$\frac{d\sigma}{dk_2^2} = \pi\frac{(C_A\alpha_S)^2}{2(k_2^2)^2}\left(1+O(\alpha_S)\right)$$

This is to be compared with the gluon-gluon cross section we computed in lecture 2. There we said that

$\displaystyle\frac{d\sigma}{d\hat{t}} = \frac{|M_{gg}|^2}{16\pi\hat{s}^2}$ and that in the high-energy limit

$|M_{gg\to gg}|^2 = \dfrac{9}{2}\dfrac{\hat{s}^2}{\hat{t}^2}(4\pi\alpha_S)^2$. So $\displaystyle\frac{d\sigma}{d\hat{t}} = \frac{1}{16\pi\hat{s}^2}\,\frac{9}{2}\,\frac{\hat{s}^2}{\hat{t}^2}(4\pi\alpha_S)^2 = \frac{9\pi\alpha_S^2}{2\,\hat{t}^2}$

in agreement with the leading order term of the singlet solution.

Finally, by writing the two-gluon cross section as

$$\frac{d\hat{\sigma}_{gg}}{d^2k_{a_\perp} d^2k_{b_\perp}} = \left(\frac{C_A\alpha_s}{k_{a_\perp}^2}\right) f(q_a, q_b, y) \left(\frac{C_A\alpha_s}{k_{b_\perp}^2}\right) \qquad \begin{aligned} q_a &= -k_a \\ q_b &= k_b \end{aligned}$$

we stress that it is a convolution of the singlet ladder, which is process independent, with the gluon-production vertices on each side of the rapidity interval.

In order to obtain the BFKL resummation to a production cross section at LL, one must convolute the BFKL singlet ladder with the appropriate production vertices

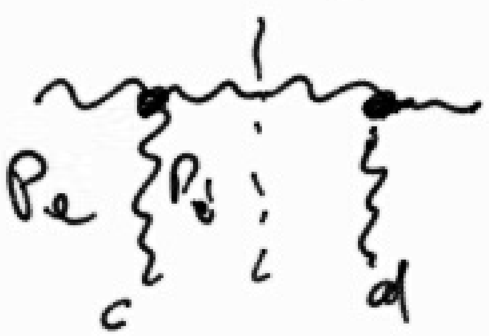
LL factorisation has in fact a modular structure.

Let us see how it works. The 2-gluon amplitude at tree level is

$$M^{aa'bb'}_{\lambda_a\lambda_{a'}\lambda_b\lambda_{b'}} = 2\hat{s}\left(i g f^{aa'c} C^{\lambda_a\lambda_{a'}}_{gg}(p_a,p_{a'})\right)$$

$$\frac{1}{t}\left(i g f^{bb'c} C^{\lambda_b\lambda_{b'}}_{gg}(p_b,p_{b'})\right)$$

Let's square separately the two helicity-conserving bits.



We define the impact factor as the square of the helicity-cons. vertex, summed (averaged) over final (initial) colours and helicities

$$\underline{I}_{g;g}(p_a, p_{a'}) = \frac{1}{2(N_c^2-1)} \overline{\sum_{\text{col, hel}}} \left(i g f^{aa'e} C_{gg}^{\lambda_a \lambda_{a'}}(p_a, p_{a'})\right.$$

$$\cdot \left(-i g f^{aa'd} \left(C_{gg}^{\lambda_a \lambda_{a'}}(p_a, p_{a'})\right)^*\right)$$

$$= g^2 \frac{C_A}{N_c^2-1} \delta^{cd}$$

We do it also for the lower vertex.

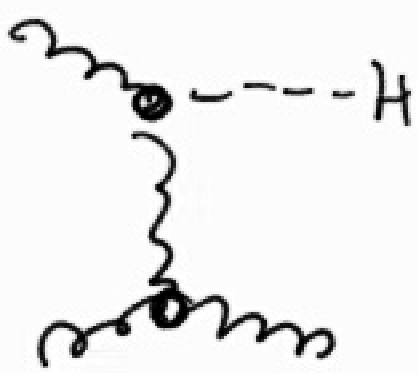
Then the squared amplitude (summed and averaged) is

