G. Da Prato, G. Putzolu: RADIATIVE CORRECTIONS TO $\tilde{\eta} \rightarrow \eta^0 + e^- + \bar{\nu}$ DECAY.
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$\pi^- \rightarrow \pi^0 + e^- + \bar{\nu}$ DECAY.

1. INTRODUCTION

Feynmann and Gell-Mann(1) introduced the hypothesis of conserved current to explain the absence of renormalization effects in the V part of the β decay. In their scheme the weak vector current is identified with the (+) component of the isotopic spin current $J^{(+)}_K$. One of the suggested tests of the theory is an accurate measurement of the decay rate for the leptonic decay of the pion:

$$\pi^- \rightarrow \pi^0 + e^- + \bar{\nu} \quad (1)$$

In fact, neglecting electromagnetic corrections, the corresponding matrix element is given by:

$$i g f_2 (\bar{\nu} \gamma^\mu (\gamma + i \gamma^5) \nu) (\pi^0 | j^{(+)}_K | \pi^-) \quad (2)$$

and we have a simple connection between the relevant matrix element of the vector current and the electromagnetic form factor of the pion $F_\pi$:

$$(\pi^0 | j^{(\mu)}_K | \pi^-) = (\pi^0 | \sigma^-_K | \pi^-) \frac{F_\pi(k^2)}{k^2} \quad (3)$$
where $\kappa^2 = (\vec{p} - \vec{p}^\prime)^2$ is the momentum transfer to the lepton pair. In the actual process (1) this momentum transfer is very small, so that one can safely put $F^2 = 1$.

In this work we propose to evaluate the radiative corrections (to order $\alpha^2$) to process (1). This would be important for a comparison of an accurate experimental result and the prediction of the Feynman and Gell-Mann Theory.

Since it is difficult to introduce the pion form factor in a gauge-invariant way for vertices with virtual pion lines, we will use a local Lagrangian and a Feynman cutoff in the calculation of radiative corrections. The results will not depend critically on this cutoff, since the divergence will be found to be only logarithmic.

2. FORMULATION

In the following we shall use the notations and the conventions of the textbook of Bogoliubov and Shirkov (2). Let $P_1, P_2, P_3, P_4$ be the momenta and $m_1, m_2, m_3, m_4 = 0$ the masses of the $\pi^-, e^-, \pi^0, \nu$.

We put also:

$$\Theta = \frac{(p_2, p_2)}{m_1, m_2}$$

(4)

All the calculations will be performed in the center of mass system.

The Lagrangian responsible for the process is:

$$\mathcal{L} = \sqrt{2} \int d^4x \left( \bar{\nu} \gamma^\mu \gamma^\nu \partial^\mu \nu - \bar{\nu} \gamma^\nu \gamma^\mu \partial^\mu \nu \right) + h.c.$$  

(5)

Following the principle of the minimal electromagnetic interaction, the Lagrangian that takes into account the electromagnetic interactions as well, is obtained from the complete Lagrangian without them:
\[ \mathcal{L}_0 = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{free}} + \mathcal{L}_f \]  

by the substitution:

\[ \partial_\mu \phi(x) \longrightarrow \left( \partial_\mu - ieA_\mu(x) \right) \phi(x) \]

where \( A(x) \) and \( \phi(x) \) are respectively the field operators of the photon and of the generic charged particle that appears in the process (1).

In this way we obtain: for the complete interaction Lagrangian \( \mathcal{L} \):

\[ \mathcal{L} = \mathcal{L}_l + e : \bar{\psi} \gamma^\mu A_\mu \psi + ie : \bar{\psi} \gamma^\mu \pi^i \gamma^\nu \gamma^\mu \pi_i \gamma_\mu : A_\nu - e^2 : \pi^\mu \pi^\nu A_\mu A_\nu : + \left[ \frac{g}{\sqrt{2}} \left( \bar{\psi} \gamma^\mu \gamma^i \pi \right) \psi \bar{\psi} A_\mu \pi^i + \text{h.c.} \right] \]

The last term is a new direct interaction between the five particles \( \pi^- , \pi^0 , e^- , \nu \) and \( \bar{\nu} \).

