We have a multigluon amplitude, a color decomposition in the $\hat{t}$ channel, and some dispersion relations. Now, the goal is to use $\hat{s}$-channel unitarity and the multigluon amplitude with exchange of a gluon ladder in the $\hat{t}$ channel in order to evaluate the discontinuity of the amplitude with the exchange of two gluons in the $\hat{t}$ channel.

To evaluate the discontinuity, we shall take an $(n+1)$-loop diagram, which is cut in the s channel, i.e. we shall put the internal legs on shell, by replacing the $n+2$ internal propagators with the cut ones $\frac{i}{k^2} \to 2\pi\delta(k^2)$

so the $(n+1)$-loop integrals yield the phase space for the production of $(n+2)$ partons

$$d\Phi_{n+2} = \prod_{i=0}^{n+1} \frac{dy_i \, d^2k_{\perp i}}{4\pi\,(2\pi)^2} (2\pi)^4 \delta^4\left(p_a + p_b - \sum_{i=0}^{2} k_i\right)$$

However, we have seen that in MRK, light-cone momentum conservation is

$$p_a^+ = x_a \sqrt{s} \simeq k_{0\perp} e^{y_0}$$

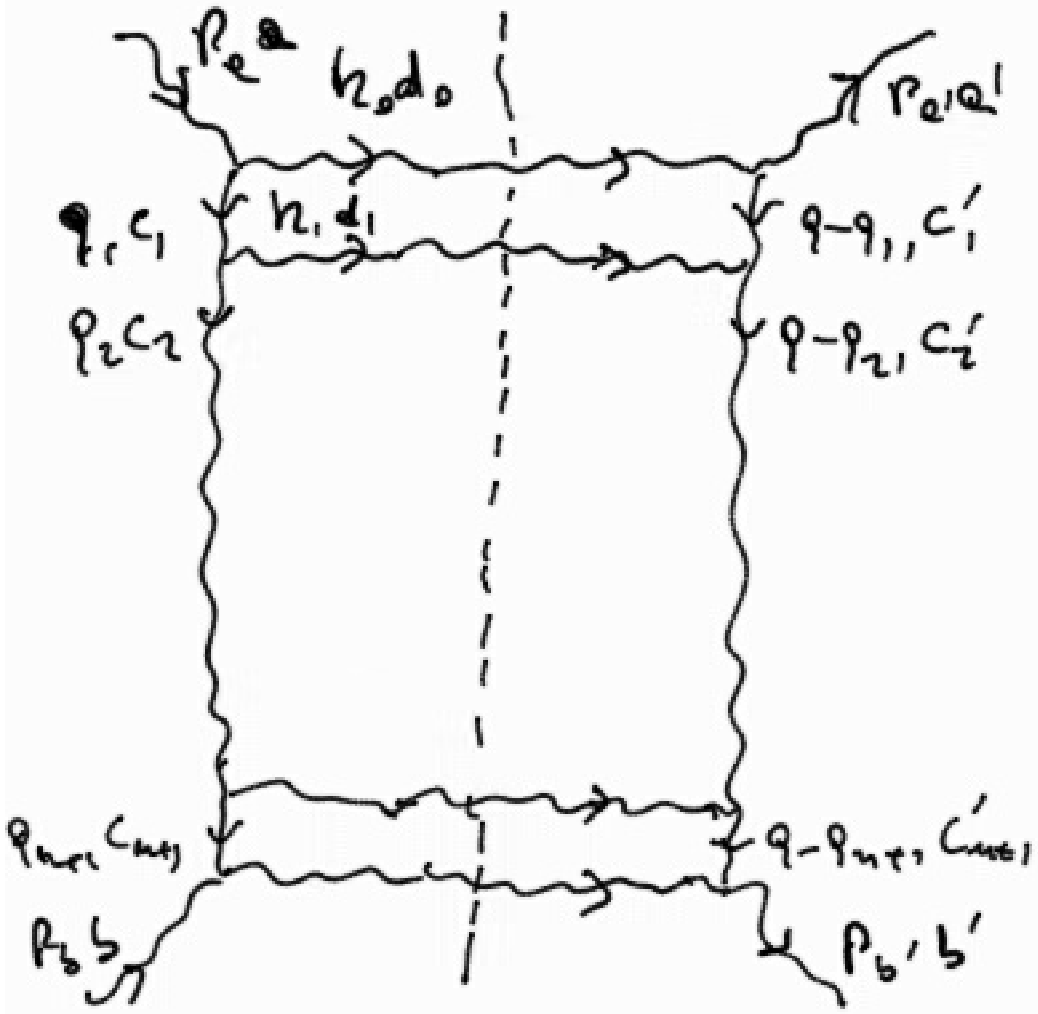
$$p_b^- = x_b \sqrt{s} \simeq k_{n+1\perp} e^{-y_{n+1}}$$

