Supersymmetry invariants and anomalies in $\mathcal{N} = 4$ supergravity

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$\mathcal{N} = 8$ supergravity

Is $\mathcal{N} = 8$ supergravity a consistent quantum field theory?

$\hookrightarrow$ Free of ambiguities associated to logarithmic divergences?

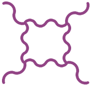
Explanation for the excellent ultra-violet behaviour of the 4-graviton amplitudes

⋆ Supersymmetry and $E_{7(7)}$ duality symmetry

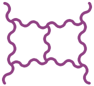
⋆ Some more hidden symmetry of the quantum theory?

⋆ Is there a non-perturbative completion that does not require the embedding in string theory?

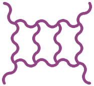
$\mathcal{N} = 8$ supergravity



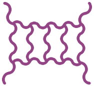
$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{k^8}{k^8} \sim p^8 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^8} \qquad \mathcal{L}^{(3)} = ef_{-42}(\phi) C^2 \bar{C}^2 + \ldots$$



$$\sim \int \frac{d^8k}{(2\pi)^8} \frac{k^{12}}{k^{14}} \sim p^{12} \int \frac{d^8k}{(2\pi)^8} \frac{1}{k^{14}} \qquad \mathcal{L}^{(5)} = ef_{-60}(\phi) \nabla^2 C^2 \nabla^2 \bar{C}^2 + \ldots$$



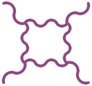
$$\sim \int \frac{d^{12}k}{(2\pi)^{12}} \frac{k^{16}}{k^{20}} \sim p^{14} \int \frac{d^{12}k}{(2\pi)^{12}} \frac{1}{k^{18}} \qquad \mathcal{L}^{(6)} = ef_{-60}(\phi) \nabla^3 C^2 \nabla^3 \bar{C}^2 + \ldots$$



$$\sim \int \frac{d^{16}k}{(2\pi)^{16}} \frac{k^{20}}{k^{26}} \sim p^{16} \int \frac{d^{16}k}{(2\pi)^{16}} \frac{1}{k^{22}} \qquad \mathcal{L}^{(7)} = e \nabla^4 C^2 \nabla^4 \bar{C}^2 + \ldots$$

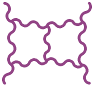
[Bern, Carrasco, Dixon, Johansson, Kosower and Roiban]

Maximal supergravity in higher dimensions



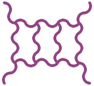
$$\sim \int \frac{d^8 k}{(2\pi)^8} \frac{k^8}{k^8} \sim p^8 \ln[\Lambda]$$

$$\mathcal{L}_8^{(1)} = eR^4 + \ldots$$



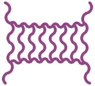
$$\sim \int \frac{d^{2\times 7} k}{(2\pi)^{14}} \frac{k^{12}}{k^{14}} \sim p^{12} \ln[\Lambda]$$

$$\mathcal{L}_7^{(2)} = e\nabla^4 R^4 + \ldots$$



$$\sim \int \frac{d^{3\times 6} k}{(2\pi)^{18}} \frac{k^{16}}{k^{20}} \sim p^{14} \ln[\Lambda]$$

$$\mathcal{L}_6^{(3)} = e\nabla^6 R^4 + \ldots$$



$$\sim \int \frac{d^{7\times 4} k}{(2\pi)^{28}} \frac{k^{32}}{k^{44}} \sim p^{16} \int \frac{d^{28} k}{(2\pi)^{28}} \frac{1}{k^{28}}$$

$$\mathcal{L}_4^{(7)} = e\nabla^4 C^2 \nabla^4 \bar{C}^2 + \ldots$$

Maximal supergravity in higher dimensions

The 4-graviton amplitude seems to factorise at ℓ-loop

$$\int \frac{d^{\ell D} k}{(2\pi)^{\ell D}} \frac{1}{k^{2\ell-2}} \sim p^{8+2\ell} \int \frac{d^{\ell D} k}{(2\pi)^{\ell D}} \frac{1}{k^{4\ell+6}} \ ,$$

would suggest that the theory is finite in four dimensions.

	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^8 R^4$	$\partial^{10} R^4$	$\partial^{12} R^4$
1-loop	$D=8$					
2-loop	$D=5$	$D=7$	$D=8$			
3-loop	$D=4$		$D=6$			$D=8$
4-loop			$D=5$		$D=6$	
5-loop		$D=4$				
6-loop			$D=4$			$D=5$
7-loop				$D=4$		
8-loop					$D=4$	

Maximal supergravity in higher dimensions

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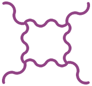
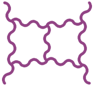
5-loop		$D=4$				
6-loop			$D=4$			$D=5$
7-loop				$D=4$		
8-loop					$D=4$	

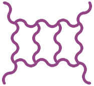
Half-maximal supergravity in higher dimensions



$$\sim \int \frac{d^8 k}{(2\pi)^8} \frac{k^8}{k^8} \sim p^8 \ln[\Lambda] \qquad \mathcal{L}_8^{(1)} = eR^4 + \ldots$$



$$\sim \int \frac{d^{2\times 5} k}{(2\pi)^{10}} \frac{k^{12}}{k^{14}} \sim p^8 \int \frac{d^{10} k}{(2\pi)^{10}} \frac{1}{k^{10}} \qquad \mathcal{L}_5^{(2)} = eR^4 + \ldots$$



$$\sim \int \frac{d^{3\times 4} k}{(2\pi)^{12}} \frac{k^{16}}{k^{20}} \sim p^8 \int \frac{d^{12} k}{(2\pi)^{12}} \frac{1}{k^{12}} \qquad \mathcal{L}_4^{(3)} = eC^2\bar{C}^2 + \ldots$$

But the explicit computations give them finite.

[Bern, Davies, Dennen, Huang]

[Vanhove, Tourkine]

Outline

- The $SL(2,\mathbb{R})$ anomaly
- Restoring modular invariance
- Ultra-violet divergences
- Conclusion and outlook

[G. Bossard, P. S. Howe and K. S. Stelle, 1212.0841]
[G. Bossard, P. S. Howe and K. S. Stelle, to appear]

[G. Bossard, C. Hillmann and H. Nicolai, 1007.5472]
[G. Bossard, P. S. Howe, K. S. Stelle and P. Vanhove, 1105.6087]

$\mathcal{N} = 4$ supergravity

$\mathcal{N} = 4$ supergravity includes

* 1 complex scalar field τ parametrizing $SL(2,\mathbb{R})/SO(2)$
* 2×4 $U(4)$-Majorana fermions χ^i_α
* 2×6 vectors $A_{\mu ij}$
* 2×4 gravitinos $\psi_{\mu\alpha i}$
* 2 gravitons $g_{\mu\nu}$

of $SL(2,\mathbb{C}) \times U(4)$.

Duality symmetry

$SL(2,\mathbb{R})/SO(2)$ is an axial symmetry, with $\tau \equiv a + ie^{-2\phi}$ and $e^{-2\phi} \star F_{ij} + aF_{ij} = dB_{ij}$.

$$\tau' = \frac{a\tau + b}{d + c\tau}$$

$$\chi_\alpha^{i\,\prime} = \left(\frac{d + c\bar{\tau}}{|d + c\tau|}\right)^{\frac{3}{2}} \chi_\alpha^i$$

$$\begin{pmatrix} B'_{ij} \\ A'_{ij} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B_{ij} \\ A_{ij} \end{pmatrix}$$

$$e^{\phi'}\left(B'_{ij} + \tau' A'_{ij}\right) = \frac{d + c\bar{\tau}}{|d + c\tau|} e^{\phi}\left(B_{ij} + \tau A_{ij}\right)$$

$$\psi'_{\alpha i} = \left(\frac{d + c\bar{\tau}}{|d + c\tau|}\right)^{\frac{1}{2}} \psi_{\alpha i}$$

Henneaux–Teitelboim action

Duality covariant complex field strength

$$\mathcal{F}_{ij} \equiv e^{\phi}\big(\partial_i B_j - \partial_j B_i + \tau(\partial_i A_j - \partial_j A_i)\big) \; ,$$

