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TWO-PHOTON REACTIONS BEYOND ONE-LOOP

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TWO-PHOTON REACTIONS BEYOND ONE-LOOP

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Abstract

We review the recent progress in the calculation of the amplitude for $\gamma\gamma \to \pi^0\pi^0$ to two loops in chiral perturbation theory. We match the low-energy amplitude in chiral perturbation theory with the result of the dispersion theoretic analysis. The neutral pion polarizabilities are also given to two-loop accuracy. Then, the results are compared with the dispersion relation calculation of the pion polarizabilities.

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The process producing a pion pair in the fusion of two photons has received a lot of attention in the recent years. It provides a very important test of chiral perturbation theory (CHPT) [1]-[5]. In this sense, the production of neutral pion pairs is the most interesting channel. This is because the Born amplitude vanishes in this case. Hence the one-loop scattering amplitude is finite and does not depend on the free parameters of the chiral lagrangian [6, 7]. It is by now a well-known fact that the one-loop cross section for $\gamma\gamma \to \pi^0\pi^0$ in CHPT [6, 7] does not agree with the experimental measurements at Crystal Ball [8], as well as with calculations based on dispersion relations [9]-[15], even at low-energy. We calculated recently the low-energy $\gamma\gamma \to \pi^0\pi^0$ amplitude to two-loops in CHPT [16] and obtained a prediction that agrees with the Crystal Ball data. Also, the low-energy CHPT amplitude compares very well with the dispersive analysis of $\gamma\gamma \to \pi^0\pi^0$ by Donoghue and Holstein [12].

The sum of the electric and magnetic polarizabilities of the neutral pion vanishes to lowest order in CHPT [17]. The value of this sum was estimated also by a sum rule, and it turned out to be different from zero [18]. We showed some time ago [19] that a vector dominance model preserving the chiral symmetry of QCD at low energy yields a value close to the one obtained from the sum rule. We obtained recently a refinement of this prediction, by calculating the two-loop Compton scattering amplitude in CHPT [16]. The largest modification in the polarizabilities, with respect to the one-loop order value, is given by the omega resonance exchange which accounts for a large fraction of the neutral pion sum rule [19], and the contribution from the chiral logarithms is small [16].

The $\gamma\gamma \to \pi^0\pi^0$ cross section for off-shell photons has been calculated recently [20] in the framework of CHPT. There, it has been shown that the measurement of the azimuthal correlations in the process $e^+e^- \to e^+e^-\pi^0\pi^0$ allows to test the higher order CHPT corrections independently from the measurement of the cross-section.

Recently, CHPT has been reformulated to include [21] into each order additional terms which in the standard CHPT are of higher order. Within this generalization of the chiral expansion of the amplitude, the process $\gamma\gamma \to \pi^0\pi^0$ has been analyzed [22].

Gauge symmetry and Lorentz invariance can be used to write the scattering matrix element

$$<\pi^0(p_1)\pi^0(p_2)_{\text{out}}|\gamma(q_1)\gamma(q_2)_{\text{in}}>=i(2\pi)^4\delta^4(P_f-P_i)T^N,$$  \hspace{1cm} (0.1)
with
\[ T^{N} = e^{2}e_{
u}^{\mu}e_{2}^{\nu}V_{\mu\nu}, \]
\[ V_{\mu\nu} = i \int dx e^{-i(q_{1}x+q_{2}y)} < \pi^{0}(p_{1})\pi^{0}(p_{2})\text{out} | T_{j\mu}(x)j_{\nu}(y) | 0 >, \]
\[ (0.2) \]

where \( j_{\mu} \) is the electromagnetic current, and \( \alpha = e^{2}/4\pi \simeq 1/137 \), as follows:
\[ V_{\mu\nu} = A(s,t,u)T_{1\mu\nu} + B(s,t,u)T_{2\mu\nu} + , \]
\[ T_{1\mu\nu} = \frac{s}{2}g_{\mu\nu} - q_{1\nu}q_{2\mu} , \]
\[ T_{2\mu\nu} = 2s\Delta_{\mu}\Delta_{\nu} - \nu^{2}g_{\mu\nu} - 2\nu(q_{1\nu}\Delta_{\mu} - q_{2\mu}\Delta_{\nu}) , \]
\[ \Delta_{\mu} = (p_{1} - p_{2})_{\mu} , \]
\[ (0.3) \]

in terms of the standard Mandelstam variables
\[ s = (q_{1} + q_{2})^{2}, t = (p_{1} - q_{1})^{2}, u = (p_{2} - q_{1})^{2} , \]
\[ \nu = t - u. \]
\[ (0.4) \]