3. — FEYNMANN DIAGRAMS.

Using the Lagrangian (8) we have seven diagrams corresponding to the process (1) up to the order \( g^2 e^2 \) (fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Fig. 1}
\end{figure}

Besides them we have also to consider three diagrams (fig. 2) relative to the same process with bremsstrahlung:
They give to the transition probability a contribution of the order $g^2 e^2$, depending on the experimental situation (see par. 5).

4. - **TRANSITION PROBABILITIES (Virtual photons).**

We put:

$$ (e^-, \pi^0, g | S-1 \pi^-) = \delta(p_1 - p_2 - p_3 - p_4) F $$

(10)

and we have:

$$ F = \sum_i F_i $$

(11)

where the $F_i$s refer to the various diagrams of fig. 1. Calling $dP_1$ the corresponding differential transition probability, summed over the final spins, we have:

$$ dP_1 = \frac{1}{\mathcal{M}^2} \sum_{spin} |F|^2 \delta(p_1 - p_2 - p_3 - p_4) dp_1 dp_2 dp_3 $$

(12)

where to the order $g^2 e^2$ we have:

$$ |F|^2 = |F_0|^2 + 2 \Re \sum_i F_i F_i^* $$

(13)

4.1 - **Calculation of $F_0$**

$$ F_0 = \frac{\xi^2 V^2}{2(2\pi)^2 \sqrt{\rho_{10} \rho_{30}}} \left( \frac{1}{2} \left[ \rho_{10} + \rho_{13} \right] \rho_{12} + \rho_{13} \rho_{12} \right) $$

(14)
\[ \sum_{\rho_{\alpha\beta}} |F_\alpha|^2 = \frac{\hat{g}^2}{(2\pi)^4 \rho_1 \rho_2 \rho_3 \rho_4} \left\{ \frac{1}{\rho_1 + \rho_2 + \rho_3 + \rho_4} \right\} \]

\[ + \left[ \frac{m_1^2 - \rho_2}{\rho_2 + \rho_4} \right] \frac{1}{\rho_2 + \rho_4} \]  

\[(15)\]

4.2 - Calculation of \((F_1 + F_2 + F_3)\)

\[ F_1 + F_2 + F_3 = -\frac{\hat{g}^2}{(2\pi)^4} \frac{1}{2\rho_1 \rho_3} \bar{u}_2^+ \int d\kappa [I^0(\rho_1 - \kappa)(\rho_1 + \kappa)]^{1/2} \left[ \frac{m_2^2 - \rho_2}{\rho_2 + \rho_4} \right] \times \]

\[ + (4m_2^2 - \rho_2^2 + 2\kappa)(\rho_1^2 + 2\rho_1 \kappa)(\rho_1 - \kappa)(\rho_1 + \kappa) \right] \times \frac{1}{\rho_2 + \rho_4} \]  

\[ + \hat{\rho}_2 + \hat{\rho}_2 \]  

\[ \kappa \left[ \kappa^2 - 2(\rho_2 \kappa) \right]^{-1} \]  

\[(16)\]

This integral shows ultraviolet and infrared divergences; consequently it has been evaluated introducing a Feynmann cutoff \((\kappa^2 / \kappa - \kappa^2)\) and a fictitious photon mass \(\kappa\). The same will be made, where necessary, for the other diagrams.