irrespective of how many partons are produced.
Thus we can still fix two rapidities, as we have done for two partons, using light-cone momenta

$$\lim_{y_0 - y_2 \gg 1} \int dy_0 \, dy_2 \, \delta(x_a \sqrt{s} - \dots) \, \delta(x_b \sqrt{s} - \dots) = \frac{1}{\hat{s}}$$

and the $(n+2)$-parton phase space becomes

$$dP_{n+2} = \frac{1}{2\hat{s}} \frac{d^2 k_{0_\perp}}{(2\pi)^2} \left(\prod_{i=1}^{n} \frac{dy_i \, d^2 k_{i_\perp}}{4\pi \, (2\pi)^2} \right) \frac{d^2 k_{n+1_\perp}}{(2\pi)^2} (2\pi)^2 \delta^2\left(\sum_{i=0}^{n+1} k_{i_\perp} \right)$$



Next, we want to compute the discontinuity of the amplitude $ff \to ff$ with exchange of two gluons in the $\hat{t}$ channel. q is the momentum exchanged in the $\hat{t}$ channel

$$\mathrm{Disc}\left[i M^{a a' b b'}_{\mu_a \mu_{a'} \mu_b \mu_{b'}}(\hat{s},\hat{t}) \right]$$

$$= \sum_{n=0}^{\infty} \int \frac{1}{2\hat{s}} \frac{d^2 k_{0\perp}}{(2\pi)^2} \left(\prod_{i=1}^{n} \frac{dy_i\, d^2 k_{i\perp}}{4\pi\, (2\pi)^2} \right) \frac{d^2 k_{n+1\perp}}{(2\pi)^2} (2\pi)^2 \delta^2\left(\sum_{i=0}^{n+1} k_{i\perp} \right)$$

$$(2is)^2 \delta_\perp^{\mu_a \mu_{a'}} \delta_\perp^{\mu_b \mu_{b'}} \left(i g f^{a d_0 c_1} \right) \left(i g f^{c_1' d_0 a'} \right)$$

$$\frac{1}{t_1} \left(\frac{s_1}{-t_1} \right)^{\alpha(t_1)} \frac{1}{t_1'} \left(\frac{s_1}{-t_1'} \right)^{\alpha(t_1')}$$

$$\left(i g f^{c_1 d_1 c_2} \right) \left(i g f^{c_1' d_1 c_2'} \right) C^{\mu_1}(q_1, q_2)\, C^{\mu_1'}(q - q_1, q - q_2) \left(-g_{\mu_1 \mu_1'} \right)$$

$$\vdots$$

$$\left(i g f^{c_n d_n c_{n+1}} \right) \left(i g f^{c_n' d_n c_{n+1}'} \right) C^{\mu_n}(q_n, q_{n+1})\, C^{\mu_n'}(q - q_n, q - q_{n+1}) \left(-g_{\mu_n \mu_n'} \right)$$

$$\frac{1}{t_{n+1}} \left(\frac{s_{n+1}}{-t_{n+1}} \right)^{\alpha(t_{n+1})} \frac{1}{t_{n+1}'} \left(\frac{s_{n+1}}{-t_{n+1}} \right)^{\alpha(t_{n+1}')}$$

$$\left(i g f^{b d_{n+1} c_{n+1}} \right) \left(i g f^{c_{n+1}' d_{n+1} b'} \right)$$

The central-emission vertex is

$$C^\mu(q_i, q_{i+1}) = (q_i + q_{i+1})_\perp^\mu + \left(\frac{\hat{s}_{bi}}{\hat{s}} + 2\frac{\hat{t}_i}{\hat{s}_{ai}}\right) p_a^\mu - \left(\frac{\hat{s}_{ai}}{\hat{s}} + 2\frac{\hat{t}_{i+1}}{\hat{s}_{bi}}\right) p_b^\mu$$

$C^\mu(q - q_i, q - q_{i+1})$ can be obtained by direct calculation, or by inverting the sign of the momentum k_i above. We get

$$C^\mu(q - q_i, q - q_{i+1})$$
$$= \left((q-q_i) + (q - q_{i+1})\right)_\perp - \left(\frac{\hat{s}_{bi}}{\hat{s}} + 2\frac{\hat{t}'_i}{\hat{s}_{ai}}\right) p_a^\mu + \left(\frac{\hat{s}_{ai}}{\hat{s}} + 2\frac{\hat{t}'_{i+1}}{\hat{s}_{bi}}\right) p_b^\mu$$