The $SL(2,\mathbb{R})$ invariant Lagrangian [J. Schwarz and A. Sen]

$$\mathcal{L} = \frac{1}{2}\varepsilon^{ijk}\big(\partial_0 A_i \partial_j B_k - \partial_0 B_i \partial_j A_k + 4N^l \partial_{[i}A_{l]}\partial_j B_k\big) - \frac{1}{4}N\sqrt{h}\, h^{ik} h^{jl} \bar{\mathcal{F}}_{ij}\mathcal{F}_{kl}$$

leads to the twisted selfduality equation

$$\mathcal{F}_{\mu\nu} - \frac{i}{2\sqrt{\text{-}g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}\mathcal{F}_{\sigma\rho} = 0$$

Henneaux–Teitelboim action

Duality covariant complex field strength

$$\mathcal{F}_{ij} \equiv e^{\phi}\big(\partial_i B_j - \partial_j B_i + \tau(\partial_i A_j - \partial_j A_i)\big) ,$$

The $SL(2,\mathbb{R})$ invariant Lagrangian [J. Schwarz and A. Sen]

$$\mathcal{L}_{\mathrm{PV}} = \frac{1}{2}\varepsilon^{ijk}\big(\partial_0 A_i \partial_j B_k - \partial_0 B_i \partial_j A_k + 4N^l \partial_{[i} A_{l]} \partial_j B_k\big) - M\varepsilon^{ijk} A_i \partial_j B_k - \frac{1}{4} N\sqrt{h}\, h^{ik} h^{jl} \bar{\mathcal{F}}_{ij} \mathcal{F}_{kl}$$

leads to the twisted selfduality equation

$$\mathcal{F}_{\mu\nu} - \frac{i}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}\mathcal{F}_{\sigma\rho} = 0$$

Wess–Zumino consistency condition

According to the quantum action principle

$\hookrightarrow$ Consistent anomalies $\Leftrightarrow$ cohomology classes

For a compact group $K \cong U(4)$ in 4 dimensions

$$\mathcal{H}^1(\mathfrak{k}|d) \cong \mathcal{H}^5(K,\mathbb{R}) \oplus \mathcal{H}^1(K,\mathbb{R})$$

Such that

$$\mathrm{Tr}\; dC_{\mathfrak{k}}\left(B_{\mathfrak{k}}dB_{\mathfrak{k}} + \frac{1}{2}{B_{\mathfrak{k}}}^3\right) \quad \Leftrightarrow \quad \mathrm{Tr}\;(g^{-1}dg)^5$$

$$\mathrm{Tr}\; C_{\mathfrak{k}}\, R^{ab} \wedge R_{ab} \quad \Leftrightarrow \quad \mathrm{Tr}\;(g^{-1}dg)$$

Atiyah–Singer family's index

For $M \to Z \to S^2$ with S^2 a two-parameters family of gauge field source $B_{\mathfrak{k}}$ for $J_{\mathfrak{k}}$

$$\mathrm{ch}(\mathrm{Ind}(\nabla_s)) = \int_M \hat{A}(Z)\mathrm{ch}(EZ)$$

which gives for

$$\begin{aligned} a_3 \mathrm{Tr}\,\mathbf{x}^3 &= -\mathrm{Tr}_{R_{\frac{1}{2}}}\,\mathbf{x}^3 + 2\mathrm{Tr}_{R_1}\,\mathbf{x}^3 - 3\mathrm{Tr}_{R_{\frac{3}{2}}}\,\mathbf{x}^3 \\ a_1 \mathrm{Tr}\,\mathbf{x} &= \frac{1}{24}\mathrm{Tr}_{R_{\frac{1}{2}}}\,\mathbf{x} + \frac{1}{6}\mathrm{Tr}_{R_1}\,\mathbf{x} - \frac{7}{8}\mathrm{Tr}_{R_{\frac{3}{2}}}\,\mathbf{x} \end{aligned}$$

such that

$$\delta\Gamma = \frac{a_1}{8\pi^2}\mathrm{Tr}\ C_{\mathfrak{k}}\, R^{ab} \wedge R_{ab} + \frac{a_3}{24\pi^2}\mathrm{Tr}\ dC_{\mathfrak{k}}\Big(B_{\mathfrak{k}}\, dB_{\mathfrak{k}} + \frac{1}{2}B_{\mathfrak{k}}{}^3\Big)\ .$$

Wess–Zumino consistency condition

Riemannian symmetric G/K is topologically trivial

$\hookrightarrow$ Trivial equivariant cohomology

$$\mathcal{H}^n_K(d) \cong \{0\} \text{ for } n > 0$$

So we have the homotopy equivalence

$$G \cong K \times \mathbb{R}^{\dim(\mathfrak{g})-\dim(\mathfrak{k})}$$

and in particular $\mathcal{H}^{5[4]}(G,\mathbb{R}) \cong \mathcal{H}^{5[4]}(K,\mathbb{R})$

Similarly, we prove that $\mathcal{H}^n_K(\delta^{\mathfrak{g}}|d) \cong \{0\}$ for $n > 0$

$$\mathcal{H}^1(\delta^{\mathfrak{g}}|d) \cong \mathcal{H}^1(\delta^{\mathfrak{k}}|d)$$

$SL(2,\mathbb{R})$ anomaly in $\mathcal{N}=4$ supergravity

The Atiyah–Singer family's index theorem gives [Marcus]

$$
\begin{aligned}
dJ^{\mathfrak{u}(1)} &= \frac{-4\times 3\times 1/24 - 6\times 2\times 1/6 + 4\times 1\times 7/8}{16\pi^2} R^{ab}{}_{\wedge} R_{ab} \\
&= \frac{1}{16\pi^2} R^{ab}{}_{\wedge} R_{ab}
\end{aligned}
$$

Which corresponds to the non-linear anomaly

$$
\mathbf{f}\Gamma^{\text{1-loop}} = \frac{1}{16\pi^2}\int \left(e^{-2\phi} R^{ab}{}_{\wedge} R_{ab} + \ldots \right)
$$

where

$$
\mathbf{f}\tau = -\tau^2 \quad \mathbf{h}\tau = 2\tau \quad \mathbf{e}\tau = 1
$$

$SL(2,\mathbb{R})$ anomaly in $\mathcal{N}=4$ supergravity

The rigid $SL(2)$ symmetry is anomalous, and is broken to its parabolic subgroup.

$$\mathbf{f}\,\Gamma^{(1)} = \frac{2+n}{32\pi^2}\int \left(e^{-2\phi}R^{ab}\wedge R_{ab} + \ldots\right)$$

where $\mathbf{f}\tau = -\tau^2$ with $\tau = a + ie^{-2\phi}$.

Correspondingly at 1-loop

$$\langle g_{\mu\nu}(p_1)g_{\sigma\rho}(p_2)a(-p_1-p_2)\rangle = \frac{2+n}{16\pi^2}\varepsilon_{\sigma)(\mu}{}^{\kappa\lambda}\left(\eta_{\nu)(\rho}p_1\cdot p_2 - p_{2\nu)}p_{1(\rho}\right)p_{1\kappa}p_{2\lambda}$$

$\mathcal{N} = 4$ supergravity's 1-loop anomaly

The rigid $SL(2)$ symmetry is anomalous, and is broken to its parabolic subgroup.

$$\mathbf{f}\,\Gamma^{(1)} = \frac{1}{16\pi^2}\int \left(e^{-2\phi} R^{ab} \wedge R_{ab} + \ldots\right)$$

where $\mathbf{f}\tau = -\tau^2$ with $\tau = a + ie^{-2\phi}$.

Correspondingly at 1-loop

$$M_3(h_1^+, h_2^+, a_3) = \frac{i}{16\pi^2}[12]^4$$

[Dixon,Bern]

[Carrasco,Kallosh,Roiban,Tsetlin]

Consistency with supersymmetry

For consistency, the anomaly must be supersymmetric.

