After having solved the BFKL equation for octet exchange, we consider now the one for singlet exchange.

The Laplace transform for the discontinuity of the amplitude is

$$\mathbb{F}_{\omega}^{(1)}(\hat{t}) = -2i\,\hat{t}\,(4\pi\alpha_s)^2 N_c^2 \int \frac{d^2 q_{1_2}}{(2\pi)^2} \, \frac{1}{q_{1_2}^2 \, (q-q_1)_2^2} \, f_\omega^{(1)}(q_1, \hat{t})$$

where $\omega = \ell - 1$ and the $C^{sing} = N_c$.

The function $f_\omega^{sing}(q_1, \hat{t})$ satisfies the (BFKL) eq.:

$$\left[\omega - \alpha(\hat{t}_1) - \alpha(\hat{t}_1')\right] f_\omega^{(1)}(q_1, \hat{t}) = 1 - 2\alpha_s N_c \int \frac{d^2 q_{2_2}}{(2\pi)^2} \, \frac{K(q_1, q_2)}{q_{2_2}^2 (q - q_2)_2^2} \, f_\omega (q_2, \hat{t})$$

Firstly, let us examine the UV behaviour:

As $q_2 \to \infty$, the kernel $K(q_1, q_2) = q_\perp^2 - \dfrac{(q-q_1)_\perp^2 q_{2_\perp}^2 + (q-q_2)_\perp^2 q_{1_\perp}^2}{(q_1-q_2)_\perp^2}$

is regular: $\lim\limits_{q_2 \to \infty} K(q_1, q_2) = q_\perp^2 - [(q-q_1)_\perp^2 + q_{1_\perp}^2]$

The RHS of the BFKL eq. goes like $\displaystyle\int d^2 q_2 \, \frac{q_\perp^2 - q_{1_\perp}^2 - (q-q_1)_\perp^2}{(q_{2_\perp}^2)^2}$

and is UV finite as $q_2 \to \infty$

As $q_1 \to \infty$, $\lim\limits_{q_1 \to \infty} K(q_1, q_2) = q_\perp^2 - (q-q_2)_\perp^2 - q_{2_\perp}^2$

so the RHS of the BFKL eq. is regular as $q_1 \to \infty$

Now, the IR behaviour: the kernel vanishes at $q_{2_\perp}^2 \to 0$

$$\lim_{q_{2_\perp}^2 \to 0} K(q_1, q_2) = 0 \quad \text{and} \quad \lim_{(q-q_2)_\perp^2 \to 0} K(q_1, q_2) = 0$$

thus $\displaystyle\int \frac{d^2 q_{2_\perp}}{(2\pi)^2} \frac{K(q_1, q_2)}{q_{2_\perp}^2 (q - q_{2_\perp})^2}$ is regular at $q_{2_\perp}^2 \to 0$, $(q-q_2)_\perp^2 \to 0$

The kernel is singular at $q_{1_\perp}^2 = q_{2_\perp}^2$,

and so is the Regge Trajectory,

however note that the integrand of the Regge Trajectory can be re-written through the partial fractioning

$$\frac{1}{q_{2_\perp}^2 (q_1 - q_2)_\perp^2} = \frac{1}{q_{2_\perp}^2 + (q_1 - q_2)_\perp^2} \left(\frac{1}{(q_1 - q_2)_\perp^2} + \frac{1}{q_{2_\perp}^2} \right)$$

integrating over q_2

$$\int d^2q_2 \frac{1}{q_2^2 (q_1-q_2)^2} = \int d^2q_2 \frac{1}{q_2^2 + (q_1-q_2)^2} \left(\frac{1}{(q_1-q_2)^2} + \frac{1}{q_2^2} \right)$$

$$= 2 \int d^2q_2 \frac{1}{[q_2^2 + (q_1-q_2)^2](q_1-q_2)^2}$$

after shifting $q_2 \to q_1 - q_2$ in the 2nd term

then one can see that the divergent pieces at $q_1^2 = q_2^2$ cancel between the kernel and the Regge trajectory.

