Amplitudes

(in the Standard Model and beyond)

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Cortona 25 May 2018

The study of scattering amplitudes is on some level the study of classes of special functions

incipit
Bourjaily He McLeod von Hippel Wilhelm *today*

- At the LHC, particles are produced through the head-on collisions of protons, and in particular through the collisions of quarks and gluons within the protons

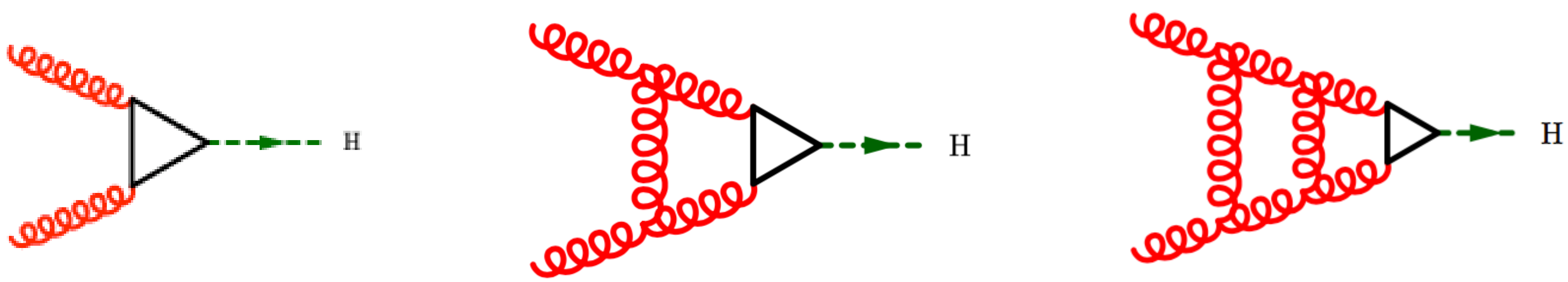
- The probability of a collision event is computed through the cross section, which is given as an integral of (squared) scattering amplitudes over the phase space of the produced particles

- The scattering amplitudes are given as a power series (loop expansion) in the strong and/or electroweak couplings

→ The scattering amplitudes are fundamental objects of particle physics

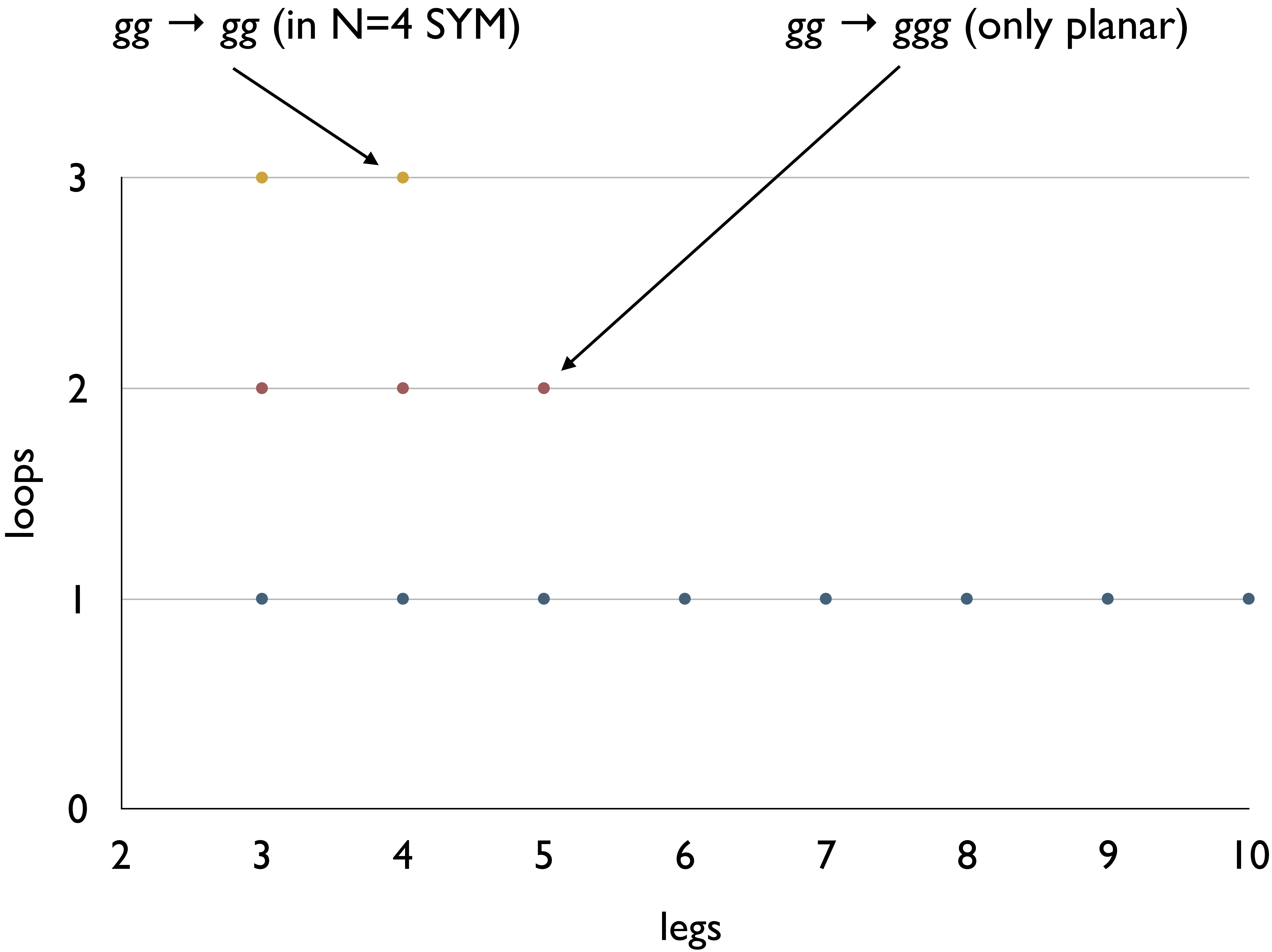
Scattering amplitudes

- The scattering amplitudes are given as a loop-momentum expansion in the strong and/or electroweak couplings



- The more terms we know in the loop expansion, the more precisely we can compute the cross section

- As a matter of fact, we know any amplitude of interest at one loop; several ($2 \to 2$) amplitudes and one ($2 \to 3$) amplitude (but only planar) at two loops; a couple ($2 \to 1$) amplitudes and one ($2 \to 2$) amplitude (in N=4 SYM) at three loops, and nothing beyond that



gg → gg (in N=4 SYM)
gg → ggg (only planar)
3
2
1
0
loops
2
3
4
5
6
7
8
9
10
legs

Computing amplitudes

- Integration by parts (IBP) to reduce the Feynman integral to a suitable basis of master integrals (MI)

  It used to be the less problematic part of the workflow.
  With ($2 \to 3$) amplitudes at two loops it has become a challenge.
  Now, several groups work on it

  ETH ITP organises a workshop dedicated only to that
  Taming the Complexity of Multiloop Integrals, ETH, 4-8 June 2018

- Differential Equation method to solve the MIs

  f: N-vector of MIs, A_i: NxN matrix, i=1,…,# external parameters

  $$\partial_i f(x_n;\varepsilon) = A_i(x_n;\varepsilon) f(x_n;\varepsilon)$$

  but in some cases ε-independent form

  $$\partial_i f(x_n;\varepsilon) = \varepsilon A_i(x_n) f(x_n;\varepsilon)$$

  Henn 2013

- analysing the space of functions of the Feynman integral

  where most of the progress has been → topic of this talk

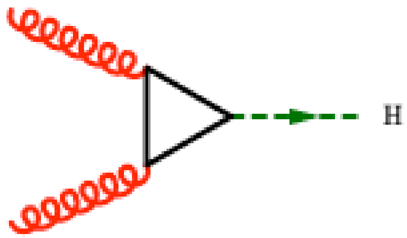
- What outcome do we expect from the loop expansion of an amplitude?

- From renormalisability and the infrared structure of the amplitude, we expect that the divergent parts are (poly)logarithmic functions of the external momenta (beyond 2 loops)

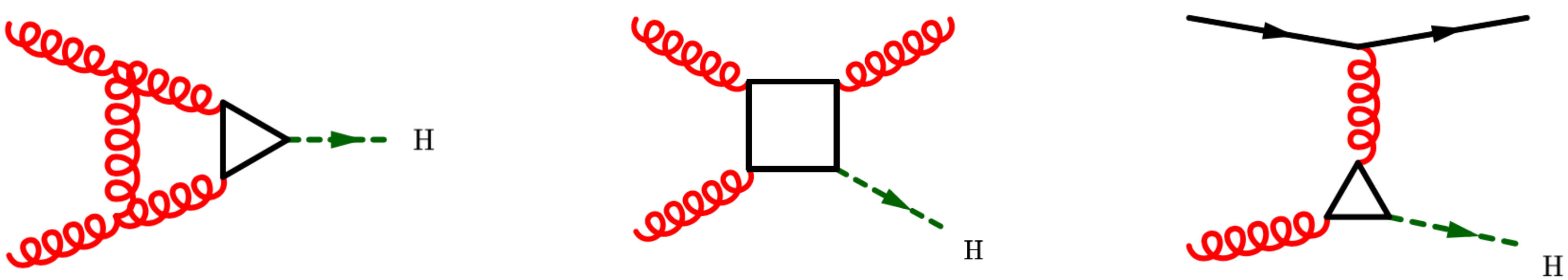
- But, except for unitarity, we have little guidance for the finite parts. Heuristically, we know that:
  — at one loop, logarithmic and dilogarithmic functions of
  the external momenta occur
  — beyond one loop, higher polylogarithmic functions appear
  and elliptic functions may appear (usually associated to several
  massive propagators)

Higgs production at LHC

- In proton collisions, the Higgs boson is produced mostly via gluon fusion
  The gluons do not couple directly to the Higgs boson
  The coupling is mediated by a heavy quark loop
  The largest contribution comes from the top loop
  The production mode is (roughly) proportional to the top Yukawa coupling y_t



- QCD NLO corrections



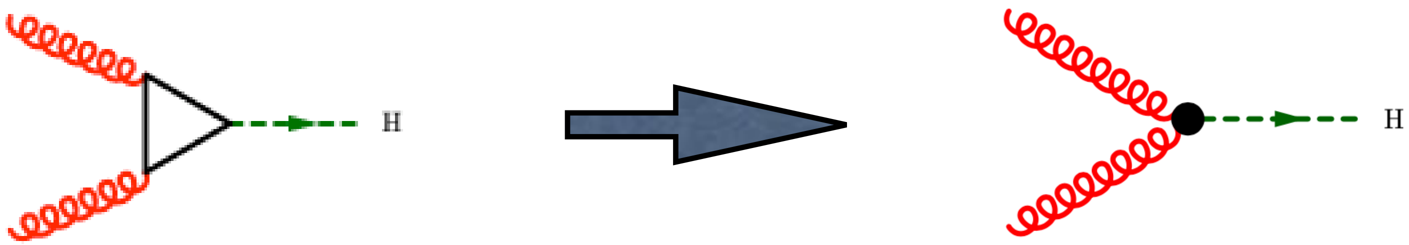
Djouadi Graudenz Spira Zerwas 1993-1995

  - QCD NLO corrections are about 100% larger than leading order

- QCD NNLO corrections are not known

Higgs production in HEFT

- $m_H << 2m_t$



all amplitudes are reduced by one loop

- … but, beware of quark mass effects

σ^{LO}_{EFT}	15.05 pb	σ^{NLO}_{EFT}	34.66 pb
$R_{LO}\,\sigma^{LO}_{EFT}$	16.00 pb	$R_{LO}\,\sigma^{NLO}_{EFT}$	36.84 pb
$\sigma^{LO}_{ex;t}$	16.00 pb	$\sigma^{NLO}_{ex;t}$	36.60 pb
$\sigma^{LO}_{ex;t+b}$	14.94 pb	$\sigma^{NLO}_{ex;t+b}$	34.96 pb
$\sigma^{LO}_{ex;t+b+c}$	14.83 pb	$\sigma^{NLO}_{ex;t+b+c}$	34.77 pb

Anastasiou Duhr Dulat Furlan Gehrmann Herzog Lazopoulos Mistlberger 2016

- $R_{LO} = \frac{\sigma^{LO}_{ex:t}}{\sigma^{LO}_{EFT}} = 1.063$

rescaled EFT (rEFT) does a good job (< 1%) in approximating the exact (only top) NLO σ but misses the *t-b* interference

Higgs production in HEFT

- QCD corrections have been computed at N^3LO

Anastasiou Duhr Dulat Herzog Mistlberger 2015

$$\sigma = 48.58\,\text{pb}^{+2.22\,\text{pb}\,(+4.56\%)}_{-3.27\,\text{pb}\,(-6.72\%)}\ (\text{theory}) \pm 1.56\,\text{pb}\,(3.20\%)\ (\text{PDF}+\alpha_s)$$

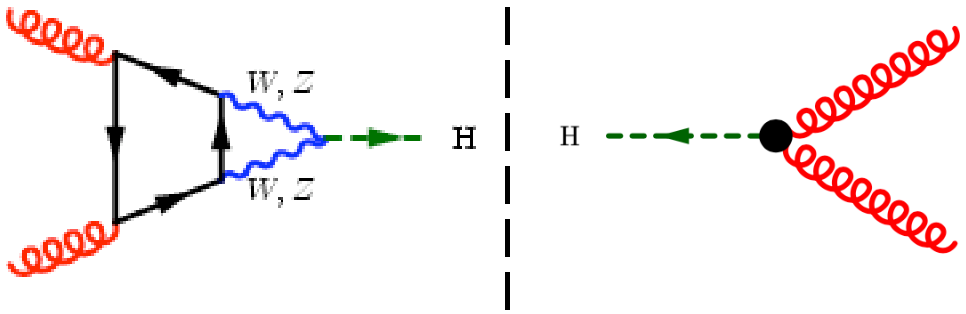
- The breakdown of the cross section

$$\begin{aligned}
48.58\,\text{pb} = {} & 16.00\,\text{pb} & (+32.9\%) & \quad (\text{LO, rEFT}) \\
& +20.84\,\text{pb} & (+42.9\%) & \quad (\text{NLO, rEFT}) \\
& -\ 2.05\,\text{pb} & (-4.2\%) & \quad ((t,b,c),\ \text{exact NLO}) \\
& +\ 9.56\,\text{pb} & (+19.7\%) & \quad (\text{NNLO, rEFT}) \\
& +\ 0.34\,\text{pb} & (+0.2\%) & \quad (\text{NNLO},\ 1/m_t) \\
& +\ 2.40\,\text{pb} & (+4.9\%) & \quad (\text{EW, QCD-EW}) \\
& +\ 1.49\,\text{pb} & (+3.1\%) & \quad (\text{N}^3\text{LO, rEFT})
\end{aligned}$$