In the linearised approximation, all fields appear in a chiral superfield

$$W = \tau + \theta_i^\alpha \chi_\alpha^i + \frac{1}{2}\varepsilon^{ijkl}\theta_i^\alpha\theta_j^\beta F_{\alpha\beta kl} + \frac{1}{6}\varepsilon^{ijkl}\theta_i^\alpha\theta_j^\beta\theta_k^\gamma \rho_{\alpha\beta\gamma l} + \frac{1}{24}\varepsilon^{ijkl}\theta_i^\alpha\theta_j^\beta\theta_k^\gamma\theta_l^\delta C_{\alpha\beta\gamma\delta} \ ,$$

and

$$\int d^8\theta W^3 \sim \tau C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + \dots \ .$$

But in the nonlinear theory

$$\{\bar{D}_{\dot\alpha i}, \bar{D}_{\dot\beta j}\}\mathcal{F} = \varepsilon_{\dot\alpha\dot\beta}\,\varepsilon_{ijkl}\,\chi^{\alpha k} D_\alpha^l \mathcal{F} \ ,$$

and the chiral measure does not exist.

The supersymmetric R^2

If we consider $\bar{C}^2$ as

$$\mathcal{L}_4[1] = \frac{1}{2} R^{ab} \wedge R_{ab} - \frac{i}{4} \varepsilon_{abcd} R^{ab} \wedge R^{cd} \ ,$$

Its integral is clearly supersymmetric, because it does not depend on a specific metric.

It is d-closed as a superform.

Ectoplasm superform

The supersymmetry invariant $\mathcal{L}_4[\mathcal{F}]$ is not defined as a superspace integral, but only as a d-closed super 4-form. [Gates]

$$d\mathcal{L}_4 = \frac{1}{24} E^E \wedge E^D \wedge E^C \wedge E^B \wedge E^A \big(D_A \mathcal{L}_{BCDE} + 2 T_{AB}{}^F \mathcal{L}_{FCDE} \big)$$

In the tangent frame, the indices A decompose as $\{{}_A, {}^i_\alpha, {}_{\dot\alpha i}\}$ with respect to $SL(2,\mathbb{C}) \times U(4)$.

$$\mathcal{L}_4[\mathcal{F}] = \frac{1}{24} \varepsilon_{abcd} E^a \wedge E^b \wedge E^c \wedge E^d L + \cdots$$
$$+ \frac{1}{24} E^\delta_l \wedge E^\gamma_k \wedge E^\beta_j \wedge E^\alpha_i M^{ijkl}_{\alpha\beta\gamma\delta} + \frac{1}{6} E^{\dot\delta l} \wedge E^\gamma_k \wedge E^\beta_j \wedge E^\alpha_i M^{ijk}_{\alpha\beta\gamma\dot\delta l} \, .$$

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In the tangent frame, the indices A decompose as $\{_A,{}^i_\alpha\, ,{}_{\dot\alpha i}\}$ with respect to $SL(2,\mathbb{C}) \times U(4)$.

$$\iota^* \mathcal{L}_4[\mathcal{F}] = \frac{1}{24} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d L\big|_{\theta=0} + \cdots$$
$$+ \frac{1}{24} \psi^\delta_l \wedge \psi^\gamma_k \wedge \psi^\beta_j \wedge \psi^\alpha_i M^{ijkl}_{\alpha\beta\gamma\delta}\big|_{\theta=0} + \frac{1}{6} \psi^{\dot\delta l} \wedge \psi^\gamma_k \wedge \psi^\beta_j \wedge \psi^\alpha_i M^{ijk}_{\alpha\beta\gamma\dot\delta l}\big|_{\theta=0}$$

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Focusing on the lowest dimensional components

($[r_1, r_2, r_3]$ such that $r_1 + 2r_2 + r_3 = 5$)

$$\begin{aligned}
D^p_\eta M^{ijkl}_{\alpha\beta\gamma\delta} + 2 T_{\eta\,\alpha}^{pi\ \dot{\varsigma} q} M^{jkl}_{\beta\gamma\delta\dot{\varsigma} q} + \circlearrowleft \quad &= \quad 0 \\
D_{\dot{\eta} p} M^{ijkl}_{\alpha\beta\gamma\delta} + 4 D^i_\alpha M^{jkl}_{\beta\gamma\delta\dot{\eta} p} + \circlearrowleft \quad &\approx \quad 0 \\
2 D_{\dot{\eta} p} M^{ijk}_{\alpha\beta\gamma\dot{\delta} l} + T_{\dot{\eta} p \dot{\delta} l q}{}^{\varsigma} M^{ijkq}_{\alpha\beta\gamma\varsigma} + \circlearrowleft \quad &\approx \quad 0 \\
T_{\dot{\eta} p \dot{\gamma} l q}{}^{\varsigma} M^{ijq}_{\alpha\beta\varsigma\dot{\delta} l} + \circlearrowleft \quad &\approx \quad 0
\end{aligned}$$

Ectoplasm superform

The supersymmetry invariant $\mathcal{L}_4[\mathcal{F}]$ is not defined as a superspace integral, but only as a *d*-closed super 4-form.

$$d\mathcal{L}_4 = \frac{1}{24} E^E \wedge E^D \wedge E^C \wedge E^B \wedge E^A \big(D_A \mathcal{L}_{BCDE} + 2 T_{AB}{}^F \mathcal{L}_{FCDE} \big)$$

For an arbitrary anti-holomorphic function $\mathcal{F}[\bar{T}]$

with $U\bar{U}(1 - T\bar{T}) = 1$, $\begin{pmatrix} U & UT \\ \bar{U}\bar{T} & \bar{U} \end{pmatrix} \in SU(1,1)$, and $\tau = i\frac{1-T}{1+T}$

$$M^{ijkl}_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\Big(\mathcal{F}[\bar{T}] F^{ij}_{\dot{\alpha}\dot{\beta}} F^{\dot{\alpha}\dot{\beta}kl} - \bar{U}^{-2}\bar{\partial}\mathcal{F}[\bar{T}]\varepsilon^{ijpq}\chi_{\dot{\alpha}p}\chi_{\dot{\beta}q}F^{\dot{\alpha}\dot{\beta}kl}$$
$$+\frac{1}{6}\bar{U}^{-4}\Big(\bar{\partial} - \frac{2T}{1-T\bar{T}}\Big)\bar{\partial}\mathcal{F}[\bar{T}]\varepsilon^{ijpq}\varepsilon^{klrs}\chi_{\dot{\alpha}p}\chi_{\dot{\beta}q}\chi^{\dot{\alpha}}_r\chi^{\dot{\beta}}_s\Big) + \circlearrowleft$$

and

$$M^{ijk}_{\alpha\beta\gamma\dot{\delta}l} = -\varepsilon_{\alpha\beta}\varepsilon^{\dot{\eta}\dot{\varsigma}}\chi^k_\gamma\chi_{\dot{\eta}l}\Big(\mathcal{F}[\bar{T}]F^{ij}_{\dot{\delta}\dot{\varsigma}} - \frac{1}{3}\bar{U}^{-2}\bar{\partial}\mathcal{F}[\bar{T}]\varepsilon^{ijpq}\chi_{\dot{\delta}p}\chi_{\dot{\varsigma}q}\Big) + \circlearrowleft$$

The supersymmetric anomaly

For $\mathcal{F}[T] = 1$

$$\mathcal{L}_4[1] = \frac{1}{2} R^{ab} \wedge R_{ab} + \frac{i}{4} \varepsilon_{abcd} R^{ab} \wedge R^{cd}$$

therefore for $\mathcal{F}[T] = \tau[T] \equiv i\frac{1-T}{1+T}$

$$i\mathcal{L}_4[\bar{\tau}] - i\bar{\mathcal{L}}_4[\tau] = \mathrm{Im}[\tau] R^{ab} \wedge R_{ab} + \mathrm{Re}[\tau] \frac{1}{2} \varepsilon_{abcd} R^{ab} \wedge R^{cd} + \dots$$

defines the anomaly.

$$\delta\Gamma = \frac{1}{8\pi^2} \int \iota^* \mathrm{Im}\Big(\mathcal{L}_4[f\tau - h]\Big) \ ,$$

with $f\tau - h = ife^{-2\phi} + \delta\phi$.