Finally,

$$\lim_{q_1^2 \to 0} K(q_1, q_2) = 0 \quad \text{and} \quad \lim_{(q-q_1)^2 \to 0} K(q_1, q_2) = 0$$

i.e. the kernel vanishes at $q_1^2 \to 0$ and $(q-q_1)^2 \to 0$ and the singlet BFKL eq. is reduced to

$$\left[\omega - \alpha(\hat{t})\right] \lim_{q_1^2 \to 0} f^{(1)}(q_1, t) = 1$$

which is the eq. of the octet one-gluon ladder, which is known to be IR sensitive. However, this divergence comes from the exchanged momentum q_1 coming into the ladder, so under a suitable behaviour of the boundary terms (as e.g. in onium scattering) this IR divergence can be tamed

Back to the singlet BFKL eq, we write $f_\omega^{(1)}$ differentially

$$f_\omega^{(1)}(q_1, \hat{t}) = \int \frac{d^2k_2}{(2\pi)^2} \bar{f}_\omega(q_1, k, \hat{t})$$

then the BFKL equation becomes

$$\left[\omega - \alpha(\hat{t}_1) - \alpha(\hat{t}_1')\right] \bar{f}_\omega(q_1, k, t) =$$

$$= (2\pi)^2 \delta^2(q_1 - k) - 2\alpha_s N_c \int \frac{d^2q_2}{(2\pi)^2} \frac{K(q_1, q_2)}{q_{2_\perp}^2 (q - q_2)_\perp^2} \bar{f}_\omega(q_2, k, t)$$

We are mostly interested in the forward scattering, and through the optical theorem to the total cross section. Then we set $q_\perp = 0$

The kernel becomes $K(q_1, q_2)\Big|_{q=0} = -2 \dfrac{q_{1_\perp}^2 q_{2_\perp}^2}{(q_1 - q_2)_\perp^2}$

and the BFKL eq. becomes ,

$$[\omega - 2\alpha(\hat{t}_1)]\, \bar{f}_\omega(q_1, k) =$$

$$= (2\pi)^2 \delta^2(q_1 - k) + 4\alpha_s N_c \int \frac{d^2 q_2}{(2\pi)^2} \frac{q_{1_\perp}^2}{q_{2_\perp}^2 (q_1 - q_2)_\perp^2} \bar{f}_\omega(q_2, k)$$

where $\alpha(\hat{t}_1) = -\alpha_s N_c\, q_{1_\perp}^2 \displaystyle\int \frac{d^2 q_2}{(2\pi)^2} \frac{1}{q_{2_\perp}^2 (q_1 - q_2)_\perp^2}$

using the partial fractioning above, to regulate the divergence at $q_1 = q_2$

$$\omega \bar{f}_\omega(q_1, k) = (2\pi)^2 \delta^2(q_1 - k)$$

kernel   Regge traject

$$+ 4\alpha_s N_c \int \frac{d^2 q_{2_\perp}}{(2\pi)^2} \left[\frac{q_{1_\perp}^2}{q_{2_\perp}^2 (q_1 - q_2)_\perp^2} \bar{f}_\omega(q_2, k) - \frac{q_{1_\perp}^2}{(q_1 - q_2)_\perp^2 [q_{2_\perp}^2 + (q_1 - q_2)_\perp^2]} \bar{f}_\omega(q_1, k) \right]$$

Next, we set $f_\omega(q, k) = \frac{1}{8\pi^2} \frac{k_\perp^2}{q_\perp^2} \bar{f}_\omega(q, k)$

then the BFKL eq. becomes

$$\omega f_\omega(q_1, k) = \frac{1}{2} \delta^2(q_1 - k)$$

$$+ \frac{\alpha_s N_c}{\pi^2} \int d^2 q_{2\perp} \left[\frac{1}{(q_1 - q_2)_\perp^2} f_\omega(q_2, k) - \frac{q_{1\perp}^2}{(q_1 - q_2)_\perp^2 \left[q_{2\perp}^2 + (q_1 - q_2)_\perp^2 \right]} f_\omega(q_1, k) \right]$$

The homogeneous equation (i.e. without the δ term) can be written as

$$\omega f_\omega(q_1, k)$$

$$= \frac{\alpha_s N_c}{\pi^2} \int d^2 q_{2\perp} \left[\frac{f_\omega(q_2, k)}{(q_1 - q_2)_\perp^2} - \frac{q_1^2}{q_2^2} \left(\frac{1}{(q_1 - q_2)_\perp^2} - \frac{1}{q_{2\perp}^2 + (q_1 - q_2)_\perp^2} \right) f_\omega(q_1, k) \right]$$

where we partial fractioned the last term.