Anastasiou Duhr Dulat Furlan Gehrmann Herzog Lazopoulos Mistlberger 2016

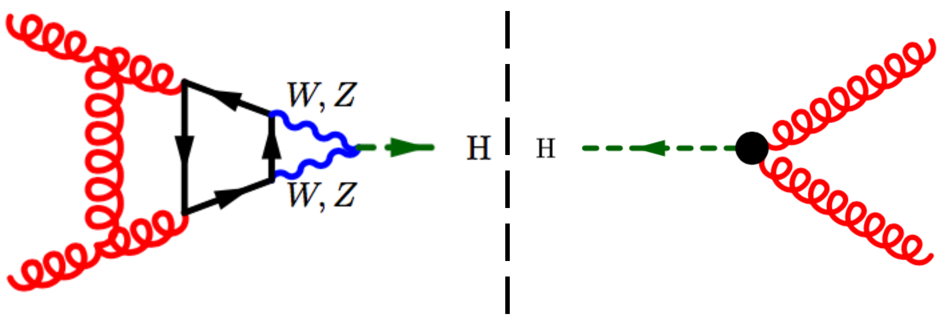
- Largest uncertainties come from quark mass effects at NNLO and from NLO corrections to QCD-EW interference

QCD-EW interference

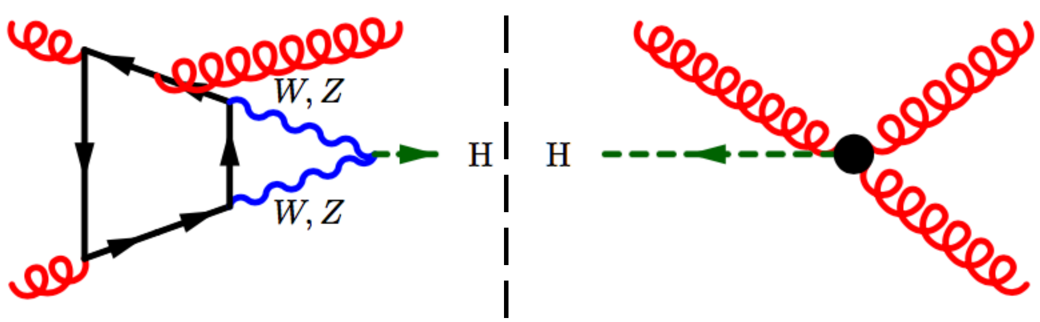


Aglietti Bonciani Degrassi Vicini 2004
(light fermion loop)
Actis Passarino Sturm Uccirati 2008
(heavy fermion loop)

- QCD NLO corrections



Bonetti Melnikov Tancredi 2017



computed in:

— $m_{w,z} \to \infty$ limit Anastasiou Boughezal Petriello 2009

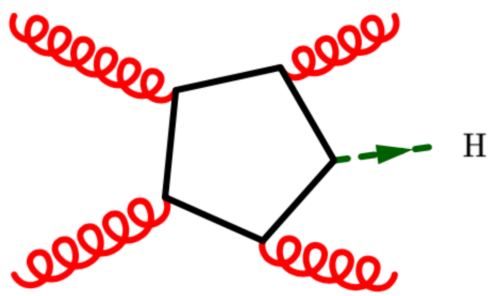
— soft approximation Bonetti Melnikov Tancredi 2018

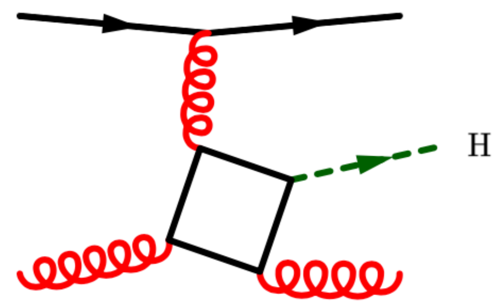
— $m_{w,z} \to 0$ limit Anastasiou VDD Furlan Mistlberger Moriello Schweitzer Specchia, to appear

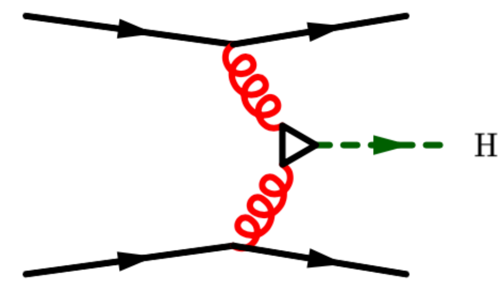
and found to be about 5.3-5.5% both at LO and NLO

QCD NNLO corrections

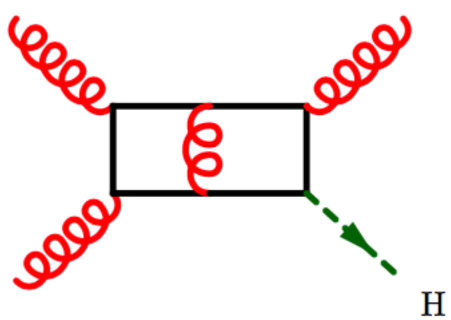
- Higgs+2jets amplitudes at one loop — VDD Kilgore Oleari Schmidt Zeppenfeld 2001

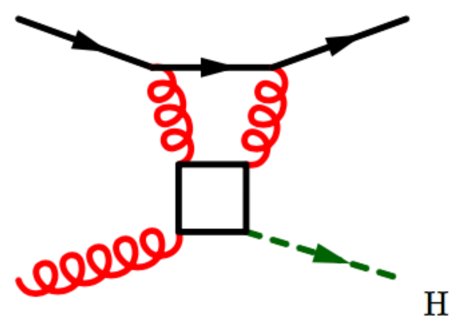






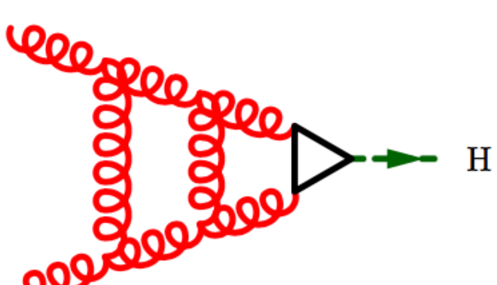
- Higgs+1jet amplitudes at two loops — Bonciani VDD Frellesvig Henn Moriello V. Smirnov 2016 (only planar diagrams)





multi-scale problem with complicated analytic structure
elliptic iterated integrals appear

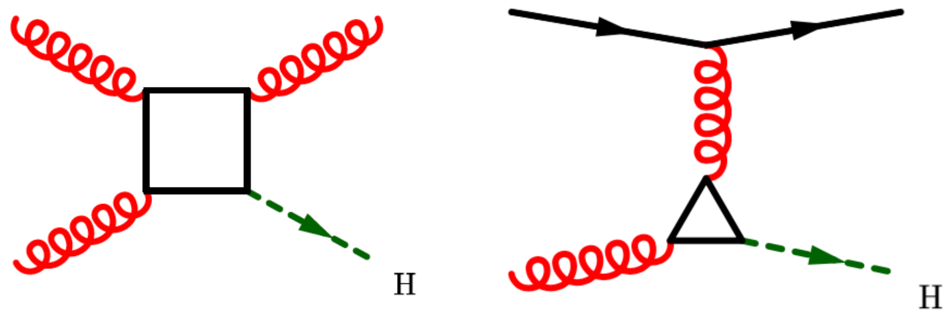
- $gg \to H$ amplitudes at three loops



one-scale problem, but with more elliptic iterated integrals …

Higgs p_T distribution at LHC

- leading order



K. Ellis Hinchliffe Soldate van der Bij 1988

- high-p_T tail of the Higgs p_T distribution is sensitive to the structure of the loop-mediated Higgs-gluon coupling New Physics particles circulating in the loop would modify it

- Full (=t+b+c) QCD NLO corrections are not known

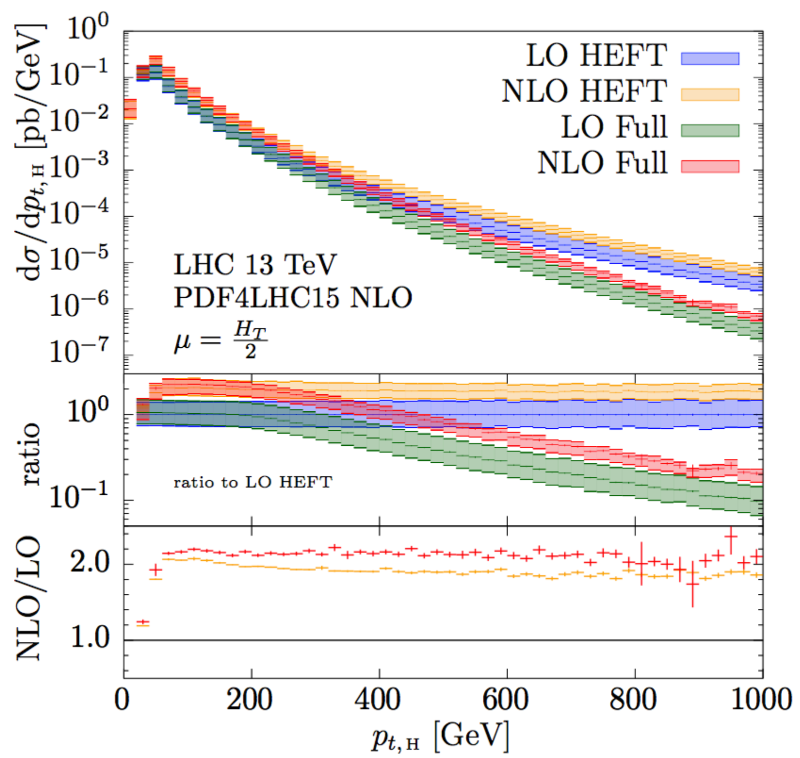
- HEFT $m_H << 2m_t$ and $p_T << m_t$

QCD corrections are known at NNLO in HEFT, and yield a 15% increase wrt NLO

Boughezal Caola Melnikov Petriello Schulze 2015
Boughezal Focke Giele Liu Petriello 2015
Chen Cruz-Martinez Gehrmann Glover Jaquier 2016

Higgs p_T distribution at LHC

- QCD (top) NLO corrections have been computed numerically



Jones Kerner Luisoni 2018

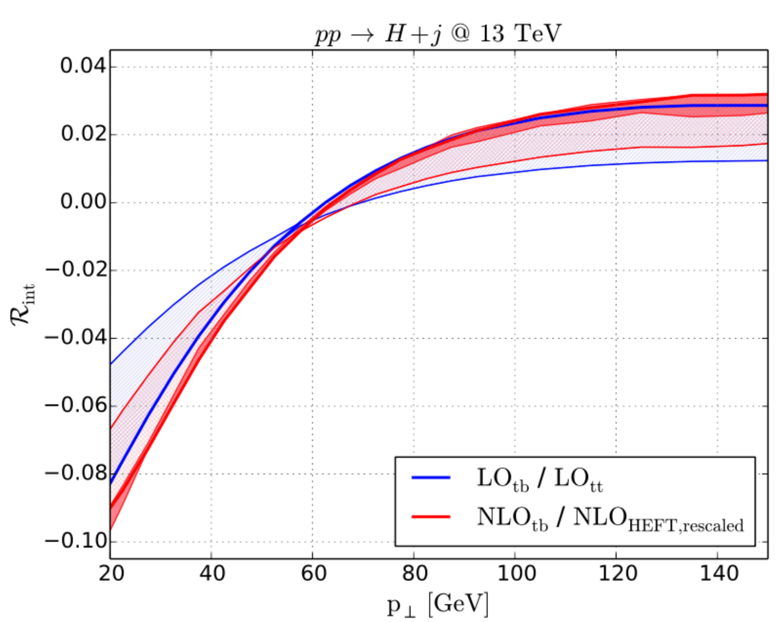
$$\frac{d\sigma}{dp_T^2} \propto \frac{1}{p_T^2}$$ in HEFT NLO corrections

$$\frac{d\sigma}{dp_T^2} \propto \frac{1}{(p_T^2)^2}$$ in top NLO corrections

No *t*-*b* interference

- QCD NLO corrections to *t*-*b* interference, using top loop in HEFT and *b*-quark loop in small m_b limit

Lindert Melnikov Tancredi Wever 2017



- In the last few years, a lot of progress has been made in understanding the analytic structure of multi-loop amplitudes, in particular on how the polylogarithmic functions appear at any loop level

- In particular, a lot of progress has been made:
  — in *N*=4 Super Yang-Mills (SYM)
  — in the Regge limit of QCD
  — in the Regge limit of *N*=4 SYM

- This progress has deep implications on how we view scattering amplitudes in the Standard Model

One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane …

incipit
The analytic S-matrix
Eden Landshoff Olive Polkinghorne 1966

$\mathcal{N}$=4 Super Yang Mills

- maximal supersymmetric theory (without gravity)
  conformally invariant, β fn. = 0
  - spin 1 gluon
    4 spin 1/2 gluinos
    6 spin 0 real scalars
- 't Hooft limit: $N_c \to \infty$ with $\lambda = g^2 N_c$ fixed
  - only planar diagrams
- AdS/CFT duality   Maldacena 97
  - large-λ limit of 4dim CFT ↔ weakly-coupled string theory
    (aka weak-strong duality)

N=4 Super Yang Mills

- amplitudes in planar *N*=4 SYM are much simpler than in Standard Model processes
- use *N*=4 SYM as a computational lab:
  - to learn techniques and tools to be used in Standard Model calculations
  - to learn about the bases of special functions which may occur in realistic scattering processes

N=4 Super Yang Mills

- In the last years, a huge progress has been made in understanding the analytic structure of the *S*-matrix of planar *N*=4 SYM
- Besides the ordinary conformal symmetry, in the planar limit the *S*-matrix exhibits a dual conformal symmetry

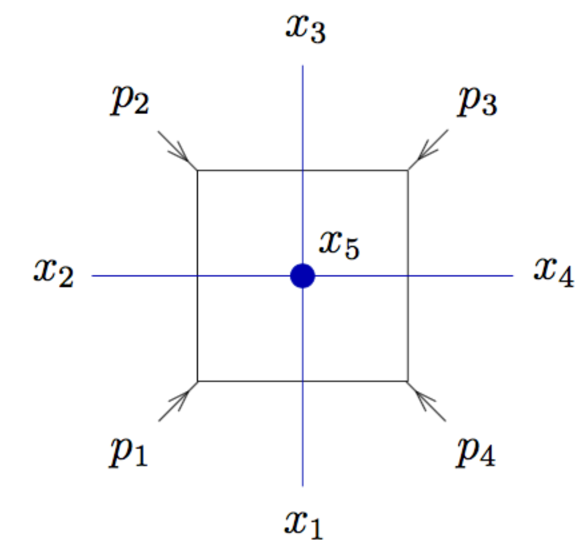
  Drummond Henn Smirnov Sokatchev 2006
- Accordingly, the analytic structure of the scattering amplitudes is highly constrained
- 4- and 5-point amplitudes are fixed to all loops by the symmetries in terms of the one-loop amplitudes and the cusp anomalous dimension

