DUALITIES, EHLERS and FORM COSETS

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When Einstein theory of gravity is formulated in the light-cone gauge the theory has been shown (Goroff, Schwarz) to have a manifest $SL(D-2,R)$ symmetry and the graviton can be associated to the space-time coset

$$SL(D-2,R)/SO(D-2,R)$$

although the Einstein action is more complicated than the naïve σ-model action.

$SL(D-2,R)$ is the so-called Ehlers group in D-dimensions ($SL(2,R)$ at $D=4$)

In the light-cone gauge one makes
D gauge choices

$$g_{i-} = g_{--} = 0 , \quad g_{+-} = (\det g_{ij})^{1/2(D-2)}$$

$(i, j = 1 \ldots D-2)$

In this way one can show that
D eqs. of motions for g^{i-}, g^{--}, g_{+-}
can be solved and one is left with

$$D(D+1)/2 - 2D = \frac{1}{2} D(D-3)$$

dynamical equations which exactly
correspond to the eqs. of
motion for the physical degrees
of freedom of the "graviton"

Dynamical symmetry.

In the light cone gauge the Einstein action is invariant under

$$\delta g_{ij} = M_k{}^l x^k \partial_l g_{ij} + M_i{}^k g_{kj} + M_j{}^k g_{ik}$$

$M_k{}^l \in SL(D-2, R)$.

$$g_{ij} \to SL(D-2,R)/SO(D-2)$$

It is the purpose of the present seminar to report on the extension of these considerations to supergravity and in particular to relate the $SL(D-2,R)$ symmetry to U-duality symmetry in D dimensions for a theory with N supersymmetries.

(N is the active number of supercharges for massless particles so $N=16$ for maximal supergravities).

The first observation to make is the fact that

$$SL(D-2,R)/SO(D-2)$$

is also the coset giving rise to a

σ-model lagrangian compactified to three-dimension on a $D-3$ torus.

In fact at $D=3$ the toroidal symmetry $GL(D-3)$ is enhanced to $SL(D-2)$ since, other than g_{ij}, $(D-3)$ vectors $g_{i\mu}$ can be dualized to scalars so that

$$m_T = \frac{D(D-3)}{2} = \dim SL(D-2)/SO(D-2) -$$

The very same coset describes the graviton transverse space at D dimension (light-cone) where now $SO(D-2)$ is the Wigner helicity symmetry and not an internal symmetry as in the $D=3$ case. $SL(D-2)$ is the remnant, in the light-cone gauge, of general covariance.

The main results of the present investigation rely on the connection between the 3-Dimensional U-duality symmetry and the Ehlers symmetry in D-dimension.

(A. Marrani, M. Trigiante, S.F. arXiv:1206.1255)

(A. Marrani, B. Zumino, S.F. arXiv:1208.0347).

A main fact is the existence of the following embedding (Keurentjes) (Cremmer, Julia; Cremmer, Julia, Lu, Pope)

$$G_N^3 \supset G_N^D \times SL(D-2, R)$$

when G_N^D is the U-duality symmetry in D dimensions with N supersymmetries and $SL(D-2, R)$ is the Ehlers group.

This embedding has the following general properties:

it is regular i.e. it preserves the rank of the group

$$\text{rank } G_N^3 = \text{rank } G_N^D + D - 3$$

It also preserves the non-compact rank i.e.

$$\text{rank } G_N^3 / H_N^3 = \text{rank } G_N^D / H_N^D + D - 3$$

where H_N^3 and H_N^D are the maximal compact subgroups of G_N^3, G_N^D respectively.

Moreover the coset spaces

$$M_N^D = \frac{G_N^3}{G_N^D \times SL(D-2,R)}$$

are Lorentzian spaces with vanishing character (Lorentzian signature : compact − noncompact generators).
In Physical term this is the space of supergravity "form" fields and their dual, where metric and scalar degrees of freedom have been modded out.