By standard methods one obtains:

\[ F_1 + F_2 + F_3 = -\frac{\hat{g}^2}{(2\pi)^4} \frac{1}{2\rho_1 \rho_3} \bar{u}_2^+ \left[ I_1 \hat{\rho}_1 + I_2 \hat{\rho}_2 + m_2 \right] \left[ \hat{\rho}_1 + \hat{\rho}_2 \right] \]

\[ - I_2 \left[ \hat{\rho}_1 \hat{\rho}_2 + m_2 \hat{\rho}_2 + \hat{\rho}_2 \hat{\rho}_2 - \hat{\rho}_2 \hat{\rho}_2 \right] - 2 \left[ I_1 \hat{\rho}_2 \hat{\rho}_2 - m_2 \hat{\rho}_2 \right] I_2 \]

\[ - 2 \left[ I_2 \hat{\rho}_2 \hat{\rho}_2 + m_2 \hat{\rho}_2 \right] + I_3 \left[ \hat{\rho}_1 \hat{\rho}_1 + 2m_2 \right] + 2 I_4 \right\} \left[ 1 + i \kappa \right] \bar{u}_2^+ \]

\[(17)\]

where

\[ I_1 = \frac{i \eta^2}{2 J_1} \]

\[ I_2 = \frac{i \eta^2}{2 J_1} \sum_\alpha \sqrt{\alpha} \hat{\alpha} \]

\[ I_3 = \frac{i \eta^2}{2 J_2} \sum_\alpha \sqrt{\alpha} \hat{\alpha} \]

\[ I_4 = \frac{i \eta^2}{2 J_3} \sum_\alpha \sqrt{\alpha} \hat{\alpha} \hat{\alpha} \]

\[(18)\]

\( J_1, J_2, J_3 \) and \( J_3 \) have been defined and evaluated from Behrend, Finkelstein and Sirlin(3) (formulae (7a) and following).
Substituting the expressions for the $j$'s, we obtain:

\[ \frac{\sigma}{(2\pi)^4} \frac{1}{2} \frac{\ell n^2}{\ell^2} \frac{1}{2} \frac{\ell n^2}{\ell^2} \left( A m_2^2 + 2 B A^2 \right) \left( 1 + i \frac{1}{2} \right) \nu_4^* \]  

where:

\[ A = -4 I_{21} (p_1 p_2) - 2 I_{22} (m_i^2 - (p_1 p_2)) + 2 I_{23} \left[ 4 (p_1 p_2) m_i^2 - 2 m_i^2 \right] + 3 I_{12} - 2 I_{42} \]

\[ B = 4 I_{21} (p_1 p_2) - 2 I_{22} \left[ 3 (p_1 p_2) m_i^2 \right] - I_{12} \left[ 4 m_i^2 + 4 (p_1 p_2) \right] + 2 I_{23} - I_{43} \]

$I_{21}, I_{22}, I_{41}$ and $I_{42}$ are defined by the positions:

\[ \hat{I}_2 = I_{21} \hat{p}_1 + I_{22} \hat{p}_2 \]

\[ \hat{I}_4 = I_{41} \hat{p}_1 + I_{42} \hat{p}_2 \]

Finally the contribution of $(F_1 + F_5 + F_6)$ to the transition probability is:

\[ \frac{S_{\ell}}{2} \left[ (F_1 + F_5 + F_6) F_6^* \right] = - \frac{\epsilon^2}{(2\pi)^4} \frac{\alpha}{\pi} \left[ \delta^3 (p_1 p_4) (p_1 p_2) + 2 (A - \delta) m_i^2 (p_1 p_2) - (A m_i^2 + 4 B m_i^2) (p_2 p_4) \right] \]

and the fractional correction $\delta^{(1,5,6)}$ is:

\[ \delta^{(1,5,6)} = \frac{S_{\ell}}{2} \left[ (F_1 + F_5 + F_6) F_6^* \right] = \frac{S_{\ell}}{2} \left( F_6^2 \right) \]

\[ = \frac{\epsilon^2}{(2\pi)^4} \frac{\alpha}{\pi} \left[ \delta^3 (p_1 p_4) (p_1 p_2) + 2 (A - \delta) m_i^2 (p_1 p_2) - (A m_i^2 + 4 B m_i^2) (p_2 p_4) \right] \]