$$\sum_{\text{col, hel}} |\overline{M}|^2 = g^4 \frac{C_A^2}{(N_c^2-1)^2} \delta^{cd} \delta^{cd} \frac{4\hat{s}^2}{\hat{t}^2}$$

$$= g^4 \frac{C_A^2}{2} \frac{\hat{s}^2}{\hat{t}^2}$$

which is the correct result

Now, suppose we want to compute Higgs production in the high-energy limit. We consider the amplitude for $gg \to gH$ scattering



$$M^{abb'}_{\lambda_a \lambda_b \lambda_{b'}} = 2\hat{s}\left(\delta^{ac} C^{\lambda_a}_{g;H}(p_a, p_H)\right) \frac{1}{\hat{t}}\left(i g f^{bb'c} C^{\lambda_b \lambda_{b'}}_{gg}(p_b, p_{b'})\right)$$

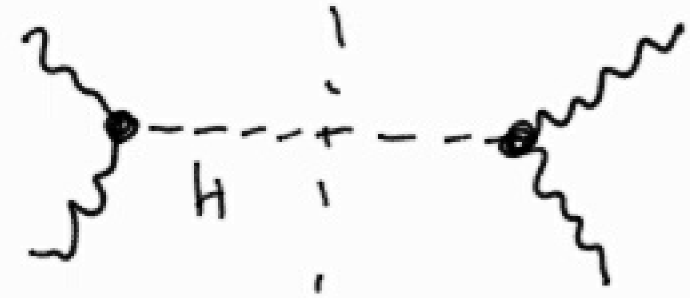
where $C^{\lambda_a}_{g;H}$ is the effective vertex for Higgs production

which can be either exact or in the large top mass limit

The impact factor for Higgs production is

$$I^{g;H}(p_a,p_H) = \frac{1}{2(N_c^2-1)} \left(\delta^{ac}\, C^{\lambda_a}_{g;H}(p_a,p_H)\right)\left(\delta^{ad}\, C^{\lambda_a}_{g;H}(p_a,p_H)^*\right)$$

$$= \frac{1}{2(N_c^2-1)}\, \delta^{cd} \left|C_{g;H}(p_a,p_H)\right|^2$$



Then the squared amplitude is

$$\sum_{col,hel} |\overline{M}|^2 = g^2 \frac{C_A}{N_c^2-1} \frac{1}{2\cancel{(N_c^2-1)}} \cancel{\delta^{cd}}\, \delta^{cd} \left|C_{g;H}(p_a,p_H)\right|^2 \frac{4\hat{s}^2}{\hat{t}^2}$$

$$= g^2 \frac{C_A}{4} \left|C_{g;H}\right|^2 \frac{\hat{s}^2}{\hat{t}^2}$$

where the detail of Higgs production are in the squared $|C_{gH}|^2$

Let us examine the modular structure of LL factorization at the cross section level

$$d\hat{\sigma} = \frac{1}{2\hat{s}}\, dP_2\, |\bar{M}|^2$$

where the 2-parton phase space in the high-energy limit is $dP_2 = \frac{1}{2\hat{s}} \frac{d^2k_a}{(2\pi)^2} \frac{d^2k_b}{(2\pi)^2} (2\pi)^2 \delta^2(k_a + k_b)$

For $gg \to gg$ scattering, the squared amplitude is

$$|\bar{M}_{gg\to gg}|^2 = \frac{C_A^2}{2} (4\pi\alpha_s)^2 \frac{\hat{s}^2}{t^2}$$

which we derived previously.

so
$$\frac{d\hat\sigma}{d^2k_a\, d^2k_b} = \frac{1}{4\cancel{\hat s^2}}\frac{1}{(2\pi)^2}\delta^2(k_a+k_b)\cdot\frac{C_A^2}{2}(4\pi\alpha_s)^2\frac{\cancel{\hat s^2}}{\hat t^2}$$

$$= \left(\frac{C_A\alpha_s}{k_{a_\perp}^2}\right)\frac{1}{2}\delta^2(k_a+k_b)\left(\frac{C_A\alpha_s}{k_{b_\perp}^2}\right) \qquad \text{since } \hat t = -k_a^2 = -k_b^2$$

