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M. Puglisi, I. F. Quercia: ELECTRONS LOADING ON A RADIO FREQUENCY ACCELERATING CAVITY OF A 1 GeV SYNCHROTRON.
Electrons in a Synchrotron are accelerated by azimuthal electric fields, obtained by one or more radio frequency accelerating gaps, along the trajectory of electrons.

Such accelerating gaps are, generally speaking, part of radio frequency resonating structures, in which power is injected by suitable electronic devices. From the point of view of the radio frequency supply, the system may be considered as one in which radio frequency power is transformed in direct current power of the accelerated beam of electrons.

If the number of electrons in the beam is very low, the power picked up from the resonator by the beam is only a small fraction of the power lost in the resonator by resistive losses, then it may be neglected. When the number of electrons in the beam increases, the beam overloads the resonator and this overloading must be taken into account in the design of the Radio frequency accelerating system.

Let us suppose that electrons are already bunched, with some azimuthal distribution. With respect to the radio frequency electric field, the bunches of electrons are circulating with different phases, according to their azimuthal distribution. In general the bunches can be considered as a periodic series of current pulses, which can be represented by a Fourier expansion. The fundamental term of the series is of course an a.c. current having the same frequency of the

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(1) S. Ramo, J. Whinnery: FIELD AND WAVES IN MODERN RADIO - (J. Wiley and Sons, Inc. - 1953) - Ch.X, pag.442-444.

radio frequency system, but shifted in phase with respect to
the radio frequency voltage at the accelerating gap. This
fact means that the fundamental component of the electrons
current may be considered as the vectorial composition of
two components: one resistive in phase, the other reactive
and 90° out of phase.

The former increases the resistive losses of the resonator,
the latter drives the resonator out of resonance; both these
effects lower the value of the electric field at the accele-
rating gap of the resonator.

The problem arises of calculating the peak radio frequen-
cy voltage at the gap of a cavity driven by a radio frequency
amplifier, and loaded by a beam of electrons. In the follo-
wing we shall write the equation for the more general case,
and we shall give a numerical solution for the particular ca-
se of a low voltage, frequency modulated, cavity of the 1 GeV
Italian Electronsynchrotron.

§ 1 - We have to calculate the following vectorial functions
inside the resonating cavity:

\[ \vec{H} = \text{magnetic field} \]
\[ \vec{E} = \text{electric field} \]
\[ \vec{J} = \text{current density} \]

It can be shown that the solenoidal part of each of these
fields can be developed by a series of solenoidal orthogonal
functions, \( \vec{E}_n \), \( \vec{H}_n \), and \( \vec{J}_n \):(3)

\[ \vec{E} = \sum_n \int_V \vec{E} \times \vec{E}_n \, dv \]
\[ \vec{H} = \sum_n \int_V \vec{H} \times \vec{H}_n \, dv \]
\[ \vec{J} = \sum_n \int_V \vec{J} \times \vec{J}_n \, dv \]

(1)

where the integrations are extended to the volume of the ca-
vity.

(3) J.C.Slater: MICROWAVE ELECTRONICS - (D.Van Nostrand Co.,
Inc. - 1950 (1951 reprinted) -) Ch.IV - We shall follow
in this paper the general calculations and the notations
used by J.C.Slater in the book referred.
The cavity can be described by a volume defined by two kinds of surfaces:

I - Surfaces $S'$ having perfect dielectric properties.

For such surfaces the tangential component of $\vec{n}$ vanishes, so that:

$$\vec{n} \wedge \vec{E}_n = 0 \quad \text{on } S'$$

(2)

where $\vec{n}$ is the unity vector normal to $S'$.

II - Surfaces $S$ having non perfect conducting properties.

For such surfaces the tangential component of $\vec{E}_n$ does not vanish then we have:

$$\vec{n} \wedge \vec{E}_n = (1 + j) \sqrt{\frac{\omega_n \mu}{2 \sigma}} \vec{H}_n$$

(3)

where $\mu$ is the permeability of the medium; $\sigma$ is the surface conductivity of $S$; $\omega_n$ is the angular frequency of the fields, If we write the Maxwell equations for the fields and currents in the cavity, taking into account the series expansion (1) and the boundary conditions (2) and (3), we obtain the following equation:

$$\frac{\omega_n}{Q_w} \frac{x_n}{x_n} + \omega_0 x_n = -j \frac{\omega_0}{\epsilon} \int_V \vec{J} \times \vec{E}_n \, dv$$

(4)

where:

$$x_n \equiv \int_V \vec{E} \times \vec{E}_n \, dv;$$

$$\omega_0 \equiv \text{resonance angular frequency of the free cavity;}$$

$$Q_w \equiv \text{quality factor of the cavity taking into account only the wall losses;}$$

$$\epsilon \equiv \text{dielectric constant of the medium.}$$

The solutions of this equation $x_n$ are obviously the coefficients of the series expansion of the electric field $\vec{E}$ according to the first of (1).

