The Regge limit in QCD and in N=4 super-Yang-Mills

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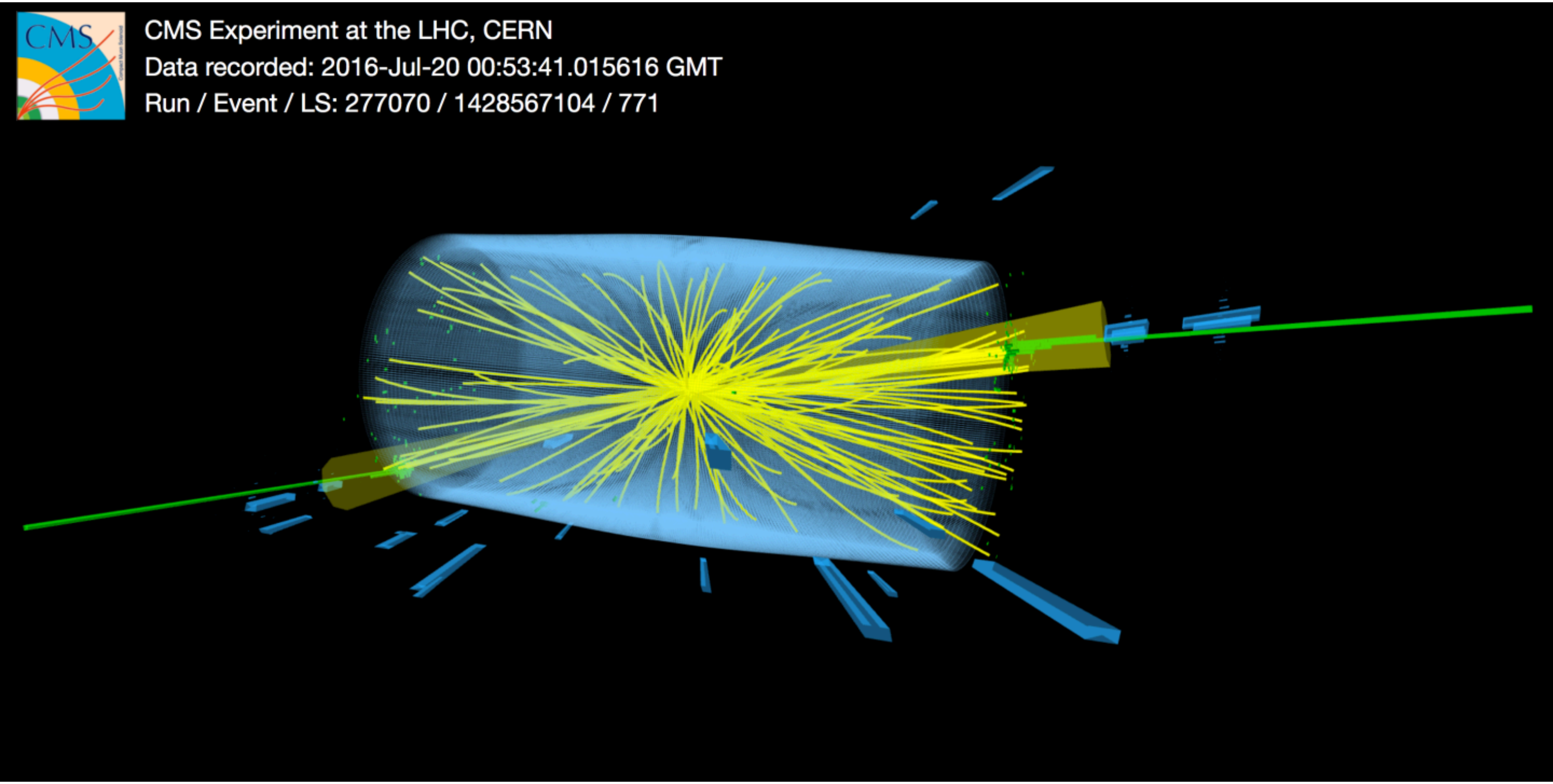


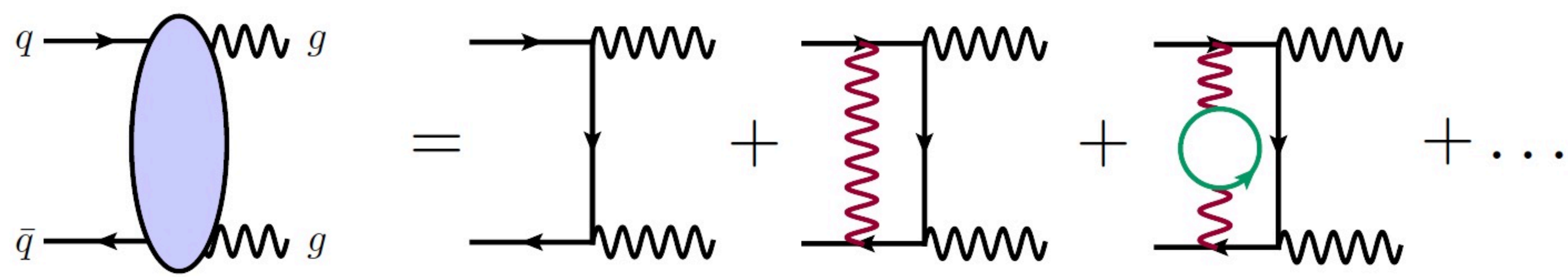
Production of two high-energy jets as recorded by CMS at the LHC

- At the LHC, particles are produced through the head-on collisions of protons, and in particular through the collisions of quarks and gluons within the protons
- The probability of a collision event is computed through the cross section, which is given as an integral of (squared) scattering amplitudes over the phase space of the produced particles
- The scattering amplitudes are given as a power series (loop expansion) in the strong and/or electroweak couplings

→ The scattering amplitudes are fundamental objects of particle physics

Scattering amplitudes

- The scattering amplitudes are given as a loop-momentum expansion in the strong and/or electroweak couplings



- The more terms we know in the loop expansion, the more precisely we can compute the cross section

- As a matter of fact, we know any amplitude of interest at one loop, several (2 → 2) amplitudes at two loops, just a couple (2 → 1) at three loops, and nothing beyond that

- What outcome do we expect from the loop expansion of an amplitude?
- From renormalisability and the infrared structure of the amplitude, we expect that the divergent parts are logarithmic functions of the external momenta
- But, except for unitarity, we have little guidance for the finite parts.
  Heuristically, we know that:
  — at one loop, logarithmic and dilogarithmic functions of the external momenta occur
  — beyond one loop, higher polylogarithmic functions appear and elliptic functions may appear

- In the last few years, a lot of progress has been made in understanding the analytic structure of multi-loop amplitudes, in particular on how the polylogarithmic functions appear at any loop level

- In particular, a lot of progress has been made:
— in $\mathcal{N}$=4 Super Yang-Mills (SYM)
— in the Regge limit of QCD
— in the Regge limit of $\mathcal{N}$=4 SYM

One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane …

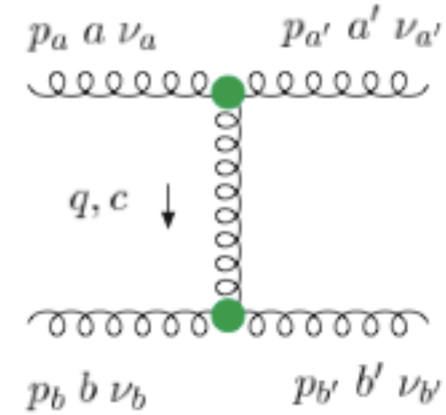
incipit
The analytic S-matrix
Eden Landshoff Olive Polkinghorne 1966

Regge limit
in the
leading logarithmic approximation

Regge limit of QCD

- In perturbative QCD, in the Regge limit $s \gg t$, any scattering process is dominated by gluon exchange in the t channel

- For a 4-gluon tree amplitude, we obtain



$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2 \left[(T^c)_{aa'} C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right] \frac{s}{t} \left[(T_c)_{bb'} C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

$C_{\nu_a\nu_{a'}}(p_a,p_{a'})$ are called *impact factors*

- leading logarithms of s/t are obtained by the substitution $\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$

- $\alpha(t)$ is the Regge gluon trajectory, with infrared coefficients

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right) \qquad \alpha_s(-t,\epsilon) = \left(\frac{\mu^2}{-t}\right)^{\epsilon} \alpha_s(\mu^2)$$

$$\alpha^{(1)} = C_A \frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A \frac{2}{\epsilon} \qquad \alpha^{(2)} = C_A\left[-\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)}\frac{2}{\epsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]$$

- in the Regge limit, the amplitude is invariant under $s \leftrightarrow u$ exchange.

  To NLL accuracy, the amplitude is given by — Fadin Lipatov 1993

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^c)_{aa'} C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[(T_c)_{bb'} C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

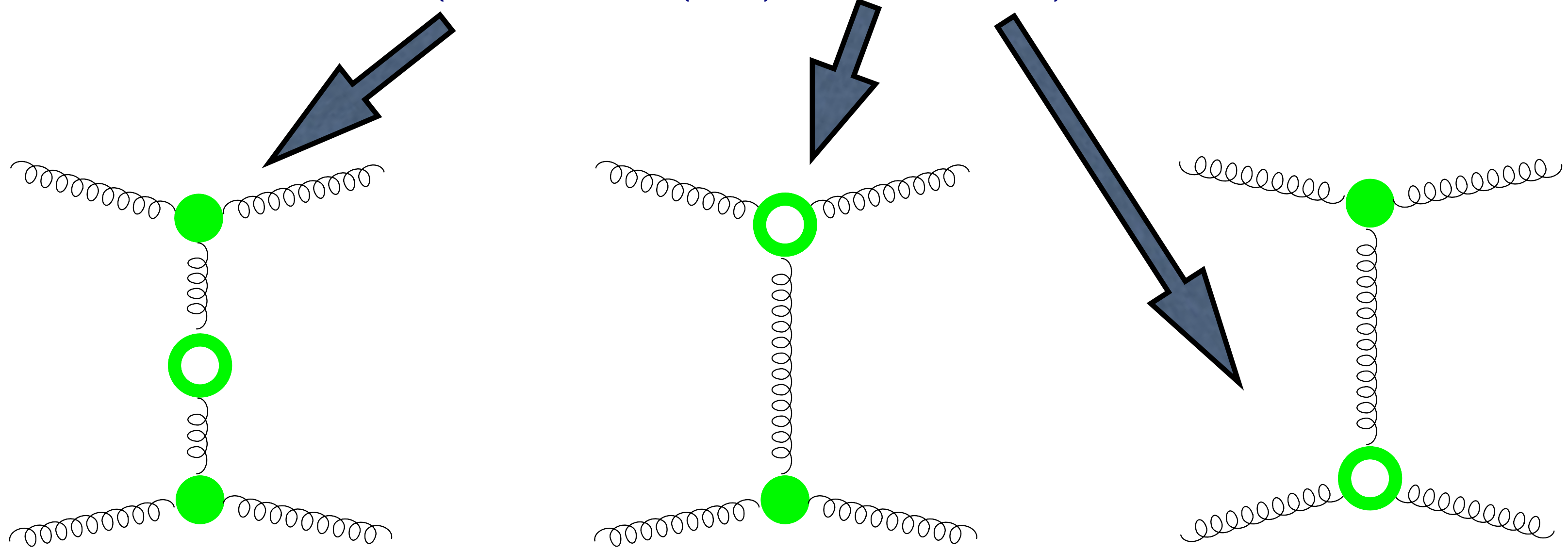
Amplitudes in the Regge limit

- Regge limit of the gluon-gluon amplitude

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^c)_{aa'}C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[(T_c)_{bb'}C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

strip colour off & expand at one loop

$$m^{(1)}_{gg\to gg} = 2\,g_s^2\,\frac{s}{t}\left(\alpha^{(1)}(t)\ln\left(\frac{s}{-t}\right)+2\,C^{(1)}_{gg}(t)\right)$$