in agreement with the LO term of the BFKL ladder

Likewise, for $gg \to gH$ production, the cross section is

$$\frac{d\sigma}{d^2k_b\, d^2k_H} = \frac{1}{4\hat s^2}\frac{1}{(2\pi)^2}\delta^2(k_b+k_H)\,\cancel{4}\pi\alpha_s\frac{C_A}{\cancel{4}}|C_{g;H}|^2\frac{\hat s^2}{\hat t^2}$$

$$= \frac{C_A\alpha_s}{16\pi}|C_{g;H}|^2\frac{1}{\hat t^2}\delta^2(k_b+k_H) \quad \text{which we can rewrite}$$

as
$$= \left(\frac{C_A\alpha_s}{k_{a_\perp}^2}\right)\frac{1}{2}\delta^2(k_b+k_H)\left(\frac{|C_{g;H}|^2}{8\pi\, k_{b_\perp}^2}\right)$$

So, in order to get the $gg \to gH$ cross section with the BFKL resummation, we just need replace $\frac{1}{2}\delta^2(k_3+k_H) \to f(q_1, q_2, y)$ with $k_{H_2} = -q_{2_2}$

$k_{3_2} = q_{2_2}$

$$\frac{d\hat{\sigma}}{d^2k_3\, d^2k_H} = \left(\frac{C_A \alpha_S}{k_{a_2}^2}\right) f(q_1, q_2, y) \left(\frac{|C_{g;H}|^2}{8\pi k_{b_2}^2}\right)$$

Let us examine in detail the 2-jet production cross section.

I write again the factorization formula for 2-jet production

$$\frac{d\sigma}{dy_a\, dy_b\, d^2k_{a_\perp} d^2k_{b_\perp}} = \sum_{ij} \int dx_a\, dx_b\, f_{i/A}(x_a,\mu_F^2)\, f_{j/B}(x_b,\mu_F^2) \frac{d\hat{\sigma}_{ij}}{dy_a\, dy_b\, d^2k_{a_\perp} d^2k_{b_\perp}}$$

with $d\hat{\sigma}_{ij} = \frac{1}{2\hat{s}}\, dP_{n+2} \left| M_{ij} \right|^2$

Because the exact $(n+2)$-parton phase space is

$$dP_{n+2} = \prod_{i=0}^{n+1} \frac{dy_i\, d^2k_{i_\perp}}{4\pi\,(2\pi)^2} (2\pi)^4\, \delta^4\Big(p_a + p_b - \sum_{i=0}^{n+1} k_i\Big)$$

I can re-write the cross section as

$$\frac{d\sigma}{dy_a\, dy_b\, d^2k_a\, dk_b^2} = \sum_{n=0}^{\infty} \int \left(\prod_{j=1}^{n} \frac{dy_j\, d^2k_{j_\perp}}{4\pi\, (2\pi)^2} \right) \sum_{ij} dx_a\, dx_b\, f_{i/A}(x_a)\, f_{j/B}(x_b)$$

$$\cdot \frac{1}{2\hat{s}} \frac{|M_{ij}|^2}{16\pi^2}\, \delta^4\left(p_a + p_b - \sum_{i=0}^{n+1} k_i\right) \qquad \text{with } \begin{array}{l} a \equiv 0 \\ b \equiv n+1 \end{array}$$

then we use

$$\delta^4\left(p_a + p_b - \sum_i k_i\right) = 2\delta(x_a\sqrt{s} \ldots)\, \delta(x_b\sqrt{s} \ldots)\, \delta^2\left(\sum_i k_{i_\perp}\right)$$

$$= \frac{2}{s}\, \delta(x_a - \ldots)\, \delta(x_b - \ldots)\, \delta^2\left(\sum_i k_{i_\perp}\right)$$

$$\frac{2x_a x_b}{\hat{s}}$$

so the X-section is

$$\frac{d\sigma}{dy_a\, dy_b\, d^2k_a\, d^2k_b} = \sum_n \int \prod_{j=1}^{n} \frac{dy_j\, d^2k_{j_\perp}}{4\pi\, (2\pi)^2} \sum_{ij} x_a f_{i/A}(x_a)\, x_b f_{j/B}(x_b)\, \frac{|M_{ij}|^2}{16\pi^2 \hat{s}^2}$$