  Anastasiou Bern Dixon Kosower 2003, Bern Dixon Smirnov 2005
  Drummond Henn Korchemsky Sokatchev 2007
- Beyond 5 points, the finite part of the amplitudes is given in terms of a remainder function *R*. The symmetries only fix the variables of *R* (some conformally invariant cross ratios) but not the analytic dependence of *R* on them

Dual conformal symmetry

- Dual space $p_i = x_i - x_{i+1} \equiv x_{i,i+1}$ $x_{n+1} = x_1$

- one-loop scalar box



$$I^{(1)} = \int \frac{d^4k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2}$$

$$p_1 = x_{12}, \quad p_2 = x_{23}, \quad p_3 = x_{34}, \quad p_4 = x_{41}, \quad k = x_{15}$$

$$\Rightarrow \quad I^{(1)} = \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \frac{1}{x_{13}^2 x_{24}^2}\,\Phi^{(1)}(s,t)$$

$$s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad t = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

't Hooft Veltman 1979
Usyukina Davydychev 1993

- conformal inversion

$$x_i^\mu \to -\frac{x_i^\mu}{x_i^2}, \qquad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}, \qquad d^4x_5 \to \frac{d^4x_5}{x_5^8}$$

$$\frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \to x_1^2 x_2^2 x_3^2 x_4^2 \,\frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

conformally covariant

- drawing dual graphs is only possible for planar diagrams

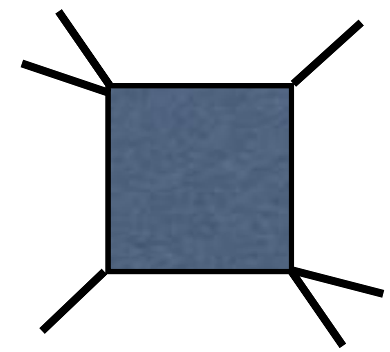
MHV amplitudes in planar N=4 SYM

- at any order in the coupling, colour-ordered maximally helicity violating MHV (- - ++…+) amplitudes in planar N=4 SYM can be written as the tree-level amplitude times a momentum dependent loop coefficient

$$M_n^{(L)} = M_n^{(0)} m_n^{(L)}$$

- at 1 loop Bern Dixon Dunbar Kosower 94

$$m_n^{(1)} = \sum_{pq} F^{2\mathrm{me}}(p,q,P,Q) \qquad n \geq 6$$



- at 2 loops, iteration formula for the *n*-pt amplitude

remainder function

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + R$$

Anastasiou Bern Dixon Kosower 03

- at all loops, ansatz for a resummed exponent

$$m_n^{(L)} = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon)\right)\right] + R$$

Bern Dixon Smirnov 05

ABDK/BDS ansatz

- ABDK/BDS ansatz is valid at all loops for 4-pt and 5-pt amplitudes

$$
\begin{aligned}
M_n &= M_n^{(0)} \left[1 + \sum_{L=1}^{\infty} a^L m_n^{(L)}(\epsilon) \right] \\
&= M_n^{(0)} \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right]
\end{aligned}
$$

Bern Dixon Smirnov 05

coupling $a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}$ $\lambda = g^2 N$ 't Hooft parameter

$$f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \, \frac{l}{2} \hat{G}^{(l)} + \epsilon^2 f_2^{(l)} \qquad E_n^{(l)}(\epsilon) = O(\epsilon)$$

$\hat{\gamma}_K^{(l)}$ cusp anomalous dimension, known to all orders of *a*

Korchemsky Radyuskin 86
Beisert Eden Staudacher 06

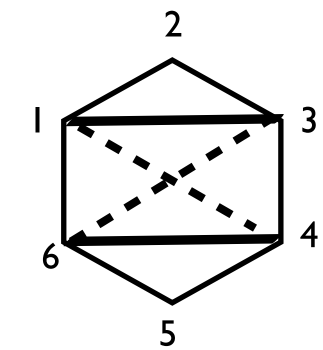
$\hat{G}^{(l)}$ collinear anomalous dimension, known through O(a^4)

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 07

Conformally invariant cross ratios

- for n = 6, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

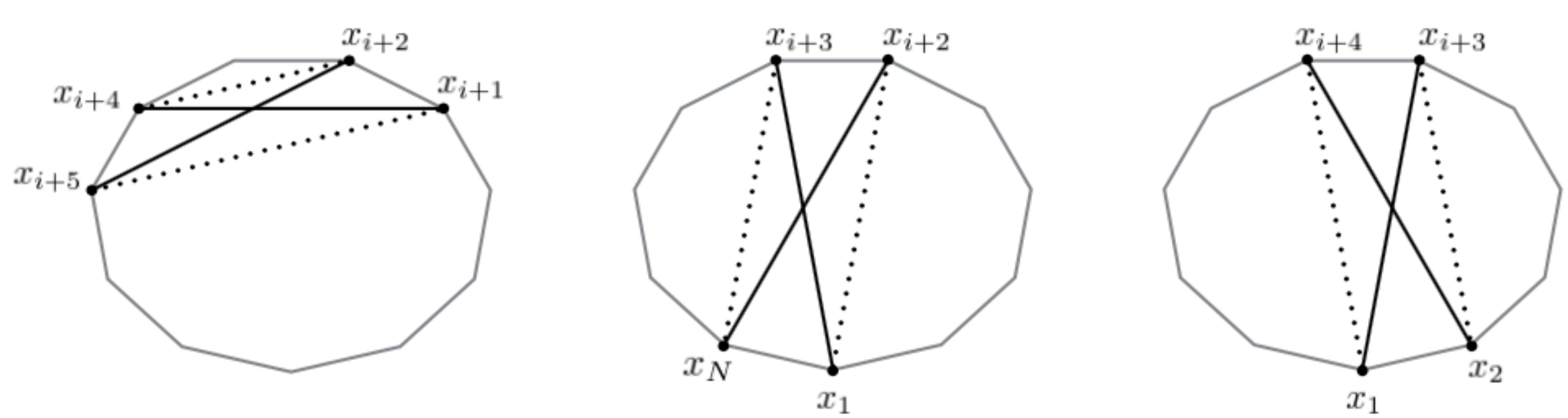


x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

thus $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$

- for n points, dual conformal invariance implies dependence on 3n-15 independent cross ratios

$$u_{1i} = \frac{x_{i+1,i+5}^2 x_{i+2,i+4}^2}{x_{i+1,i+4}^2 x_{i+2,i+5}^2}, \qquad u_{2i} = \frac{x_{N,i+3}^2 x_{1,i+2}^2}{x_{N,i+2}^2 x_{1,i+3}^2}, \qquad u_{3i} = \frac{x_{1\,i+4}^2 x_{2,i+3}^2}{x_{1,i+3}^2 x_{2,i+4}^2}$$



u_{1i} u_{2i} u_{3i}

Amplitudes in planar *N*=4 SYM

- The progress in understanding the analytic structure of the *S*-matrix in planar *N*=4 SYM is also due to an improved understanding of the mathematical structures underlying the scattering amplitudes

- *n*-point amplitudes are expected to be written in terms of iterated integrals (on the space of configurations of points in 3-dim projective space $\mathrm{Conf}_n(P^3)$)

  Golden Goncharov Spradlin Vergu Volovich 2013

- The simplest case of iterated integrals are the iterated integrals over rational functions, i.e. the multiple polylogarithms (MPL)

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t) \qquad G(a;z) = \log\left(1-\frac{z}{a}\right) \qquad (a_1,\ldots,a_n) \in \mathbb{C}$$

Goncharov 2001

- It is thought that maximally helicity violating (MHV) and next-to-MHV (NMHV) amplitudes can be expressed in terms of multiple polylogarithms of uniform transcendental weight

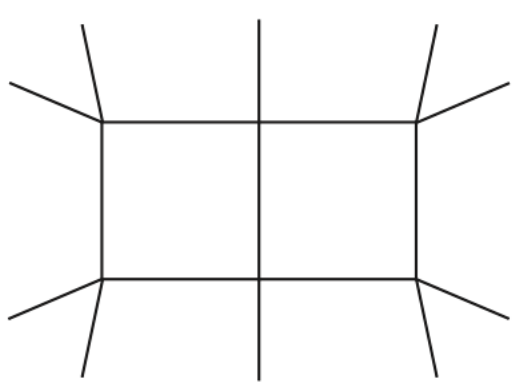
  Arkani-Hamed Bourjaily Cachazo Goncharov Postnikov Trnka 2012

Amplitudes in planar *N*=4 SYM

- MHV and NMHV amplitudes feature *maximal transcendentality*, i.e. *L*-loop amplitudes are expressed in terms of multiple polylogarithms of weight 2*L* only

- MHV amplitudes are *pure*, i.e. the coefficients of the multiple polylogarithms are (rational) numbers

- 2-loop 10-pt N^3MHV amplitude features elliptic iterated integrals



Caron-Huot Larsen 2012
Bourjaily McLeod Spradlin von Hippel Wilhelm 2017

Amplitudes in planar $\mathcal{N}$=4 SYM

- 6-pt (N)MHV amplitudes are known analytically up to 5(4) loops

  Duhr Smirnov VDD 2009
  Goncharov Spradlin Vergu Volovich 2010
  Dixon Drummond Henn 2011
  Dixon Drummond von Hippel Pennington 2013
  Dixon Drummond Duhr Pennington 2014
  Caron-Huot Dixon von Hippel McLeod 2016

  Dixon Drummond Henn 2011
  Dixon von Hippel 2014
  Dixon von Hippel McLeod 2015

- 7-pt MHV amplitudes are known analytically at two loops

  Golden Spradlin 2014

- No analytic result is known beyond 7 points
  (the algebra of the iterated integrals is infinite starting from 8 points)

  Golden Goncharov Spradlin Vergu Volovich 2013

Taxonomy
of
logarithmic functions

Polylogarithms

- classical polylogarithms $$\mathrm{Li}_m(z) = \int_0^z dt \frac{\mathrm{Li}_{m-1}(t)}{t} = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad \mathrm{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1-z)$$

Euler 1768
Spence 1809

- harmonic polylogarithms (HPLs)

$$H(a,\vec{w};z) = \int_0^z \mathrm{d}t\, f(a;t)\, H(\vec{w};t) \qquad f(-1;t) = \frac{1}{1+t}, \quad f(0;t) = \frac{1}{t}, \quad f(1;t) = \frac{1}{1-t}$$

with $\{a,\vec{w}\} \in \{-1,0,1\}$

Remiddi Vermaseren 1999

- classical polylogarithms are multiple polylogarithms with specific roots

$$G(\vec{0}_n;x) = \frac{1}{n!} \ln^n x \qquad G(\vec{a}_n;x) = \frac{1}{n!} \ln^n \left(1 - \frac{x}{a}\right) \qquad G(\vec{0}_{n-1},a;x) = -\,\mathrm{Li}_n\left(\frac{x}{a}\right)$$

- when the root equals +1,-1,0 multiple polylogarithms become HPLs

Multiple polylogarithms

- MPLs form a shuffle algebra

$$G_{\omega_1}(z)G_{\omega_2}(z) = \sum_{\omega} G_{\omega}(z)$$ with ω the shuffle of ω_1 and ω_2

example

$$\begin{aligned} G(a;z)\,G(b;z) &= \int_0^z \frac{dt_1}{t_1-a} \int_0^z \frac{dt_2}{t_2-b} \\ &= \int_0^z \frac{dt_1}{t_1-a} \int_0^{t_1} \frac{dt_2}{t_2-b} + \int_0^z \frac{dt_2}{t_2-a} \int_0^{t_2} \frac{dt_1}{t_1-b} \\ &= G(a,b;z) + G(b,a;z) \end{aligned}$$

- $\lim_{z\to 0} G(a_1,\ldots,a_n;z) = 0$ unless $\vec{a} = \vec{0}$

- $$\frac{\partial}{\partial z} G(a_1,\ldots,a_k;z) = \frac{1}{z-a_1}\, G(a_2,\ldots,a_k;z)$$

- MPLs can be represented as nested harmonic sums

$$\sum_{n_1=1}^{\infty} \frac{u_1^{n_1}}{n_1^{m_1}} \sum_{n_2=1}^{n_1-1} \cdots \sum_{n_k=1}^{n_{k-1}-1} \frac{u_k^{n_k}}{n_k^{m_k}} = (-1)^k\, G\left(\underbrace{0,\ldots,0}_{m_1-1}, \frac{1}{u_1}, \ldots, \underbrace{0,\ldots,0}_{m_k-1}, \frac{1}{u_1\ldots u_k}; 1\right)$$

Hopf algebra and the coproduct

- multiple polylogarithms form a Hopf algebra with a *coproduct*

Goncharov 2002

- algebra is a vector space with a product $\mu: A \otimes A \to A$ $\mu(a \otimes b) = a \cdot b$
  that is associative $A \otimes A \otimes A \to A \otimes A \to A$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

- coalgebra is a vector space with a coproduct $\Delta: B \to B \otimes B$
  that is coassociative $B \to B \otimes B \to B \otimes B \otimes B$

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$$

  μ puts together; Δ decomposes

- a Hopf algebra is an algebra and a coalgebra,
  such that product and coproduct are compatible $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$

- take a word, sum over ways to split it into two: *deconcatenation*

$$T = w\,x\,y\,z$$

$$\Delta(T) = w\,x\,y\,z \otimes 1 + w\,x\,y \otimes z + w\,x \otimes y\,z + w \otimes x\,y\,z + 1 \otimes w\,x\,y\,z$$