We suppose that the solution can be expanded as a Fourier series

$$f_\omega(q_1, k) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu \, a(\nu, n) \, e^{i\nu(\lambda_1-\lambda)} \, e^{in(\varphi_1-\varphi)}$$

with $\lambda_1 = \ln q_1^2/\mu^2$ $\lambda = \ln k^2/\mu^2$ μ^2 a scale factor

and $\varphi_1-\varphi$ the azimuthal angle between q_1 and k

Firstly, we fix the coefficient $a(\nu, n)$ using the inhomogeneous equation..

We want to use the integral representation of the δ function

Now $k_x = \sqrt{k^2}\cos\varphi$ $\qquad \dfrac{\partial(k_x, k_y)}{\partial(k^2, \varphi)} = \begin{vmatrix} \dfrac{\cos\varphi}{2\sqrt{k^2}} & \dfrac{\sin\varphi}{2\sqrt{k^2}} \\ -\sqrt{k^2}\sin\varphi & \sqrt{k^2}\cos\varphi \end{vmatrix} = \dfrac{1}{2}$

$k_y = \sqrt{k^2}\sin\varphi$

$$d^2k = \frac{1}{2}\, dk^2\, d\varphi \quad \Rightarrow \quad \delta^2(q_1 - k) = 2\,\delta(q_1^2 - k^2)\,\delta(\varphi_1 - \varphi)$$

then $\quad d\lambda = \dfrac{dk^2}{k^2} \quad \Rightarrow \quad \delta(\lambda_1 - \lambda) = q_1^2\,\delta(q_1^2 - k^2) = \sqrt{q_1^2 k^2}\,\delta(q_1^2 - k^2)$

so $\quad \delta^2(q_1 - k) = \dfrac{2}{\sqrt{k^2 q_1^2}}\,\delta(\lambda_1 - \lambda)\,\delta(\varphi_1 - \varphi)$

but $\quad \delta(\lambda_1 - \lambda)\,\delta(\varphi_1 - \varphi) = \dfrac{1}{(2\pi)^2} \sum_n \int d\nu\; e^{i\nu(\lambda_1 - \lambda)}\, e^{in(\varphi_1 - \varphi)}$

so

$$\delta^2(q_1 - k) = \frac{1}{2\pi^2}\,\frac{1}{\sqrt{k^2 q_1^2}} \sum_n \int d\nu\; e^{i\nu(\lambda_1 - \lambda)}\, e^{in(\varphi_1 - \varphi)}$$

Replacing the integral representation of the δ function into the inhomogeneous equation, and calling $\omega(\nu,n)$ the eigenvalue of the homogeneous eq., the BFKL eq. becomes

$$\omega\, a(\nu,n) = \frac{1}{(2\pi)^2}\frac{1}{\sqrt{q_1^2 k^2}} + \omega(\nu,n)\, a(\nu,n)$$

thus $$a(\nu,n) = \frac{1}{(2\pi)^2}\frac{1}{\sqrt{q_1^2 k^2}}\frac{1}{\omega-\omega(\nu,n)}$$

so the solution of the BFKL equation is

$$f_\omega(q_1,k) = \frac{1}{(2\pi)^2}\frac{1}{\sqrt{q_1^2 k^2}}\sum_n\int d\nu\,\frac{1}{\omega-\omega(\nu,n)}\, e^{i\nu(\lambda_1-\lambda)}\, e^{in(\varphi_1-\varphi)}$$

Next, we need to find the spectrum of the eigenvalues

To do that, we replace the solution into the homogeneous eq.

So we need to compute 3 integrals. The first (i.e. the kernel) is 1) $\int \frac{d^2q_2}{(q_2-q_1)^2} f_\omega(q_2,k)$. We need to consider in particular

$$\frac{d^2q_2}{(q_1-q_2)^2} \frac{1}{\sqrt{q_2^2 k^2}} e^{i\nu(\lambda_2-\lambda)} e^{in(\varphi_2-\varphi)}$$

(in what follows, I drop λ & φ)

$$= \frac{1}{2}\sqrt{\frac{q_1^2 q_2^2}{q_1^2 k^2}}\, d\lambda_2\, d\varphi_2 \frac{1}{(q_1-q_2)^2} e^{i\nu\lambda_2} e^{in\varphi_2}$$

since $d^2q_2 = \frac{1}{2} dq_2^2 d\varphi_2 = \frac{1}{2} q_2^2 d\lambda_2 d\varphi_2$ for $\lambda_2 = \ln q_2^2/\mu^2$