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defines the anomaly.

$$\delta\Gamma = f \frac{1}{8\pi^2} \int \iota^* \mathrm{Im}\Big(\mathcal{L}_4[\tau]\Big) - h \frac{1}{32\pi^2} \int \varepsilon_{abcd} R^{ab} \wedge R^{cd} \ ,$$

with $f\tau - h = ife^{-2\phi} + \delta\phi$.

The supersymmetric anomaly

For $\mathcal{F}[T] = 1$

$$\mathcal{L}_4[1] = \frac{1}{2} R^{ab} \wedge R_{ab} + \frac{i}{4} \varepsilon_{abcd} R^{ab} \wedge R^{cd}$$

therefore for $\mathcal{F}[T] = \tau[T] \equiv i\frac{1-T}{1+T}$

$$i\mathcal{L}_4[\bar{\tau}] - i\bar{\mathcal{L}}_4[\tau] = \mathrm{Im}[\tau] R^{ab} \wedge R_{ab} + \mathrm{Re}[\tau] \frac{1}{2} \varepsilon_{abcd} R^{ab} \wedge R^{cd} + \ldots$$

defines the anomaly.

$$\delta\Gamma = f \frac{1}{8\pi^2} \int \iota^* \mathrm{Im}\Big(\mathcal{L}_4[\tau]\Big) - h\chi(M) \ ,$$

with $f\tau - h = ife^{-2\phi} + \delta\phi$.

The supersymmetric anomaly

For $\mathcal{F}[T] = 1$

$$\mathcal{L}_4[1] = \frac{1}{2} R^{ab} \wedge R_{ab} - \frac{i}{4} \varepsilon_{abcd} R^{ab} \wedge R^{cd} + \frac{1}{4} R^{AB} \wedge R_{AB} + \frac{i}{2} \mathcal{L}_4^{(1)}$$

therefore for $\mathcal{F}[T] = \tau[T] \equiv i \frac{1-T}{1+T}$

$$i\mathcal{L}_4[\bar{\tau}] - i\bar{\mathcal{L}}_4[\tau] = \mathrm{Im}[\tau] \Big(R^{ab} \wedge R_{ab} + \frac{1}{2} \varepsilon^{ijkl} \varepsilon^{pqrs} P_{ijA} \wedge P_{klB} \wedge P_{pq}{}^{A} \wedge P_{rs}{}^{B} \Big) + \mathrm{Re}[\tau] \Big(\frac{1}{2} \varepsilon_{abcd} R^{ab} \wedge R^{cd} + \mathcal{L}_4^{(1)} \Big) + \dots$$

defines the anomaly.

$$\delta\Gamma = \frac{2+n}{16\pi^2} \int \iota^* \mathrm{Im}\Big(\mathcal{L}_4[f\tau - h] \Big) ,$$

with $f\tau - h = ife^{-2\phi} + \delta\phi$.

The supersymmetric anomaly

For $\mathcal{F}[T] = 1$

$$\mathcal{L}_4[1] = \frac{1}{2} R^{ab} \wedge R_{ab} - \frac{i}{4} \varepsilon_{abcd} R^{ab} \wedge R^{cd} + \frac{1}{4} R^{AB} \wedge R_{AB} + \frac{i}{2} \mathcal{L}_4^{(1)}$$

therefore for $\mathcal{F}[T] = \tau[T] \equiv i\frac{1-T}{1+T}$

$$i\mathcal{L}_4[\bar{\tau}] - i\bar{\mathcal{L}}_4[\tau] = \mathrm{Im}[\tau]\Big(R^{ab} \wedge R_{ab} + \frac{1}{2} \varepsilon^{ijkl} \varepsilon^{pqrs} P_{ijA} \wedge P_{klB} \wedge P_{pq}{}^{A} \wedge P_{rs}{}^{B} \Big) + \mathrm{Re}[\tau]\Big(\frac{1}{2} \varepsilon_{abcd} R^{ab} \wedge R^{cd} + \mathcal{L}_4^{(1)} \Big) + \dots$$

defines the anomaly.

$$\delta\Gamma = f\frac{2+n}{16\pi^2} \int \iota^* \mathrm{Im}\Big(\mathcal{L}_4[\tau]\Big) + h\frac{2+n}{32\pi^2} \int \iota^* \mathcal{L}_4^{(1)} - h\frac{2+n}{2} \chi(M) \ ,$$

with $f\tau - h = ife^{-2\phi} + \delta\phi$.

The 1-loop divergence

The invariant $\int \iota^* \mathcal{L}_4^{(1)}$ defines the 1-loop divergence. At the linearised level

$$\begin{aligned}\int \iota^* \mathcal{L}_4^{(1)} &\sim \int d^4x d^4u d^4\bar{u} V(u,\bar{u}) d^4\theta d^4\bar{\theta} (W_{34A} W_{34}{}^A)^2 \\ &= \tfrac{1}{16} \int d^4x \sqrt{-g} \Big(\delta_{AB}\delta_{CD} + 2\delta_{AC}\delta_{BD}\Big) F^A_{\alpha\beta} F^{\alpha\beta B} \bar{F}^C_{\dot{\alpha}\dot{\beta}} \bar{F}^{\dot{\alpha}\dot{\beta}D} + \ldots\end{aligned}$$

At the non-linear level one can define the integral

$$\int \iota^* \mathcal{L}_4^{(1)} = \int d\mu_{(4,2,2)} (V_{34A} V_{34}{}^A)^2$$

The 1-loop divergence

The dilaton shift anomaly defines precisely the 1-loop divergence [Fischler, Fradkin, Tseytlin]

$$\left(\mu\frac{\partial}{\partial\mu} - \kappa\frac{\partial}{\partial\kappa}\right)\Gamma \approx -\frac{2+n}{64\pi^2}\left[\int \iota^* \mathcal{L}_4^{(1)} \cdot \Gamma\right] + \mathcal{O}(\kappa^2) \ ,$$

which is associated to the $SL(2,\mathbb{R}) \times SO(6,n)$ invariant

$$\int \iota^* \mathcal{L}_4^{(1)} = \frac{1}{16}\int d^4x\sqrt{-g}\big(\delta_{AB}\delta_{CD} + 2\delta_{AC}\delta_{BD}\big)F^A_{\alpha\beta}F^{\alpha\beta B}\bar{F}^C_{\dot\alpha\dot\beta}\bar{F}^{\dot\alpha\dot\beta D} + \dots$$

String theory explanation [Green, Russo, Vanhove]

$$\ln(8\pi\ell_s^2 s) = \ln(\kappa^2 s) - 2\langle\phi\rangle$$

S-duality in string theory

Heterotic string theory on $\mathbb{R}^{1,3} \times T^6$

$\hookrightarrow$ non-perturbative $SL(2,\mathbb{Z})$ S-duality

$$\kappa^2 = 8\pi \langle e^{2\phi} \rangle \ell_s^2 \quad \left(= \; 8\pi \frac{\ell_s{}^6 g_s{}^2}{R^6} \ell_s{}^2 \right)$$

$\ell_s \to 0$

$R \to 0$

$\mathcal{N} = 4$ supergravity satisfies $SL(2,\mathbb{R})$ $\xrightarrow{\text{non-perturbative}}$ $SL(2,\mathbb{Z})$

$SL(2,\mathbb{Z})$ transformations

Applying a finite transformation

$$g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \ ,$$

one obtains that

$$\rho_{g+\delta g}\Gamma = \rho_g\Gamma + \frac{2+n}{16\pi^2}\int \iota^* \mathrm{Im}\Big(\mathcal{L}_4\Big[f\,\frac{a\tau+b}{c\tau+d} - h\Big]\Big) \ ,$$

where $\delta g\, g^{-1} = \begin{pmatrix} h & e \\ f & -h \end{pmatrix}$.