The contraction of the central-emission vertices yields

$$C^\mu(q_i, q_{i+1})\, C_\mu(q - q_i, q - q_{i+1}) = -2\left(q_\perp^2 - \frac{(q-q_i)_\perp^2\, q_{i+1_\perp}^2 + (q - q_{i+1_\perp})^2 q_{i_\perp}^2}{(q_i - q_{i+1})_\perp^2}\right)$$

$$\equiv -2K(q_i, q_{i+1}) \quad \text{(please check it)}$$

$$M^{aa'bb'}_{\mu_a\mu_{a'}\mu_b\mu_{b'}} = \delta_{\mu_a\mu_{a'}}\,\delta_{\mu_b\mu_{b'}} \sum_T P^{aa'}_{bb'}(T)\, A^T(\hat{s},\hat{t})$$

let us project out the colour state $T' = \underline{1},\ \underline{8}_A$

$$M^{aa'bb'}_{\mu_a\mu_{a'}\mu_b\mu_{b'}}\, P^{hh'}_{aa'}(T') = \delta_{\mu_a\mu_{a'}}\,\delta_{\mu_b\mu_{b'}}\, P^{hh'}_{bb'}(T')\, A^{T'}(s,t)$$

$$1 \Rightarrow \frac{1}{N_c^2-1}\,\delta^{hh'}\delta_{aa'} \qquad 8_A \Rightarrow \frac{1}{N_c}\, f^{aga'} f^{hgh'}$$

$$\begin{array}{llll}
f^{ad_0c_1}\, f^{c_1'd_0a'} & \to & -N_c\,\delta^{c_1c_1'}\delta^{hh'}\dfrac{1}{N_c^2-1} & -\dfrac{N_c}{2}\, f^{c_1gc_1'} f^{hgh'}\dfrac{1}{N_c} \\
f^{c_1d_1c_2}\, f^{c_1'd_1c_2'} & \to & N_c\,\delta^{c_2c_2'} & \dfrac{N_c}{2}\, f^{c_2gc_2'} \\
\vdots & & \vdots & \\
f^{c_nd_nc_{n+1}}\, f^{c_n'd_nc_{n+1}'} & \to & N_c\,\delta^{c_{n+1}c_{n+1}'} & \dfrac{N_c}{2}\, f^{c_{n+1}gc_{n+1}'} \\
f^{bd_{n+1}c_{n+1}}\, f^{c_{n+1}'d_{n+1}b'} & \to & -N_c\,\delta^{bb'} & -\dfrac{N_c}{2}\, f^{bgb'}
\end{array}$$

Note that in the 1^{st} rung of the octet, we used

$$f^{a d_0 c_1} f^{c_1' d_0 a'} f^{a g a'}$$

$$= \frac{1}{2}\left(- f^{a d_0 c_1} f^{c_1' a' d_0} - f^{a' d_0 c_1} f^{a c_1' d_0}\right) f^{a g a'}$$ using that a, a' are dummy

which through the Jacobi identity becomes

$$= \frac{1}{2} f^{c_1' d_0 c_1} f^{c' a d_0} f^{a g a'}$$

$$= -\frac{N_c}{2} f^{c_1 g c_1'}$$

So out of the structure constants, we get a colour factor

$$C_T^{n+2} \quad \text{with} \quad C_T = \begin{cases} N_c & \text{for } \mathbb{1} \\ N_c/2 & \text{for } \underline{8}_A \end{cases}$$

Stripped out of the colour projector, the discontinuity can be written as

$$\mathrm{Disc}\, A^{T}(s,\hat{t}) = 2i\hat{s} \sum_{n=0}^{\infty} \left(-g_s^2 C_T\right)^{n+2} \int \prod_{i=1}^{n} \frac{dy_i}{4\pi} \prod_{j=1}^{n+1} \frac{d^2 q_{j\perp}}{(2\pi)^2}$$

$$\cdot \prod_{k=1}^{n+1} \frac{e^{(y_{k-1}-y_k)(\alpha(\hat{t}_k)+\alpha(\hat{t}'_k))}}{\hat{t}_k \hat{t}'_k} \prod_{m=1}^{n} 2K(q_m, q_{m+1})$$

where we used transverse momentum conservation, and changed integration variables from the transverse momenta of the outgoing gluons to the transverse momenta of the gluons exchanged in the t channel.