So we compute first the azimuthal integral

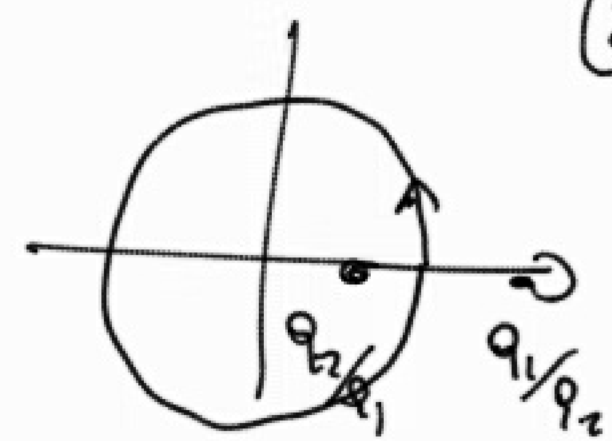
$$\int d\varphi_2 \, \frac{e^{in\varphi_2}}{(q_1-q_2)^2} = \int d\varphi_2 \, \frac{e^{in\varphi_2}}{q_1^2+q_2^2-2q_1q_2\cos(\varphi_1-\varphi_2)}$$

set $z = e^{i(\varphi_2-\varphi_1)} \Rightarrow dz = iz\,d\varphi_2$ $\quad \cos\varphi_1-\varphi_2 = \dfrac{z+z^{-1}}{2}$

This is symmetric under $q_1 \leftrightarrow q_2$ and $n \leftrightarrow -n$, so we take $n \geq 0$

$$= e^{in\varphi_1} \oint \frac{dz}{iz} \, \frac{z^{|n|}}{q_1^2+q_2^2-q_1q_2(z+z^{-1})}$$

$$= e^{in\varphi_1} \oint \frac{dz}{i} \, \frac{-z^{|n|}}{(q_1 z-q_2)(q_2 z-q_1)}$$



$\mathbb{C}$

$q_1 > q_2$

If $q_1 > q_2$, get $\dfrac{2\pi e^{in\varphi_1}}{q_1 q_2} \dfrac{-\left(q_2/q_1\right)^{|n|}}{q_2/q_1 - q_1/q_2}$

Likewise for $q_2 > q_1$

So $$\int d\varphi_2 \frac{e^{in\varphi_2}}{(q_1-q_2)^2} = 2\pi e^{in\varphi_1} \left[\frac{\left(q_2/q_1\right)^{|n|}}{q_1^2-q_2^2} \vartheta\left(q_1^2-q_2^2\right) + \frac{\left(q_1/q_2\right)^{|n|}}{q_2^2-q_1^2} \vartheta\left(q_2^2-q_1^2\right) \right]$$

Note that $$\frac{\left(q_2/q_1\right)^{|n|} \sqrt{q_1^2 q_2^2}}{q_1^2-q_2^2} = \frac{\left(q_2^2/q_1^2\right)^{\frac{|n|+1}{2}}}{1-q_2^2/q_1^2}$$

So $$\int_{-\infty}^{\infty} d\lambda_2 \, e^{i\nu(\lambda_2-\lambda_1)} \left[\frac{\left(q_2^2/q_1^2\right)^{\frac{|n|+1}{2}}}{1-q_2^2/q_1^2} \vartheta(\lambda_1-\lambda_2) + \frac{\left(q_1^2/q_2^2\right)^{\frac{|n|+1}{2}}}{1-q_1^2/q_2^2} \vartheta(\lambda_2-\lambda_1) \right]$$

Note $e^{i\nu(\lambda_2-\lambda_1)} = \left(q_2^2/q_1^2\right)^{i\nu}$

Then for $q_1 > q_2$ set $x = q_2^2/q_1^2$ $\quad dx = \frac{dq_2^2}{q_1^2} = \frac{q_2^2}{q_1^2} d\lambda_2 = x\, d\lambda_2$

$q_1 < q_2$ set $x = q_1^2/q_2^2$ $\quad dx = -\frac{q_1^2}{(q_2^2)^2} dq_2^2 = -\frac{q_1^2}{q_2^2} d\lambda_2 = -x\, d\lambda_2$

The integral yields

$$\int_0^1 dx \, \frac{1}{1-x} \left[x^{\frac{|n|-1}{2}+i\nu} + x^{\frac{|n|-1}{2}-i\nu} \right]$$

Collecting it all,

$$\int \frac{d^2 q_2}{(q_1-q_2)^2} f_\omega(q_2, k) = \frac{1}{2} \frac{e^{in\varphi_1} e^{i\nu\lambda_1}}{\sqrt{q_1^2 k^2}} 2\pi \int_0^1 dx \, \frac{1}{1-x} \left[x^{\frac{|n|-1}{2}+i\nu} + x^{\frac{|n|-1}{2}-i\nu} \right]$$