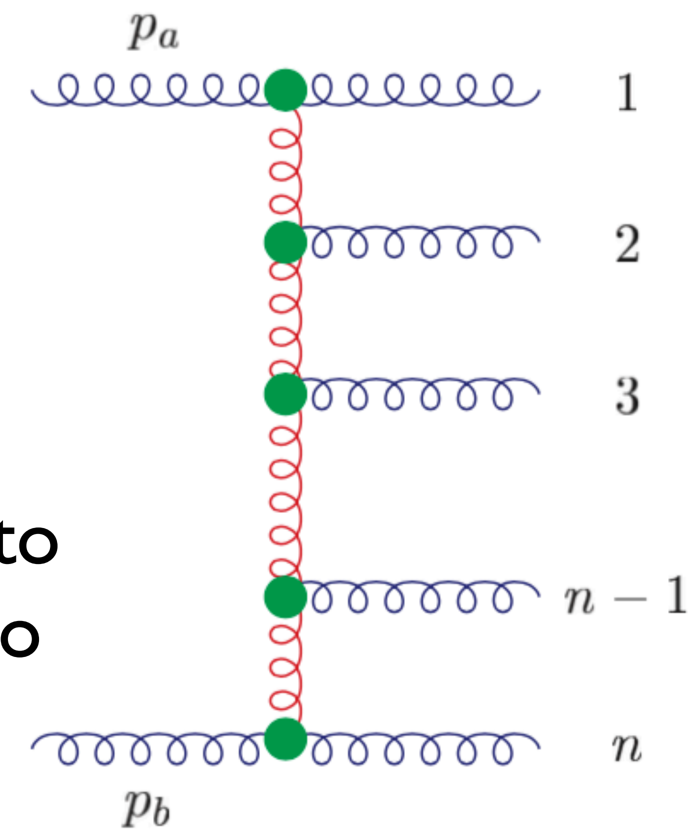
the Regge gluon trajectory is universal;
the one-loop gluon impact factor is a polynomial in t, ε, starting at $1/\varepsilon^2$

- perform the Regge limit of the quark-quark amplitude
  → get one-loop quark impact factor

- if factorisation holds, one can obtain the one-loop quark-gluon amplitude
  by assembling the Regge trajectory and the gluon and quark impact factors
  the result should match the quark-gluon amplitude in the high-energy limit: it does

Balitski Fadin Kuraev Lipatov

- BFKL is a resummation of multiple gluon radiation out of the gluon exchanged in the *t* channel



- the Leading Logarithmic (BFKL 1976-77) and Next-to-Leading Logarithmic (Fadin-Lipatov 1998) contributions in log(s/|t|) of the radiative corrections to the gluon propagator in the *t* channel are resummed to all orders in α_s

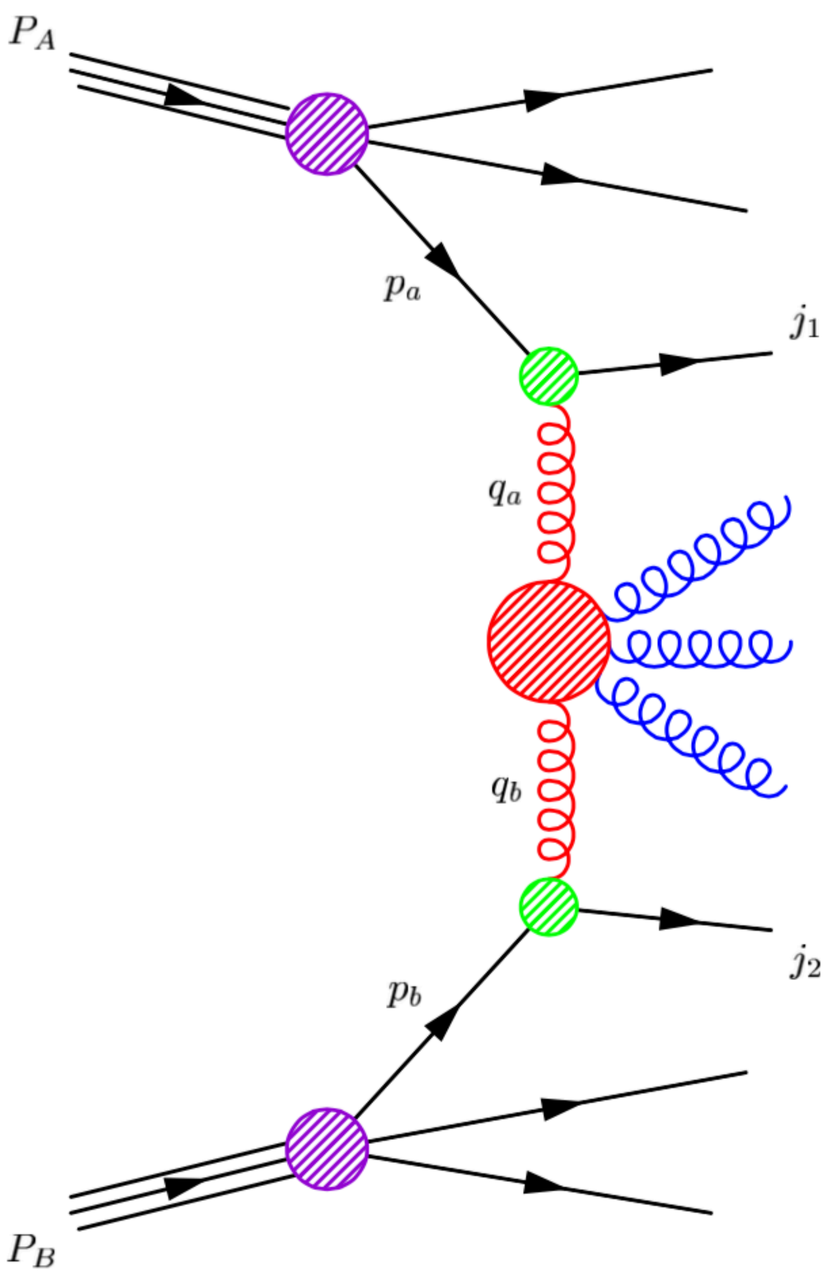
- the resummation yields an integral (BFKL) equation for the evolution of the gluon propagator in 2-dim transverse momentum space

- the BFKL equation is obtained in the limit of strong rapidity ordering of the emitted gluons, with no ordering in transverse momentum - *multi-Regge kinematics* (MRK)

- the solution is a Green's function of the momenta flowing in and out of the gluon ladder exchanged in the *t* channel

Mueller-Navelet jets

Mueller Navelet 1987



Dijet production cross section with two tagging jets in the forward and backward directions

$p_a = x_a P_A \quad p_b = x_b P_B$ incoming parton momenta

S: hadron centre-of-mass energy

$s = x_a x_b S$: parton centre-of-mass energy

E_{Tj}: jet transverse energies

$$\Delta y = |y_{j_1} - y_{j_2}| \simeq \log \frac{s}{E_{Tj_1} E_{Tj_2}}$$

is the rapidity interval between the tagging jets

gluon radiation is considered in MRK and resummed through the LL BFKL equation

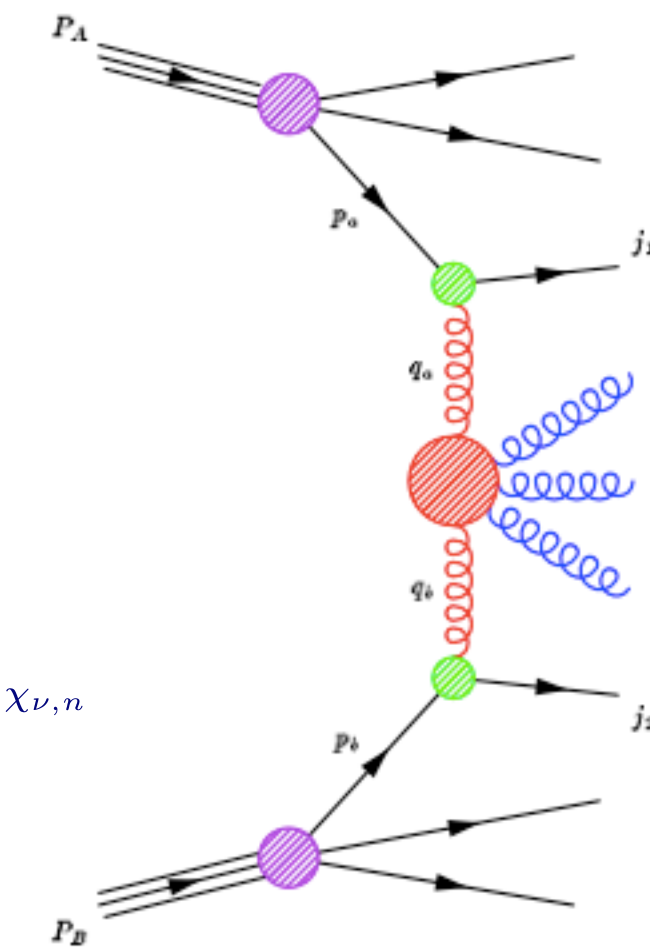
Mueller-Navelet evaluated the inclusive dijet cross section up to 5 loops

Mueller-Navelet dijet cross section

- the cross section for dijet production at large rapidity intervals $\Delta y = y_1 - y_2 = \ln\left(\frac{\hat{s}}{-t}\right) \gg 1$

  with $\hat{s} = x_a x_b S\,, \quad t = -\sqrt{p_{1\perp}^2 p_{2\perp}^2}$

$$\frac{d\hat{\sigma}_{gg}}{dp_{1\perp}^2\, dp_{2\perp}^2\, d\phi_{jj}} = \frac{\pi}{2}\left[\frac{C_A \alpha_s}{p_{1\perp}^2}\right] f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y) \left[\frac{C_A \alpha_s}{p_{2\perp}^2}\right]$$



can be described through the BFKL Green's function

$$f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y) = \frac{1}{(2\pi)^2 \sqrt{q_{1\perp}^2\, q_{2\perp}^2}} \sum_{n=-\infty}^{+\infty} e^{in\phi} \int_{-\infty}^{+\infty} d\nu \left(\frac{q_{1\perp}^2}{q_{2\perp}^2}\right)^{i\nu} e^{\eta\, \chi_{\nu,n}}$$

with $\eta \equiv \frac{C_A\, \alpha_s}{\pi} \Delta y$ and ϕ the angle between q_1^2 and q_2^2

and the LL BFKL eigenvalue

$$\chi_{\nu,n} = -2\gamma_E - \psi\left(\frac{1}{2} + \frac{|n|}{2} + i\nu\right) - \psi\left(\frac{1}{2} + \frac{|n|}{2} - i\nu\right)$$