- iterate: sum over ways to split it into three

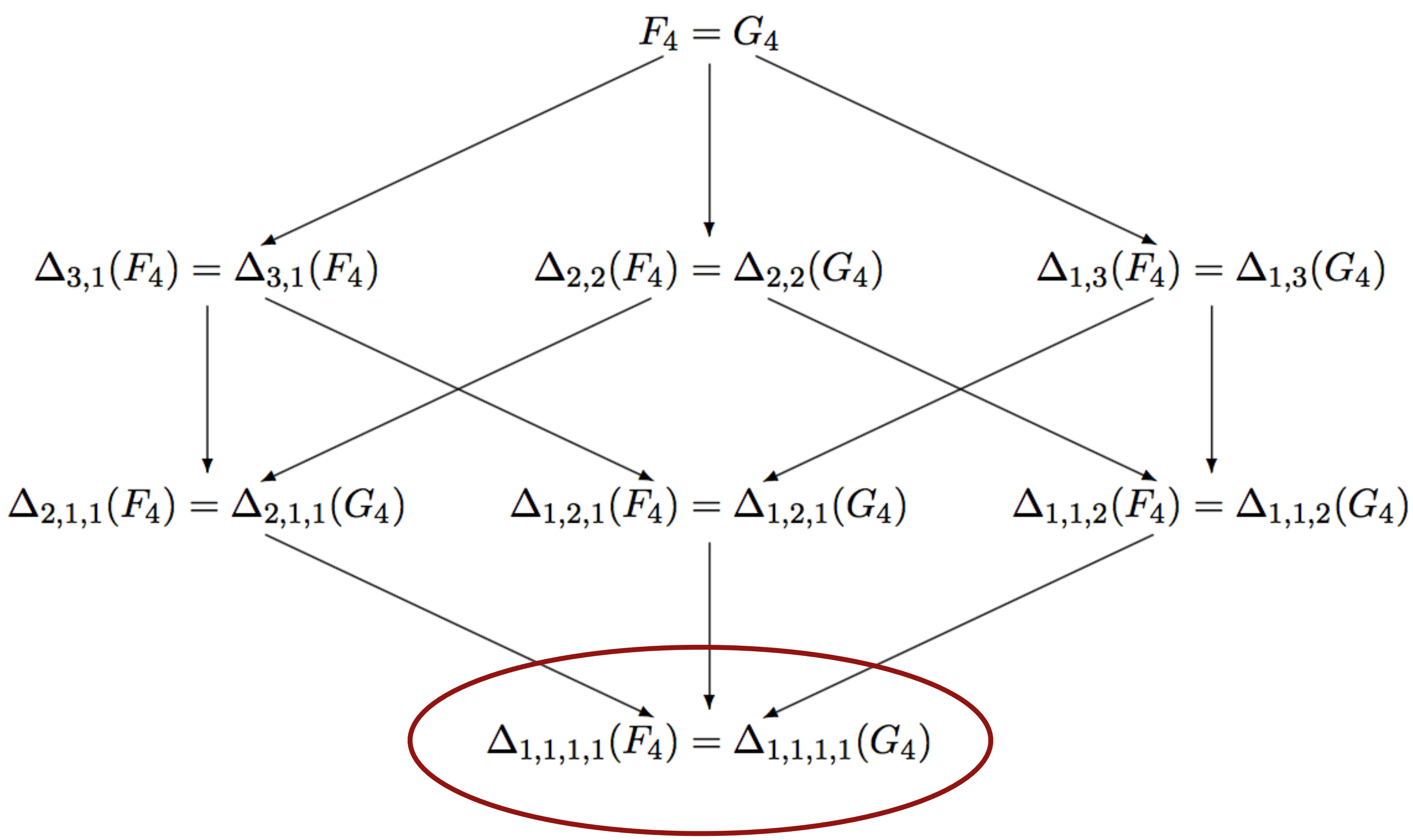
$$w\,x \otimes y\,z \to (w \otimes x) \otimes y\,z$$

$$w\,x \otimes y\,z \to w\,x \otimes (y \otimes z)$$

if sum over all possibilities,
get to the same result

example on a function of weight 4

Duhr 2012



symbols lie within the maximal iteration of a coproduct

Coproduct on polylogarithms

- coproduct on classical polylogarithms

$$\Delta(\ln z) = 1 \otimes \ln z + \ln z \otimes 1$$

$$\begin{aligned}\Delta(\ln y \ln z) &= \Delta(\ln y) \cdot \Delta(\ln z) \\ &= (1 \otimes \ln y + \ln y \otimes 1) \cdot (1 \otimes \ln z + \ln z \otimes 1) \\ &= 1 \otimes \ln y \ln z + \ln y \otimes \ln z + \ln z \otimes \ln y + \ln y \ln z \otimes 1\end{aligned}$$

$$\text{Sym}[\ln y \; \ln z] = y \otimes z + z \otimes y$$

$$\Delta\big(\text{Li}_2(z)\big) = 1 \otimes \text{Li}_2(z) + \text{Li}_2(z) \otimes 1 - \ln(1-z) \otimes \ln z$$

$$\text{Sym}[\text{Li}_2(z)] = -(1-z) \otimes z$$

$$\Delta\big(\text{Li}_n(z)\big) = 1 \otimes \text{Li}_n(z) + \text{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \text{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$$

- (n-1,1) component of the coproduct

$$\Delta_{n-1,1}\big(\text{Li}_n(z)\big) = \text{Li}_{n-1}(z) \otimes \ln z$$

iterating $\Delta_{1,\ldots,1}\big(\text{Li}_n(z)\big) = -\ln(1-z) \otimes \underbrace{\ln z \otimes \cdots \otimes \ln z}_{n-1}$

$$\text{Sym}[\text{Li}_n(z)] = -(1-z) \otimes \overbrace{z \otimes \cdots \otimes z}^{n-1}$$

- the symbol is the (1,…,1) component of the coproduct

- for the constants, define

$$\Delta(\zeta_{2n}) = \zeta_{2n} \otimes 1 \qquad \text{Brown 11}$$

$$\Delta(\pi) = \pi \otimes 1 \qquad \text{Duhr 12}$$

Coproducts and functional identities

- weight 1 $\quad \mathrm{Li}_1(\frac{1}{z}) = -\ln(1-\frac{1}{z}) = -\ln(1-z) + \ln(-z) = -\ln(1-z) + \ln z - i\pi$

- weight 2

$$\Delta_{1,1}\left(\mathrm{Li}_2\left(\frac{1}{z}\right)\right) = -\ln\left(1-\frac{1}{z}\right)\otimes\ln\left(\frac{1}{z}\right)$$
$$= \ln(1-z)\otimes\ln z - \ln z\otimes\ln z + i\pi\otimes\ln z$$
$$= \Delta_{1,1}\left(-\mathrm{Li}_2(z) - \frac{1}{2}\ln^2 z + i\pi\ln z\right)$$

$i\pi$ more than the symbol

so $\quad \mathrm{Li}_2\left(\frac{1}{z}\right) = -\mathrm{Li}_2(z) - \frac{1}{2}\ln^2 z + i\pi\ln z + c\pi^2 \qquad z=1\to c=\frac{1}{3}$

- weight 3

$$\Delta_{1,1,1}\left(\mathrm{Li}_3\left(\frac{1}{z}\right)\right) = -\ln\left(1-\frac{1}{z}\right)\otimes\ln\left(\frac{1}{z}\right)\otimes\ln\left(\frac{1}{z}\right)$$
$$= -\ln(1-z)\otimes\ln z\otimes\ln z + \ln z\otimes\ln z\otimes\ln z - i\pi\otimes\ln z\otimes\ln z$$
$$= \Delta_{1,1,1}\left(\mathrm{Li}_3(z) + \frac{1}{6}\ln^3 z - \frac{i\pi}{2}\ln^2 z\right)$$

one can do better

$$\Delta_{2,1}\left(\mathrm{Li}_3\left(\frac{1}{z}\right) - \left(\mathrm{Li}_3(z) + \frac{1}{6}\ln^3 z - \frac{i\pi}{2}\ln^2 z\right)\right) = -\frac{\pi^2}{3}\otimes\ln z$$
$$= \Delta_{2,1}\left(-\frac{\pi^2}{3}\ln z\right)$$

so $\quad \mathrm{Li}_3\left(\frac{1}{z}\right) = \mathrm{Li}_3(z) + \frac{1}{6}\ln^3 z - \frac{i\pi}{2}\ln^2 z - \frac{\pi^2}{3}\ln z + c_1\zeta_3 + c_2 i\pi^3 \qquad z=1\to c_1=c_2=0$

Symbols

- take a function defined as an iterated integral of logarithms of rational functions R_i

$$T^{(k)} = \int_a^b \mathrm{d}\, lnR_1 \circ \cdots \circ \mathrm{d}\, lnR_k = \int_a^b \left(\int_a^t \mathrm{d}\, lnR_1 \circ \cdots \circ \mathrm{d}\, lnR_{k-1} \right) \mathrm{d}\, lnR_k(t)$$

then the total differential can be written as

$$d\, T^{(k)} = \sum_i T_i^{(k-1)} d \ln R_i$$

- the symbol is defined recursively as $\mathrm{Sym}[T^{(k)}] = \sum_i \mathrm{Sym}[T_i^{(k-1)}] \otimes R_i$

Goncharov

as such, the symbol is defined on the tensor product of the group of rational functions, modulo constants

$$\cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots$$

$$\cdots \otimes (cR_1) \otimes \cdots = \cdots \otimes R_1 \otimes \cdots$$

- if T is a multiple polylogarithm G, then

$$dG(a_{n-1}, \ldots, a_1; a_n) = \sum_{i=1}^{n-1} G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n) d \ln \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}} \right)$$

the symbol is

$$\mathrm{Sym}\left(G(a_{n-1}, \ldots, a_1; a_n)\right) = \sum_{i=1}^{n-1} \mathrm{Sym}\left(G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n)\right) \otimes \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}} \right)$$

- the symbol knows about the discontinuities of *T*; if

$$\mathrm{Sym}[T^{(k)}] = R_1 \otimes \cdots \otimes R_k$$

then *T* has a branch cut at $R_l = 0$, and the symbol of the discontinuity is

$$\mathrm{Sym}[Disc_{R_1}(T^{(k)})] = R_2 \otimes \cdots \otimes R_k$$

- $$Disc(\ln x \ln y) = \begin{cases} 2\pi i \ln x & \text{along the } y \text{ cut } [-\infty, 0] \\ 2\pi i \ln y & \text{along the } x \text{ cut } [-\infty, 0] \end{cases}$$

$$\mathrm{Sym}[\ln x \, \ln y] = x \otimes y + y \otimes x$$

- in general, if $Disc(f\,g) = Disc(f)\, g + f\, Disc(g)$

and $\mathrm{Sym}[f] = \otimes_{i=1}^{n} R_i \qquad \mathrm{Sym}[g] = \otimes_{i=n+1}^{m} R_i$

then $\mathrm{Sym}[fg] = \sum_{\sigma} \otimes_{i=1}^{n} R_{\sigma(i)}$

where σ denotes the set of all shuffles of $n+(m-n)$ elements

e.g. $\mathrm{Sym}[f] = R_1 \otimes R_2 \qquad \mathrm{Sym}[g] = R_3 \otimes R_4$

$$\begin{aligned} \mathrm{Sym}[fg] = &\quad R_1 \otimes R_2 \otimes R_3 \otimes R_4 + R_1 \otimes R_3 \otimes R_2 \otimes R_4 + R_1 \otimes R_3 \otimes R_4 \otimes R_2 \\ + &\quad R_3 \otimes R_1 \otimes R_2 \otimes R_4 + R_3 \otimes R_1 \otimes R_4 \otimes R_2 + R_3 \otimes R_4 \otimes R_1 \otimes R_2 \end{aligned}$$

- symbols form a shuffle algebra, *i.e.* a vector space with a shuffle product
(also iterated integrals and multiple polylogarithms form shuffle algebras)

MPLs, coproduct and unitarity

- multiple polylogarithms form a Hopf algebra with a *coproduct* Goncharov 2002

$$\Delta(L_w) = \sum_{k=0}^{w} \Delta_{k,w-k}(L_w) = \sum_{k=0}^{w} L_k \otimes L_{w-k}$$

- the coproduct steers the functional identities among MPLs, thus it allows us to reduce a given set of MPLs of weight *n* to a (minimal) basis of MPLs of weight ≤ n, which we are then to analytically continue from Euclidean to Minkowski space, and to evaluate numerically

- the analytic structure of amplitudes is constrained by unitarity and the optical theorem $\mathrm{Disc}(M) = iMM^\dagger$

- discontinuity(derivative) acts in the first(last) entry of the coproduct

$$\Delta \mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\Delta \qquad \Delta\partial = (\mathrm{id} \otimes \partial)\Delta$$

Duhr 2012

then the coproduct of an amplitude is related to unitarity

in particular, for massless amplitudes

$$\Delta(M) = \ln(s_{ij}) \otimes \ldots$$

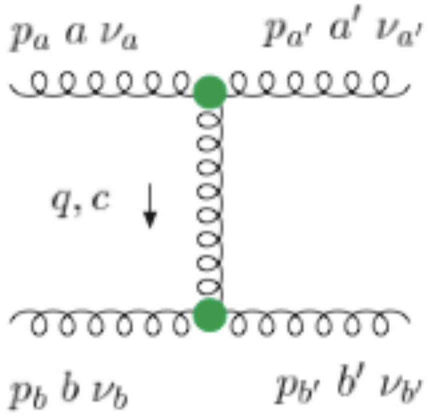
- massless amplitudes may have branch points when Mandelstam invariants vanish $s_{ij} \to 0$ or become infinite $s_{ij} \to \infty$

Regge limit

Regge limit of QCD

- In perturbative QCD, in the Regge limit *s* » *t*, any scattering process is dominated by gluon exchange in the *t* channel

- For a 4-gluon tree amplitude, we obtain



$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2\left[(T^c)_{aa'}C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right]\frac{s}{t}\left[(T_c)_{bb'}C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

$C_{\nu_a\nu_{a'}}(p_a,p_{a'})$ are called *impact factors*

- leading logarithms of *s/t* are obtained by the substitution $\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$

- $\alpha(t)$ is the Regge gluon trajectory, with infrared coefficients

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2\alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right) \qquad \alpha_s(-t,\epsilon) = \left(\frac{\mu^2}{-t}\right)^{\epsilon}\alpha_s(\mu^2)$$

$$\alpha^{(1)} = C_A\,\frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A\,\frac{2}{\epsilon} \qquad \alpha^{(2)} = C_A\left[-\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)}\,\frac{2}{\epsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]$$

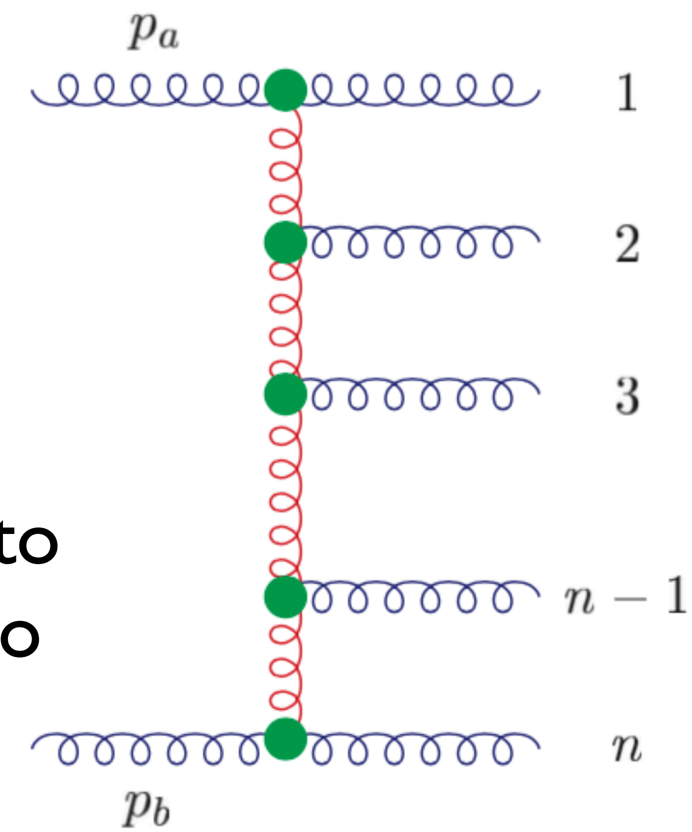
- in the Regge limit, the amplitude is invariant under *s* ↔ *u* exchange.