We have n integrations over rapidities, and $n+1$ rapidity differences

To disentangle the rapidity integrals, we take the Laplace transform $F_\ell^T(\hat{t}) = \int_0^\infty dy\, e^{-\ell y}\, \mathrm{Disc}\, A^T(\hat{s}_t, \hat{t})$

with $y = y_0 - y_{n+1}$

and change integrations over rapidities to rapidity differences

$\vartheta_1 = y_0 - y_1$

$\vartheta_2 = y_1 - y_2$

$\vdots$

$\vartheta_{n+1} = y_n - y_{n+1}$

Since $Y = y_0 - y_{n+1} = \sum_{i=0}^{n} y_i - y_{i+1} = \sum_{i=1}^{n+1} \vartheta_i$

the overall factor $\hat{s} = -\hat{t}\, e^{Y} = -\hat{t} \prod_{i=0}^{n} e^{\vartheta_i}$

so $$\int dy\, e^{-\ell y} \int \prod_{i=1}^{n} \frac{dy_i}{4\pi} (-t e^{Y}) \prod_{k=1}^{n+1} e^{(y_{k-1}-y_k)(\alpha(\hat{t}_k)+\alpha(\hat{t}'_k))}$$

$$= -t \frac{1}{(4\pi)^n} \prod_{i=1}^{n+1} \int_0^\infty d\vartheta_i\, e^{\vartheta_i[-\ell+1+\alpha(\hat{t}_i)+\alpha(\hat{t}'_i)]}$$

$$= -\frac{t}{(4\pi)^n} \prod_{i=1}^{n+1} \left[\ell - 1 - \alpha(\hat{t}_i) - \alpha(\hat{t}'_i)\right]^{-1}$$

So the Laplace Transform of the discontinuity of the amplitude becomes

$$\mathcal{F}_e^T(\hat{t}) = -2i\,\hat{t}\,(4\pi\alpha_s)^2 C_T^2 \sum_{n=0}^{\infty} \int \prod_{j=1}^{n+1} \frac{d^2 q_{j2}}{(2\pi)^2}$$

$$\cdot \frac{1}{\hat{t}_1 \hat{t}_1'} \frac{1}{\ell-1-\alpha(\hat{t}_1)-\alpha(\hat{t}_1')}$$

$$(-2\alpha_s C_T)\, K(q_1, q_2)$$

$$\cdot \frac{1}{\hat{t}_2 \hat{t}_2'} \frac{1}{\ell-1-\alpha(\hat{t}_2)-\alpha(\hat{t}_2')}$$

$$\vdots$$

$$(-2\alpha_s C_T)\, K(q_n, q_{n+1})$$

$$\frac{1}{\hat{t}_{n+1} \hat{t}_{n+1}'} \cdot \frac{1}{\ell-1-\alpha(\hat{t}_{n+1})-\alpha(\hat{t}_{n+1}')}$$

We can write $F_e^T(t)$ recursively as

$$F_e^T(\hat{t}) = -2i\hat{t}\,(4\pi\alpha_s)^2 C_T^2 \int \frac{d^2 q_{1_\perp}}{(2\pi)^2} \frac{1}{q_{1_\perp}^2 (q-q_1)_\perp^2} f_e^T(q_1, \hat{t})$$

where the function $f_e^T(q,t)$ fulfills the integral equation

$$\left[\ell - 1 - \alpha(\hat{t}_1) - \alpha(\hat{t}_1')\right] f_e^T(q_1, \hat{t}) = 1 - 2\alpha_s C_T \int \frac{d^2 q_{2_\perp}}{(2\pi)^2} \frac{K(q_1, q_2)}{q_{2_\perp}^2 (q-q_2)_\perp^2} f_e^T(q_2, \hat{t})$$

This is the BFKL equation describing the gluon ladder exchanged in the $\hat{t}$ channel. It is made of two parts:

- the central-emission vertex (squared), which, through $K(q_1, q_2)$ forms the kernel of the BFKL equation
- the gluon Regge Trajectory $\alpha(\hat{t})$, which enters the

homogeneous part of the equation.

Also, note that the equation evolves in the 2-dim transverse momentum space, so we have replaced

$$\hat{t}_i = -\vec{q}_{i_2}^{\,2} \qquad \hat{t}_i = -(\bar{q}-\bar{q}_i)_2^2$$

The equation is valid for either singlet or octet exchange.