Mueller-Navelet dijet cross section

- azimuthal angle distribution ($\phi_{jj} = \phi - \pi$)

$$\frac{d\hat{\sigma}_{gg}}{d\phi_{jj}} = \frac{\pi(C_A\alpha_s)^2}{2E_\perp^2}\left[\delta(\phi_{jj}-\pi) + \sum_{k=1}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{e^{in\phi}}{2\pi}f_{n,k}\right)\eta^k\right]$$

with $f_{n,k} = \frac{1}{2\pi}\frac{1}{k!}\int_{-\infty}^{\infty} d\nu\, \frac{\chi_{\nu,n}^k}{\nu^2+\frac{1}{4}}$

- the dijet cross section is $\hat{\sigma}_{gg} = \frac{\pi(C_A\alpha_s)^2}{2E_\perp^2}\sum_{k=0}^{\infty} f_{0,k}\,\eta^k$ Mueller Navelet 1987

with

$$\begin{aligned}
f_{0,0} &= 1\,, \\
f_{0,1} &= 0\,, \\
f_{0,2} &= 2\zeta_2\,, \\
f_{0,3} &= -3\zeta_3\,, \\
f_{0,4} &= \frac{53}{6}\,\zeta_4\,, \\
f_{0,5} &= -\frac{1}{12}\,(115\zeta_5 + 48\zeta_2\zeta_3)
\end{aligned}$$

N=4 Super Yang Mills

- maximal supersymmetric theory (without gravity)
  conformally invariant, β fn. = 0
  - spin 1 gluon
    4 spin 1/2 gluinos
    6 spin 0 real scalars
- 't Hooft limit: $N_c \to \infty$ with $\lambda = g^2 N_c$ fixed
  - only planar diagrams
- AdS/CFT duality — Maldacena 97
  - large-λ limit of 4dim CFT ↔ weakly-coupled string theory

    (aka weak-strong duality)

N=4 Super Yang Mills

- amplitudes in *N*=4 SYM are much simpler than in Standard Model processes
- use *N*=4 SYM as a computational lab:
  - to learn techniques and tools to be used in Standard Model calculations
  - to learn about the bases of special functions which may occur in the scattering processes

N=4 Super Yang Mills

- In the last years, a huge progress has been made in understanding the analytic structure of the *S*-matrix of *N*=4 SYM

- Besides the ordinary conformal symmetry, in the planar limit the *S*-matrix exhibits a dual conformal symmetry

  Drummond Henn Smirnov Sokatchev 2006

- Accordingly, the analytic structure of the scattering amplitudes is highly constraint

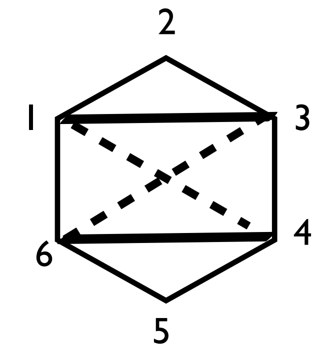
- 4- and 5-point amplitudes are fixed to all loops by the symmetries in terms of the one-loop amplitudes and the cusp anomalous dimension

  Anastasiou Bern Dixon Kosower 2003, Bern Dixon Smirnov 2005
  Drummond Henn Korchemsky Sokatchev 2007

- Beyond 5 points, the finite part of the amplitudes is given in terms of a remainder function *R*. The symmetries only fix the variables of *R* (some conformally invariant cross ratios) but not the analytic dependence of *R* on them

- for $n = 6$, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

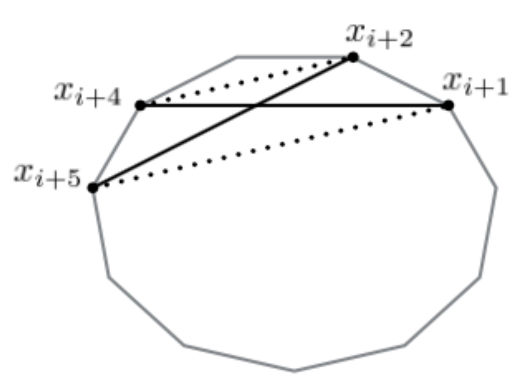


x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

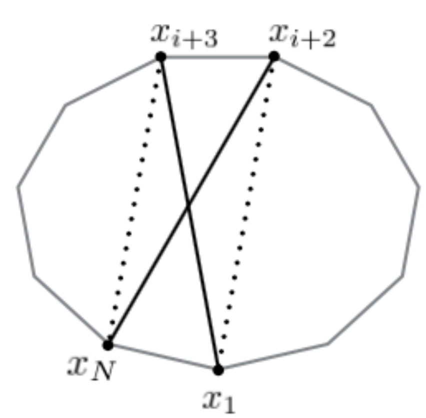
thus $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$

- for n points, dual conformal invariance implies dependence on $3n$-15 independent cross ratios

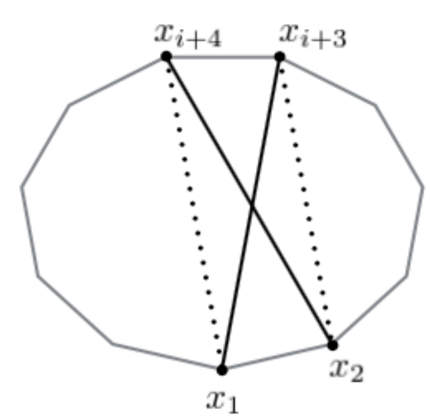
$$u_{1i} = \frac{x_{i+1,i+5}^2 x_{i+2,i+4}^2}{x_{i+1,i+4}^2 x_{i+2,i+5}^2}\,, \qquad u_{2i} = \frac{x_{N,i+3}^2 x_{1,i+2}^2}{x_{N,i+2}^2 x_{1,i+3}^2}\,, \qquad u_{3i} = \frac{x_{1\,i+4}^2 x_{2,i+3}^2}{x_{1,i+3}^2 x_{2,i+4}^2}$$



u_{1i}



u_{2i}



u_{3i}

Scattering amplitudes

- Scattering amplitudes of gluons are functions of:
  — the external momenta
  — colour
  — helicities:
    — maximally helicity violating (- - ++…+)
    — next-to-maximally helicity violating (- - - ++…+)
    — and so on

Scattering amplitudes in *N*=4 SYM

- The progress in understanding the analytic structure of the *S*-matrix in planar *N*=4 SYM is also due to an improved understanding of the mathematical structures underlying the scattering amplitudes

- *n*-point amplitudes are expected to be written in terms of iterated integrals (on the space of configurations of points in 3-dim projective space $\mathrm{Conf}_n(\mathrm{CP}^3)$)

  Golden Paulos Spradlin Volovich 2014

- The simplest case of iterated integrals are the iterated integrals over rational functions, i.e. the multiple polylogarithms

  Goncharov 2001

$$G(a,\vec{w};z) = \int_0^z \frac{\mathrm{d}t}{t-a}\, G(\vec{w};t)\,, \qquad G(a;z) = \ln\left(1-\frac{z}{a}\right) \qquad G(0,1;z) = -\mathrm{Li}_2(z)$$

- It is thought that maximally helicity violating (MHV) and next-to-MHV (NMHV) amplitudes can be expressed in terms of multiple polylogarithms of uniform transcendental weight

  Arkani-Hamed et al. *Scattering Amplitudes and the Positive Grassmannian* 2012

Scattering amplitudes in $\mathcal{N}=4$ SYM

- MHV and NMHV amplitudes feature *maximal transcendentality*, i.e. *L*-loop amplitudes are expressed in terms of multiple polylogarithms of weight 2*L* only

- MHV amplitudes are *pure*, i.e. the coefficients of the multiple polylogarithms are (rational) numbers

Scattering amplitudes in *N*=4 SYM

- 6-pt (N)MHV amplitudes are known analytically up to 5(4) loops

  Duhr Smirnov VDD 2009
  Goncharov Spradlin Vergu Volovich 2010
  Dixon Drummond Henn 2011
  Dixon Drummond von Hippel Pennington 2013
  Dixon Drummond Duhr Pennington 2014
  Caron-Huot Dixon von Hippel McLeod 2016

  Dixon Drummond Henn 2011
  Dixon von Hippel 2014
  Dixon von Hippel McLeod 2015

- 7-pt MHV amplitudes are known analytically at two loops

  Golden Spradlin 2014

- No analytic result is known beyond 7 points.
  The *cluster algebra* which defines the multiple polylogarithms
  is infinite starting from 8 points

  Golden Goncharov Spradlin Vergu Volovich 2013

Multi-Regge limit of $\mathcal{N}=4$ SYM

- In the Euclidean region (where all Mandelstam invariants are negative), amplitudes in MRK factorise completely in terms of building blocks which are expressed in terms of Regge poles and can be determined to all orders through the 4-pt and 5-pt amplitudes.
  Thus the remainder functions R vanish at all points Duhr Glover VDD 2008

- After analytic continuation to some regions of the Minkowski space, the amplitudes develops cuts which are described by a dispersion relation for octet exchange, which is similar to the singlet BFKL equation in QCD
  Bartels Lipatov Sabio-Vera 2008

- Accordingly, 6-pt amplitudes have been thoroughly examined, both at weak and at strong coupling

- In particular, 6-pt amplitudes at weak coupling can be expressed in terms of single-valued harmonic polylogarithms
  Dixon Duhr Pennington 2012

Regge factorisation of the n-pt amplitude

$$m_n(1,2,\ldots,n) = s\,[g\,C(p_2,p_3)]\,\frac{1}{t_{n-3}}\left(\frac{-s_{n-3}}{\tau}\right)^{\alpha(t_{n-3})}[g\,V(q_{n-3},q_{n-4},\kappa_{n-4})]$$
$$\cdots\times\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa_1)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_n)]$$