One can go from the analytic functions \( A \) and \( B \) of the variables \( s, t \) and \( u \), symmetric under crossing \((t,u) \rightarrow (u,t)\), to the helicity amplitudes in the following way:
\[ H_{++} = A + 2(4M_{s}^{2} - s)B , \]
\[ H_{+-} = \frac{8(M_{s}^{4} - tu)}{s}B. \]
\[ (0.5) \]

In Ref. [16] the renormalization procedure is formulated in the minimal subtraction scheme, and the expressions of the order \( E^{6} \) renormalized amplitudes involve three parameters, i.e. \( a_{1}^{*}, a_{2}^{*} \) and \( b^{*} \)
\[ A_{6} = \frac{a_{1}^{*}M^{2} + a_{2}^{*}s}{(16\pi^{2}F^{2})^{2}} + \cdots , \]
\[ (0.6) \]
\[ B_{6} = \frac{b^{*}}{(16\pi^{2}F^{2})^{2}} + \cdots , \]
\[ (0.7) \]

where the ellipses stand for finite contributions from the loop-integrals. Here \( F \) is the pion decay constant in the chiral limit, \( F_{\pi} = F(1 + O(\hat{m})) \), \( F_{\pi} \simeq 93 \text{ MeV} \), and the physical pion mass is
\[ M_{s}^{2} = M^{2}(1 + O(\hat{m})) , \]
\[ M^{2} = 2\hat{m}B, \]
\[ (0.8) \]
Table 1: Resonance contributions to the coupling constants $a_1^r, a_2^r$ and $b^r$. Column 6 contains the sums of those contributions which have a definite sign.

<table>
<thead>
<tr>
<th>$I^R$</th>
<th>$\omega$</th>
<th>$\rho^0$</th>
<th>$\phi$</th>
<th>$A(1^{+-})$</th>
<th>$\Sigma_R I^R$</th>
<th>$S(0^{++})$</th>
<th>$f_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1^r$</td>
<td>-33.2</td>
<td>-6.1</td>
<td>-0.1</td>
<td>0.0</td>
<td>-39</td>
<td>±0.8</td>
<td>±4.1</td>
</tr>
<tr>
<td>$a_2^r$</td>
<td>12.5</td>
<td>2.3</td>
<td>$\simeq$ 0</td>
<td>-1.3</td>
<td>13</td>
<td>±1.3</td>
<td>±1.0</td>
</tr>
<tr>
<td>$b^r$</td>
<td>2.1</td>
<td>0.4</td>
<td>$\simeq$ 0</td>
<td>0.7</td>
<td>3</td>
<td>0.0</td>
<td>±0.5</td>
</tr>
</tbody>
</table>

where $B$ is related to the order parameter $<0 | \bar{q}q | 0>$, and in the isospin symmetry limit $m_u = m_d = \bar{m}$.

The three unknown parameters $a_1^r$, $a_2^r$ and $b^r$ are estimated by resonance saturation [16] and the results are displayed in table 1. The $\omega$ exchange yields the dominant contribution.