Firstly, we are going to solve it for the octet. Then

$$C_T = \frac{N_c}{2} \qquad \text{Also} \quad \alpha(\hat{t}) = \alpha_s N_c \hat{t} \int \frac{d^2 k_2}{(2\pi)^2} \frac{1}{k_2^2 (q-k)_2^2}$$

and we set $\omega = l-1$

$$\left[\omega - \alpha(\hat{t}_1) - \alpha(\hat{t}_1')\right] f_\omega^{(8_A)}(q_1, \hat{t}) = 1 - \alpha_s N_c \int \frac{d^2 k_2}{(2\pi)^2} \frac{K(q_1,k)}{k_2^2 (q-k)_2^2} f_\omega^{(8_A)}(k, \hat{t})$$

let us try a constant solution $f_\omega^{(8_A)}(k,t) = f_\omega^{(8_A)}(q_1,t) = f_\omega^{(8_A)}(t)$

$$\left\{\omega + \alpha_s N_c q_{1_2}^2 \int \cancel{\frac{d^2k_2}{(2\pi)^2} \frac{1}{k_2^2 (q_1-k)_2^2}} + \alpha_s N_c (q-q_1)_2^2 \int \frac{d^2k_2}{(2\pi)^2} \frac{1}{k_2^2 (q-q_1-k)_2^2}\right\} f_\omega^{(8_A)}$$

$$= 1 - \alpha_s N_c \int \frac{d^2k_2}{(2\pi)^2} \left[q_2^2 - \frac{(q-q_1)_2^2 k_2^2 + \cancel{(q-k)_2^2 q_{1_2}^2}}{(q_1-k)_2^2} \right] \frac{1}{k_2^2 (q-k)_2^2} f_\omega^{(8_A)}(t)$$

shift $k_2 \to (q-k)_2$ then the last terms on the l.h.s and on the r.h.s. cancel. We are left with

$$\omega f_\omega^{(8_A)}(t) = 1 - \underbrace{\alpha_s N_c q_2^2 \int \frac{d^2k_2}{(2\pi)^2} \frac{1}{k_2^2 (q-k)_2}}_{\alpha(t)} f_\omega^{(8_A)}(t)$$

the solution is $f_\omega^{(8_A)}(t) = \dfrac{1}{\omega - \alpha(t)}$ a Regge pole!

Substituting it into the Laplace Transform, we have

$$\mathbb{F}_\omega^{(8_A)} = -2i\,\hat{t}\,(4\pi\alpha_s)^2 \frac{N_c^2}{4} \int \frac{d^2k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 (q-k)_\perp^2} \frac{1}{\omega - \alpha(\hat{t})}$$

using the Regge trajectory, this can be written as

$$\mathbb{F}_\omega^{(8_A)} = -8i\pi^2\alpha_s N_c \frac{\alpha(\hat{t})}{\omega - \alpha(\hat{t})}$$

then, we replace it into the Sommerfeld-Watson representation of the amplitude for octet exchange,

$$A^{(8_A)}(\hat{s},\hat{t}) = -8i\pi^2\alpha_s N_c \frac{1}{4\pi} \int_{\delta-i\infty}^{\delta+i\infty} dl\, \frac{(-)^l - 1}{\sin \pi l}\, e^{ly} \frac{\alpha(\hat{t})}{l-1-\alpha(\hat{t})}$$

where the octet parity is $(-)^T = -1$

we pick up the Regge pole (assuming that it is the rightmost singularity), and we get

$$A^{(8_a)}(\hat{s},\hat{t}) = -2i\pi\,\alpha_s N_c\, 2\pi i\, \frac{-1+(-)^{1+\alpha(t)}}{\sin\pi(1+\alpha(t))}\,\alpha(\hat{t})\, e^{y(1+\alpha(\hat{t}))}$$

$$= 4\pi\alpha_s N_c\, \frac{\pi\alpha(\hat{t})}{\sin\pi\alpha(\hat{t})}\left(1+e^{i\pi\alpha(t)}\right)\left(\frac{\hat{s}}{-\hat{t}}\right)^{1+\alpha(\hat{t})}$$

Note that

- we started with a multigluon amplitude which exchanges a gluon, i.e. an octet, in the $\hat{t}$ channel, with virtual corrections given by a Regge trajectory, and we found, consistently, that the octet solution of the BFKL equation is a Regge pole

- we started with a discontinuity at $O(\alpha_s^2)$, which computes the radiative corrections to a two-gluon ladder, but the octet solution is $O(\alpha_s)$, because the octet occurs already in one-gluon exchange

- if, in the Regge trajectory, we take $\alpha_s \ln \frac{Q^2}{\mu^2} \ll 1$, and thus approximate $\pi |\alpha(\hat{t})| \ll 1$ everywhere except in the exponent, we get

$$A^{(8_A)}(s,t) \sim 8\pi N_c \alpha_s \left(\frac{s}{-t}\right)^{1+\alpha(\hat{t})}$$

in agreement with the ansatz for the exponentiation of the one-loop amplitude