  To NLL accuracy, the amplitude is given by — Fadin Lipatov 1993

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^c)_{aa'}C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)} + \left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[(T_c)_{bb'}C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

Balitski Fadin Kuraev Lipatov

- BFKL is a resummation of multiple gluon radiation out of the gluon exchanged in the *t* channel



- the Leading Logarithmic (BFKL 1976-77) and Next-to-Leading Logarithmic (Fadin-Lipatov 1998) contributions in log(s/|t|) of the radiative corrections to the gluon propagator in the *t* channel are resummed to all orders in α_s

- the resummation yields an integral (BFKL) equation for the evolution of the gluon propagator in 2-dim transverse momentum space

- the BFKL equation is obtained in the limit of strong rapidity ordering of the emitted gluons, with no ordering in transverse momentum - *multi-Regge kinematics* (MRK)

- the solution is a Green's function of the momenta flowing in and out of the gluon ladder exchanged in the *t* channel

Multi-Regge kinematics in *N*=4 SYM

- In the Euclidean region (where all Mandelstam invariants are negative), amplitudes in MRK factorise completely in terms of building blocks which are expressed in terms of Regge poles and can be determined to all orders through the 4-pt and 5-pt amplitudes.
  Thus the remainder functions *R* vanish at all points

  Brower Nastase Schnitzer Tan 2008
  Bartels Lipatov Sabio-Vera 2008
  Duhr Glover VDD 2008

- After analytic continuation to some regions of the Minkowski space, the amplitude develops cuts. The discontinuity of the amplitude is described by a dispersion relation for octet exchange, which is similar to the singlet BFKL equation in QCD

  Bartels Lipatov Sabio-Vera 2008

- Accordingly, 6-pt amplitudes have been thoroughly examined, both at weak and at strong coupling

  Basso Caron-Huot Sever 2014

- In particular, 6-pt amplitudes at weak coupling can be expressed in terms of single-valued harmonic polylogarithms

  Dixon Duhr Pennington 2012

Regge factorisation of the *n*-pt amplitude

$$m_n(1,2,\dots,n) = s\,[g\,C(p_2,p_3)]\,\frac{1}{t_{n-3}}\left(\frac{-s_{n-3}}{\tau}\right)^{\alpha(t_{n-3})}[g\,V(q_{n-3},q_{n-4},\kappa_{n-4})]$$

$$\cdots\times\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa_1)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_n)]$$

n-pt amplitude in the multi-Regge limit

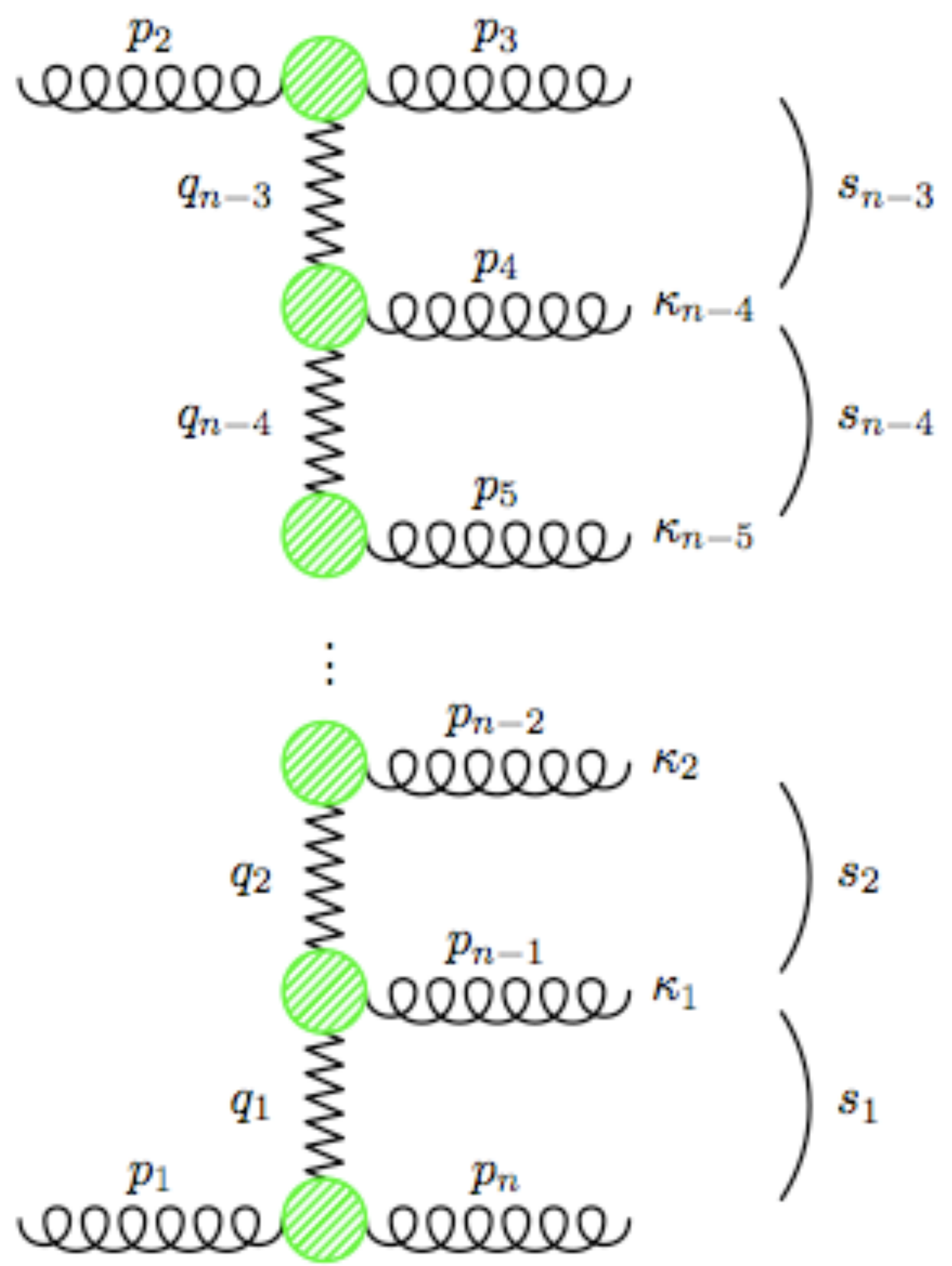
$$y_3 \gg y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|$$

$$s \gg s_1, s_2, \dots, s_{n-3} \gg -t_1, -t_2 \dots, -t_{n-3}$$

the *l*-loop *n*-pt amplitude can be assembled using the *l*-loop trajectories, vertices and coefficient functions, determined through the *l*-loop 4-pt and 5-pt amplitudes

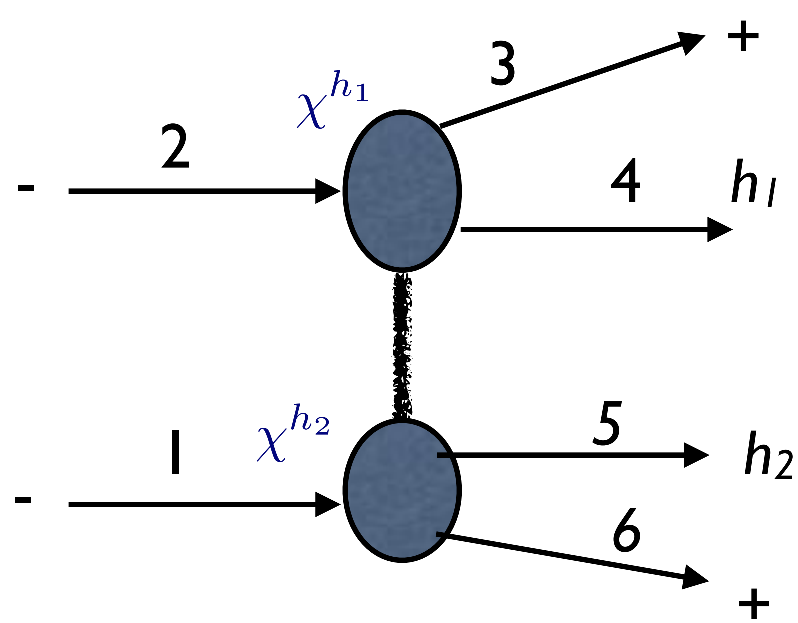
→ in Euclidean space,
no violation of the BDS ansatz can be found in the multi-Regge limit

Duhr Glover VDD 2008



Discontinuity of the amplitude in MRK

6-pt amplitude



continue to a Minkowski region

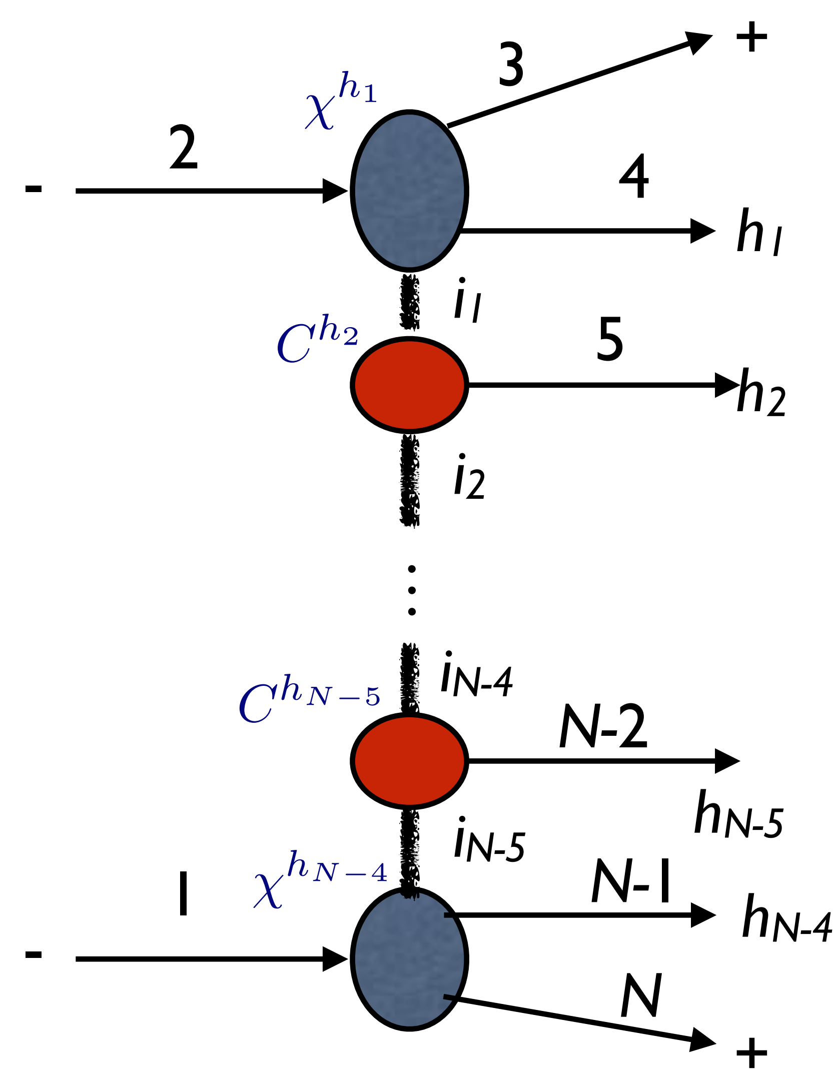
$s_{34},\ s_{56} < 0 \quad s,\ s_{45} > 0$

one cross ratio picks up a phase

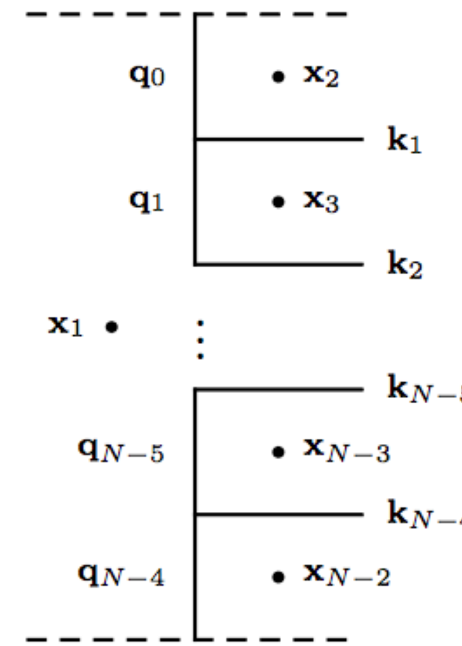
$$u_1 = \frac{s_{12}s_{45}}{s_{345}s_{456}} \to |u_1|\, e^{-2\pi i}$$

compute $\quad \mathrm{Disc}(\mathcal{M})|_{s_{45}}$

n-pt amplitude



Moduli space of Riemann spheres



- in MRK, there is no ordering in transverse momentum, i.e. only the n-2 transverse momenta are non-trivial

- dual conformal invariance in transverse momentum space implies dependence on n-5 cross ratios of the transverse momenta

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})} = -\frac{\mathbf{q}_{i+1}\,\mathbf{k}_i}{\mathbf{q}_{i-1}\,\mathbf{k}_{i+1}} \qquad i = 1, \ldots, n-5$$

- $\mathcal{M}_{0,p}$ = space of configurations of p points on the Riemann sphere

  Because we can fix 3 points at 0, 1, ∞, its dimension is dim($\mathcal{M}_{0,p}$)= p-3

- $\mathcal{M}_{0,n-2}$ is the space of the MRK, with dim($\mathcal{M}_{0,n-2}$) = n-5

  Its coordinates can be chosen to be the z_i's,
  i.e. the cross ratios of the transverse momenta

VDD Drummond Druc Duhr Dulat Marzucca Papathanasiou Verbeek 2016

- on $\mathcal{M}_{0,n-2}$, the singularities are associated to degenerate configurations when two points merge $x_i \to x_{i+1}$
  i.e. when momentum p_i becomes soft $p_i \to 0$

Iterated integrals on $\mathcal{M}_{0,n-2}$

- iterated integrals on $\mathcal{M}_{0,p}$ can be written as multiple polylogarithms

Brown 2006

→ amplitudes in MRK can be written in terms of multiple polylogarithms

- unitarity implies that for massless amplitudes

  $\Delta(M) = \ln(s_{ij}) \otimes \ldots$

  in particular, for amplitudes in MRK

  $\Delta(M) = \ln|\mathbf{x}_i - \mathbf{x}_j|^2 \otimes \ldots$

- except for the soft limit $p_i \to 0$, in MRK the transverse momenta never vanish

  $|\mathbf{x}_i - \mathbf{x}_j|^2 \neq 0$ → single-valued functions

- therefore, *n*-point amplitudes in MRK of planar *N*=4 SYM can be written in terms of single-valued iterated integrals on $\mathcal{M}_{0,n-2}$