The 2^{nd} integral (from the Regge trajectory) is

2) $\displaystyle \int d^2q_2 \, \frac{q_1^2}{q_2^2 (q_1-q_2)^2} f_w(q_1)$ in particular, we need evaluate

$$\frac{1}{2} \, d\lambda_2 \, d\varphi_2 \, \frac{1}{\sqrt{q_1^2 k^2}} \, \frac{q_1^2}{(q_1-q_2)^2} \, e^{i\nu\lambda_1} e^{in\varphi_1}$$

$$\int d\varphi_2 \, \frac{q_1^2}{(q_1-q_2)^2} = 2\pi \left[\frac{q_1^2}{q_1^2-q_2^2} \, \theta(q_1^2-q_2^2) + \frac{q_1^2}{q_2^2-q_1^2} \, \theta(q_2^2-q_1^2) \right]$$

and setting as before $x = q_2^2/q_1^2 \quad dx = x\, d\lambda_2$ for $q_1^2 > q_2^2$

$x = q_1^2/q_2^2 \quad dx = -x\, d\lambda_2$ for $q_2^2 > q_1^2$

the integral over λ_2 yields:

$$\int_0^1 \frac{dx}{x} \left(\frac{1}{1-x} + \frac{1}{1/x - 1} \right) = \int_0^1 \frac{dx}{x} \, \frac{1+x}{1-x} = \int_0^1 dx \left(\frac{1}{x} + \frac{2}{1-x} \right)$$

So $\int d^2 q_2 \frac{q_1^2}{q_2^2 (q_1 - q_2)^2} f_\omega(q_1, k) = \frac{1}{2} \frac{e^{i\nu\lambda_1} e^{in\varphi_1}}{\sqrt{q_1^2 k^2}} 2\pi \int_0^1 dx \left(\frac{1}{x} + \frac{2}{1-x} \right)$

The 3rd integral (also from the Regge trajectory) is

3) $\int d^2 q_2 \frac{q_1^2}{q_2^2 \left[q_2^2 + (q_1 - q_2)^2 \right]} f_\omega(q_1, k)$ and we need evaluate

$$\frac{1}{2} \int d\lambda_2 d\varphi_2 \frac{1}{\sqrt{q_1^2 k^2}} \frac{q_1^2}{\left[q_2^2 + (q_1 - q_2)^2 \right]} e^{i\nu\lambda_1} e^{in\varphi_1}$$

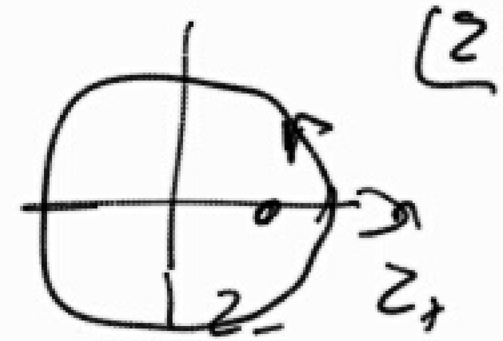
So, $\int d\varphi_2 \frac{q_1^2}{q_2^2 + (q_1 - q_2)^2} = \oint \frac{dz}{iz} \frac{q_1^2}{q_1^2 + 2q_2^2 - q_1 q_2 (z + z^{-1})}$

$$= -\oint \frac{dz}{i} \frac{q_1^2}{q_1 q_2 z^2 - (q_1^2 + 2q_2^2) z + q_1 q_2}$$

The roots are at $z_\pm = \dfrac{q_1^2 + 2q_2^2 \pm \sqrt{q_1^4 + 4q_2^4}}{2q_1 q_2}$

Now $z_+ z_- = 1$ $\quad z_+ + z_- = \dfrac{q_1}{q_2} + 2\dfrac{q_2}{q_1}$ $\quad$ so $z_+ + \dfrac{1}{z_+} > 1$

the integral is



$$-\int \frac{dz}{i}\,\frac{q_1^2}{q_1 q_2 (z-z_+)(z-z_-)} = -2\pi\,\frac{q_1^2}{q_1 q_2 (z_- - z_+)}$$

$$= 2\pi\,\frac{q_1^2}{\sqrt{q_1^4 + 4q_2^4}}$$

So

$$\int d^2 q_2\,\frac{q_1^2 f_w(q_1,k)}{q_2^2\left[q_2^2 + (q_1 - q_2)^2\right]} = \frac{1}{2}\,\frac{e^{i\nu\lambda_1}e^{in\varphi_1}}{\sqrt{q_1^2 k^2}}\,2\pi \int_0^1 \frac{dx}{x}\left(\frac{1}{\sqrt{1+4x^2}} + \frac{x}{\sqrt{4+x^2}}\right)$$