n-pt amplitude in the multi-Regge limit

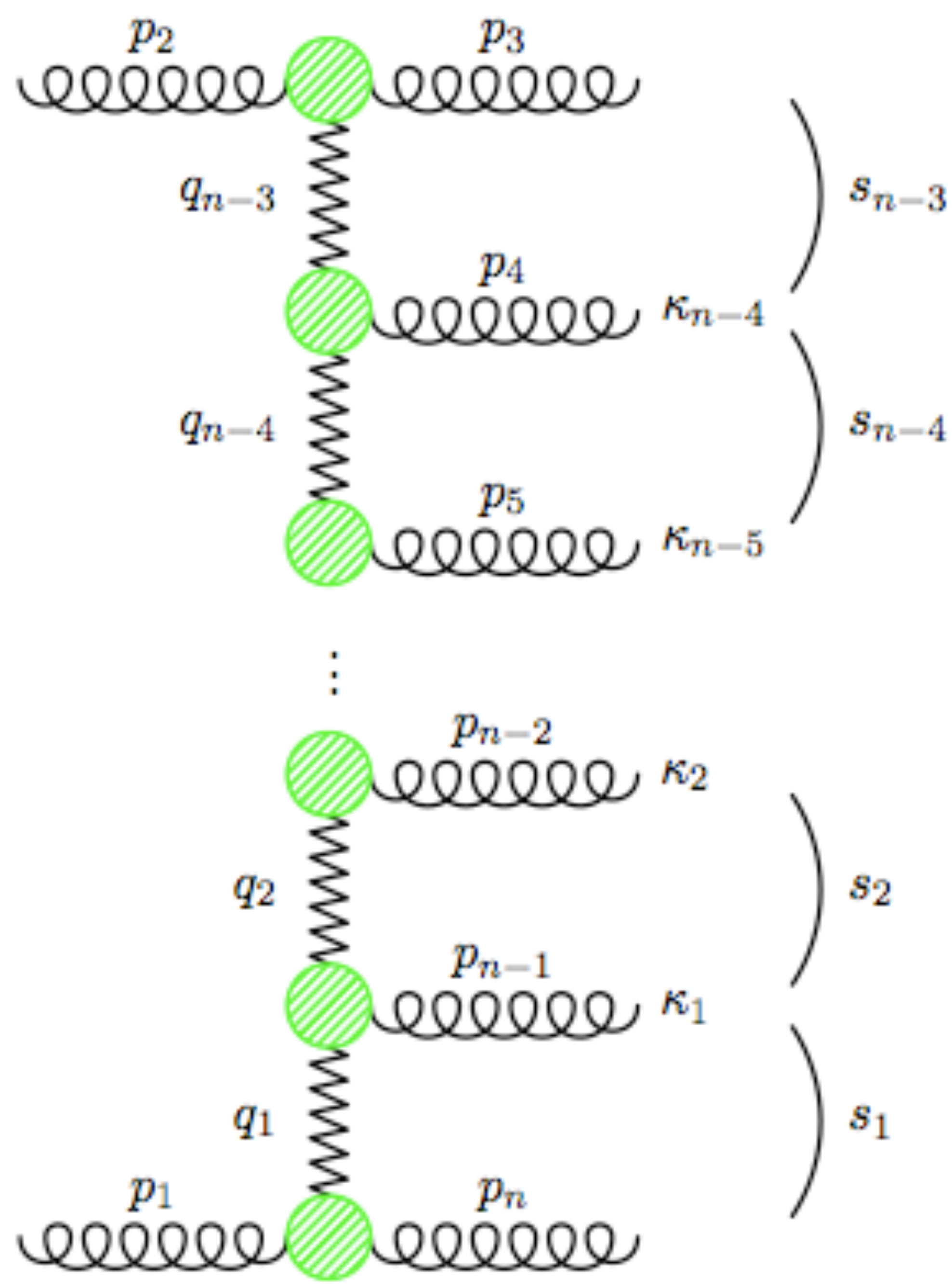
$$y_3 \gg y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|$$

$$s \gg s_1, s_2, \ldots, s_{n-3} \gg -t_1, -t_2 \ldots, -t_{n-3}$$

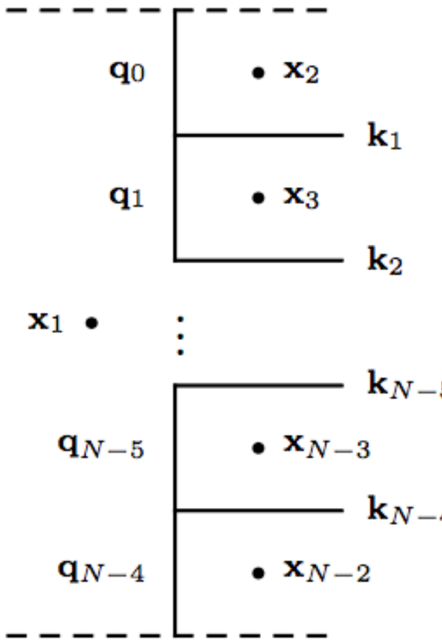
the *l*-loop *n*-pt amplitude can be assembled using the *l*-loop trajectories, vertices and coefficient functions, determined through the *l*-loop 4-pt and 5-pt amplitudes

➡ in Euclidean space,
no violation of the BDS ansatz can be found in the multi-Regge limit

Duhr Glover VDD 2008



Moduli space of Riemann spheres



- in MRK, there is no ordering in transverse momentum, i.e. only the n-2 transverse momenta are non-trivial
- dual conformal invariance in transverse momentum space implies dependence on n-5 cross ratios of the transverse momenta

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})} = -\frac{\mathbf{q}_{i+1}\,\mathbf{k}_i}{\mathbf{q}_{i-1}\,\mathbf{k}_{i+1}} \qquad i = 1, \ldots, n-5$$

- $\mathcal{M}_{0,p}$ = space of configurations of p points on the Riemann sphere
- Because we can fix 3 points at 0, 1, ∞, its dimension is $\dim(\mathcal{M}_{0,p}) = p-3$
- $\mathcal{M}_{0,n-2}$ is the space of the MRK, with $\dim(\mathcal{M}_{0,n-2}) = n-5$
- Its coordinates can be chosen to be the z_i's, i.e. the cross ratios of the transverse momenta

Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek VDD 2016

Iterated integrals on $\mathcal{M}_{0,n-2}$

- on $\mathcal{M}_{0,n-2}$, the singularities are associated to degenerate configurations when two points merge $x_i \to x_{i+1}$
  i.e. when momentum p_i becomes soft $p_i \to 0$

- iterated integrals on $\mathcal{M}_{0,p}$ can be written as multiple polylogarithms
  Brown 2006

➡ amplitudes in MRK can be written in terms of multiple polylogarithms

- analytic structure of amplitudes is constrained by unitarity and the optical theorem $\mathrm{Disc}(M) = iMM^\dagger$

- massless amplitudes may have branch points when Mandelstam invariants vanish $s_{ij} \to 0$ or become infinite $s_{ij} \to \infty$, but branch cuts are constrained by unitarity

Hopf algebra and the coproduct

- algebra is a vector space with a product $\mu: A \otimes A \to A$ $\quad \mu(a \otimes b) = a \cdot b$
  that is associative $A \otimes A \otimes A \to A \otimes A \to A$ $\quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- coalgebra is a vector space with a coproduct $\Delta: B \to B \otimes B$
  that is coassociative $B \to B \otimes B \to B \otimes B \otimes B$

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$$

- μ puts together; Δ decomposes
- a Hopf algebra is an algebra and a coalgebra,
  such that product and coproduct are compatible $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$
- multiple polylogarithms form a Hopf algebra with a *coproduct* — Goncharov 2002

$$\Delta(L_w) = \sum_{k=0}^{w} \Delta_{k,w-k}(L_w) = \sum_{k=0}^{w} L_k \otimes L_{w-k}$$

- in particular, on classical polylogarithms

$$\Delta(\ln z) = 1 \otimes \ln z + \ln z \otimes 1$$

$$\Delta\big(\mathrm{Li}_2(z)\big) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \ln(1-z) \otimes \ln z$$

- the two entries of the coproduct are the discontinuity and the derivative

$$\Delta \mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\Delta \qquad \Delta \partial = (\mathrm{id} \otimes \partial)\Delta$$

Duhr 2012

- then the coproduct of an amplitude is related to unitarity

- thus, for massless amplitudes

$$\Delta(M) = \ln(s_{ij}) \otimes \ldots$$

- in particular, for amplitudes in MRK

$$\Delta(M) = \ln |\mathbf{x}_i - \mathbf{x}_j|^2 \otimes \ldots$$

- except for the soft limit $p_i \to 0$, in MRK the transverse momenta never vanish

$|\mathbf{x}_i - \mathbf{x}_j|^2 \neq 0$ ⟹ single-valued functions

- therefore, *n*-point amplitudes in MRK of planar *N*=4 SYM can be written in terms of single-valued iterated integrals on $\mathcal{M}_{0,n-2}$

Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek VDD 2016

- for n=6, iterated integrals on $\mathcal{M}_{0,4}$ are harmonic polylogarithms
thus, 6-point amplitudes in MRK of can be written in terms of single-valued harmonic polylogarithms (SVHPL)

Dixon Duhr Pennington 2012

Harmonic polylogarithms

- classical polylogarithms $\mathrm{Li}_m(z) = \int_0^z dz' \frac{\mathrm{Li}_{m-1}(z')}{z'}$

- harmonic polylogarithms (HPLs)

$$H(a,\vec{w};z) = \int_0^z \mathrm{d}t\, f(a;t)\, H(\vec{w};t) \qquad f(-1;t) = \frac{1}{1+t}\,, \quad f(0;t) = \frac{1}{t}\,, \quad f(1;t) = \frac{1}{1-t}$$

with $\{a,\vec{w}\} \in \{-1,0,1\}$ Remiddi Vermaseren 1999

- HPLs obey the differential equations

$$\frac{d}{dz} H_{0\,\omega}(z) = \frac{H_\omega(z)}{z}\,, \qquad \frac{d}{dz} H_{1\,\omega}(z) = \frac{H_\omega(z)}{1-z}$$

subject to the constraints

$$H(z) = 1, \qquad H_{\vec{0}_n}(z) = \frac{1}{n!}\ln^n z, \qquad \lim_{z\to 0} H_{\omega\neq\vec{0}_n}(z) = 0$$

- HPLs form a shuffle algebra

$$H_{\omega_1}(z)\, H_{\omega_2}(z) = \sum_\omega H_\omega(z)$$ with ω the shuffle of ω_1 and ω_2

- HPLs are multi-valued functions on the complex plane

Single-valued polylogarithms

- Single-valued functions are real analytic functions on the complex plane

- Because the discontinuities of the classical polylogarithms are known

$$\Delta \mathrm{Li}_n(z) = 2\pi i \frac{\log^{n-1} z}{(n-1)!}$$

one can build combinations of classical polylogarithms
such that all branch cuts cancel on the punctured plane C/{0,1}
(Riemann sphere with punctures)

- An example is the Bloch-Wigner dilogarithm

$$D_2(z) = \mathrm{Im}[\mathrm{Li}_2(z)] + \arg(1-z)\log|z|$$

Single-valued harmonic polylogarithms

- define a function $\mathscr{L}$ that is real-analytic and single-valued on $\mathbb{C}/\{0,1\}$ and that has the same properties as the HPLs

  it obeys the differential equations

$$\frac{\partial}{\partial z}\mathcal{L}_{0\,\omega}(z) = \frac{\mathcal{L}_\omega(z)}{z} \qquad \frac{\partial}{\partial z}\mathcal{L}_{1\,\omega}(z) = \frac{\mathcal{L}_\omega(z)}{1-z}$$

  subject to the constraints

$$\mathcal{L}_e(z) = 1\,, \qquad \mathcal{L}_{\vec{0}_n}(z) = \frac{1}{n!}\ln^n|z|^2 \qquad \lim_{z\to 0}\mathcal{L}_{\omega\neq\vec{0}_n}(z) = 0$$

  the SVHPLs $\mathscr{L}_\omega(z)$ also form a shuffle algebra

$$\mathcal{L}_{\omega_1}(z)\,\mathcal{L}_{\omega_2}(z) = \sum_\omega \mathcal{L}_\omega(z)$$ with ω the shuffle of ω_1 and ω_2

- SVHPLs can be explicitly expressed as combinations of HPLs such that all the branch cuts cancel