$$\cdot\, \delta^2\left(\sum_i k_{i_\perp}\right)$$

but since the $(n+2)$-parton phase space in the high-energy limit is

$$dP_{n+2} = \frac{1}{2\hat{s}} \frac{d^2k_a}{(2\pi)^2} \left(\prod_{j=1}^{n} \frac{dy_j\, d^2k_{j_\perp}}{4\pi\,(2\pi)^2} \right) \frac{d^2k_{b_\perp}}{(2\pi)^2} (2\pi)^2\, \delta^2\left(\sum_{i=0}^{n+1} k_{i_\perp} \right)$$

and $d\sigma_{ij} = \frac{1}{2\hat{s}}\, dP_{n+2}\, |M_{ij}|^2$

I can also re-write the cross section as

$$\frac{d\sigma}{dy_a\, dy_b\, d^2k_a\, d^2k_b} = \sum_{ij} x_a f_{i/A}(x_a)\, x_b f_{j/B}(x_b) \frac{d\hat{\sigma}_{ij}}{d^2k_a\, d^2k_b}$$

this is the Mueller-Navelet cross section, where I can use

$$\frac{d\hat{\sigma}_{gg}}{d^2k_a\, d^2k_b} = \left(\frac{C_A \alpha_s}{k_{a_\perp}^2} \right) f(q_a, q_b, y) \left(\frac{C_A \alpha_s}{k_{b_\perp}^2} \right) \qquad \begin{array}{l} q_a = -k_a \\ q_b = k_b \end{array}$$

(The original Mueller-Navelet cross section was at fixed incoming parton momentum fractions

$$\frac{d\sigma}{dx_a\, dx_b\, d^2k_a\, d^2k_b} = \sum_{ij} f_{i/A}(x_a)\, f_{j/B}(x_b)\, \frac{d\hat{\sigma}_{ij}}{d^2k_a\, d^2k_b}$$

but the differences are about the exp. set-up, and we'll not discuss them here).

Also, because the difference in the jet-production vertices, whose square is the impact factor, is in the colour strength

$$I^{gg} = g^2 \frac{C_A}{N_c^2-1}\, \delta^{cd} \qquad I^{qq} = g^2 \frac{C_F}{N_c^2-1}\, \delta^{cd}$$

then we can write the dijet cross section as

$$\frac{d\sigma}{dy_a\, dy_b\, d^2k_a\, dk_b^2} = x_a f_{eff}(x_a)\, x_b f_{eff}(x_b) \frac{d\hat{\sigma}_{gg}}{d^2k_a\, d^2k_b}$$

through the gg parton cross section, and include the quarks through the effective PDF:

$$f_{eff}(x, \mu_F^2) = G(x, \mu_F^2) + \frac{C_F}{C_A} \sum_f \left[Q_f(x, \mu_F^2) + \bar{Q}_f(x, \mu_F^2) \right]$$

As we said in the parton cross section, we use the inverse Laplace transform of the BFKL solution

$$f(q_1, q_2, \Delta y) = \frac{1}{(2\pi)^2} \frac{1}{\sqrt{q_{1\perp}^2 q_{2\perp}^2}} \sum_n \int d\nu\, e^{\eta \chi_{\nu,n}} \left(\frac{q_1^2}{q_2^2}\right)^{i\nu} e^{in(\varphi_1 - \varphi_2)}$$

$\varphi_1 - \varphi_2$ is the angle between $\vec{q}_1$ and $\vec{q}_2$, so $\varphi = \pi - \varphi_{jj}$

and $\eta \chi_{\nu,n} = \omega(\nu,n)\, y$ since $\omega(\nu,n) = \frac{C_A \alpha_s}{\pi} \chi_{\nu,n} \Rightarrow \eta = \frac{C_A \alpha_s}{\pi} y$

$$\chi_{\nu,n} = -\psi\left(\frac{|n|+1}{2} + i\nu\right) - \psi\left(\frac{|n|+1}{2} - i\nu\right) - 2\gamma_E$$

If we integrate out the jet transverse momenta over $E_\perp \leq p_\perp \leq \infty$, then the parton cross section becomes

$$\frac{d\hat{\sigma}_{gg}}{d\varphi_{jj}} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \left[\delta(\varphi_{jj} - \pi) + \sum_{k=1}^{\infty} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in(\varphi_1 - \varphi_2)}}{2\pi} f_{n,k} \right) \eta^k \right]$$

with $f_{n,k} = \frac{1}{2\pi} \frac{1}{k!} \int_{-\infty}^{\infty} d\nu \frac{\chi_{\nu,n}^k}{\nu^2 + \frac{1}{4}}$ (please check it!)