So the spectrum of eigenvalues is given by

$$\omega(\nu,n) = \frac{\alpha_s N_c}{\pi} \int_0^1 dx \left[\frac{1}{1-x} \left(x^{\frac{|n|-1}{2}+i\nu} + x^{\frac{|n|-1}{2}-i\nu} - 2 \right) \right.$$

$$\left. - \frac{1}{x} + \frac{1}{x} \frac{1}{\sqrt{1+4x^2}} + \frac{1}{\sqrt{4+x^2}} \right]$$

Now, we show that the last 3 terms cancel out,

$$\int_0^1 dx \frac{1}{\sqrt{x^2+4}} = \int_0^{1/2} dy \frac{1}{\sqrt{1+y^2}} = \left. \operatorname{arc\,sinh} y \right|_0^{1/2} = \left. \ln\left(y + \sqrt{1+y^2}\right) \right|_0^{1/2}$$

$$= \ln \frac{1+\sqrt{5}}{2}$$

in $\int_0^1 dx \frac{1}{x} \left(\frac{1}{\sqrt{1+4x^2}} - 1 \right)$ each of the 2 terms diverges at $x \to 0$, but the difference is regular

$$\lim_{\epsilon\to 0}\int_\epsilon^1 dx\,\frac{1}{x}\left(\frac{1}{\sqrt{1+4x^2}}-1\right) = \lim_{\epsilon\to 0}\left(\int_{2\epsilon}^{2} dy\,\frac{1}{y\sqrt{1+y^2}} - \int_\epsilon^1 \frac{dx}{x}\right)$$

$$= -\left|\ln\frac{1+\sqrt{1+y^2}}{y}\right|_{2\epsilon}^{2} - \ln\frac{1}{\epsilon}$$

since $\frac{d}{dy}\ln\frac{1+\sqrt{1+y^2}}{y} = \frac{1}{1+\sqrt{1+y^2}}\frac{y}{\sqrt{1+y^2}} - \frac{1}{y}$

$$= \frac{\cancel{y^2} - \sqrt{1+y^2} - (1+\cancel{y^2})}{y\sqrt{1+y^2}\left(1+\sqrt{1+y^2}\right)}$$

$$= -\frac{1}{y\sqrt{1+y^2}}$$

$$= -\ln\frac{1+\sqrt{5}}{2} + \ln\frac{2}{2\epsilon} - \ln\frac{1}{\epsilon}$$

$$= -\ln\frac{1+\sqrt{5}}{2}$$

so $$\omega(\nu,n) = \frac{\alpha_s N_c}{\pi}\int_0^1 dx\,\frac{1}{1-x}\left(x^{\frac{|n|-1}{2}+i\nu} + x^{\frac{|n|-1}{2}-i\nu} - 2\right)$$

which is regular at $x=0$ & $x=1$, because the BFKL equation is regular at $q_2^2=0$, $q_2^2=q_1^2$, $q_2\to\infty$.

using the log derivative of the Γ function,

$$\frac{d\ln\Gamma(y)}{dy} = \psi(y) = \int_0^1 dx\, \frac{x^{y-1}-1}{x-1} + \psi(1)$$

where $\gamma_E = -\psi(1)$ is the Euler constant

the eigenvalue of the BFKL equation becomes

$$\omega(\nu, n) = -\frac{\alpha_s N_c}{\pi}\left[\psi\left(\frac{|n|+1}{2}+i\nu\right) + \psi\left(\frac{|n|+1}{2}-i\nu\right) - 2\psi(1)\right]$$