A general analysis for all D, N values is reported in the published papers, here we will only consider few cases, just to show features which turn out to be of general validity.

The cases considered here are ($N=16$) maximal supergravities and few examples with lower N.

For maximal supergravities the embedding is

$$E_{8(8)} \supset E_{11-D,\,11-D} \times SL(D-2)$$

with an extra case for $D=10$ (corresponding to type II B)

$$E_{8(8)} \supset SL(2,R) \times E_{7(7)} \supset SL(2,R) \times SL(8,R)$$

(when actually the Ehlers group is enhanced)

* $D=11$ $E_{8(8)} \to SL(9)$ max, ns

$D=10A$ $E_{8(8)} \to SL(9) \to SO(1,1) \times SL(8)$ nm

$D=10B$ $E_{8(8)} \to SL(2) \times E_{7(7)} \to SL(2) \times SL(8)$ nm

$D=9$ $E_{8(8)} \to SL(9) \to GL(2) \times SL(7)$ nm

or

$E_{8(8)} \to SL(2) \times E_{7(7)} \to GL(2) \times SL(7)$ nm

$D=8$ $E_{8(8)} \to SL(2) \times E_{7(7)} \to SL(2) \times SL(3) \times SL(6)$ nm

* $D=7$ $E_{8(8)} \to SL(5) \times SL(5)$ max, ns

$D=6$ $E_{8(8)} \to SO(8,8) \to SO(5,5) \times SL(4)$ nm

* $D=5$ $E_{8(8)} \to E_{6(6)} \times SL(3)$ max, ns

** $D=4$ $E_{8(8)} \to E_{7(7)} \times SL(2)$ max, s

$G^3_N \to G^4_N \times SL(2,R)$

$D=3$ (de Wit, Tollsten, Nicolai)

$G^3_{16} = E_{8(8)}$	$G^4_{16} = E_{7(7)}$
$G^3_{12} = E_{7-5}$	$G^4_{12} = SO^*(12)$
$G^3_{10} = E_{6(-14}$	$G^4_{10} = SU(5,1)$
$G^3_{8} = SO(8,2+n)$	$G^4_{8} = SL(2,R) \times SO(6,n)$
$G^3_{6} = SU(4,1+n)$	$G^4_{6} = U(3,m)$
$G^3_{4} = SO(4,2+n)$	$G^4_{4} = SL(2,R) \times SO(2,n)$
$G^3_{4} = SU(2,1+n)$	$G^4_{4} = U(1,m)$
$E_{8(-24)}$	$E_{7(-25)}$
$E_{7(-5)}$	$SO^*(12)$
$E_{6(2)}$	$SU(3,3)$
$F_{4(4)}$	$Sp(6,R)$

In order to examine the particle content of the theory let's take the maximal compact subgroups of left and right hand side

$$H^3_{16} \rightarrow H^D_{16} \times SO(D-2)$$

Here one finds that H^D_{16} is nothing but the R-symmetry of the corresponding theory.

The decomposition of a chiral spinor 128 of $SO(16)$ following the embedding $16 \rightarrow 16_s$ of $SO(16)$ into $SO(9)$ precisely gives

$$128 \rightarrow 44 + 84$$

and the chiral spinor of opposite chirality of SO(16) gives

$$128' \to \underset{(8\times 16)}{128} \ (\text{spin } \tfrac{3}{2} \text{ of } \underset{(\text{spinor})}{SO(9)})$$

Let's now consider the decomposition of $E_{8(8)}$ into $SO(9)$ through $SO(16)$ (Spin 16)

$$E_{8(8)} \to 120^- + 128^+ \to$$

$$\to 36^- + 84^- + 44^+ + 84^+$$

($nc - c = 8$). This reflects the fact that $E_{8(8)}/SO(16)$ has euclidean signature.