The Born term is obtained by doing the integral in $f_{n,0}$ which gives $f_{n,0} = 1$ and by using

$$\frac{1}{2\pi} \sum_n e^{in(\varphi_1 - \varphi_2)} = \delta(\varphi_{jj'} - \pi)$$

Using the formula for $\chi_{\nu,n}$, recursive over n, (see Tutorials) one can obtain a recursive formula for $f_{n,k}$ in terms of a one-fold integral over ν.

If we integrate out also the azimuthal angle over $0 < \varphi_{jj'} < 2\pi$, only the zero mode survives, and we obtain the Mueller-Navelet total dijet cross section

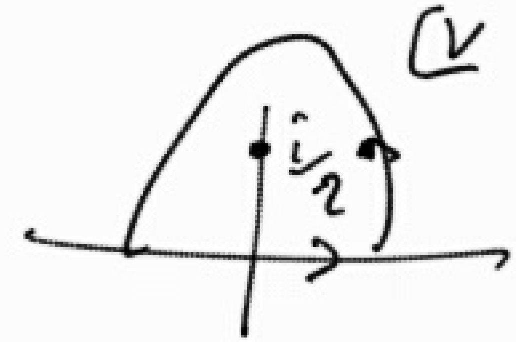
$$\hat{\sigma}_{gg} = \frac{\pi\,(C_A\alpha_s)^2}{2E_\perp^2}\sum_{k=0}^{\infty} f_{0,k}\,\eta^k$$

clearly, $f_{0,0} = 1$

$$f_{0,1} = \frac{1}{2\pi}\int d\nu\,\frac{\chi_{\nu,0}}{\nu^2+\frac{1}{4}}$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\,\frac{-2\gamma_E-\psi\left(\frac{1}{2}+i\nu\right)-\psi\left(\frac{1}{2}-i\nu\right)}{\nu^2+1/4}$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu\,\frac{-\gamma_E-\psi\left(\frac{1}{2}-i\nu\right)}{(\nu+i/2)(\nu-i/2)}$$



$$= -\frac{1}{\pi}\,2\pi i\,\frac{-\gamma_E-\psi(1)}{2i/2} = 0$$

So $f_{0,1} = 0$

Mueller-Navelet then computed

$$f_{0,2} = 2\zeta_2 \qquad f_{0,3} = -3\zeta_3 \qquad f_{0,4} = \frac{53}{6}\zeta_4 \qquad f_{0,5} = -\frac{115\zeta_5 + 48\zeta_2\zeta_3}{12}$$

If we introduce a complex variable $w = \frac{p_{1_\perp}}{p_{2_\perp}}$

with $p_\perp = p_x + i\, p_y$

such that $|w|^2 = \frac{|p_{1_\perp}|^2}{|p_{2_\perp}|^2} = \frac{q_{1_\perp}^2}{q_{2_\perp}}$ and $\left(\frac{w}{w^*}\right)^{1/2} = e^{-i\varphi_{jj}} = -e^{i\varphi}$

then the Green's function can also be written as

$$f(\vec{q}_{1_2}, \vec{q}_{2_2}, y) = \frac{1}{2}\delta^2(q_{1_2} - q_{2_2}) + \frac{1}{2\pi\sqrt{q_{1_2}^2 q_{2_2}^2}} \sum_{k=1}^{\infty} \eta^k f_k(w, w^*)$$

where the coefficients f_k are

$$f_k(w, w^*) = \frac{1}{k!} \sum_{n=-\infty}^{\infty} (-)^n \left(\frac{w}{w^*}\right)^{n/2} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} |w|^{2i\nu} \chi_{\nu,n}^k$$

It can be shown that the f_k's are analytic functions of w, because one can show that the f_k's are singular whenever one of the ladder gluons is soft $k_{i_\perp} \to 0$. However, this never occurs on the BFKL ladder, thus the f_k's are analytic, and can be described in terms of single-valued (SV) functions.