The low-energy constants $h_{\pm}^r$ and $h_{s}^r$ corresponding to the helicity amplitudes $H_{++}$ and $H_{+-}$ read

\[
H_{++}^{2\text{loops}} = \frac{1}{(16\pi^2 F^2)^2} \left\{ h_{+}^r M^2 + h_{s}^r s \right\} + \cdots ,
\]

\[
H_{+-}^{2\text{loops}} = \frac{8(M^4 - tu)}{s(16\pi^2 F^2)^2} h_{-}^r + \cdots ,
\]

\[
h_{+}^r = a_1^r + 8b^r , \quad h_{s}^r = a_2^r - 2b^r , \quad h_{-}^r = b^r . \tag{0.9}
\]

From column 6 in table 1 the central values of these couplings are obtained in Ref. [16], where a 30% uncertainty is associated to the contributions generated by the vector and axial-vector exchange, and a 100% error to the contributions from scalars and from $f_2$. Adding these errors in quadrature, one finds [16]

\[
h_{+}^r(M_\rho) = -14 \pm 5 ,
\]

\[
h_{s}^r(M_\rho) = 7 \pm 3 ,
\]

\[
h_{-}^r(M_\rho) = 3 \pm 1 . \tag{0.10}
\]

The values of $h_{+}^r$ and $h_{s}^r$ are not affected by the tensor exchange, since the corresponding coupling is purely $D$-wave. Scalars do not contribute to the value of $h_{-}^r$.

In Ref. [23], these couplings have been determined i) from vector-meson exchange
and using nonet-symmetry, and ii) from the chiral quark model, with the result

\[(h^r_+, h^r_s, h^r_-)_{\mu=M_\rho} = \begin{cases} (-18,9,2) & \text{vector-mesons (nonet)} \\ (-12,6,2) & \text{chiral quark model} \end{cases}\]  

(0.11)

which agrees within the uncertainties with the values in (0.10).

The following analytic results are obtained in Ref. [16] for the amplitude $A$ to two loops:

\[A = \frac{4G_\pi(s)}{sF^2_\pi}(s - M^2_\pi) + U_A + P_A + O(E^4).\]  

(0.12)

The unitary part $U_A$ contains $s, t$ and $u$-channel cuts, and $P_A$ is a linear polynomial in $s$. Explicitly,

\[U_A = \frac{2}{sF^2_\pi} \tilde{G}(s) \left[ (s^2 - M^2_\pi) \tilde{J}(s) + C(s, \bar{t}_i) \right] + \frac{\bar{I}_\Delta}{24\pi^2 F^4_\pi} (s - M^2_\pi) \tilde{J}(s) \]

\[+ \frac{\bar{I}_2 - 5/6}{144\pi^2 sF^4_\pi} (s - 4M^2_\pi) \left\{ \bar{H}(s) + 4 \left[ s\bar{G}(s) + 2M^2_\pi \bar{G}(s) - 3 \bar{J}(s) \right] \right\} \]

\[+ \Delta_A(s, t, u),\]  

(0.13)

with

\[C(s, \bar{t}_i) = \frac{1}{48\pi^2} \left\{ 2(\bar{I}_1 - 4/3)(s - 2M^2_\pi)^2 + (\bar{I}_2 - 5/6)(4s^2 - 8sM^2_\pi + 16M^4_\pi)/3 \right.\]

\[- 3M^4_\pi \bar{t}_3 + 12M^2_\pi(s - M^2_\pi)\bar{t}_4 - 12sM^2_\pi + 15M^4_\pi \right\},\]

\[d^2_{\bar{t}0} = \frac{1}{2}(3\cos \theta^2 - 1).\]  

(0.14)

The loop-functions $\tilde{J}$ etc. are given in appendix C of Ref. [16].

The polynomial part is

\[P_A = \frac{1}{(16\pi^2 F^2_\pi)^2} [a_1 M^2_\pi + a_2 s],\]

\[a_1 = a_1' + \frac{1}{18} \left\{ 4l^2 + l(8\bar{I}_2 + 12\bar{I}_\Delta) - \frac{4}{3} - \frac{20}{3} \bar{I}_2 + 12\bar{I}_\Delta + \frac{110}{9} \right\},\]

\[a_2 = a_2' - \frac{1}{18} \left\{ l^2 + l(2\bar{I}_2 + 12\bar{I}_\Delta + \frac{2}{3} - \frac{5}{3} \bar{I}_2 + 12\bar{I}_\Delta + \frac{697}{144} \right\},\]

\[l = \ln \frac{M^2}{\mu^2}.\]  

(0.15)