We note that had we not regulated the $q_2^2 = q_1^2$ divergence with the substitution in the Regge trajectory

$$\int d^2 q_2 \frac{q_1^2}{q_2^2 (q_1 - q_2)^2} = 2\int d^2 q_2 \frac{q_1^2}{(q_1-q_2)^2[q_2^2 + (q_1-q_2)^2]}$$

the BFKL equation would have been

$$\omega f_\omega(q_1,k) = \frac{1}{2}\,\delta^2(q_1-k) + \frac{\alpha_s N_c}{\pi^2}\int d^2q_2\, \frac{1}{(q_1-q_2)^2}\left[f_\omega(q_2,k) - \frac{q_1^2}{2q_2^2} f_\omega(q_1,k)\right]$$

the eigenvalue would have been

$$\omega(\nu,n) = \frac{\alpha_s N_c}{\pi} \int_0^1 dx \left[\frac{1}{1-x}\left(x^{\frac{|n|-1}{2}+i\nu} + x^{\frac{|n|-1}{2}-i\nu} - 1\right) - \frac{2}{x}\right]$$

which has divergences at $x=0 \Leftrightarrow q_2^2 = 0$ & $x=1 \Leftrightarrow q_2^2 = q_1^2$, which need to be carefully regulated (see app. A, hep-ph/9503226).

Then one can show that in fact the BFKL eigenvalue agrees with the one computed above.

The solution of the BFKL equation is

$$f_\omega(q_1, k) = \frac{1}{(2\pi)^2} \frac{1}{\sqrt{k^2 q_1^2}} \sum_n \int d\nu \, \frac{1}{\omega - \omega(\nu, n)} \, e^{i\nu(\lambda_1 - \lambda)} \, e^{in(\varphi_1 - \varphi)}$$

with $\lambda_1 = \ln\left(q_1^2/\mu^2\right)$ $\quad \lambda = \ln\left(k^2/\mu^2\right)$

and $\omega(\nu, n)$ the BFKL eigenvalue we computed above.

The Laplace transform of the discontinuity of the amplitude was written as

$$F_\omega^{(1)}\Big|_{t=0} = -2 i t \,(4\pi\alpha_s)^2 N_c^2 \int \frac{d^2 q_1}{(2\pi)^2} \, \frac{1}{q_1^2 (q - q_1)^2} f_\omega^{(1)}(q_1)\Big|_{t=0}$$

where we used $s = -t e^y$. Since we are now considering $t = 0$, we should have replaced $s = k_c^2 e^y$ instead. Also,

$$f_\omega^{(1)}(q_1)\Big|_{t=0} = \int \frac{d^2 k}{(2\pi)^2} \, \bar{f}_\omega(q_1, k) = \int \frac{d^2 k}{(2\pi)^2} \, 8\pi^2 \frac{q_1^2}{k^2} f_\omega(q_1, k) = 2 \int d^2 k \, \frac{q_1^2}{k^2} f_\omega(q_1, k)$$

So $$\Gamma_\omega^{(1)}\Big|_{t=0} = 2i k_\perp^2 (4\pi\alpha_s)^2 N_c^2 \int \frac{d^2q_1}{(2\pi)^2} \frac{1}{(q_1^2)^2}\, 2\int d^2k\, \frac{q_1^2}{k^2} f_\omega(q_1, k)$$

$$= 16 i k_\perp^2 \alpha_s^2 N_c^2 \int d^2q_1\, d^2q_2\, \frac{1}{q_1^2 q_2^2} f_\omega(q_1, q_2)$$

Then the inverse Laplace transform is

$$\operatorname{Disc} A(s, t=0) = \int \frac{d\ell}{2\pi i} e^{\ell y}\, \Gamma_\omega^{(1)}\Big|_{t=0} \qquad \ell = 1+\omega$$

$$= 16 i \hat{s}\, \alpha_s^2 N_c^2 \int d^2q_1\, d^2q_2\, \frac{1}{q_1^2 q_2^2} f(q_1, q_2, y)$$

where $f(q_1, q_2, y)$ is the inverse Laplace transform of the BFKL solution w.r.t. ω

$$f(q_1, q_2, y) = \int \frac{d\omega}{2\pi i} e^{\omega y} f_\omega(q_1, q_2)$$

$$= \frac{1}{(2\pi)^2} \frac{1}{\sqrt{q_1^2 q_2^2}} \sum_n \int d\nu\, e^{\omega(\nu, n) y}\, e^{i\nu \ln q_1^2/q_2^2}\, e^{in(\varphi_1 - \varphi_2)}$$

$\varphi_1 - \varphi_2$ is the azimuthal angle in $k_{1_\perp} \cdot k_{2_\perp} = k_{1_\perp} k_{2_\perp} \cos(\varphi_1 - \varphi_2)$