  Brown 2004

- examples

$$\mathcal{L}_0(z) = H_0(z) + H_0(\bar{z}) = \ln|z|^2$$

$$\mathcal{L}_1(z) = H_1(z) + H_1(\bar{z}) = -\ln|1+z|^2$$

$$\mathcal{L}_{0,1}(z) = \frac{1}{4}\left[-2H_{1,0} + 2\bar{H}_{1,0} + 2H_0\bar{H}_1 - 2\bar{H}_0H_1 + 2H_{0.1} - 2\bar{H}_{0,1}\right]$$

$$= \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln|z|^2\left(\ln(1-z) - \ln(1-\bar{z})\right)$$

Single-valued multiple polylogarithms

- Single-valued multiple polylogarithms (SVMPL) can be constructed through a map that to each multiple polylogarithm associates its single-valued version

Brown 2004, 2013, 2015

examples of SVMPLs

$$\mathcal{G}_a(z) = G_a(z) + G_{\bar{a}}(\bar{z}) = \ln\left|1 - \frac{z}{a}\right|^2$$

$$\begin{aligned}\mathcal{G}_{a,b}(z) = &\, G_{a,b}(z) + G_{\bar{b},\bar{a}}(\bar{z}) + G_b(a)G_{\bar{a}}(\bar{z}) + G_{\bar{b}}(\bar{a})G_{\bar{a}}(\bar{z}) \\ &- G_a(b)G_{\bar{b}}(\bar{z}) + G_a(z)G_{\bar{b}}(\bar{z}) - G_{\bar{a}}(\bar{b})G_{\bar{b}}(\bar{z})\end{aligned}$$

MRK in *N*=4 SYM

- In MRK, 6-pt MHV and NMHV amplitudes are known at any number of loops

  Lipatov Prygarin 2010-2011
  Dixon Duhr Pennington 2012
  Pennington 2012
  Lipatov Prygarin Schnitzer 2012

- Beyond 6 points, only 2-loop MHV amplitudes were known in MRK at LLA

  In MRK at LLA, the 2-loop *n*-pt remainder function $R_n^{(2)}$
  can be written as a sum of 2-loop 6-pt remainder functions $R_6^{(2)}$

  Prygarin Spradlin Vergu Volovich 2011
  Bartels Kormilitzin Lipatov Prygarin 2011
  Bargheer Papathanasiou Schomerus 2015

- In MRK at LLA, we can compute all MHV amplitudes at L loops
  in terms of amplitudes with up to (L+4) points

  Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek VDD 2016

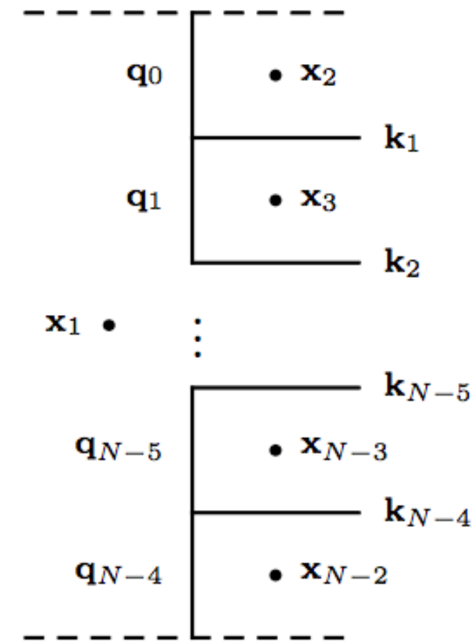
- We showed explicitly that the MHV amplitudes at:
  - 2 loops are determined by the 2-loop 6-pt amplitudes
  - 3 loops are determined by the 6- and 7-pt amplitudes through 3 loops
  - 4 loops are determined by the 6-, 7-, and 8-pt amplitudes through 4 loops
  - 5 loops are determined by the 6-, 7-, 8- and 9-pt amplitudes through 5 loops

- We computed also all non-MHV amplitudes up 8 points and 4 loops

MRK factorisation in N=4 SYM

- For the helicities $h_1, \ldots, h_{N-4}$ define the ratio

$$\mathcal{R}_{h_1,\ldots,h_{N-4}} = \left[\frac{A_N(-,+,h_1,\ldots,h_{N-4},+,-)}{A_N^{\rm BDS}(-,+,\ldots,+,-)}\right]_{|\rm MRK,\ LLA}$$

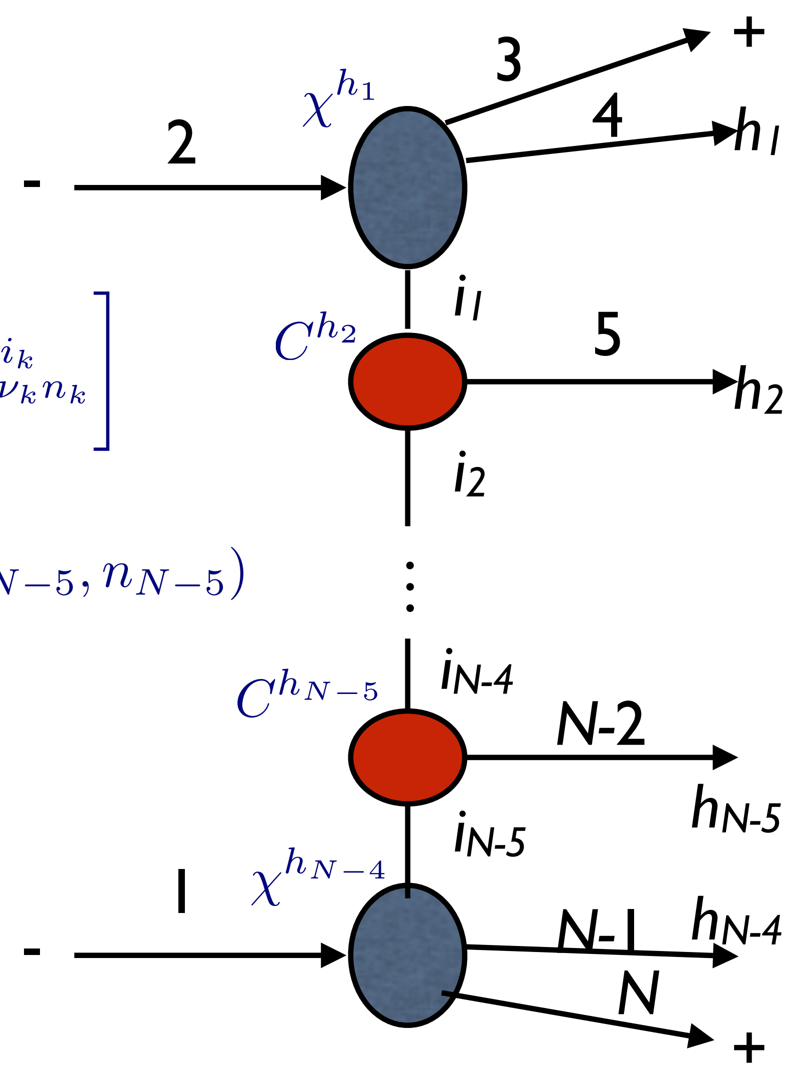


- factorisation in MRK at LLA

$$\mathcal{R}_{h_1,\ldots,h_{N-4}}(\tau_1, z_1, \ldots, \tau_{N-5}, z_{N-5})$$
$$\approx 2\pi i \sum_{i=2}^{\infty} \sum_{i_1+\ldots+i_{N-5}=i-1} a^i \left(\prod_{k=1}^{N-5} \frac{1}{i_k!} \ln^{i_k} \tau_k\right) g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5})$$

with τ_k = function of cross ratios, and with coefficients

$$g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) = \frac{(-1)^{N+1}}{2}\left[\prod_{k=1}^{N-5}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2}\int_{-\infty}^{+\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}E^{i_k}_{\nu_k n_k}\right]$$
$$\times \chi^{h_1}(\nu_1,n_1)\left[\prod_{j=2}^{N-5} C^{h_j}(\nu_{j-1},n_{j-1},\nu_j,n_j)\right]\chi^{-h_{N-4}}(\nu_{N-5},n_{N-5})$$



where:
the χ's are the 2 impact factors,
the C's are the N-6 central-emission vertices
the E's are the N-5 BFKL-like eigenvalues for octet exchange

Convolutions

- we use the Fouries-Mellin (FM) transform

$$\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \, |z|^{2i\nu} \, F(\nu,n)$$

which maps products into convolutions

$$\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = (f * g)(z) = \frac{1}{\pi} \int \frac{d^2 w}{|w|^2} \, f(w) \, g\left(\frac{z}{w}\right)$$

- we compute the integral through the residue formula

$$\int \frac{d^2 z}{\pi} \, f(z) = \mathrm{Res}_{z=\infty} F(z) - \sum_i \mathrm{Res}_{z=a_i} F(z)$$

Schnetz 2013

where *F* is the antiholomorphic primitive of *f* $\quad \bar{\partial}_z F = f$

Convolutions and factorization

- through the FM transform of the BFKL eigenvalue

  $\mathcal{E}(z) = \mathcal{F}\left[E_{\nu n}\right]$

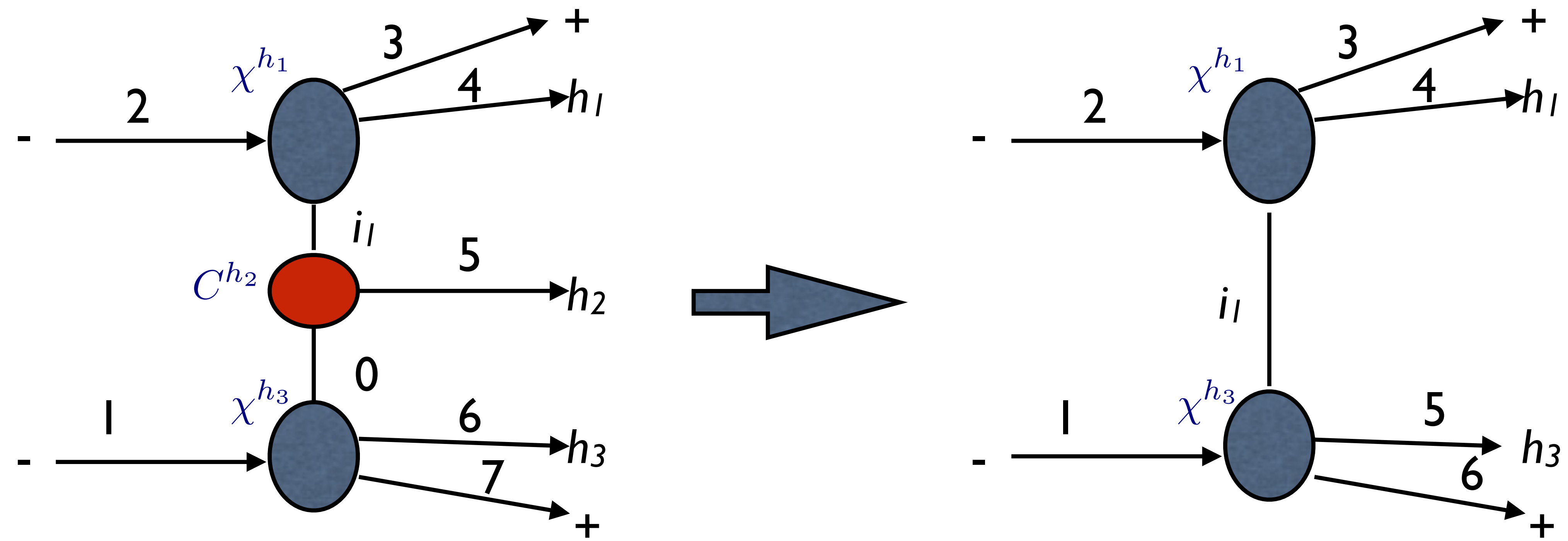
  we can write the recursion

  $$g_{+\ldots+}^{(i_1,\ldots,i_k+1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5}) = \mathcal{E}(z_k) * g_{+\ldots+}^{(i_1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5})$$

  which implies that we can drop all the propagators without a log

  $$g_{+\ldots+}^{(0,\ldots,0,i_{a_1},0,\ldots,0,i_{a_2},0,\ldots,0,i_{a_k},0,\ldots,0)}(\rho_1,\ldots,\rho_{N-5}) = g_{+\ldots+}^{(i_{a_1},i_{a_2},\ldots,i_{a_k})}(\rho_{i_{a_1}},\rho_{i_{a_2}},\ldots,\rho_{i_{a_k}})$$

example for N=7, with $h_1 = h_2$



which connects amplitudes with a different number of legs in fact, if all indices are zero except for one

$$g^{(0,\ldots,0,i_a,0,\ldots,0)}_{+\ldots+}(\rho_1,\ldots,\rho_{N-5}) = g^{(i_a)}_{++}(\rho_a)$$