It is easy to see that by exchanging $44^+ \to 84^-$ and modding out $36^- + 44^+$ it is equivalent to consider the

Coset $E_{8(8)}/SL(9,\mathbb{R})$ (non symmetric space)

Indeed under $SL(9,\mathbb{R}) \to SO(9,\mathbb{R})$

$$80 \to 36^- + 44^+ \quad (nc - c = 8)$$

so that

$$E_{8(8)}/SL(9,\mathbb{R}) = 84 + 84' = 84^+ + 84^-$$

(under $SO(9,\mathbb{R})$) $\quad nc - c = 0$

The physical interpretation of these results is that the above coset describes a form field and its magnetic dual.

Another simple example (when G_N^D is empty) is pure $N=4$ supergravity in $D=5$

In this case $G_4^3 = G_{2(2)}$

and the decomposition is

$G_{2(2)} \to SL(3,R)$ (non symmetric embedding)

Restricting to the m.c.s. of both sides

one has

$SU(2)_L \times SU(2)_R \to SU(2)_D$

$14 \to (3_L,1)^- + (1,3_R)^- + (2_R, 4_L)^+$

$(nc-c=2)$

The (bosonic) particle spectrum is

obtained by decomposing $(2_L, 4_R)$

under $SU(2)_D$ $\quad 8 = 3+5$

(spin 1 + spin 2)

$14 \to 3^- + 3^- + 3^+ + 5^+$

(by interchanging 3^- with 5^+) we

get

$G_{2(2)}/SL(3,R) = 3^- + 3^+ \quad (nc-c=0)$

It should be pointed out that the embeddings considered above are maximal and symmetric only at $D=4$ where

$$E_{8(8)} \to E_{7(7)} \times SL(2,R)$$

They are also maximal (but not symmetric) for $D=5,7,11$. All other cases are non-maximal. This means that for $D=6,8,9,10$ the duality and/or the Ehlers symmetry get enhanced.

The $N=16$, $D=4$ case

$$E_{8(8)} \to E_{7(7)} \times SL(2,R)$$

$$SO(16) \longrightarrow SU(8) \times U(1)$$

$$248 = (133,1) + (1,3) + (56,2)$$
$$= (70,0)^+ + (63,0)^- + (1,+2)^+ + (1,-2)^+ + (1,0)^-$$
$$+ \underset{\downarrow}{(56,2)}$$
$$(28+\overline{28},+1) + (28+\overline{28},-1)$$

$$248 = 120^- + 128^+ =$$

$$= \{(63,0)^- + (1,0)^-\} + (28,-1) + (\overline{28},1) \to \frac{SO(16)}{U(8)}$$
$$+ (1,+2)^+ + (1,-2)^+ + (28,1)^+ + (\overline{28},-1)^+ + (70,0)^+$$

$$\frac{E_{8(8)}}{E_{7(7)} \times SL(2,R)} = (56,2) = (28,1)^+ + (\overline{28},-1)^+ + (28,-1)^- + (\overline{28},1)^-$$

$$G_{16}^3 / H_{16}^3 + H_{16}^3 = H_{16}^4 \oplus \underset{J}{U(1)} \oplus \left[\frac{E_{8(8)}}{E_{7(7)} \times SL(2,R)}\right]$$
$$\oplus \frac{E_{7(7)}}{SU(8)} \oplus \frac{SL(2,R)}{U(1)}$$

Two more cases, the $N=16$ $D=5$ and

$N=4$ $D=5$ based on E_8 3-d groups

Embeddings:

$$N=16 \quad E_{8(8)} \to E_{6(6)} \times SL(3,R) \qquad (\text{max}, \text{ns})$$