4.3 Calculation of $F_2, F_3$ and $F_4$:

These diagrams describe self-energy effects; after mass and wave function renormalisation the diagram $F_4$ does not give any contribution, while the diagrams $F_2$ and $F_3$ give the following fractional corrections to the transition
probability: \( \delta^{(1)} = -\frac{\Delta}{\pi} \left( \frac{1}{2} \ln \frac{\lambda}{m^2} + \ln \frac{\lambda m}{m^2} + \frac{2}{3} \right) \)
\( \delta^{(2)} = -\frac{\Delta}{\pi} \left(- \ln \frac{\lambda}{m_1} + \frac{\lambda m}{m_1} + 1 \right) \)  \(\text{(24)}\)

5. TRANSITION PROBABILITIES (real photons).

As is well known, when treating approximations to the order \( g^2 e^2 \), one must consider the corrections due to real photons of energy inferior to a maximum value \( \Xi \) depending upon the experimental resolution, as well as the corrections due to virtual photons. In our case we have assumed \( \Xi \ll m_2 \). We put:

\[
\left( \xi, \gamma, \chi, \eta \right) \left( \xi, \eta \right) = \delta \left( p_1 - p_2 - p_3 - p_4 - n \right) G
\]  \(\text{(25)}\)

Where \( K \) is the momentum of the emitted photon, and \( G = F_7 + F_8 + F_9 \) (see fig. 2).

Calling \( dP_2 \) the corresponding differential transition probability, summed over the final spins and polarizations, and integrated over \( K \) with \( |K| \leq \Xi \), one obtains:

\[
dP_2 = \frac{1}{\Xi^2} \sum_{\text{spins}} \sum_{\text{polarizations}} \int \frac{dK}{|K| \leq \Xi} |\langle \xi, \eta | \delta \left( p_1 - p_2 - p_3 - p_4 - n \right) \rangle|^2 dP_2 \, dp_2 \, dp_3 \, dp_4
\]  \(\text{(26)}\)

Because \( \Xi \ll m_2 \), we can neglect the diagram \( F_9 \), which gives corrections proportional to \( \Xi \) and \( \Xi^2 \); for the correction relative to \( (F_7 + F_8) \) one obtains by standard methods:

\[
\text{d}P_2 = \delta^{(7,8)} \left( \frac{\Xi^2}{\Xi^2} \sum_{\text{spins}} \sum_{\text{polarizations}} \delta \left( p_1 - p_2 - p_3 - p_4 \right) \right) dP_2 \, dp_2 \, dp_3 \, dp_4
\]  \(\text{(27)}\)

where \( \delta^{(7,8)} \), that is consequently the fractional correction due to the bremsstrahlung, is given by:

\[
\delta^{(7,8)} = -\frac{\Delta}{\pi} \left[ 2 \ln \frac{2e}{\lambda m} \left( 1 - \cos \omega \theta \right) - \omega \theta \right] +
\left( p_1 \rho_1 \right) \int dz \frac{1}{2} \ln \frac{1 - v_2}{1 - v_2}
\]

\(\text{(28)}\)

where \( \rho_2 = \frac{1}{2} \left[ p_1 (1 - z) + p_2 (1 - z) \right] \) and \( v_2 = \frac{|p_2 |}{p_2 \omega} \).
6. **TOTAL CORRECTION AND APPROXIMATIONS.**

The total percentual correction is given by:

\[ \delta = \delta^{(\nu, \delta)} + \delta^{(\nu, \nu)} + \delta^{(\nu, \nu_2)} \]

(29)

where the \( \delta^{(\nu)} \)'s are given from (23), (24) and (28). As it must be \( \delta \) does not contain \( \lambda_m \).

In the very good approximation \( m_2 \ll m_1 \), \( \delta^{(\nu, \nu)} \) and \( \delta^{(\nu, \nu_2)} \) may be written as follows:

\[
\delta^{(\nu)} = -\frac{\alpha}{\pi} \left[ \theta \alpha h \theta \left( \frac{2}{\lambda_m} \ln \left( \frac{\lambda_m}{m_2} - \frac{1}{2} \right) - \ln \left( \frac{m_1}{m_2} \right) + \frac{5}{2} \ln \left( \frac{m_1}{\lambda_m} \right) - \frac{13}{8} \right] 
\]