- which implies that

$$\mathcal{R}^{(2)}_{+\ldots+} = \sum_{1\le i\le N-5} \ln\tau_i\, g^{(1)}_{++}(\rho_i)$$

with

$$g^{(1)}_{++}(\rho_1) = -\frac{1}{4}\mathcal{G}_{0,1}(\rho_1) - \frac{1}{4}\mathcal{G}_{1,0}(\rho_1) + \frac{1}{2}\mathcal{G}_{1,1}(\rho_1)$$

which shows, as previously stated, that in MRK at LLA, the 2-loop *n*-pt remainder function $R_n^{(2)}$ can be written as a sum of 2-loop 6-pt amplitudes, in terms of SVHPLs

- At 3 loops, the *n*-pt remainder function $R_n^{(3)}$ can be written as a sum of 3-loop 6-pt and 7-pt amplitudes

$$\mathcal{R}^{(3)}_{+\ldots+} = \frac{1}{2}\sum_{1\leq i\leq N-5} \ln^2\tau_i\, g^{(2)}_{++}(\rho_i) + \sum_{1\leq i<j\leq N-5} \ln\tau_i\, \ln\tau_j\, g^{(1,1)}_{+++}(\rho_i,\rho_j)$$

with

$$\begin{aligned} g^{(2)}_{++}(\rho_1) = &-\frac{1}{8}\mathcal{G}_{0,0,1}(\rho_1) - \frac{1}{4}\mathcal{G}_{0,1,0}(\rho_1) + \frac{1}{2}\mathcal{G}_{0,1,1}(\rho_1) - \frac{1}{8}\mathcal{G}_{1,0,0}(\rho_1) \\ &+ \frac{1}{2}\mathcal{G}_{1,0,1}(\rho_1) + \frac{1}{2}\mathcal{G}_{1,1,0}(\rho_1) - \mathcal{G}_{1,1,1}(\rho_1) \end{aligned}$$

$$\begin{aligned} g^{(1,1)}_{+++}(\rho_1,\rho_2) = &-\frac{1}{8}\mathcal{G}_{0,1,\rho_2}(\rho_1) - \frac{1}{8}\mathcal{G}_{0,\rho_2,1}(\rho_1) + \frac{1}{8}\mathcal{G}_{1,1,\rho_2}(\rho_1) - \frac{1}{8}\mathcal{G}_{1,\rho_2,0}(\rho_1) \\ &- \frac{1}{8}\mathcal{G}_{\rho_2,1,0}(\rho_1) + \frac{1}{8}\mathcal{G}_{\rho_2,1,1}(\rho_1) + \frac{1}{4}\mathcal{G}_{1,\rho_2,1}(\rho_1) - \frac{1}{4}\mathcal{G}_1(\rho_2)\,\mathcal{G}_{1,\rho_2}(\rho_1) \\ &+ \frac{1}{8}\mathcal{G}_1(\rho_1)\,\mathcal{G}_{0,0}(\rho_2) - \frac{1}{8}\mathcal{G}_0(\rho_2)\,\mathcal{G}_{0,1}(\rho_1) + \frac{1}{8}\mathcal{G}_1(\rho_2)\,\mathcal{G}_{0,1}(\rho_1) - \frac{1}{8}\mathcal{G}_{\rho_2}(\rho_1)\,\mathcal{G}_{0,1}(\rho_2) \\ &+ \frac{1}{8}\mathcal{G}_1(\rho_2)\,\mathcal{G}_{0,\rho_2}(\rho_1) - \frac{1}{8}\mathcal{G}_0(\rho_2)\,\mathcal{G}_{1,0}(\rho_1) + \frac{1}{8}\mathcal{G}_1(\rho_2)\,\mathcal{G}_{1,0}(\rho_1) + \frac{1}{8}\mathcal{G}_0(\rho_2)\,\mathcal{G}_{1,1}(\rho_1) \\ &- \frac{1}{8}\mathcal{G}_1(\rho_2)\,\mathcal{G}_{1,1}(\rho_1) - \frac{1}{8}\mathcal{G}_1(\rho_1)\,\mathcal{G}_{1,1}(\rho_2) + \frac{1}{8}\mathcal{G}_{\rho_2}(\rho_1)\,\mathcal{G}_{1,1}(\rho_2) + \frac{1}{8}\mathcal{G}_0(\rho_2)\,\mathcal{G}_{1,\rho_2}(\rho_1) \\ &+ \frac{1}{8}\mathcal{G}_1(\rho_2)\,\mathcal{G}_{\rho_2,0}(\rho_1) - \frac{1}{8}\mathcal{G}_1(\rho_2)\,\mathcal{G}_{\rho_2,1}(\rho_1) \end{aligned}$$

Note that $R_n^{(3)}$ cannot be written only in terms of SVHPLs, but SVMPLs are necessary

- At 4 loops, the *n*-pt remainder function $R_n^{(4)}$ can be written as a sum of 4-loop 6-pt, 7-pt and 8-pt amplitudes, and in general, we can compute all MHV amplitudes at *L* loops in terms of amplitudes with up to (*L*+4) points

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- We displayed that explicitly up to 5 loops, showing that the *n*-pt remainder function $R_n^{(5)}$ can be written as a sum of 5-loop 6-, 7-, 8- and 9-pt amplitudes

- Note also that because the convolutions with pure functions are pure, so are the MHV amplitudes

- MRK factorisation works also for non-MHV amplitudes, however at each loop the number of building blocks is infinite, and the helicity flips make the non-MHV amplitudes not pure

- We constructed explicitly all non-MHV amplitudes through 4 loop and 8 points

Mueller-Navelet jets and SVHPLs

- The singlet LL BFKL ladder in QCD, and thus the dijet cross section in the high-energy limit, can also be expressed in terms of SVHPLs, i.e. in terms of single-valued iterated integrals on $\mathcal{M}_{0,4}$

  Dixon Duhr Pennington VDD 2013

- Mueller & Navelet evaluated analytically the inclusive dijet cross section up to 5 loops. We evaluated it analytically up to 13 loops

- Also, we could evaluate analytically the dijet cross section differential in the jet transverse energies or the azimuthal angle between the jets (up to 6 loops)

BFKL Green's function and single-valued functions

- use complex transverse momentum $\tilde{q}_k \equiv q_k^x + i q_k^y$

  and a complex variable $z \equiv \frac{\tilde{q}_1}{\tilde{q}_2}$

  the Green's function can be expanded into a power series in $\eta_\mu = \overline{\alpha}_\mu \, y$

$$f^{LL}(q_1, q_2, \eta_\mu) = \frac{1}{2}\delta^{(2)}(q_1 - q_2) + \frac{1}{2\pi\sqrt{q_1^2 \, q_2^2}} \sum_{k=1}^{\infty} \frac{\eta_\mu^k}{k!} \, f_k^{LL}(z)$$

  where the coefficient functions f_k are given by the Fourier-Mellin transform

$$f_k^{LL}(z) = \mathcal{F}\left[\chi_{\nu n}^k\right] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \, |z|^{2i\nu} \, \chi_{\nu n}^k$$

  - the f_k have a unique, well-defined value for every ratio of the magnitudes of the two jet transverse momenta and angle between them.
    So, they are real-analytic functions of w

generating functional of SVHPLs

- to all orders in η the BFKL Green's function can be written in terms of a generating functional of SVHPLs — Dixon Duhr Pennington VDD 2013

writing the coefficient function f_k as

$$f_k^{LL}(z) = \frac{|z|}{2\pi\,|1-z|^2}\,F_k(z)$$

we obtain that the first few functions F_k are

$$\begin{aligned}
F_1(z) &= 1\,,\\
F_2(z) &= 2\,\mathcal{G}_1(z) - \mathcal{G}_0(z)\,,\\
F_3(z) &= 6\,\mathcal{G}_{1,1}(z) - 3\,\mathcal{G}_{0,1}(z) - 3\,\mathcal{G}_{1,0}(z) + \mathcal{G}_{0,0,0}(z)\,,\\
F_4(z) &= 24\,\mathcal{G}_{1,1,1}(z) + 4\,\mathcal{G}_{0,0,1}(z) + 6\,\mathcal{G}_{0,1,0}(z) - 12\,\mathcal{G}_{0,1,1}(z) + 4\,\mathcal{G}_{1,0,0}(z)\\
&\quad - 12\,\mathcal{G}_{1,0,1}(z) - 12\,\mathcal{G}_{1,1,0}(z) - \mathcal{G}_{0,0,0}(z) + 8\,\zeta_3
\end{aligned}$$

note that f_k has weight k-1

Azimuthal angle distribution

- this allows us to write the azimuthal angle distribution as

$$\frac{d\hat{\sigma}_{gg}}{d\phi_{jj}} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \left[\delta(\phi_{jj} - \pi) + \sum_{k=1}^{\infty} \frac{a_k(\phi_{jj})}{\pi} \eta^k \right]$$

where the contribution of the k^{th} loop is

$$a_k(\phi_{jj}) = \int_0^\infty \frac{d|w|}{|w|} f_k(w, w^*) = \frac{\text{Im}\, A_k(\phi_{jj})}{\sin \phi_{jj}}$$