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- for n=6, iterated integrals on $\mathcal{M}_{0,4}$ are harmonic polylogarithms
  thus, 6-point amplitudes in MRK of can be written in terms of
  single-valued harmonic polylogarithms (SVHPL)

Dixon Duhr Pennington 2012

MRK in $\mathcal{N}$=4 SYM

- In MRK, 6-pt MHV and NMHV amplitudes are known at any number of loops

Lipatov Prygarin 2010-2011
Dixon Duhr Pennington 2012
Lipatov Prygarin Schnitzer 2012

- knowing the space of functions of the *n*-point amplitudes in MRK, (i.e. that is made of single-valued iterated integrals on $\mathcal{M}_{0,n-2}$) allowed us to compute all MHV amplitudes at ℓ loops in LLA in terms of amplitudes with up to (ℓ+4) points, in practice up to 5 loops, and all non-MHV amplitudes in LLA up 8 points and 4 loops

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for MHV amplitudes in MRK at LLA at:

- at 2 loop, the *n*-pt remainder function $R_n^{(2)}$ can be written as a sum of 2-loop 6-pt remainder functions $R_6^{(2)}$
- …
- …
- at 5 loops, the *n*-pt remainder function $R_n^{(5)}$ can be written as a sum of 5-loop 6-, 7-, 8- and 9-pt amplitudes

Prygarin Spradlin Vergu Volovich 2011
Bartels Kormilitzin Lipatov Prygarin 2011
Bargheer Papathanasiou Schomerus 2015

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- … extended to 7-pt (N)MHV amplitudes at 5(4) loops in NLLA

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Single-valued polylogarithms

- Single-valued functions are real analytic functions on the complex plane

- Because the discontinuities of the classical polylogarithms are known

$$\Delta \mathrm{Li}_n(z) = 2\pi i \frac{\log^{n-1} z}{(n-1)!}$$

one can build combinations of classical polylogarithms
such that all branch cuts cancel on the punctured plane C/{0,1}
(Riemann sphere with punctures)

- An example is the Bloch-Wigner dilogarithm

$$SVP_2(z) = \mathrm{Im}[\mathrm{Li}_2(z)] - \log|z| \arg(1-z)$$

in general

$$SVP_n(z) = R_n \left[\sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \log^k |z| \mathrm{Li}_{n-k}(z) \right]$$

B_k Bernouilli numbers $\quad R_n = \begin{cases} \mathrm{Re} & \text{odd } n \\ \mathrm{Im} & \text{even } n \end{cases}$

Single-valued harmonic polylogarithms

- define a function $\mathscr{L}$ that is real-analytic and single-valued on $\mathbb{C}/\{0,1\}$ and that has the same properties as the HPLs

  the SVHPLs $\mathscr{L}_\omega(z)$ also form a shuffle algebra

  $$\mathcal{L}_{\omega_1}(z)\,\mathcal{L}_{\omega_2}(z) = \sum_{\omega} \mathcal{L}_\omega(z)$$ with ω the shuffle of ω_1 and ω_2

- SVHPLs can be explicitly expressed as combinations of HPLs such that all the branch cuts cancel

Brown 2004

$$\mathcal{L}_0(z) = H_0(z) + H_0(\bar{z}) = \ln|z|^2$$
$$\mathcal{L}_1(z) = H_1(z) + H_1(\bar{z}) = -\ln|1+z|^2$$
$$\mathcal{L}_{0,1}(z) = \frac{1}{4}\left[-2H_{1,0} + 2\bar{H}_{1,0} + 2H_0\bar{H}_1 - 2\bar{H}_0H_1 + 2H_{0.1} - 2\bar{H}_{0,1}\right]$$
$$= \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln|z|^2\left(\ln(1-z) - \ln(1-\bar{z})\right)$$

Single-valued multiple polylogarithms

- Single-valued multiple polylogarithms (SVMPL) can be constructed through a map that to each multiple polylogarithm associates its single-valued version

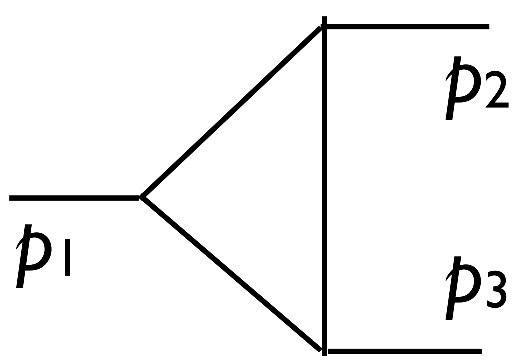
Brown 2004, 2013, 2015

examples of SVMPLs

$$\mathcal{G}_a(z) = G_a(z) + G_{\bar{a}}(\bar{z}) = \ln\left|1 - \frac{z}{a}\right|^2$$

$$\mathcal{G}_{a,b}(z) = G_{a,b}(z) + G_{\bar{b},\bar{a}}(\bar{z}) + G_b(a)G_{\bar{a}}(\bar{z}) + G_{\bar{b}}(\bar{a})G_{\bar{a}}(\bar{z}) - G_a(b)G_{\bar{b}}(\bar{z}) + G_a(z)G_{\bar{b}}(\bar{z}) - G_{\bar{a}}(\bar{b})G_{\bar{b}}(\bar{z})$$

- 3-mass triangles with massless propagators

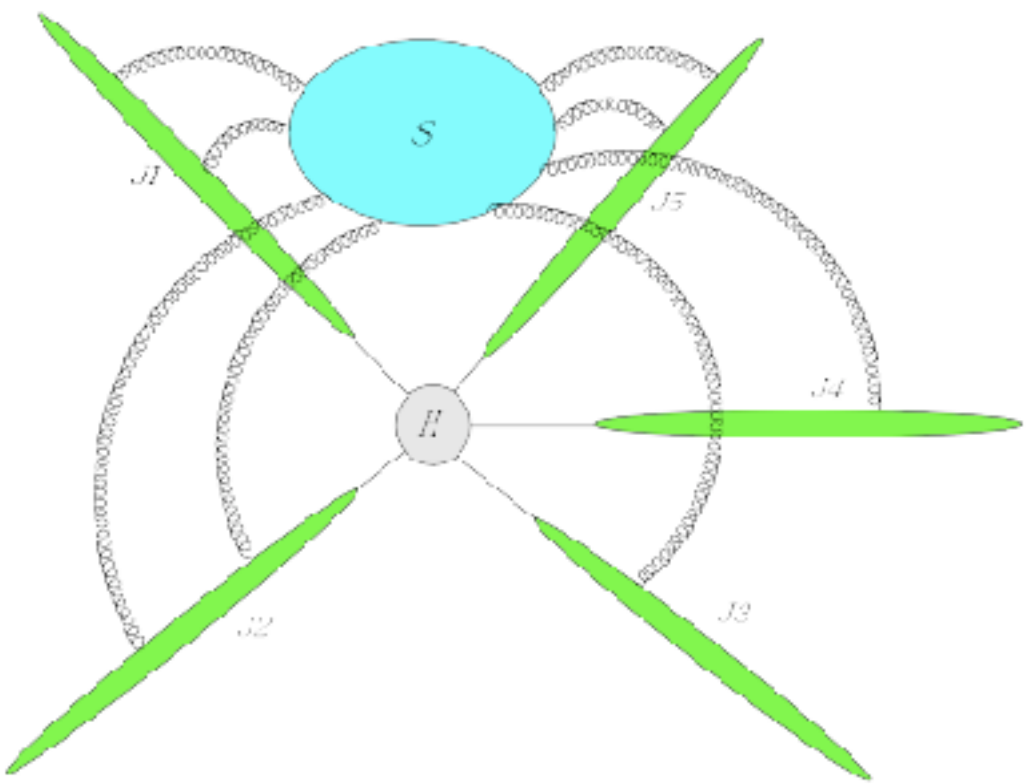


$$z\bar{z} = \frac{p_1^2}{p_3^2} \qquad (1-z)(1-\bar{z}) = \frac{p_2^2}{p_3^2}$$

can be written in terms of SVMPLs

Chavez Duhr 2012

IR structure of a QCD amplitude with *n* massless partons



$$\mathcal{M}_n(\{p_i\},\alpha_s) = Z_n(\{p_i\},\alpha_s,\mu)\,\mathcal{H}_n(\{p_i\},\alpha_s,\mu)$$

Z_n is solution to the RGE equation

$$Z_n = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma_n(\{p_i\},\lambda,\alpha_s(\lambda^2))\right\}$$

Γ_n is the soft anomalous dimension

$$\Gamma_n(\{p_i\},\lambda,\alpha_s) = \Gamma_n^{\mathrm{dip}}(\{p_i\},\lambda,\alpha_s) + \Delta_n(\{\rho_{ijkl}\},\alpha_s)$$

$$\Gamma_n^{\mathrm{dip}}(\{p_i\},\lambda,\alpha_s) = -\frac{1}{2}\hat{\gamma}_K(\alpha_s)\sum_{i<j}\log\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T_i}\cdot\mathbf{T_j} + \sum_{\mathbf{i=1}}^{\mathbf{n}}\gamma_{\mathbf{J_i}}(\alpha_\mathbf{s})$$

$$\rho_{ijkl} = \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

Becher Neubert; Gardi Magnea 2009

At 2 loops, $\Delta^{(2)} = 0$, Γ_2 : Catani 1998; Aybat Dixon Sterman 2006

At 3 loops,

$$\Delta_4^{(3)}(\rho_{1234},\rho_{1432},\alpha_s) = 16\mathbf{T}_1^{a_1}\mathbf{T}_2^{a_2}\mathbf{T}_3^{a_3}\mathbf{T}_4^{a_4}\left\{f^{a_1a_2b}f^{a_3a_4b}\left[F\left(1-\frac{1}{z}\right)-F\left(\frac{1}{z}\right)\right]\right.$$
$$\left. + f^{a_1a_3b}f^{a_4a_2b}\left[F(z)-F(1-z)\right] + f^{a_1a_4b}f^{a_2a_3b}\left[F\left(\frac{1}{1-z}\right)-F\left(\frac{z}{z-1}\right)\right]\right\}$$

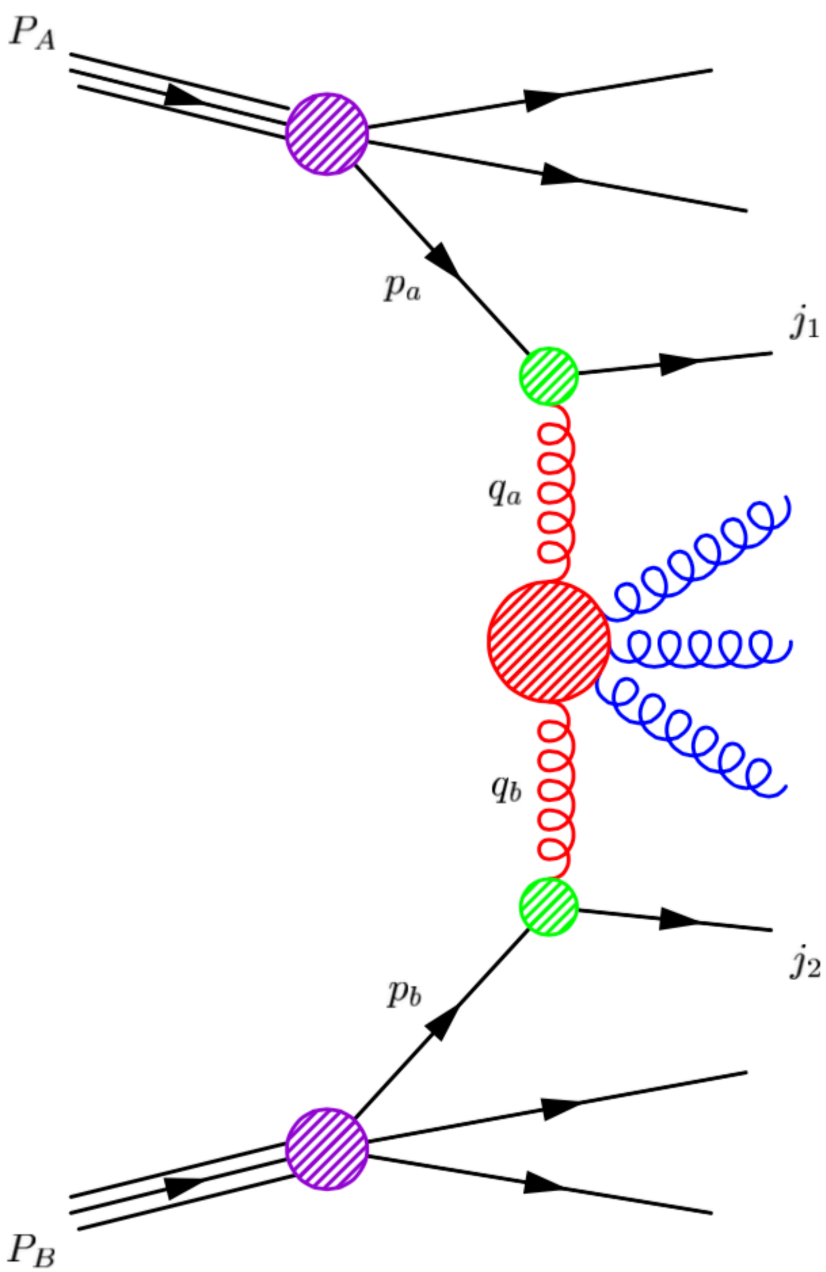
$$\rho_{1234} = z\bar{z} \qquad \rho_{1432} = (1-z)(1-\bar{z}) \qquad F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2[\mathcal{L}_{001}(z)+\mathcal{L}_{100}(z)] + 6\zeta_4\mathcal{L}_1(z)$$

is given in terms of SVHPLs

Almelid Duhr Gardi 2015

Mueller-Navelet jets

Mueller Navelet 1987



Dijet production cross section with two tagging jets in the forward and backward directions

$p_a = x_a P_A \quad p_b = x_b P_B$ incoming parton momenta

S: hadron centre-of-mass energy

$s = x_a x_b S$: parton centre-of-mass energy

E_{Tj}: jet transverse energies

$$\Delta y = |y_{j_1} - y_{j_2}| \simeq \log \frac{s}{E_{Tj_1} E_{Tj_2}}$$

is the rapidity interval between the tagging jets

gluon radiation is considered in MRK and resummed through the LL BFKL equation

Mueller-Navelet evaluated the inclusive dijet cross section up to 5 loops

Mueller-Navelet dijet cross section

- the cross section for dijet production at large rapidity intervals $\Delta y = y_1 - y_2 = \ln\left(\frac{\hat{s}}{-t}\right) \gg 1$

  with $\hat{s} = x_a x_b S\,, \quad t = -\sqrt{p_{1\perp}^2 p_{2\perp}^2}$

$$\frac{d\hat{\sigma}_{gg}}{dp_{1\perp}^2 dp_{2\perp}^2 d\phi_{jj}} = \frac{\pi}{2}\left[\frac{C_A\alpha_s}{p_{1\perp}^2}\right] f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y)\left[\frac{C_A\alpha_s}{p_{2\perp}^2}\right]$$