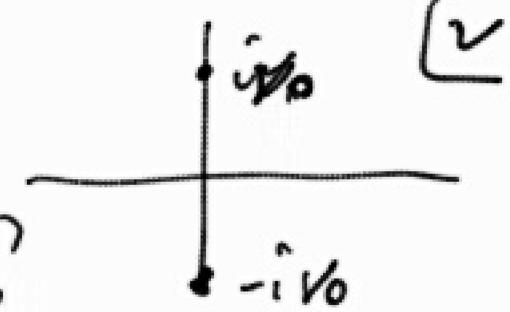
We note that the leading contribution to $\omega(\nu, n)$ comes from $\nu = 0$, $n = 0$, and that for small ν the eigenvalue is expanded as a power series in ν as (see the Tutorial)

$$\omega(\nu, n=0) = 2\frac{\alpha_s N_c}{\pi}\left(2\ln 2 - 7\zeta_3 \nu^2 + \dots\right)$$

$$= A - B\nu^2 + \dots \qquad \text{with } A = 4\ln 2 \frac{\alpha_s N_c}{\pi}$$

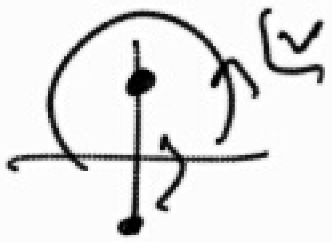
Let us replace it into the BFKL solution $\qquad B = -14\zeta_3 \frac{\alpha_s N_c}{\pi}$

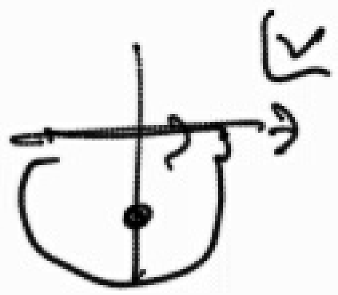
$$f_\omega(k_a, k_b) \simeq \frac{1}{(2\pi)^2}\frac{1}{\sqrt{k_a^2 k_b^2}}\int d\nu \, \frac{1}{\omega - A + B\nu^2}\, e^{i\nu \ln k_a^2/k_b^2}$$



In the complex plane of ν, the poles are at $\nu_0 = \sqrt{\frac{\omega - A}{B}}$

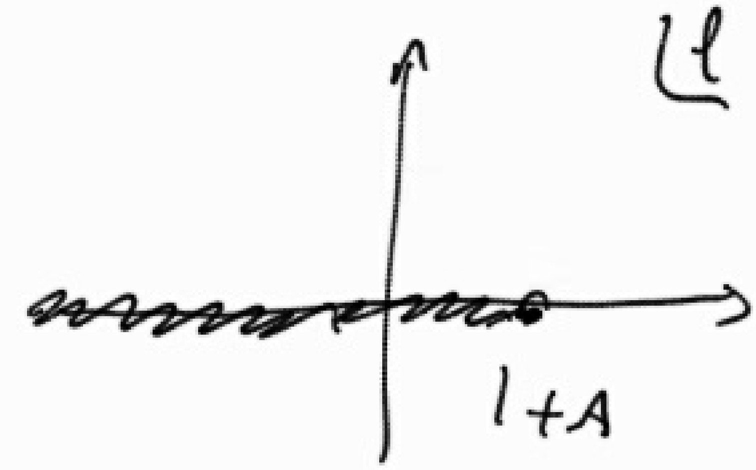
So $f_\omega(k_e,k_3) = \frac{1}{(2\pi)^2}\frac{1}{\sqrt{k_e^2 k_3^2}}\int d\nu \, \frac{1}{B(\nu+i\nu_0)(\nu-i\nu_0)}\, e^{i\nu \ln k_e^2/k_3^2}$





$$\Rightarrow \begin{cases} 2\pi i \dfrac{1}{B\, 2i\nu_0}\left(\dfrac{k_e^2}{k_3^2}\right)^{-\nu_0} & k_e^2 > k_3^2 \\[2ex] -2\pi i \dfrac{1}{B(-2i\nu_0)}\left(\dfrac{k_e^2}{k_3^2}\right)^{\nu_0} & k_e^2 < k_3^2 \end{cases}$$

So $f_\ell(k_e,k_3) = \frac{1}{(2\pi)^2}\frac{1}{\sqrt{k_e^2 k_3^2}}\frac{\pi}{\sqrt{B(\ell-1-A)}}\, e^{-\nu_0\left|\ln k_e^2/k_3^2\right|}$



i.e. in the complex plane of ℓ,
the leading singularity is a branch cut
extending from $-\infty$ to $\ell = 1+A$