with

$$\begin{aligned}
A_1(\phi_{jj}) &= -\frac{1}{2} H_0 \,, \\
A_2(\phi_{jj}) &= H_{1,0} \,, \\
A_3(\phi_{jj}) &= \frac{2}{3} H_{0,0,0} - 2H_{1,1,0} + \frac{5}{3}\zeta_2 H_0 - i\pi\,\zeta_2 \,, \\
A_4(\phi_{jj}) &= -\frac{4}{3} H_{0,0,1,0} - H_{0,1,0,0} - \frac{4}{3} H_{1,0,0,0} + 4H_{1,1,1,0} - \zeta_2 \left(2H_{0,1} + \frac{10}{3} H_{1,0} \right) + \frac{4}{3}\zeta_3\, H_0 + i\pi \left(2\zeta_2 H_1 - 2\zeta_3 \right) , \\
A_5(\phi_{jj}) &= -\frac{46}{15} H_{0,0,0,0,0} + \frac{8}{3} H_{0,0,1,1,0} + 2H_{0,1,0,1,0} + 2H_{0,1,1,0,0} + \frac{8}{3} H_{1,0,0,1,0} + 2H_{1,0,1,0,0} \\
&\quad + \frac{8}{3} H_{1,1,0,0,0} - 8H_{1,1,1,1,0} - \zeta_2 \left(\frac{33}{5} H_{0,0,0} - 4H_{0,1,1} - 4H_{1,0,1} - \frac{20}{3} H_{1,1,0} \right) \\
&\quad - \zeta_3 \left(2H_{0,1} + \frac{8}{3} H_{1,0} \right) + \frac{217}{15}\zeta_4 H_0 + i\pi \left[\zeta_2 \left(\frac{10}{3} H_{0,0} - 4H_{1,1} \right) + 4\zeta_3 H_1 - \frac{10}{3}\zeta_4 \right]
\end{aligned}$$

where $H_{i,j,\ldots} \equiv H_{i,j,\ldots}(e^{-2i\phi_{jj}})$

Dixon Duhr Pennington VDD 2013

Transverse momentum distribution

$$\frac{d\hat{\sigma}_{gg}}{dp_{1\perp}^2\, dp_{2\perp}^2} = \frac{\pi (C_A \alpha_s)^2}{2 p_{1\perp}^2 p_{2\perp}^2} \left[\delta(p_{1\perp}^2 - p_{2\perp}^2) + \frac{1}{2\pi \sqrt{p_{1\perp}^2\, p_{2\perp}^2}}\, b(\rho;\eta) \right]$$

where $\rho = |w|$ $\qquad$ $b(\rho;\eta) = \frac{2\,\pi\,\rho}{1-\rho^2} \sum_{k=1}^{\infty} B_k(\rho)\,\eta^k$

with

$$\begin{aligned}
B_1(\rho) &= 1\,,\\
B_2(\rho) &= -\frac{1}{2}\, H_0 - 2H_1\,,\\
B_3(\rho) &= \frac{1}{6} H_{0,0} + 2H_{0,1} + H_{1,0} + 4H_{1,1}\,,\\
B_4(\rho) &= -\frac{1}{24} H_{0,0,0} - \frac{4}{3} H_{0,0,1} - H_{0,1,0} - 4H_{0,1,1} - \frac{1}{3} H_{1,0,0} - 4H_{1,0,1} - 2H_{1,1,0} - 8H_{1,1,1} + \frac{1}{3}\,\zeta_3\,,\\
B_5(\rho) &= \frac{1}{120} H_{0,0,0,0} + \frac{2}{3} H_{0,0,0,1} + \frac{2}{3} H_{0,0,1,0} + \frac{8}{3} H_{0,0,1,1} + \frac{1}{3} H_{0,1,0,0} + 4H_{0,1,0,1}\\
&\quad + 2H_{0,1,1,0} + 8H_{0,1,1,1} + \frac{1}{12} H_{1,0,0,0} + \frac{8}{3} H_{1,0,0,1} + 2H_{1,0,1,0} + 8H_{1,0,1,1}\\
&\quad + \frac{2}{3} H_{1,1,0,0} + 8H_{1,1,0,1} + 4H_{1,1,1,0} + 16H_{1,1,1,1} + \zeta_3 \left(-\frac{1}{12} H_0 - \frac{2}{3} H_1 \right),
\end{aligned}$$

where $H_{i,j,\ldots} \equiv H_{i,j,\ldots}(\rho^2)$

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Mueller-Navelet dijet cross section reloaded

- the MN dijet cross section is $$\hat{\sigma}_{gg} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \sum_{k=0}^{\infty} f_{0,k}\, \eta^k$$

the first 5 loops were computed by Mueller-Navelet. We computed it through the 13 loops

Dixon Duhr Pennington VDD 2013

$$f_{0,6} = \frac{13}{4}\zeta_3^2 + \frac{3737}{120}\zeta_6\,,$$
$$f_{0,7} = -\frac{87}{5}\zeta_3\zeta_4 - \frac{116}{9}\zeta_2\zeta_5 - \frac{3983}{144}\zeta_7\,,$$
$$f_{0,8} = -\frac{37}{75}\zeta_{5,3} + \frac{64}{15}\zeta_2\zeta_3^2 + \frac{369}{20}\zeta_5\zeta_3 + \frac{50606057}{453600}\zeta_8\,,$$
$$f_{0,9} = -\frac{139}{60}\zeta_3^3 - \frac{15517}{252}\zeta_6\zeta_3 - \frac{3533}{63}\zeta_4\zeta_5 - \frac{557}{15}\zeta_2\zeta_7 - \frac{5215361}{60480}\zeta_9\,,$$
$$f_{0,10} = -\frac{2488}{4725}\zeta_{5,3}\zeta_2 - \frac{94721}{211680}\zeta_{7,3} + \frac{1948}{105}\zeta_4\zeta_3^2 + \frac{2608}{105}\zeta_2\zeta_5\zeta_3 + \frac{12099}{224}\zeta_7\zeta_3 + \frac{1335931}{47040}\zeta_5^2 + \frac{25669936301}{63504000}\zeta_{10}$$
$$f_{0,11} = \frac{62}{315}\zeta_{5,3}\zeta_3 + \frac{83}{120}\zeta_{5,3,3} - \frac{2872}{945}\zeta_2\zeta_3^3 - \frac{13211}{672}\zeta_5\zeta_3^2 - \frac{661411}{3024}\zeta_8\zeta_3 - \frac{242776937}{725760}\zeta_{11} - \frac{605321}{3024}\zeta_5\zeta_6 - \frac{2583643}{16200}\zeta_4\zeta_7 - \frac{28702763}{340200}\zeta_2\zeta_9\,,$$
$$f_{0,12} = \frac{74711}{162000}\zeta_{5,3}\zeta_4 - \frac{13793}{7560}\zeta_{6,4,1,1} + \frac{3965011}{793800}\zeta_{7,3}\zeta_2 - \frac{33356851}{4082400}\zeta_{9,3} + \frac{252163}{181440}\zeta_3^4 + \frac{620477}{10080}\zeta_6\zeta_3^2 + \frac{8101339}{75600}\zeta_4\zeta_5\zeta_3 + \frac{342869}{3780}\zeta_2\zeta_7\zeta_3 + \frac{101571047}{680400}\zeta_9\zeta_3 + \frac{71425871}{1587600}\zeta_2\zeta_5^2 + \frac{904497401571619}{620606448000}\zeta_{12} + \frac{484414571}{2721600}\zeta_5\zeta_7\,,$$
$$f_{0,13} = \frac{4513}{1890}\zeta_{5,3}\zeta_5 + \frac{27248}{23625}\zeta_{5,3,3}\zeta_2 - \frac{97003}{235200}\zeta_{5,5,3} + \frac{13411}{75600}\zeta_{7,3}\zeta_3 + \frac{7997743}{12700800}\zeta_{7,3,3} - \frac{187318}{14175}\zeta_4\zeta_3^3 - \frac{125056}{4725}\zeta_2\zeta_5\zeta_3^2 - \frac{17411413}{302400}\zeta_7\zeta_3^2 - \frac{5724191}{100800}\zeta_5^2\zeta_3 - \frac{1874972477}{2376000}\zeta_{10}\zeta_3 - \frac{2418071698069}{2235340800}\zeta_{13} - \frac{2379684877}{6048000}\zeta_{11}\zeta_2 - \frac{297666465053}{523908000}\zeta_6\zeta_7 - \frac{1770762319}{2494800}\zeta_5\zeta_8 - \frac{229717224973}{628689600}\zeta_4\zeta_9$$

Regge limit
in the
next-to-leading logarithmic
approximation

BFKL theory

- the BFKL equation describes the evolution of the gluon propagator in 2-dim transverse momentum space

$$\omega\, f_\omega(q_1,q_2) = \frac{1}{2}\,\delta^{(2)}(q_1-q_2) + \int d^2k\, K(q_1,k)\, f_\omega(k,q_2)$$

- the solution is given in terms of eigenfunctions $\Phi_{\nu n}$ and an eigenvalue $\omega_{\nu n}$

$$f_\omega(q_1,q_2) \;=\; \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu\, \frac{1}{\omega-\omega_{\nu n}}\, \Phi_{\nu n}(q_1)\, \Phi^*_{\nu n}(q_2)$$

as a function of rapidity, the solution is

$$f(q_1,q_2,y) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu\, \Phi_{\nu n}(q_1)\, \Phi^*_{\nu n}(q_2)\, e^{y\,\omega_{\nu n}}$$

- we expand kernel K, eigenfunctions $\Phi_{\nu n}$ and eigenvalue $\omega_{\nu n}$ in powers of $\overline{\alpha}_\mu = \frac{N_C}{\pi}\alpha_S(\mu^2)$

$$K(q_1,q_2) = \overline{\alpha}_\mu \sum_{l=0}^{\infty} \overline{\alpha}_\mu^l\, K^{(l)}(q_1,q_2) \qquad \omega_{\nu n} = \overline{\alpha}_\mu \sum_{l=0}^{\infty} \overline{\alpha}_\mu^l\, \omega_{\nu n}^{(l)} \qquad \Phi_{\nu n}(q) = \sum_{l=0}^{\infty} \overline{\alpha}_\mu^l\, \Phi_{\nu n}^{(l)}(q)$$

- At LLA

$$\omega_{\nu n}^{(0)} = -2\gamma_E - \psi\left(\frac{|n|+1}{2}+i\nu\right) - \psi\left(\frac{|n|+1}{2}-i\nu\right) \qquad \Phi_{\nu n}^{(0)}(q) = \frac{1}{2\pi}\,(q^2)^{-1/2+i\nu}\, e^{in\theta}$$

note that in N=4 SYM the eigenfunctions and the eigenvalue are the same

BFKL eigenvalue at NLLA

- At NLLA in QCD and in N=4 SYM, the eigenvalue is

Fadin Lipatov 1998
Kotikov Lipatov 2000, 2002
Chirilli Kovchegov 2013
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$$\omega_{\nu n}^{(1)} = \frac{1}{4}\,\delta_{\nu n}^{(1)} + \frac{1}{4}\,\delta_{\nu n}^{(2)} + \frac{1}{4}\,\delta_{\nu n}^{(3)} + \gamma_K^{(2)}\chi_{\nu n} - \frac{1}{8}\beta_0\chi_{\nu n}^2 + \frac{3}{2}\zeta_3$$

with one-loop beta function and two-loop cusp anomalous dimension

$$\beta_0 = \frac{11}{3} - \frac{2N_f}{3N_c} \qquad \gamma_K^{(2)} = \frac{1}{4}\left(\frac{64}{9} - \frac{10N_f}{9N_c}\right) - \frac{\zeta_2}{2}$$

and with

$$\delta_{\nu n}^{(1)} = \partial_\nu^2 \chi_{\nu n} \qquad \chi_{\nu n} = \omega_{\nu n}^{(0)}$$

$$\delta_{\nu n}^{(2)} = -2\Phi(n,\gamma) - 2\Phi(n, 1-\gamma)$$

$$\delta_{\nu n}^{(3)} = -\frac{\Gamma(\frac{1}{2}+i\nu)\Gamma(\frac{1}{2}-i\nu)}{2i\nu}\left[\psi\left(\frac{1}{2}+i\nu\right) - \psi\left(\frac{1}{2}-i\nu\right)\right]$$
$$\times\left[\delta_{n0}\left(3 + \left(1+\frac{N_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right) - \delta_{|n|2}\left(\left(1+\frac{N_f}{N_c^3}\right)\frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)}\right)\right]$$