(30)

\[
\delta^{(\nu, \nu_2)} = -\frac{\alpha}{\pi} \left[ 2 \ln \left( \frac{\lambda_m}{m_2} \right) \left( \theta \alpha h \theta \right) - \theta \alpha h \theta - 1 \right] 
\]

Then:

\[
\delta = -\frac{\alpha}{\pi} \left[ \theta \alpha h \theta - \frac{1}{2} \theta \alpha h \theta - \frac{5}{2} \ln \left( \frac{m_1}{m_2} \right) + 3 \ln \left( \frac{\lambda_m}{m_1} \right) - \frac{13}{8} \right] 
\]

(31)

\[
+ \ln \left( \frac{\lambda_m}{m_2} \right) + \ln \left( \frac{\lambda_m}{m_2} \right) \left( 1 - 2 \theta \alpha h \theta \right) \right] 
\]

7. **INTEGRAL TRANSITION PROBABILITY.**

We call \( P_0 \) and \( P \) the integral transition probabilities to the order \( g^2 e^2 \), and \( \delta_P = (P - P_0)/P_0 \) the corresponding fractional correction.

By definition:

\[
P_0 = \frac{1}{2 \pi} \int \sum \left[ \frac{\alpha h}{\pi} \delta \left( \frac{m_1}{m_2} \right) \right] \frac{d \rho_1}{d \rho_2} d \rho_3 d \rho_4
\]

(32)

\[
P = \frac{1}{2 \pi} \int \sum \left[ \frac{\alpha h}{\pi} \delta \left( \frac{m_1}{m_2} \right) \right] \frac{d \rho_1}{d \rho_2} d \rho_3 d \rho_4
\]

Performing the integrations considering also the pion's recoil, and neglecting only terms like \( m_2^2 \) in comparison with \( m_1^2 \), one obtains:

\[
P_0 = \frac{2 g^2 e^2}{\pi \lambda_m} \left[ \frac{\alpha h}{\pi} \delta \left( \frac{m_1}{m_2} \right) \right] \frac{d \rho_1}{d \rho_2} d \rho_3 d \rho_4
\]

(33)

\[
\delta_P = -\frac{\alpha}{\pi} \left[ \frac{2 \lambda_m}{\pi} \right] \left[ \frac{\alpha h}{\pi} \delta \left( \frac{m_1}{m_2} \right) \right] \frac{d \rho_1}{d \rho_2} d \rho_3 d \rho_4
\]

(34)
Where \( \Delta = \frac{m_2^2 - m_1^2}{2m} \),

Finally one obtains numerically:

\[
\rho_0 = 1.105 \, \text{sec}^{-1} \tag{35}
\]

\[
\delta \rho = 0.027 \pm 0.005 \, \text{ln} \frac{2E}{m_2} \pm 0.007 \, \text{ln} \frac{\lambda}{m_1} \tag{36}
\]

8. - DISCUSSION.

The value (35) has been obtained using the following numerical values:

\[
m_1 = (139,59 \pm 0,05) \, \text{MeV} \tag{4}
\]

\[
m_3 = (135,00 \pm 0,05) \, \text{MeV} \tag{4}
\]

\[
gm^2_{\text{proton}} = (1,204 \pm 0,001) \cdot 10^{-5} \tag{5}
\]

From the errors on \( m_1, m_3 \) and \( g \) follows an error of 1.6% on \( \rho_0 \), essentially determined by the incertitude on the masses; hence this error is about one half of the correction (36).

In (36) the cutoff for short wavelengths is not cancelled; however, the result depends only logarithmically on this cutoff.

Assuming for example \( \Sigma = 1/20 \, m_2 \) and \( \lambda = 10 \, \text{m} \), we find:

\[
\delta \rho \approx 0.032 \tag{38}
\]

Finally for the lifetime \( \tau \) we get:

\[
\tau = 0.904 \frac{(1 + 0.032) (1 \pm 0.016)}{(1 + 0.016)} \, \text{sec} = (0.933 \pm 0.015) \, \text{sec} \tag{39}
\]

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