Here $\mu$ denotes the renormalization scale, the $\bar{I}_i$ are the renormalized coupling constants of the $O(E^4)$ lagrangian [2], and $\bar{I}_\Delta = \bar{I}_6 - \bar{I}_5$. The values of these coupling
constants can be found in column 2 of table 1 of Ref. [16]. The result for $B$ is

$$B = U_B + P_B + O(E^2),$$

(0.16)

with unitary part

$$U_B = \frac{(l_2 - 5/6)\bar{H}(s)}{288\pi^2F_\pi^4s} + \Delta_B(s, t, u).$$

(0.17)

The polynomial is obtained in Ref. [16]

$$P_B = \frac{b}{(16\pi^2F_\pi^4)^2},$$

$$b = b* - \frac{1}{36}\left[t^2 + l(2l_2 + \frac{2}{3}) - \frac{l_2}{3} + \frac{393}{144}\right].$$

(0.18)

The integrals $\Delta_{A,B}(s, t, u)$ contain contributions that very small for the cross sections below $\sqrt{s} \leq 400$ MeV, both for $\gamma\gamma \rightarrow \pi^0\pi^0$ (0.1% at 400 MeV) and for the crossed channel $\gamma\pi^0 \rightarrow \gamma\pi^0$ (1.5% at 400 MeV) [16].

The cross section $\gamma\gamma \rightarrow \pi^0\pi^0$ receives a substantial correction near threshold due to $\pi\pi$ final-state interactions – which are absent in Compton scattering. Yet, it is shown in Ref. [16] that the two-loop contributions are not small in this channel. Since in the one-loop approximation the amplitude $H_{++}$ is one order of magnitude larger in the $\gamma\gamma \rightarrow \pi^0\pi^0$ channel than at Compton threshold, even very small corrections in $\gamma\gamma \rightarrow \pi^0\pi^0$ may appear large in Compton scattering [19]. The result of the two-loop calculation and the one-loop approximation differ by one order of magnitude already near threshold [16]. This is mainly due to the effect of the low-energy constant $h_\gamma$ in $H_{+-}$ (omega-exchange in the language of resonance saturation [19]).

The low-energy limit of the coupling with the photon in the Compton amplitude for a composite system is characterized (among other parameters) by the electric and magnetic polarizabilities. One can test the hadron dynamics through experiments on the hadron polarizabilities [18]. The expansion of the amplitude for charged pion Compton scattering,

$$\gamma(q_1)\pi^+(p_1) \rightarrow \gamma(q_2)\pi^+(p_2),$$

(0.19)

near threshold reads

$$T^C = 2\left[\hat{c}_1 \cdot \hat{c}_2^* \left(\frac{\alpha}{M_\pi} - \bar{\alpha}_\pi\omega_1\omega_2\right)\bar{\beta}_\pi(q_1 \times \hat{c}_1) \cdot (q_2 \times \hat{c}_2^*) + \cdots\right]$$

(0.20)

with $q_i^\mu = (\omega_i, q_i)$. For neutral pions, one has, in terms of $A$ and $B$,

$$\bar{\alpha}_{\pi^0} = \frac{\alpha}{2M_\pi}(A + 16M_\pi^2B)|_{s=0, t=M_\pi^2},$$

$$\bar{\beta}_{\pi^0} = -\frac{\alpha}{2M_\pi}A|_{s=0, t=M_\pi^2}. $$

(0.21)
Below we denote

\[
(\alpha \pm \beta)^C = \bar{\alpha}_\pi \pm \bar{\beta}_\pi ,
\]
\[
(\alpha \pm \beta)^N = \bar{\alpha}_{\pi^0} \pm \bar{\beta}_{\pi^0} .
\]  

(0.22)

The pion polarizabilities have been estimated\(^1\) through dispersion sum rules [18]

\[
(\alpha + \beta)^C = 0.39 \pm 0.04 ,
\]
\[
(\alpha - \beta)^C = 10 \pm 3 ,
\]
\[
(\alpha + \beta)^N = 1.04 \pm 0.07 ,
\]
\[
(\alpha - \beta)^N = -10 \pm 4 .
\]  

(0.23)