$\Phi(n,\gamma)$ is a sum over linear combinations of ψ functions
and γ is a shorthand $\gamma = 1/2 + i\nu$

In blue we labeled the terms which occur only in QCD,
in red the ones which occur in QCD and in N=4 SYM

BFKL eigenfunctions at NLLA

Chirilli Kovchegov 2013
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- At NLLA in QCD, the eigenfunction is

$$\Phi_{\nu n}(q) = \Phi^{(0)}_{\nu n}(q)\left[1 + \overline{\alpha}_\mu \frac{\beta_0}{8}\,\ln\frac{q^2}{\mu^2}\left(\partial_\nu P\frac{\chi_{\nu n}}{\partial_\nu \chi_{\nu n}} + i\,\ln\frac{q^2}{\mu^2}\,P\frac{\chi_{\nu n}}{\partial_\nu \chi_{\nu n}}\right) + \mathcal{O}(\overline{\alpha}^2_\mu)\right]$$

- At NLLA, the expansion of the BFKL ladder is

$$f(q_1,q_2,y) = f^{LL}(q_1,q_2,\eta_\mu) + \overline{\alpha}_\mu\, f^{NLL}(q_1,q_2,\eta_\mu) + \ldots\,, \qquad \eta_\mu = \overline{\alpha}_\mu\, y$$

f^{NLL} contains the NLO corrections to the eigenvalue *and* to the eigenfunctions, however if we use the scale of the strong coupling to be the geometric mean of the transverse momenta at the ends of the ladder, then we can use the LO eigenfunctions instead of the NLO ones

$$f(q_1,q_2,y) = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\nu\,\Phi^{(0)}_{\nu n}(q_1)\,\Phi^{(0)*}_{\nu n}(q_2)\,e^{y\,\overline{\alpha}_S(s_0)[\omega^{(0)}_{\nu n}+\overline{\alpha}_S(s_0)\omega^{(1)}_{\nu n}]} + \ldots$$

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with $\quad \mu^2 = s_0 = \sqrt{q_1^2 q_2^2}$

Fourier-Mellin transform

- At NLLA, the BFKL ladder is

$$f^{NLL}(q_1, q_2, \eta_{s_0}) = \frac{1}{2\pi\sqrt{q_1^2 q_2^2}} \sum_{k=1}^{\infty} \frac{\eta_{s_0}^k}{k!} f_{k+1}^{NLL}(z) \qquad \eta_{s_0} = \overline{\alpha}_S(s_0)\, y$$

with coefficients given by the Fourier-Mellin transform

$$f_k^{NLL}(z) = \mathcal{F}\left[\omega_{\nu n}^{(1)}\, \chi_{\nu n}^{k-2}\right] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} |z|^{2i\nu}\, \omega_{\nu n}^{(1)}\, \chi_{\nu n}^{k-2} \qquad \chi_{\nu n} = \omega_{\nu n}^{(0)}$$

using the explicit form of the eigenvalue

$$\omega_{\nu n}^{(1)} = \frac{1}{4}\delta_{\nu n}^{(1)} + \frac{1}{4}\delta_{\nu n}^{(2)} + \frac{1}{4}\delta_{\nu n}^{(3)} + \gamma_K^{(2)}\chi_{\nu n} - \frac{1}{8}\beta_0\chi_{\nu n}^2 + \frac{3}{2}\zeta_3$$

the coefficients can be written as

$$f_k^{NLL}(z) = \frac{1}{4}C_k^{(1)}(z) + \frac{1}{4}C_k^{(2)}(z) + \frac{1}{4}C_k^{(3)}(z) + \gamma_K^{(2)}\, f_{k-1}^{LL}(z) - \frac{1}{8}\beta_0\, f_k^{LL}(z) + \frac{3}{2}\zeta_3\, f_{k-2}^{LL}(z)$$

with $\quad C_k^{(i)}(z) = \mathcal{F}\left[\delta_{\nu n}^{(i)}\, \chi_{\nu n}^{k-2}\right]$

the weight of $f^{NLL}{}_k$ is

weight($f^{NLL}{}_k$)= $\quad k \quad\quad k \quad\quad 0 \le w \le k \quad\quad k-2 \le w \le k \quad\quad k-1 \quad\quad k$

SV functions

- $C_k^{(1)}(z)$ are SVHPLs of uniform weight k with singularities at z=0 and z=1

  $C_k^{(3)}(z)$ are MPLs of type $G(a_1,\ldots,a_n;|z|)$ with $a_k \in \{-i,0,i\}$

  they are SV functions of z because they have no branch cut on the positive real axis, and have weight $0 \leq w \leq k$

  For $C_k^{(2)}(z)$ one needs Schnetz' generalised SVMPLs with singularities at

  $$z = \frac{\alpha\,\bar{z}+\beta}{\gamma\,\bar{z}+\delta}\,, \qquad \alpha,\beta,\gamma,\delta \in \mathbb{C}$$

  Schnetz 2016

  then one can show that $C_k^{(2)}(z)$ are Schnetz' generalised SVMPLs

  $\mathcal{G}(a_1,\ldots,a_n;z)$ with singularities at $a_i \in \{-1,0,1,-1/\bar{z}\}$

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- In moment space, the maximal weight of the BFKL eigenvalue and of the anomalous dimensions of the leading twist operators which control the Bjorken scaling violations in QCD is the same as the corresponding quantities in N=4 SYM

  Kotikov Lipatov 2000, 2002
  Kotikov Lipatov Velizhanin 2003

  Interestingly, in transverse momentum space at NLLA, the maximal weight of the BFKL ladder in QCD is *not* the same as the one of the ladder in *N*=4 SYM

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BFKL ladder in a generic SU(N_c) gauge theory

- one can consider the BFKL eigenvalue at NLLA in a SU(N_c) gauge theory with scalar or fermionic matter in arbitrary representations

$$\omega^{(1)}_{\nu n} = \frac{1}{4}\delta^{(1)}_{\nu n} + \frac{1}{4}\delta^{(2)}_{\nu n} + \frac{1}{4}\delta^{(3)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) + \frac{3}{2}\zeta_3 + \gamma^{(2)}(\tilde{n}_f, \tilde{n}_s)\,\chi_{\nu n} - \frac{1}{8}\beta_0(\tilde{n}_f, \tilde{n}_s)\,\chi^2_{\nu n}$$

Kotikov Lipatov 2000

with $\beta_0(\tilde{n}_f, \tilde{n}_s) = \frac{11}{3} - \frac{2\tilde{n}_f}{3N_c} - \frac{\tilde{n}_s}{6N_c}$ $\quad \gamma^{(2)}(\tilde{n}_f, \tilde{n}_s) = \frac{1}{4}\left(\frac{64}{9} - \frac{10\,\tilde{n}_f}{9N_c} - \frac{4\,\tilde{n}_s}{9N_c}\right) - \frac{\zeta_2}{2}$

$$\tilde{n}_f = \sum_R n_f^R T_R \qquad \tilde{n}_s = \sum_R n_s^R T_R \qquad \mathrm{Tr}(T_R^a T_R^b) = T_R\,\delta^{ab} \qquad T_F = \frac{1}{2}$$

$\tilde{n}_s(\tilde{n}_f) =$ number of scalars (Weyl fermions) in the representation *R*

$$\delta^{(3)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) = \delta^{(3,1)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) + \delta^{(3,2)}_{\nu n}(\tilde{N}_f, \tilde{N}_s)$$

with $\tilde{N}_x = \frac{1}{2}\sum_R n_x^R\, T_R\,(2C_R - N_c)\,, \qquad x = f, s$

- Necessary and sufficient conditions for a SU(N_c) gauge theory to have a BFKL ladder of maximal weight are:

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— the one-loop beta function must vanish
— the two-loop cusp AD must be proportional to ζ_2
— $\delta^{(3,2)}_{\nu n}$ must vanish → $2\tilde{N}_f = N_c^2 + \tilde{N}_s$

⇒ There is no theory whose BFKL ladder has uniform maximal weight which agrees with the maximal weight terms of QCD

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Matter in the fundamental and in the adjoint

- We solve the conditions above for matter in the fundamental *F* and in the adjoint *A* representations. We obtain:

$$2\,n_f^F = n_s^F \qquad\qquad 2\,n_f^A = 2 + n_s^A$$

which describes the spectrum of a gauge theory with *N* supersymmetries and $n^F = n_f^F$ chiral multiplets in *F* and $n^A = n_f^A - N$ chiral multiplets in *A*

- There are four solutions to those conditions

$\mathcal{N}$	4	2	1	1
n_A	0	0	0	2
n_F	0	$4N_c$	$6N_c$	$2N_c$

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— the first is *N*=4 SYM
— the second is N=2 superconformal QCD with $N_f = 2N_c$ hypermultiplets
— the third is *N*=1 superconf. QCD

- because the one-loop beta function is fixed by matter loops in gluon self-energies, we are only sensitive to the matter content of a theory, and not to its details (like scalar potential or Yukawa couplings)

Conclusions at LLA

- n-point amplitudes in the LLA of the Multi-Regge limit of N=4 SYM can be fully expressed in terms of single-valued iterated integrals on $\mathcal{M}_{0,n-2}$ i.e. in terms of single-valued multiple polylogarithms
- We showed that one can compute all MHV amplitudes at L loops in terms of amplitudes with up to (L+4) points
- We displayed that explicitly up to 5 loops, showing that the n-pt remainder function $R_n^{(5)}$ can be written as a sum of 5-loop 6-, 7-, 8- and 9-pt amplitudes
- We constructed explicitly all non-MHV amplitudes through 4 loop and 8 points
- The singlet BFKL ladder, and thus the dijet cross section in the LLA of the high-energy limit of QCD, can also be expressed in terms of of single-valued iterated integrals on $\mathcal{M}_{0,4}$, i.e. in terms of SVHPLs
- We computed the Mueller-Navelet dijet cross section through 13 loops
- We expect that the n-jet cross section in the LLA of the high-energy limit of QCD can be written in terms of single-valued iterated integrals on $\mathcal{M}_{0,n+2}$

Conclusions at NLLA

- The singlet BFKL ladder at NLLA in QCD and in N=4 SYM can also be expressed in terms of SVMPLs, but generalised to non-constant values of the roots
- In transverse momentum space at NLLA, the maximal weight of the BFKL ladder in QCD is *not* the same as the one of the ladder in *N*=4 SYM
- There is no $SU(N_c)$ gauge theory whose BFKL ladder has uniform maximal weight which agrees with the maximal weight terms of QCD
- We found four theories with matter in the fundamental and in the adjoint representations, whose BFKL ladder has uniform maximal weight