Integrating this equation, one computes that

$$\rho_g\Gamma = \Gamma + \frac{2+n}{16\pi^2}\int \iota^* \mathrm{Im}\Big(\mathcal{L}_4\big[\ln(c\tau+d)\big]\Big) \ .$$

$SL(2,\mathbb{Z})$ transformations

To recover modular invariance one must add the counterterm

$$\Gamma^{\mathrm{S}} = \Gamma - \frac{2+n}{8\pi^2}\int \iota^* \mathrm{Im}\Big(\mathcal{L}\big[\ln(\eta(\tau))\big]\Big) \ ,$$

Using the property that the function $\eta(\tau)$ transforms as

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = e^{\frac{i\pi\tilde{b}}{12}}(c\tau+d)^{\frac{1}{2}}\eta(\tau)$$

one gets [Rohkhlin]

$$\begin{aligned}
\rho_g\Gamma^{\mathrm{S}} &= \Gamma^{\mathrm{S}} - \frac{2+n}{8\pi^2}\frac{\pi\tilde{b}}{12}\int \iota^*\mathrm{Re}\Big(\mathcal{L}[1]\Big) \\
&= \Gamma^{\mathrm{S}} + 2\pi\frac{2+n}{2}\tilde{b}\frac{-1}{24}\int\frac{1}{8\pi^2}R^{ab}\wedge R_{ab} \\
&= \Gamma^{\mathrm{S}} + 2\pi(2+n)\tilde{b}\frac{\mathrm{Ind}(\nabla\!\!\!/)}{2} \ .
\end{aligned}$$

Twisted self-duality equation

The modified equations of motion of the vector fields

$$\bar{F}^{ij}_{\alpha\beta} = \kappa^2 \frac{2+n}{48\pi}\Big(\mathrm{Im}[\tau]E_2(\tau)C_{\alpha\beta\gamma\delta}\tfrac{1}{2}\varepsilon^{ijkl}F^{\gamma\delta}{}_{kl}$$
$$+\frac{\pi}{3}\mathrm{Im}[\tau]^2\big(E_2(\tau)^2-\tfrac{6}{\pi\mathrm{Im}[\tau]}E_2(\tau)-E_4(\tau)\big)\tfrac{1}{2}\varepsilon^{ijkl}\tfrac{1}{2}\varepsilon^{pqrs}\big(F_{\alpha\beta kl}F_{\beta\gamma pq}F^{\beta\gamma}{}_{rs}+2F_{\alpha\beta pq}F_{\beta\gamma rs}F^{\beta\gamma}{}_{kl}\big)\Big)$$

are supersymmetric, and

$$\bar{F}^{ij}_{\alpha\beta} = \kappa^2 \frac{2+n}{48\pi}\Big(\mathrm{Im}[\tau]\hat{E}_2(\tau,\bar{\tau})C_{\alpha\beta\gamma\delta}\tfrac{1}{2}\varepsilon^{ijkl}F^{\gamma\delta}{}_{kl}$$
$$+\frac{\pi}{3}\mathrm{Im}[\tau]^2\big(\hat{E}_2(\tau,\bar{\tau})^2-E_4(\tau)\big)\tfrac{1}{2}\varepsilon^{ijkl}\tfrac{1}{2}\varepsilon^{pqrs}\big(F_{\alpha\beta kl}F_{\beta\gamma pq}F^{\beta\gamma}{}_{rs}+2F_{\alpha\beta pq}F_{\beta\gamma rs}F^{\beta\gamma}{}_{kl}\big)\Big)$$

are $SL(2,\mathbb{Z})$ invariant.

Anomalous Ward identity

At higher order, the anomaly is renormalised consistently

$$\delta\Gamma = f\frac{2+n}{16\pi^2}\int\left[\iota^*\mathrm{Im}\big(\mathcal{L}_4[\tau]\big)\cdot\Gamma\right] + h\frac{2+n}{32\pi^2}\int\left[\iota^*\mathcal{L}_4^{(1)}\cdot\Gamma\right]$$

Therefore if there is a logarithmic divergence

$$\begin{aligned}
\Big(\frac{\partial}{\partial\mu}-\kappa\frac{\partial}{\partial\kappa}\Big)\Gamma &\approx -\beta_{\ell+1}\kappa^{2\ell}[S^{(\ell)}\cdot\Gamma]+\mathcal{O}(\kappa^{2\ell+2})\\
\Big(\frac{\partial}{\partial\mu}-\kappa\frac{\partial}{\partial\kappa}\Big)\left[\iota^*\mathrm{Im}\big(\mathcal{L}_4[\tau]\big)\cdot\Gamma\right] &\approx -\gamma_\ell\kappa^{2\ell}[\mathcal{A}_f^{(\ell)}\cdot\Gamma]+\mathcal{O}(\kappa^{2\ell+2})\\
\Big(\frac{\partial}{\partial\mu}-\kappa\frac{\partial}{\partial\kappa}\Big)\left[\iota^*\mathcal{L}_4^{(1)}\cdot\Gamma\right] &\approx -\gamma'_\ell\kappa^{2\ell}[\mathcal{A}_h^{(\ell)}\cdot\Gamma]+\mathcal{O}(\kappa^{2\ell+2})
\end{aligned}$$

Then

$$\beta_\ell\delta S^{(\ell)} = \gamma_\ell f\frac{2+n}{16\pi^2}\int\mathcal{A}_f^{(\ell)} + \gamma'_\ell h\frac{2+n}{32\pi^2}\int\mathcal{A}_h^{(\ell)}\ .$$

Slavnov–Taylor

Non-linear supersymmetry is realised through Slavnov–Taylor

$$(\Sigma^{(0)}, \Sigma^{(1)}) = 0 \iff \delta^{(0)} S^{(1)} + \delta^{(1)} S^{(0)} = 0$$

Therefore $S^{(1)}$ is an on-shell supersymmetry invariant.

Question of completion of $\Sigma = \Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)} + \dots$ is

$$(\Sigma^{(1)}, \Sigma^{(1)}) + 2(\Sigma^{(0)}, \Sigma^{(2)}) = 0 \iff \delta^{(1)} S^{(1)} + \delta^{(0)} S^{(2)} + \delta^{(2)} S^{(0)} = 0$$

If $\Sigma^{(2)}$ does not exist, there is an algebraic anomaly.

$\hookrightarrow$ non-existence of $S^{(2)}$ does not prevent the divergence

$\hookrightarrow$ Is associated to a supersymmetry anomaly $\mathcal{A} = (\Sigma^{(1)}, \Sigma^{(1)})$!

$$(\Sigma^{(0)}, \mathcal{A}) = 0 \ , \qquad \mathcal{A} \neq (\Sigma^{(0)}, -2\Sigma^{(2)})$$

Higher order invariants

The supersymmetry invariants satisfying $SO(6,n)$ symmetry

- ⋆ 1-loop $\mathcal{F}(\tau)(R^2 + F^2\bar{F}^2)$
- ⋆ 2-loop $\mathcal{F}(\tau)\nabla F^2\nabla\bar{F}^2 + (\tau-\bar{\tau})\partial\mathcal{F}(\tau)\, R^2F^2$ **?**
- ⋆ 3-loop $\mathcal{F}(\tau)\nabla^2F^2\nabla^2\bar{F}^2 + (\tau-\bar{\tau})\partial\mathcal{F}(\tau)\, R^2\nabla^2F^2$

  $(\tau-\bar{\tau})\mathcal{F}_{(2)}(\tau)\nabla^2\tau\nabla^2\tau R^2$

  $(\Delta-2)G(\tau,\bar{\tau})R^4 + G(\tau,\bar{\tau})(\nabla^2F^2\nabla^2\bar{F}^2 + R^2\nabla F\nabla\bar{F})$

  $\Delta\, G_{(s)}(\tau,\bar{\tau})R^4 + G_{(s)}(\tau,\bar{\tau})\nabla^2F^2\nabla^2\bar{F}^2$

Anomalous Ward identity

At higher order, the anomaly is renormalised consistently

$$\delta\Gamma = f\frac{2+n}{16\pi^2}\int\Big[\iota^*\mathrm{Im}\big(\mathcal{L}_4[\tau]\big)\cdot\Gamma\Big] + h\frac{2+n}{32\pi^2}\int\Big[\iota^*\mathcal{L}_4^{(1)}\cdot\Gamma\Big]$$