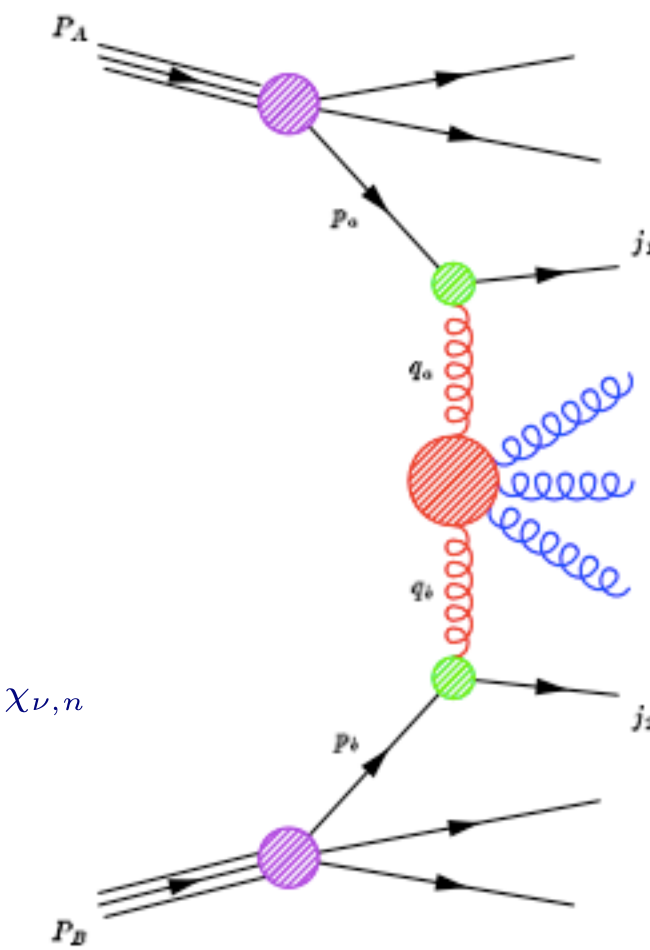
can be described through the BFKL Green's function

$$f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y) = \frac{1}{(2\pi)^2\sqrt{q_{1\perp}^2\, q_{2\perp}^2}} \sum_{n=-\infty}^{+\infty} e^{in\phi} \int_{-\infty}^{+\infty} d\nu \left(\frac{q_{1\perp}^2}{q_{2\perp}^2}\right)^{i\nu} e^{\eta\, \chi_{\nu,n}}$$

with $\eta \equiv \frac{C_A\, \alpha_s}{\pi}\Delta y$ and ϕ the angle between $q_1{}^2$ and $q_2{}^2$

and the LL BFKL eigenvalue

$$\chi_{\nu,n} = -2\gamma_E - \psi\left(\frac{1}{2} + \frac{|n|}{2} + i\nu\right) - \psi\left(\frac{1}{2} + \frac{|n|}{2} - i\nu\right)$$



Mueller-Navelet dijet cross section

- azimuthal angle distribution ($\phi_{jj} = \phi - \pi$)

$$\frac{d\hat{\sigma}_{gg}}{d\phi_{jj}} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \left[\delta(\phi_{jj} - \pi) + \sum_{k=1}^{\infty} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in\phi}}{2\pi} f_{n,k} \right) \eta^k \right]$$

with $f_{n,k} = \frac{1}{2\pi} \frac{1}{k!} \int_{-\infty}^{\infty} d\nu \, \frac{\chi_{\nu,n}^k}{\nu^2 + \frac{1}{4}}$

- the dijet cross section is $\hat{\sigma}_{gg} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \sum_{k=0}^{\infty} f_{0,k} \, \eta^k$ Mueller Navelet 1987

with

$$\begin{aligned} f_{0,0} &= 1\,, \\ f_{0,1} &= 0\,, \\ f_{0,2} &= 2\zeta_2\,, \\ f_{0,3} &= -3\zeta_3\,, \\ f_{0,4} &= \frac{53}{6}\zeta_4\,, \\ f_{0,5} &= -\frac{1}{12}(115\zeta_5 + 48\zeta_2\zeta_3) \end{aligned}$$

Mueller-Navelet jets and SVHPLs

- The singlet LL BFKL ladder in QCD, and thus the dijet cross section in the high-energy limit, can also be expressed in terms of SVHPLs, i.e. in terms of single-valued iterated integrals on $\mathscr{M}_{0,4}$

  Dixon Duhr Pennington VDD 2013

- Mueller & Navelet evaluated analytically the inclusive dijet cross section up to 5 loops. We evaluated it analytically up to 13 loops

- Also, we could evaluate analytically the dijet cross section differential in the jet transverse energies or the azimuthal angle between the jets (up to 6 loops)

BFKL Green's function and single-valued functions

- use complex transverse momentum $\tilde{q}_k \equiv q_k^x + i q_k^y$

  and a complex variable $z \equiv \frac{\tilde{q}_1}{\tilde{q}_2}$

  the BFKL Green's function can be expanded into a power series in $\eta_\mu = \overline{\alpha}_\mu \, y$

$$f^{LL}(q_1, q_2, \eta_\mu) = \frac{1}{2}\delta^{(2)}(q_1 - q_2) + \frac{1}{2\pi\sqrt{q_1^2\, q_2^2}} \sum_{k=1}^{\infty} \frac{\eta_\mu^k}{k!} f_k^{LL}(z)$$

  where the coefficient functions f_k are given by the Fourier-Mellin transform

$$f_k^{LL}(z) = \mathcal{F}\left[\chi_{\nu n}^k\right] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} |z|^{2i\nu} \chi_{\nu n}^k$$

  - the f_k have a unique, well-defined value for every ratio of the magnitudes of the two jet transverse momenta and angle between them.
    So, they are real-analytic functions of z

Azimuthal angle distribution

- this allows us to write the azimuthal angle distribution as

$$\frac{d\hat{\sigma}_{gg}}{d\phi_{jj}} = \frac{\pi(C_A\alpha_s)^2}{2E_\perp^2}\left[\delta(\phi_{jj}-\pi) + \sum_{k=1}^{\infty}\frac{a_k(\phi_{jj})}{\pi}\eta^k\right]$$

where the contribution of the *k*th loop is

$$a_k(\phi_{jj}) = \int_0^\infty \frac{d|w|}{|w|} f_k(w,w^*) = \frac{\text{Im}\, A_k(\phi_{jj})}{\sin\phi_{jj}}$$

with

$$\begin{aligned}
A_1(\phi_{jj}) &= -\frac{1}{2}H_0\,,\\
A_2(\phi_{jj}) &= H_{1,0}\,,\\
A_3(\phi_{jj}) &= \frac{2}{3}H_{0,0,0} - 2H_{1,1,0} + \frac{5}{3}\zeta_2 H_0 - i\pi\,\zeta_2\,,\\
A_4(\phi_{jj}) &= -\frac{4}{3}H_{0,0,1,0} - H_{0,1,0,0} - \frac{4}{3}H_{1,0,0,0} + 4H_{1,1,1,0} - \zeta_2\left(2H_{0,1} + \frac{10}{3}H_{1,0}\right) + \frac{4}{3}\zeta_3\,H_0 + i\pi\left(2\zeta_2 H_1 - 2\zeta_3\right),\\
A_5(\phi_{jj}) &= -\frac{46}{15}H_{0,0,0,0,0} + \frac{8}{3}H_{0,0,1,1,0} + 2H_{0,1,0,1,0} + 2H_{0,1,1,0,0} + \frac{8}{3}H_{1,0,0,1,0} + 2H_{1,0,1,0,0}\\
&\quad + \frac{8}{3}H_{1,1,0,0,0} - 8H_{1,1,1,1,0} - \zeta_2\left(\frac{33}{5}H_{0,0,0} - 4H_{0,1,1} - 4H_{1,0,1} - \frac{20}{3}H_{1,1,0}\right)\\
&\quad - \zeta_3\left(2H_{0,1} + \frac{8}{3}H_{1,0}\right) + \frac{217}{15}\zeta_4 H_0 + i\pi\left[\zeta_2\left(\frac{10}{3}H_{0,0} - 4H_{1,1}\right) + 4\zeta_3 H_1 - \frac{10}{3}\zeta_4\right]
\end{aligned}$$

where $H_{i,j,\ldots} \equiv H_{i,j,\ldots}(e^{-2i\phi_{jj}})$

Dixon Duhr Pennington VDD 2013

Mueller-Navelet dijet cross section reloaded

- the MN dijet cross section is $$\hat{\sigma}_{gg} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \sum_{k=0}^{\infty} f_{0,k}\, \eta^k$$

the first 5 loops were computed by Mueller-Navelet. We computed it through the 13 loops

Dixon Duhr Pennington VDD 2013

$$f_{0,6} = \frac{13}{4}\zeta_3^2 + \frac{3737}{120}\zeta_6\,,$$

$$f_{0,7} = -\frac{87}{5}\zeta_3\zeta_4 - \frac{116}{9}\zeta_2\zeta_5 - \frac{3983}{144}\zeta_7\,,$$

$$f_{0,8} = -\frac{37}{75}\zeta_{5,3} + \frac{64}{15}\zeta_2\zeta_3^2 + \frac{369}{20}\zeta_5\zeta_3 + \frac{50606057}{453600}\zeta_8\,,$$

$$f_{0,9} = -\frac{139}{60}\zeta_3^3 - \frac{15517}{252}\zeta_6\zeta_3 - \frac{3533}{63}\zeta_4\zeta_5 - \frac{557}{15}\zeta_2\zeta_7 - \frac{5215361}{60480}\zeta_9\,,$$

$$f_{0,10} = -\frac{2488}{4725}\zeta_{5,3}\zeta_2 - \frac{94721}{211680}\zeta_{7,3} + \frac{1948}{105}\zeta_4\zeta_3^2 + \frac{2608}{105}\zeta_2\zeta_5\zeta_3 + \frac{12099}{224}\zeta_7\zeta_3 + \frac{1335931}{47040}\zeta_5^2 + \frac{25669936301}{63504000}\zeta_{10}$$

$$f_{0,11} = \frac{62}{315}\zeta_{5,3}\zeta_3 + \frac{83}{120}\zeta_{5,3,3} - \frac{2872}{945}\zeta_2\zeta_3^3 - \frac{13211}{672}\zeta_5\zeta_3^2 - \frac{661411}{3024}\zeta_8\zeta_3 - \frac{242776937}{725760}\zeta_{11} - \frac{605321}{3024}\zeta_5\zeta_6 - \frac{2583643}{16200}\zeta_4\zeta_7 - \frac{28702763}{340200}\zeta_2\zeta_9\,,$$

$$f_{0,12} = \frac{74711}{162000}\zeta_{5,3}\zeta_4 - \frac{13793}{7560}\zeta_{6,4,1,1} + \frac{3965011}{793800}\zeta_{7,3}\zeta_2 - \frac{33356851}{4082400}\zeta_{9,3} + \frac{252163}{181440}\zeta_3^4 + \frac{620477}{10080}\zeta_6\zeta_3^2 + \frac{8101339}{75600}\zeta_4\zeta_5\zeta_3 + \frac{342869}{3780}\zeta_2\zeta_7\zeta_3 + \frac{101571047}{680400}\zeta_9\zeta_3 + \frac{71425871}{1587600}\zeta_2\zeta_5^2 + \frac{904497401571619}{620606448000}\zeta_{12} + \frac{484414571}{2721600}\zeta_5\zeta_7\,,$$

$$f_{0,13} = \frac{4513}{1890}\zeta_{5,3}\zeta_5 + \frac{27248}{23625}\zeta_{5,3,3}\zeta_2 - \frac{97003}{235200}\zeta_{5,5,3} + \frac{13411}{75600}\zeta_{7,3}\zeta_3 + \frac{7997743}{12700800}\zeta_{7,3,3} - \frac{187318}{14175}\zeta_4\zeta_3^3 - \frac{125056}{4725}\zeta_2\zeta_5\zeta_3^2 - \frac{17411413}{302400}\zeta_7\zeta_3^2 - \frac{5724191}{100800}\zeta_5^2\zeta_3 - \frac{1874972477}{2376000}\zeta_{10}\zeta_3 - \frac{2418071698069}{2235340800}\zeta_{13} - \frac{2379684877}{6048000}\zeta_{11}\zeta_2 - \frac{297666465053}{523908000}\zeta_6\zeta_7 - \frac{1770762319}{2494800}\zeta_5\zeta_8 - \frac{229717224973}{628689600}\zeta_4\zeta_9$$

Regge limit
in the
next-to-leading logarithmic
approximation

BFKL eigenvalue at NLLA

- At NLLA in QCD and in N=4 SYM, the eigenvalue is

$$\omega_{\nu n}^{(1)} = \frac{1}{4}\delta_{\nu n}^{(1)} + \frac{1}{4}\delta_{\nu n}^{(2)} + \frac{1}{4}\delta_{\nu n}^{(3)} + \gamma_K^{(2)}\chi_{\nu n} - \frac{1}{8}\beta_0\chi_{\nu n}^2 + \frac{3}{2}\zeta_3$$

Fadin Lipatov 1998
Kotikov Lipatov 2000, 2002

with one-loop beta function and two-loop cusp anomalous dimension

$$\beta_0 = \frac{11}{3} - \frac{2N_f}{3N_c} \qquad \gamma_K^{(2)} = \frac{1}{4}\left(\frac{64}{9} - \frac{10N_f}{9N_c}\right) - \frac{\zeta_2}{2}$$

and with

$$\delta_{\nu n}^{(1)} = \partial_\nu^2 \chi_{\nu n} \qquad \chi_{\nu n} = \omega_{\nu n}^{(0)}$$

$$\delta_{\nu n}^{(2)} = -2\Phi(n,\gamma) - 2\Phi(n, 1-\gamma)$$

$$\delta_{\nu n}^{(3)} = -\frac{\Gamma(\frac{1}{2}+i\nu)\Gamma(\frac{1}{2}-i\nu)}{2i\nu}\left[\psi\left(\frac{1}{2}+i\nu\right) - \psi\left(\frac{1}{2}-i\nu\right)\right]$$
$$\times\left[\delta_{n0}\left(3 + \left(1+\frac{N_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right) - \delta_{|n|2}\left(\left(1+\frac{N_f}{N_c^3}\right)\frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)}\right)\right]$$

$\Phi(n,\gamma)$ is a sum over linear combinations of ψ functions
and γ is a shorthand $\gamma = 1/2 + i\nu$

In blue we labeled the terms which occur only in QCD,
in red the ones which occur in QCD and in N=4 SYM