Given a classical polylogarithm

$$\mathrm{Li}_m(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m}$$

$$= \int_0^z dz' \, \frac{\mathrm{Li}_{m-1}(z')}{z'}$$

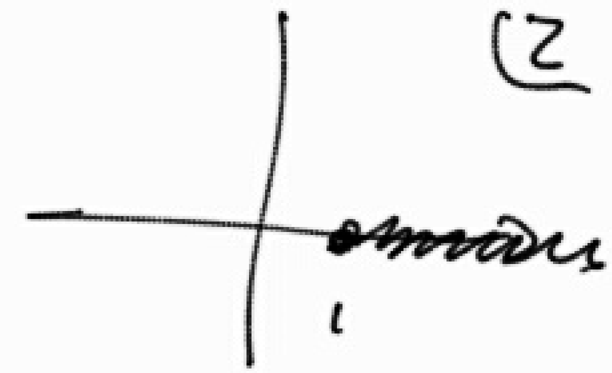
For $m=1$, it is defined to be the logarithm $\mathrm{Li}(z) = -\ln(1-z)$

then $\mathrm{Li}_2(z) = -\int_0^z dz' \, \frac{\ln(1-z')}{z'}$ is the dilogarithm

For $z=1$, the polylogarithm equals the Riemann ζ fn.

$$\mathrm{Li}_m(1) = \sum_{k=1}^{\infty} \frac{1}{k^m} = \zeta_m$$

Because $Li_1(z) = -\ln(1-z)$, the polylog is multi-valued and the branch cut is along the real axis $(z > 1)$ for all n. The discontinuity across the cut is



$$\Delta Li_n(z) = 2\pi i \, \frac{\log^{n-1} z}{(n-1)!}$$

Because $\Delta Li_n(z)$ is known, one can construct linear combinations of products of $Li_n(z)$ such that all branch cuts cancel. For example, take the functions

$$D_n(z) = R_n \left\{ \sum_{k \geq 1}^{n} \frac{(-\log|z|)^{n-k}}{(n-k)!} Li_k(z) + \frac{\log^n |z|}{2n!} \right\}$$

where $\mathcal{R}_m = \begin{cases} \mathrm{Im} & m \text{ even} \\ \mathrm{Re} & m \text{ odd} \end{cases}$ (Zagier 1990)

The discontinuity is

$$\Delta D_m(z) = 2\pi i \sum_{k=1}^{m} \frac{(-\log|z|)^{m-k}}{(m-k)!} \frac{\log^{k-1}(z)}{(k-1)!}$$

$$= 2\pi \frac{i^m}{(m-1)!} (\arg z)^{m-1}$$

Because $\Delta D_m(z)$ is real for even m, and imaginary for odd m, then $D_m(z)$ is single-valued. A particular case is the Bloch-Wigner dilogarithm

$$D_2(z) = \mathrm{Im}\left[\mathrm{Li}_2(z)\right] + \arg(1-z)\log|z|$$

$\mathrm{Li}_m(z)$ are generalised to iterated integrals over rational functions, called multiple polylogarithms (Goncharov 2001)

$$G(a,\vec{w};z) = \int_0^z \frac{dt}{t-a}\, G(\vec{w};t) \quad \text{with} \quad G(a;z) = \log\left(1-\frac{z}{a}\right)$$

and $a \in \mathbb{C}$. If $a=0$, one gets back the $\mathrm{Li}_m(z)$.

If $\{a,\vec{w}\} \in \{-1,0,1\}$, that is

$$H(a,\vec{w};z) = \int_0^z dt\, f(a;t)\, H(\vec{w};t)$$

(Remiddi, Vermaseren 1999)

$$f(-1;t) = \frac{1}{1+t} \qquad f(0;t) = \frac{1}{t} \qquad f(1;t) = \frac{1}{1-t}$$

these are called harmonic polylogarithms (HPL)

Just like for $\mathrm{Li}_m(z)$, it is possible to construct single-valued versions of the HPLs (Brown 2004). It has been shown (Dixon, Duhr, Pennington, VDD 2013) that the coefficients f_k of the singlet BFKL ladder can be written in terms of SVHPLs.