Therefore if there is a logarithmic divergence

$$\begin{aligned}
\Big(\frac{\partial}{\partial\mu} - \kappa\frac{\partial}{\partial\kappa}\Big)\Gamma &\approx -\beta_{\ell+1}\kappa^{2\ell}[S^{(\ell)}\cdot\Gamma] + \mathcal{O}(\kappa^{2\ell+2}) \\
\Big(\frac{\partial}{\partial\mu} - \kappa\frac{\partial}{\partial\kappa}\Big)\big[\iota^*\mathrm{Im}\big(\mathcal{L}_4[\tau]\big)\cdot\Gamma\big] &\approx -\gamma_\ell\kappa^{2\ell}[\mathcal{A}_f^{(\ell)}\cdot\Gamma] + \mathcal{O}(\kappa^{2\ell+2}) \\
\Big(\frac{\partial}{\partial\mu} - \kappa\frac{\partial}{\partial\kappa}\Big)\big[\iota^*\mathcal{L}_4^{(1)}\cdot\Gamma\big] &\approx -\gamma'_\ell\kappa^{2\ell}[\mathcal{A}_h^{(\ell)}\cdot\Gamma] + \mathcal{O}(\kappa^{2\ell+2})
\end{aligned}$$

Then

$$\beta_\ell\delta S^{(\ell)} = \gamma_\ell f\frac{2+n}{16\pi^2}\int\mathcal{A}_f^{(\ell)} + \gamma'_\ell h\frac{2+n}{32\pi^2}\int\mathcal{A}_h^{(\ell)}\ .$$

Higher order invariants

The supersymmetry invariants satisfying $SO(6,n)$ symmetry

- ⋆ 1-loop $\mathcal{F}(\tau)(R^2 + F^2\bar{F}^2)$
- ⋆ 2-loop $\mathcal{F}(\tau)\nabla F^2\nabla\bar{F}^2 + (\tau-\bar{\tau})\partial\mathcal{F}(\tau)\, R^2F^2$ **?**
- ⋆ 3-loop $\mathcal{F}(\tau)\nabla^2F^2\nabla^2\bar{F}^2 + (\tau-\bar{\tau})\partial\mathcal{F}(\tau)\, R^2\nabla^2F^2$

  $(\tau-\bar{\tau})\mathcal{F}_{(2)}(\tau)\nabla^2\tau\nabla^2\tau R^2$

  $(\Delta-2)G(\tau,\bar{\tau})R^4 + G(\tau,\bar{\tau})(\nabla^2F^2\nabla^2\bar{F}^2 + R^2\nabla F\nabla\bar{F})$

  $\Delta G_{(s)}(\tau,\bar{\tau})R^4 + G_{(s)}(\tau,\bar{\tau})\nabla^2F^2\nabla^2\bar{F}^2$

Higher order invariants

The supersymmetry invariants satisfying $SO(6,n)$ symmetry

- ⋆ 1-loop $F^2\bar{F}^2$
- ⋆ 2-loop $\nabla F^2 \nabla \bar{F}^2$ **?**
- ⋆ 3-loop $\nabla^2 F^2 \nabla^2 \bar{F}^2$

$$(\Delta - 2)G(\tau,\bar{\tau})R^4 + G(\tau,\bar{\tau})(\nabla^2 F^2 \nabla^2 \bar{F}^2 + R^2 \nabla F \nabla \bar{F})$$

$$\Delta G_{(s)}(\tau,\bar{\tau})R^4 + G_{(s)}(\tau,\bar{\tau})\nabla^2 F^2 \nabla^2 \bar{F}^2$$

Anomalous Ward identity

At higher order, the anomaly is renormalised consistently

$$\delta\Gamma = f\frac{2+n}{16\pi^2}\int\left[\iota^*\mathrm{Im}\big(\mathcal{L}_4[\tau]\big)\cdot\Gamma\right] + h\frac{2+n}{32\pi^2}\int\left[\iota^*\mathcal{L}_4^{(1)}\cdot\Gamma\right]$$

Moreover $\mathcal{L}_4^{(1)}$ is duality invariant.

$$\left(\frac{\partial}{\partial\mu} - \kappa\frac{\partial}{\partial\kappa}\right)\left[\iota^*\mathcal{L}_4^{(1)}\cdot\Gamma\right] \approx -\gamma_\ell\kappa^{2\ell}[\mathcal{A}_h^{(\ell)}\cdot\Gamma] + \mathcal{O}(\kappa^{2\ell+2})$$

Therefore $\mathcal{A}_h^{(\ell)}$ will also be duality invariant at 1 and 2-loop. Then

$$(\tau-\bar{\tau})\frac{\partial G(\tau,\bar{\tau})}{\partial\tau} = c_1\ .$$

Higher order invariants

The supersymmetry invariants satisfying $SO(6, n)$ symmetry

- ⋆ 1-loop $F^2 \bar{F}^2$
- ⋆ 2-loop $\nabla F^2 \nabla \bar{F}^2$ **?**
- ⋆ 3-loop $\nabla^2 F^2 \nabla^2 \bar{F}^2$

$$R^4 - \frac{1}{2}(\nabla^2 F^2 \nabla^2 \bar{F}^2 + R^2 \nabla F \nabla \bar{F})$$

$$(\frac{1}{2} - 2\phi)R^4 + \phi(\nabla^2 F^2 \nabla^2 \bar{F}^2 + R^2 \nabla F \nabla \bar{F})$$

Higher order invariants

The supersymmetry invariants satisfying $SO(6)$ symmetry

- ⋆ 1-loop
- ⋆ 2-loop
- ⋆ 3-loop

$$R^4 - \frac{1}{2}(\nabla^2 F^2 \nabla^2 \bar{F}^2 + R^2 \nabla F \nabla \bar{F})$$

$(4,2,2)$ harmonic superspace

Harmonics variables ${u^r}_i, {u^{\hat{r}}}_i$ of $(U(2)\times U(2))\backslash U(4)$ such that the vector fields

$$
\begin{aligned}
D^r_\alpha &= {u^r}_i\big(D^i_\alpha - {\Omega^i_\alpha}^j{}_k({u^s}_j {u^k}_{\hat{t}} {d_s}^{\hat{t}} + {u^{\hat{s}}}_j {u^k}_t {d_{\hat{s}}}^t)\big) \ , \\
\bar{D}_{\dot{\alpha}\hat{r}} &= {u^i}_{\hat{r}}\big(\bar{D}_{\dot{\alpha}i} - {\Omega_{\dot{\alpha}i}}^j{}_k({u^s}_j {u^k}_{\hat{t}} {d_s}^{\hat{t}} + {u^{\hat{s}}}_j {u^k}_t {d_{\hat{s}}}^t)\big) \ ,
\end{aligned}
$$

Consistency requires

$$
{u^r}_i {u^s}_j {u^t}_k T^{ij\ \dot{\gamma}k}_{\alpha\beta} = 0 \ , \quad {u^i}_{\hat{r}} {u^j}_{\hat{s}} {u^k}_{\hat{t}} T_{\dot{\alpha}i\dot{\beta}jk}{}^{\gamma} = 0 \ ,
$$

and

$$
{u^r}_i {u^s}_j {u^t}_k {u^l}_{\hat{r}} R^{ij\ \ k}_{\alpha\beta\ \ l} = 0 \ , \quad {u^r}_i {u^j}_{\hat{s}} {u^t}_k {u^l}_{\hat{r}} R^{i\ \ \ k}_{\alpha\dot{\beta}j\ l} = 0 \ ,
$$

and complex conjugates.