$$N=4 \quad E_{8(-24)} \to E_{6(-26)} \times SL(3,R)$$

The maximal case $(N=16)$

by taking the mcs of both sides

we have

$$120^- + 128^+ = (36,1)^- + (1,3)^- + (27,3)^- + (1,5)^+_{J=2} + (27,3)^+ + (42,1)^+_{J=0}$$

$$= (36,1)^- + (42,1)^+ + (1,3)^- + (1,5)^+ + (27,3)^- + (27,3)^+$$

$$E_{8(8)} / E_{6(6)} \times SL(3,R) = (27,3)^+ + (27',3')^-$$

The bose-fermi spectrum at $D=5$ $N=16$ is obtained by the decomposition of the mcs of both sides

$$SO(16) \to USp(8) \times SU_J(2)$$

$$128 \to (1,5^{+})^{+} + (27,3)^{+}$$

$$128' \to (8,4)^{+} + (48,2)^{-}$$

The octonionic magical model ($N=4$)

$$E_{8(8)} \to E_{6(-26)} \times SL(3,R)$$

$$248 \to (78,1) + (1,8) + (27,3) + (27',3')$$

taking the mcs on both sides we have

$$E_7 \times SU_H(2) \to F_4 \times SU_L(2) \times SU_H(2)$$

$$248 \to (133, 1_H)^- + (1, 3_H)^- + (56, 2)_H^+ =$$

Further decomposing $E_7 \to F_4 \times SU_L(2)$

$$248 \to (52, 1_L, 1_H)^- + (1, 3_L, 1_H)^- + (26, 3_L, 1_H)^-$$
$$+ (1, 4_L, 2_H)^+ + (26, 2_L, 2_H)^+ + (1, 1_L, 3_H)^-$$

$$= (52, 1)^- + (1,3)^- + (26,3)^- + (1,3)^+ + (1,5)^+$$
$$+ (1,3)^- + (26,3)^+ + (26,1)^+$$

$$= (52,1)^- + (1,3)^- + (26,1)^+ + (1,5)^+$$
$$+ (1,3)^+ + (1,3)^- + (26,3)^+ + (26,3)^-$$

The bosonic spectrum corresponds to the + signs

$(1,5)^+ + (1,3)^+ + (26,3)^+ + (26,1)^+$.

The form content is obtained by replacing $(1,5)^+ + (26,1)^+$ with $(1,3)^-$ and $(26,3)^-$ so we have $27 = 1+26$

$$E_{8(-24)}/E_{6(-26)} \times SL(3) = (27,3)^+ + (27,3)^-$$

exactly as in the maximal case.

black holes and black strings coupled to one form and their dual -

Fermions in $N=4$ theories. (Octonionic model)

we introduce an extra $SU(2)'$
under which the bosons (gravity
+ vector matter) are singlets while
the gauginos and gravitinos are doublets.

The bosonic spectrum is

$$(56, 2_H, 1) \to (1, 4_L, 2_H, 1) + (26, 2_L, 2_H, 1)$$

where the first three entries on the
right hand side correspond to

$$SO(16) \to E_7 \times SU(2)_H \to F_4 \times SU(2)_L \times SU(2)_H$$

and $SU(2)_J = \text{diag}\; SU(2)_L \times SU(2)_H$

$$(56, 2_H) \to (1,3) + (1,5) + (26,3) + (26,1)$$

For fermions, exchange $SU(2)_H \to SU(2)' \to$

$$(56, 1, 2_H) \to (1, 4_J, 2_H) + (26, 2_J, 2_H)$$

as it should be.

$$\frac{G_N^3}{H_N^3} \oplus \frac{H_N^3}{H_N^D \times SO(D-2)} = \frac{G_N^3}{G_N^D \times SL(D-2)} \oplus \left(\frac{G_N^D}{H_N^D} + \frac{SL(D-2)}{SO(D-2)} \right)$$

$$+++ \qquad --- \qquad\qquad\qquad + \;+++$$

$(++++)$ = physical spectrum with gravita and scalars deleted

$(---)$ carrying [illegible] reps of $SO(D-2)$ due to the embedding $N \to$ spinor $SO(D-2)$ of $SO(D-2)$ in $SO(N)$. [illegible] $(nC = C)$ -