The charged pion polarizabilities have been determined in an experiment on the radiative pion-nucleus scattering \(\pi^- A \rightarrow \pi^- \gamma A\) [24] and in the pion photoproduction process \(\gamma p \rightarrow \gamma \pi^+ n\) [25]. Assuming the constraint \((\alpha + \beta)^C = 0\) the two experiments yield

\[
(\alpha - \beta)^C = \begin{cases} 
13.6 \pm 2.8 & [24] \\
40 \pm 24 & [25] . 
\end{cases}
\]  

(0.24)

Relaxing the constraint \((\alpha + \beta)^C = 0\), one obtains from the Serpukhov data

\[
(\alpha + \beta)^C = 1.4 \pm 3.1(\text{stat.}) \pm 2.5(\text{sys.}) \ [26] ,
\]
\[
(\alpha - \beta)^C = 15.6 \pm 6.4(\text{stat.}) \pm 4.4(\text{sys.}) \ [26] .
\]  

(0.25)

At one-loop one has [17, 27]

\[
\bar{\alpha}_{\pi^0} = -\bar{\beta}_{\pi^0} = -\frac{\alpha}{96\pi^2 M_\pi F^2} = -0.50 .
\]  

(0.26)

At order \(O(E^6)\) we found [16]

\[
\bar{\alpha}_{\pi^0} = -0.50 + 0.21 - 0.07 \approx -0.35 ,
\]
\[
\bar{\beta}_{\pi^0} = 0.50 + 0.79 + 0.24 \approx 1.50
\]  

(0.27)

where the three contributions that add up to the final results on the r.h.s. are the one-loop, the resonance, and the two-loop contributions, respectively. There is a

\(^1\)The values of the polarizabilities are in units of \(10^{-4}\)fm\(^3\) in what follows.
large contribution from the resonance exchange. Our results saturates the forward sum rule \((\alpha + \beta)^N = 1.04 \pm 0.07\) in (0.23)

\[
(\alpha + \beta)^N = 1.0 + 0.16 \simeq 1.15 \quad .
\] (0.28)

Information on the charged pion polarizabilities may be obtained from \(\gamma \gamma \rightarrow \pi^+\pi^-\) data [27]. The low-energy constant \(\vec{l}_\Delta\) appears as the only free parameter order in the \(O(E^4)\) amplitude, as well as in the leading-order expression for \(\bar{\alpha}_\pi\) and \(\bar{\beta}_\pi\). A fit to the cross section then determines \(\bar{\alpha}_\pi\) and \(\bar{\beta}_\pi\). The result [27]

\(\vec{l}_\Delta = 2.3 \pm 1.7\) corresponds to numerical value for the leading-order \(\bar{\alpha}_\pi = 2.7 \pm 0.4\), plus systematic uncertainties due to the \(O(E^6)\) corrections. For the charged pions, a two-loop calculation is not yet available. The charged pion polarizabilities are given beyond the one-loop order by including the meson resonance contribution in Refs. [28, 19]

The construction of unitarized \(S\)-wave amplitudes for \(\gamma \gamma \rightarrow \pi\pi\) which contain \((\alpha - \beta)^{C,N}\) as adjustable parameters has been carried out in Ref. [15]. In this case, only \((\alpha - \beta)^{C,N}\) can be determined from the data [8, 29], with the result

\[
(\alpha - \beta)^C = 4.8 \pm 1.0 \quad [15] ,
\]

\[
(\alpha - \beta)^N = -1.1 \pm 1.7 \quad [15] .
\] (0.29)

The value (0.29) for \((\alpha - \beta)^N\) is consistent with the two-loop result for the neutral pion, whereas the corresponding calculation for charged pions is not available and so it cannot be compared with the value (0.29) for \((\alpha - \beta)^C\).

Finally, it is interesting to compare the chiral expansion [16] with the dispersive calculation carried out by Donoghue and Holstein [12]. The two representations of the \(S\)-wave amplitude agree numerically very well below \(E = 0.4\ GeV\). In the dispersive method, higher order terms are partially summed up. The agreement indicates that yet higher orders in the chiral expansion do not affect much the threshold amplitude.

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