$(4,2,2)$ harmonic superspace

One defines Grassmann normal coordinates $\zeta_r^\alpha, \bar{\zeta}^{\dot{\alpha}\hat{r}}$ associated to D_α^r and $D_{\dot{\alpha}\hat{r}}$. The superfielbein Berezinian admits a normal coordinate expansion

$$\mathrm{Ber}(E) = \mathcal{E}_{(4,2,2)}\Big(1 + \zeta_r^\alpha B_\alpha^r + \bar{\zeta}^{\dot{\alpha}\hat{r}}\bar{B}_{\dot{\alpha}\hat{r}} + \cdots + \zeta^{3\,r}{}_\alpha \bar{\zeta}^3{}_{\dot{\beta}\hat{s}} L^{\alpha\dot{\beta}}{}_r{}^{\hat{s}}\Big)$$

One defines G-analytic superfields

$$D_\alpha^r \mathcal{L}_{(4,2,2)} = 0 \ , \quad \bar{D}_{\dot{\alpha}\hat{r}}\mathcal{L}_{(4,2,2)} = 0 \ .$$

The $(4,2,2)$ harmonic superspace integral

$$\int d\mu_{(4,2,2)}\ \mathcal{L}_{(4,2,2)} \equiv \int d^4x d^4u d^4\bar{u} V(u,\bar{u}) d^4\theta d^4\bar{\theta}\mathcal{E}_{(4,2,2)}\ \mathcal{L}_{(4,2,2)}$$

is supersymmetric.

$(4,1,1)$ harmonic superspace

Harmonics variables $u^1{}_i, u^r{}_i, u^4{}_i$ of $(U(1)\times U(2)\times U(1))\backslash U(4)$ such that the vector fields

$$\begin{aligned} D^1_\alpha &= u^1{}_i\big(D^i_\alpha - \Omega^i_\alpha{}^j{}_k u^I{}_j u^k{}_J d_I{}^J\big) \ , \\ \bar{D}_{\dot\alpha 4} &= u^i{}_4\big(\bar{D}_{\dot\alpha i} - \Omega_{\dot\alpha i}{}^j{}_k u^I{}_j u^k{}_J d_I{}^J\big) \ , \end{aligned}$$

Consistency requires

$$u^1{}_i u^1{}_j u^1{}_k T^{i\,j}_{\alpha\beta}{}^{\dot\gamma k} = 0 \ , \quad u^i{}_4 u^j{}_4 u^k{}_4 T_{\dot\alpha i \dot\beta j}{}^{\gamma}_{k} = 0 \ ,$$

and

$$u^1{}_i u^1{}_j u^1{}_k R^{i\,j}_{\alpha\beta}{}^k{}_l = 0 \ , \quad u^1{}_i u^j{}_4 u^1{}_k R^i_{\alpha\dot\beta j}{}^k{}_l = -\tfrac{1}{2} u_l{}^1 \chi^1_\alpha \bar\chi_{\dot\beta 4} \ ,$$

and complex conjugates.

$(4,1,1)$ harmonic superspace

One defines Grassmann normal coordinates $\zeta_1^\alpha, \bar{\zeta}^{\dot{\alpha}4}$ associated to D_α^1 and $D_{\dot{\alpha}4}$. The superfielbein Berezinian admits a normal coordinate expansion

$$\text{Ber}(E) = \mathcal{E}_{(4,1,1)}\Big(1 + \zeta_1^\alpha \bar{\zeta}^{\dot{\beta}4} \chi_\alpha^1 \bar{\chi}_{\dot{\beta}4}\Big)$$

One defines G-analytic superfields

$$D_\alpha^1 \mathcal{L}_{(4,1,1)} = 0 \ , \quad \bar{D}_{\dot{\alpha}4} \mathcal{L}_{(4,1,1)} = 0 \ .$$

The $(4,1,1)$ harmonic superspace integral

$$\int d\mu_{(4,1,1)} \ \mathcal{L}_{(4,1,1)} \equiv \int d^4x d^5u d^5\bar{u} V(u,\bar{u}) d^6\theta d^6\bar{\theta} \mathcal{E}_{(4,1,1)} \ \mathcal{L}_{(4,1,1)}$$

is supersymmetric.

$(4,1,0)$ harmonic superspace

Harmonics variables ${u^1}_i, {u^r}_i$ of $(U(1) \times U(3))\backslash U(4)$ such that the vector fields

$$D^r_\alpha = {u^1}_i\big(D^i_\alpha - {\Omega^i_\alpha}^j{}_k({u^1}_j {u^k}_r {d_1}^r + {u^r}_j {u^k}_1 {d_r}^1)\big) \ ,$$

Consistency requires

$$ {u^1}_i {u^1}_j {u^1}_k T^{ij\,\dot\gamma k}_{\alpha\beta} = 0 \ ,$$

and

$$ {u^1}_i {u^1}_j {u^1}_k {R^{ij\,k}_{\alpha\beta}}_l = 0 \ .$$

$(4,1,0)$ harmonic superspace

One defines Grassmann normal coordinates ζ_1^α associated to D_α^1.
The superfielbein Berezinian is itself G-analytic

$$\mathrm{Ber}(E) = \mathcal{E}_{(4,1,0)}$$

One defines G-analytic superfields

$$D_\alpha^1 \mathcal{L}_{(4,1,0)} = 0 \ .$$

The $(4,1,0)$ harmonic superspace integral

$$\int d\mu_{(4,1,0)} \ \mathcal{L}_{(4,1,0)} \equiv \int d^4x d^6u d^6\bar{u} V(u,\bar{u}) d^6\theta d^8\bar{\theta} \mathcal{E}_{(4,1,0)} \ \mathcal{L}_{(4,1,0)}$$

is supersymmetric.

R^4 type invariants

Full-superspace integrals

$$\int d^4x d^{16}\theta \mathrm{Ber}(E)\, K(T,\bar{T}) = \frac{1}{4}\int d\mu_{(4,1,1)}\Big(\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\chi^1_\alpha\chi^1_\beta\bar\chi_{\dot\alpha 4}\bar\chi_{\dot\beta 4}\,((1-T\bar T)^2\partial\bar\partial - 2)$$
$$-\frac{1}{2}\varepsilon^{\alpha\beta}\chi^1_\alpha\chi^1_\beta\varepsilon^{\gamma\delta}\lambda_{\gamma 4A}\lambda_{\delta 4}{}^A\frac{1}{\bar U^2}\bar\partial - \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\alpha 4}\bar\chi_{\dot\beta 4}\varepsilon^{\dot\gamma\dot\delta}\bar\lambda^1_{\dot\gamma A}\bar\lambda^{1A}_{\dot\delta}\frac{1}{U^2}\partial$$
$$+2\varepsilon^{\alpha\beta}\lambda_{\alpha 4A}\chi^1_\beta\varepsilon^{\dot\alpha\dot\beta}\bar\lambda^{1A}_{\dot\alpha}\bar\chi_{\dot\beta 4} + \frac{1}{4}\varepsilon^{\alpha\beta}\lambda_{\alpha 4A}\lambda_{\beta 4A}\varepsilon^{\dot\alpha\dot\beta}\bar\lambda^{1B}_{\dot\alpha}\bar\lambda^1_{\dot\beta B}\Big)(1-T\bar T)^2\partial\bar\partial\, K(T,\bar T)$$
$$= \int d^4x\Big((\Delta-2)\Delta K\ C^2\bar C^2 + \Delta K\ \partial^2F^2\partial^2\bar F^2 + \Delta K\ \partial F\partial\bar F C\bar C$$
$$+ D\Delta K\ \partial F\partial F C^2 + \bar D\Delta K\ \partial\bar F\partial\bar F\bar C^2 + \dots\Big)$$