Fourier-Mellin transform

- At NLLA, the BFKL gluon ladder is

$$f^{NLL}(q_1, q_2, \eta_{s_0}) = \frac{1}{2\pi\,\sqrt{q_1^2 q_2^2}} \sum_{k=1}^{\infty} \frac{\eta_{s_0}^k}{k!}\, f_{k+1}^{NLL}(z) \qquad \eta_{s_0} = \overline{\alpha}_S(s_0)\, y$$

with coefficients given by the Fourier-Mellin transform

$$f_k^{NLL}(z) = \mathcal{F}\left[\omega_{\nu n}^{(1)}\, \chi_{\nu n}^{k-2}\right] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\, |z|^{2i\nu}\, \omega_{\nu n}^{(1)}\, \chi_{\nu n}^{k-2} \qquad \chi_{\nu n} = \omega_{\nu n}^{(0)}$$

using the explicit form of the eigenvalue

$$\omega_{\nu n}^{(1)} = \frac{1}{4}\,\delta_{\nu n}^{(1)} + \frac{1}{4}\,\delta_{\nu n}^{(2)} + \frac{1}{4}\,\delta_{\nu n}^{(3)} + \gamma_K^{(2)}\chi_{\nu n} - \frac{1}{8}\beta_0\chi_{\nu n}^2 + \frac{3}{2}\zeta_3$$

the coefficients can be written as

$$f_k^{NLL}(z) = \frac{1}{4}\,C_k^{(1)}(z) + \frac{1}{4}\,C_k^{(2)}(z) + \frac{1}{4}\,C_k^{(3)}(z) + \gamma_K^{(2)}\, f_{k-1}^{LL}(z) - \frac{1}{8}\,\beta_0\, f_k^{LL}(z) + \frac{3}{2}\,\zeta_3\, f_{k-2}^{LL}(z)$$

with $C_k^{(i)}(z) = \mathcal{F}\left[\delta_{\nu n}^{(i)}\, \chi_{\nu n}^{k-2}\right]$

the weight of f^{NLL}_k is

weight(f^{NLL}_k)= k $\quad k \quad 0 \le w \le k \quad k-2 \le w \le k \quad k-1 \quad k$

generalised SVMPLs

- $C_k^{(1)}(z)$ are SVHPLs of uniform weight k with singularities at $z=0$ and $z=1$

  $C_k^{(3)}(z)$ are MPLs of type $G(a_1,\ldots,a_n;|z|)$ with $a_k \in \{-i,0,i\}$

  they are SV functions of z because they have no branch cut on the positive real axis, and have weight $0 \leq w \leq k$

  For $C_k^{(2)}(z)$ one needs Schnetz' generalised SVMPLs with singularities at

  $$z = \frac{\alpha\,\bar{z}+\beta}{\gamma\,\bar{z}+\delta}\,, \qquad \alpha,\beta,\gamma,\delta \in \mathbb{C}$$

  Schnetz 2016

  then one can show that $C_k^{(2)}(z)$ are Schnetz' generalised SVMPLs

  $\mathcal{G}(a_1,\ldots,a_n;z)$ with singularities at $a_i \in \{-1,0,1,-1/\bar{z}\}$

  Duhr Marzucca Verbeek VDD 2017

- In moment space, the maximal weight of the BFKL eigenvalue and of the anomalous dimensions of the leading twist operators which control the Bjorken scaling violations in QCD is the same as the corresponding quantities in N=4 SYM (Principle of Maximal Transcendentality)

  Kotikov Lipatov 2000, 2002
  Kotikov Lipatov Velizhanin 2003

  Interestingly, in transverse momentum space at NLLA, the maximal weight of the BFKL ladder in QCD is *not* the same as the one of the ladder in *N*=4 SYM

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BFKL ladder in a generic SU(N$_c$) gauge theory

- one can consider the BFKL eigenvalue at NLLA in a SU(N$_c$) gauge theory with scalar or fermionic matter in arbitrary representations

$$\omega^{(1)}_{\nu n} = \frac{1}{4}\delta^{(1)}_{\nu n} + \frac{1}{4}\delta^{(2)}_{\nu n} + \frac{1}{4}\delta^{(3)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) + \frac{3}{2}\zeta_3 + \gamma^{(2)}(\tilde{n}_f, \tilde{n}_s)\,\chi_{\nu n} - \frac{1}{8}\beta_0(\tilde{n}_f, \tilde{n}_s)\,\chi^2_{\nu n}$$

Kotikov Lipatov 2000

with $\beta_0(\tilde{n}_f, \tilde{n}_s) = \frac{11}{3} - \frac{2\tilde{n}_f}{3N_c} - \frac{\tilde{n}_s}{6N_c}$ $\qquad \gamma^{(2)}(\tilde{n}_f, \tilde{n}_s) = \frac{1}{4}\left(\frac{64}{9} - \frac{10\,\tilde{n}_f}{9N_c} - \frac{4\,\tilde{n}_s}{9N_c}\right) - \frac{\zeta_2}{2}$

$$\tilde{n}_f = \sum_R n^R_f T_R \qquad \tilde{n}_s = \sum_R n^R_s T_R \qquad \mathrm{Tr}(T^a_R T^b_R) = T_R\,\delta^{ab} \qquad T_F = \frac{1}{2}$$

$\tilde{n}_s(\tilde{n}_f) =$ number of scalars (Weyl fermions) in the representation *R*

$$\delta^{(3)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) = \delta^{(3,1)}_{\nu n}(\tilde{N}_f, \tilde{N}_s) + \delta^{(3,2)}_{\nu n}(\tilde{N}_f, \tilde{N}_s)$$

with $\tilde{N}_x = \frac{1}{2}\sum_R n^R_x\, T_R\,(2C_R - N_c)\,, \qquad x = f, s$

- Necessary and sufficient conditions for a SU(N$_c$) gauge theory to have a BFKL ladder of maximal weight are:

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— the one-loop beta function must vanish
— the two-loop cusp AD must be proportional to ζ_2
— $\delta^{(3,2)}_{\nu n}$ must vanish $\rightarrow$ $2\tilde{N}_f = N_c^2 + \tilde{N}_s$

⇒ There is no theory whose BFKL ladder has uniform maximal weight which agrees with the maximal weight terms of QCD

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Matter in the fundamental and in the adjoint

- We solve the conditions above for matter in the fundamental *F* and in the adjoint *A* representations. We obtain:

$$2\,n_f^F = n_s^F \qquad\qquad 2\,n_f^A = 2 + n_s^A$$

which describes the spectrum of a gauge theory with *N* supersymmetries and $n^F = n_f^F$ chiral multiplets in *F* and $n^A = n_f^A - N$ chiral multiplets in *A*

- There are four solutions to those conditions

$\mathcal{N}$	4	2	1	1
n_A	0	0	0	2
n_F	0	$4N_c$	$6N_c$	$2N_c$

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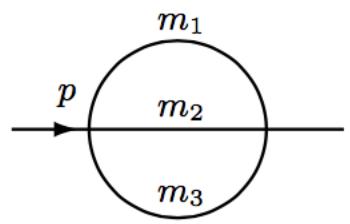
— the first is *N*=4 SYM
— the second is N=2 superconformal QCD with $N_f = 2N_c$ hypermultiplets
— the third is *N*=1 superconf. QCD

- Caveat:
because the one-loop beta function is fixed by matter loops in gluon self-energies, we are only sensitive to the matter content of a theory, and not to its details (like scalar potential or Yukawa couplings)

Hic sunt leones …

Elliptic iterated integrals

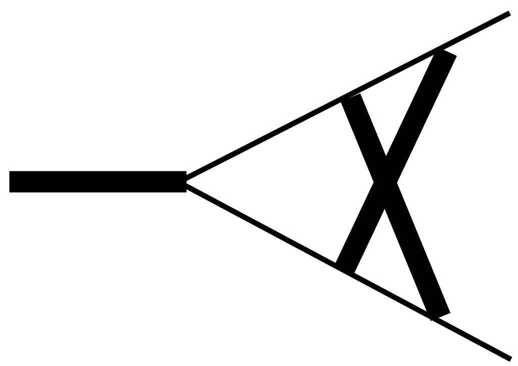
- 2-loop sunrise graph



Broadhurst 1989; …; Bloch Vanhove 2013; …

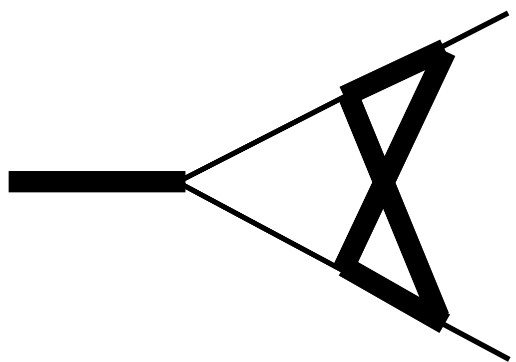
- 2-loop 3-pt functions

electroweak form factor



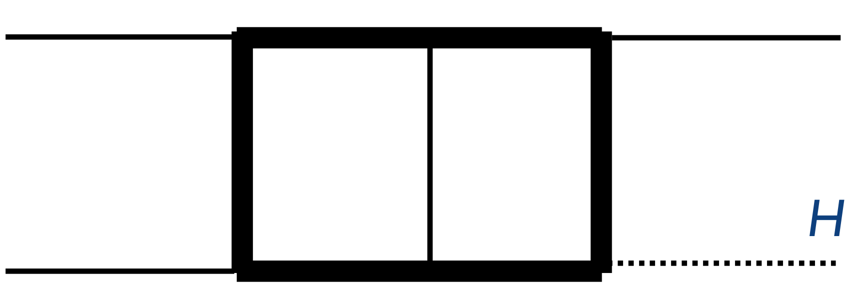
Aglietti Bonciani Grassi Remiddi 2007

t-*t*bar



von Manteuffel Tancredi 2017

- 2-loop 4-pt function for Higgs + 1 jet

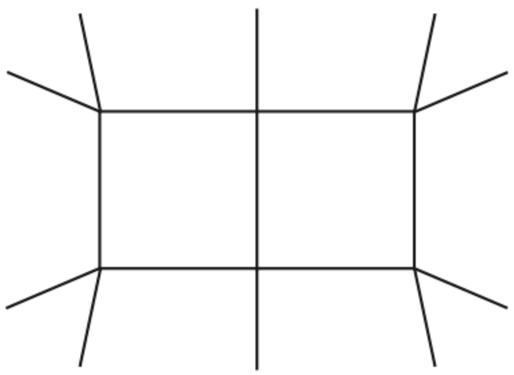


Bonciani VDD Frellesvig Henn Moriello V. Smirnov 2016

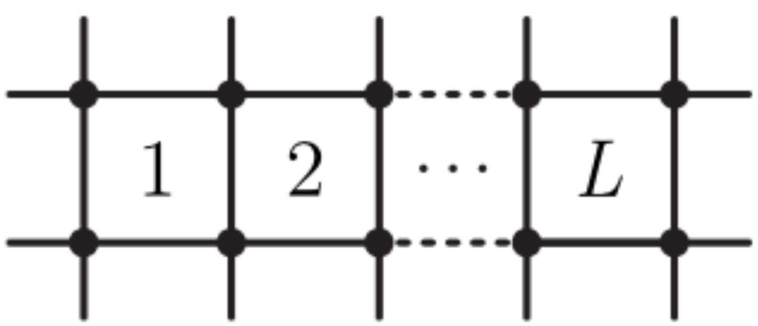
massless elliptic iterated integrals

- 2-loop 10-pt N^3MHV amplitude in planar $\mathcal{N}$=4 SYM

Caron-Huot Larsen 2012
Bourjaily McLeod Spradlin von Hippel Wilhelm 2017



- traintracks:
  L-loop Feynman integrals involving (2*L*+6) massless legs
  they occur in massless φ^4 and in planar $\mathcal{N}$=4 SYM



Bourjaily He McLeod von Hippel Wilhelm *today*

- iterated integrals on $\mathcal{M}_{0,p}$ are multiple polylogarithms — Brown 2006

$\mathcal{M}_{0,p}$ = space of configurations of p points on the Riemann sphere

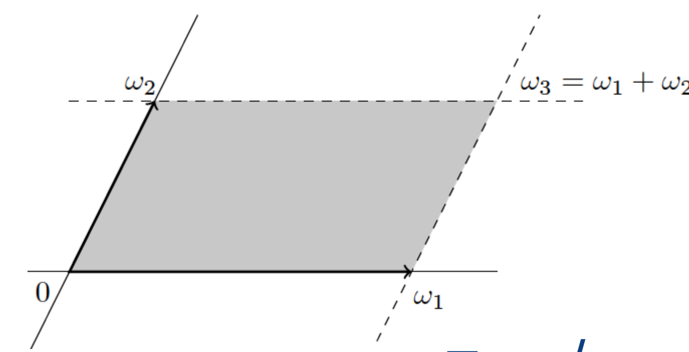
$$G(a,\vec{w};z) = \int_0^z \frac{\mathrm{d}t}{t-a} G(\vec{w};t)\,, \qquad G(a;z) = \ln\left(1-\frac{z}{a}\right) \qquad a,\vec{w}\in\mathbb{C}$$

- iterated integrals on a torus … — Brown Levin 2011

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 \dots n_k \\ z_1 \dots z_k \end{smallmatrix};z,\tau\right) = \int_0^z dt\, g^{(n_1)}(t-z_1,\tau)\,\tilde{\Gamma}\left(\begin{smallmatrix} n_2 \dots n_k \\ z_2 \dots z_k \end{smallmatrix};t,\tau\right) \qquad n_i\in\mathbb{N},\quad z_i\in\mathbb{C}$$

with kernels defined through the Eisenstein-Kronecker series

$$F(z,\alpha,\tau) = \frac{1}{\alpha}\sum_{n=0}^{\infty} g^{(n)}(z,\tau)\,\alpha^n = \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}$$



$\tau = \omega_2/\omega_1$

θ_1 Jacobi theta function; $g^{(n)}$ has at most simple poles at $z = m + n\tau$ $\quad m,n\in\mathbb{Z}$

… are elliptic multiple polylogarithms (eMPL)

$$E_3\left(\begin{smallmatrix} n_1 \dots n_k \\ z_1 \dots z_k \end{smallmatrix};z,\vec{a}\right) = \int_0^z dt\,\varphi_{n_1}(z_1,t,\vec{a})\,E_3\left(\begin{smallmatrix} n_2 \dots n_k \\ z_2 \dots z_k \end{smallmatrix};t,\vec{a}\right) \qquad n_i\in\mathbb{Z},\quad z_i\in\mathbb{C}\quad a_i\in\mathbb{R}$$

with $\vec{a}=(a_1,a_2,a_3)$ are the zeroes of the elliptic curve $y^2=(x-a_1)(x-a_2)(x-a_3)$

and $E_3\,(;z,\vec{a})=1$

- 2-loop sunrise can be written in terms of eMPLs — Brödel Duhr Dulat Penante Tancredi 2017

Conclusions

- In the last few years, a lot of progress has been made in understanding the analytic structure of multi-loop amplitudes

- we understand the analytic and algebraic properties of amplitudes, when they are written in terms of MPLs and/or SVMPLs

- an in-depth exploration of how elliptic iterated integrals arise at 2 loops and beyond has just begun