R^4 type invariants

1/4 BPS-superspace integrals

$$
\begin{aligned}
\frac{1}{4}\int d\mu_{(4,1,1)}\Big(&\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\chi^1_\alpha\chi^1_\beta\bar\chi_{\dot\alpha 4}\bar\chi_{\dot\beta 4}\left((1-T\bar T)^2\partial\bar\partial-2\right)\\
&-\frac{1}{2}\varepsilon^{\alpha\beta}\chi^1_\alpha\chi^1_\beta\varepsilon^{\gamma\delta}\lambda_{\gamma 4A}\lambda_{\delta 4}{}^{A}\frac{1}{\bar U^2}\bar\partial-\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\alpha 4}\bar\chi_{\dot\beta 4}\varepsilon^{\dot\gamma\dot\delta}\bar\lambda^1_{\dot\gamma A}\bar\lambda^{1A}_{\dot\delta}\frac{1}{U^2}\partial\\
&+2\varepsilon^{\alpha\beta}\lambda_{\alpha 4A}\chi^1_\beta\varepsilon^{\dot\alpha\dot\beta}\bar\lambda^{1A}_{\dot\alpha}\bar\chi_{\dot\beta 4}+\frac{1}{4}\varepsilon^{\alpha\beta}\lambda_{\alpha 4A}\lambda_{\beta 4A}\varepsilon^{\dot\alpha\dot\beta}\bar\lambda^{1B}_{\dot\alpha}\bar\lambda^{1}_{\dot\beta B}\Big)G(T,\bar T)\\
&=\int d^4x\Big((\Delta-2)G\;C^2\bar C^2+G\;\partial^2F^2\partial^2\bar F^2+G\;\partial F\partial\bar F C\bar C\\
&\qquad\qquad+DG\;\partial F\partial F C^2+\bar D G\;\partial\bar F\partial\bar F\bar C^2+\dots\Big)
\end{aligned}
$$

R^4 type invariants

1/2 BPS-superspace integrals

$$
\begin{aligned}
\frac{1}{4}\int d\mu_{(4,2,2)}\Big(& \big((X^{(0)})^2 + X^{(2)}X^{(-2)}\Delta + X^{(4)}X^{(-4)}\Delta(\Delta+2) + X^{(6)}X^{(-6)}\Delta(\Delta^2+10\Delta+12)\big)\,G_{(s)} \\
& + \big(X^{(2)}X^{(0)} + X^{(4)}X^{(-2)}\Delta + X^{(6)}X^{(-4)}\Delta(\Delta+4)\big)\,DG_{(s)} + \text{c.c.} \\
& + \big(X^{(4)}X^{(0)} + X^{(6)}X^{(-2)}\Delta\big)\,D^2G_{(s)} + X^{(6)}X^{(0)}D^3G_{(s)} + \text{c.c.}\Big) \\
= \int d^4x\Big(& \Delta G_{(s)}\; C^2\bar{C}^2 + G_{(s)}\; \partial^2F^2\partial^2\bar{F}^2 \\
& + DG_{(s)}\; \partial F\partial F C^2 + \bar{D}G_{(s)}\; \partial\bar{F}\partial\bar{F}\bar{C}^2 + \ldots\Big)
\end{aligned}
$$

where

$$
\begin{aligned}
X^{(6)} &= \frac{1}{12}\varepsilon^{\hat{r}\hat{s}}\varepsilon^{\hat{t}\hat{u}}\bar{\chi}_{\dot{\alpha}\hat{r}}\bar{\chi}_{\dot{\beta}\hat{s}}\bar{\chi}^{\dot{\alpha}}_{\hat{t}}\bar{\chi}^{\dot{\beta}}_{\hat{u}}\,, \\
X^{(4)} &= \bar{M}^{12}_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\alpha}}_3\bar{\chi}^{\dot{\beta}}_4 + \frac{1}{8}\varepsilon^{\hat{r}\hat{s}}\varepsilon^{\hat{t}\hat{u}}\bar{\chi}_{\dot{\alpha}\hat{r}}\bar{\chi}^{\dot{\alpha}}_{\hat{t}}\lambda^{A}_{\beta\hat{s}}\lambda^{\beta}_{\hat{u}A}\,, \\
X^{(2)} &= \frac{1}{2}\bar{M}^{12}_{\dot{\alpha}\dot{\beta}}\bar{M}^{\dot{\alpha}\dot{\beta}12} + \frac{1}{4}\varepsilon^{\hat{r}\hat{s}}\varepsilon_{rs}\bar{\chi}^{\dot{\alpha}}_{\hat{r}}\bar{\chi}^{\dot{\beta}}_{\hat{s}}\bar{\lambda}^{rA}_{\dot{\alpha}}\bar{\lambda}^{s}_{\dot{\beta}A} + iP^{A}_{\alpha\dot{\beta}34}\varepsilon^{\hat{r}\hat{s}}\bar{\chi}^{\dot{\beta}}_{\hat{r}}\lambda^{\alpha}_{\hat{s}A} - \frac{1}{32}\varepsilon^{\hat{r}\hat{s}}\varepsilon^{\hat{t}\hat{u}}\lambda^{A}_{\alpha\hat{r}}\lambda^{\alpha}_{\hat{s}A}\lambda^{B}_{\beta\hat{t}}\lambda^{\beta}_{\hat{u}B}\,, \\
X^{(0)} &= -P^{A}_{\alpha\dot{\beta}34}P^{\alpha\dot{\beta}}{}_{34A} - \frac{1}{2}M_{\alpha\beta34}\lambda^{\alpha A}_{3}\lambda^{\beta}_{4A} - \frac{1}{2}\bar{M}^{12}_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\alpha}1A}\bar{\lambda}^{\dot{\beta}2}_{A} \\
&\qquad - \frac{1}{8}\varepsilon^{\hat{r}\hat{s}}\varepsilon_{rs}\chi^{r}_{\alpha}\lambda^{\alpha A}_{\hat{r}}\bar{\chi}_{\dot{\beta}\hat{s}}\bar{\lambda}^{\dot{\beta}s}_{A} - \frac{1}{4}\lambda^{[A}_{\alpha3}\lambda^{\alpha B]}_{4}\bar{\lambda}^{1}_{\dot{\beta}[A}\bar{\lambda}^{\dot{\beta}2}_{B]}\,. \qquad (3.
\end{aligned}
$$

Protected R^4

Depending of the power of the dilaton

- ⋆ non-BPS

$$(\Delta - 2)\Delta G(\tau,\bar\tau)\, R^4 \sim \int d^{16}\theta E\, G(\tau,\bar\tau)\ .$$

- ⋆ $\frac{1}{4}$-BPS

$$(\Delta - 2) G(\tau,\bar\tau)\, R^4 \sim \int d^{12}\theta \mathcal{E} \chi^2 \bar\chi^2\, G(\tau,\bar\tau)\ .$$

- ⋆ $\frac{1}{2}$-BPS

$$\Delta G(\tau,\bar\tau)\, R^4 \sim \int d^{8}\theta \mathcal{E} F^2 \bar F^2\, G(\tau,\bar\tau)\ .$$

Heterotic string theory non-renormalisation

The corresponding R^4 coupling in heterotic string theory

$$\int d^{10}x d^{16}\theta \mathrm{Ber}(E) e^{\left(2\ell-\frac{3}{2}\right)\Phi} \sim \int d^{10}x(\ell-1)(\ell-2)(\ell-3)(\ell-4)e^{\left(2\ell-\frac{3}{2}\right)\phi}(t_8 t_8 - \tfrac{1}{8}\varepsilon\varepsilon)R^4$$

- ⋆ 0-loop non-BPS tree-loop $\zeta(3)$ term
- ⋆ 1-loop $\frac{1}{2}$-BPS 1-loop anomaly term
- ⋆ 2-loop $\frac{1}{4}$-BPS
- ⋆ 3-loop $\frac{1}{4}$-BPS
- ⋆ 4-loop $\frac{1}{2}$-BPS
- ⋆ 5-loop non-BPS

[Tseytlin] [Vanhove, Tourkine]

Conclusion

Can supersymmetry and duality invariance explain the absence of logarithm divergence at 3-loop in $\mathcal{N}=4$ $d=4$ supergravity, respectively 2-loop in 5 dimensions?

Yes for $n=0$ and $D=5$ with the rather strong assumption that an off-shell harmonic formulation with a duality invariant action exists

$\hookrightarrow$ This is a marginal case.

At 7-loop in $\mathcal{N}=8$ supergravity

$$\int \partial^8 R^4 + \cdots \sim \int d^4x d^{28}\theta \mathcal{E} \chi^4$$

is $\frac{1}{8}$-BPS. It is very similar to the R^4 invariant in $\mathcal{N}=4$.

$$\int d^{32}\theta \mathrm{Ber}(E)\, G(\Phi) = \int (\Delta + s)\Delta G(\phi)\ \partial^8 R^4 + \dots$$