It can be easily shown (4) that the coefficients of the series expansion of the field $\vec{H}$, are related to the $x_n$ by the following relation:

$$\gamma_n \equiv \int_V \vec{H} \times \vec{H}_n \, dv = j \sqrt{\frac{\epsilon}{\mu}} x_n$$

(4) See reference of note (3), Ch. IV.
§ 2 - The general solution of equation (4) is of course the sum of two parts: a transient one due to the solution of the corresponding homogeneous equation; a steady one which takes into account the forced term due to the currents injected in the cavity. Of course we are interested in the latter solution.

We shall suppose that the forced term is a time sinusoidal function with an angular frequency \( \omega \). In this case the steady solution of the equation (4) is of the type:

\[
x_n = A e^{j \omega t}
\]

By substituting (5) in (4) we have:

\[
\begin{align*}
\left\{ j \left( \frac{\omega^2 - \omega_n^2}{\omega_n^2} \right) - \frac{\omega}{\omega_n} (1 + j) \frac{1}{Q_w} \right\} x_n &= \\
&= \frac{1}{\varepsilon \omega_0} \int_V \vec{J} \times \vec{E}_n \, dv
\end{align*}
\]

Let us now remember that:

\[
x_n = \int_V \vec{E} \times \vec{E}_n \, dv
\]

Dividing both sides of (6) by \( \frac{\omega}{\omega_n} x_n \) we obtain the following equation:

\[
j \left[ \left( \frac{\omega^2 - \omega_n^2}{\omega_n^2} \right) - \frac{1}{Q_w} \right] - \frac{1}{Q_w} = \frac{1}{\varepsilon \omega} \int_V \vec{J} \times \vec{E}_n \, dv
\]

We shall base on (7) the calculations of the electronic loading.

§ 3 - Let us consider a cylindrical cavity as sketched in fig. 1. In such a cavity we may assume that at any time the electric field \( \vec{E} \) is mostly confined between the two planes of the 'condenser part' of the cavity.\(^{(5)}\)

The magnetic field can be assumed as filling, with circular lines of force, the thoroidal part of the cavity.

The value of the radio frequency voltage between the condenser plates at a distance \( d \) is therefore related to the

\(^{(5)}\) See reference of note (1), Ch.X, pag. 436-440.
value of the electric field by the simple formula:

\[ V(t) = \left| \vec{E}(t) \right| d \]

This is also the voltage by which the electrons crossing the cavity in the axial direction at a time \( t \) are accelerated.

Moreover we suppose that the cavity is driven in the fundamental mode (referred to as the "n" mode), so that:

\[ \vec{E} = \vec{E}_n, \]

all other modes giving a negligible contribution to the series expansion (1). In this case the integral at the denominator of the right side of (7) becomes:

\[ \int_{V} \vec{E} \times \vec{E}_n \, dv = (\frac{V}{d})^2 S \, d \]

(9)

\( S \) being the surface of the condenser part of the cavity.

We have now to take into account the current densities inside the cavity. In this paper we shall suppose that the power is supplied to the cavity by a direct coupling between the cavity and the anode circuit of a power triode runned as grounded grid amplifier shown schematically in fig.2.

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(6) L.B. Arguimbau: VACUUM TUBES CIRCUITS (J. Wiley and Sons, Inc. - 1948) - Ch. VI, pag 286-287.

This is just the case of the frequency modulated Radio Frequency System of the 1 GeV Italian Electronsynchrotron. In such a case two different electronic currents are circulating in the cavity:

a) the current $\overrightarrow{I}_v$ due to the electrons inside the triode;
b) the current $\overrightarrow{I}_e$ due to the electrons of the accelerated beam.

Both currents add to build up the total current density according to the formula:

$$\overrightarrow{J} = \overrightarrow{J}_e + \overrightarrow{J}_v = \frac{\overrightarrow{I}_e}{a} + \frac{\overrightarrow{I}_v}{b}$$

where $a$ and $b$ are the cross sections of the two currents. As a consequence, the integral at the numerator of the right side of (7) can be written:

$$\int_{V} \overrightarrow{J} \times \overrightarrow{E}_n \, dv = \int_{V} \overrightarrow{J}_e \times \overrightarrow{E}_n \, dv + \int_{V} \overrightarrow{J}_v \times \overrightarrow{E}_n \, dv$$

the integrals being extended to the volumes in which the two currents are confined.

Taking into account position (8) we get:

$$\begin{align*}
\int_{V} \overrightarrow{J}_e \times \overrightarrow{E}_n \, dv &= \frac{\overrightarrow{I}_e}{a} \int_{V} \overrightarrow{E} \, dv = I_e V \\
\int_{V} \overrightarrow{J}_v \times \overrightarrow{E}_n \, dv &= \frac{\overrightarrow{I}_v}{b} \int_{V} \overrightarrow{E} \, dv = I_v V
\end{align*}$$

where $I_e$ and $I_v$ are the scalar values of the beam and the triode currents.

Injecting (9) and (10) in the (7) we obtain the following equation:

$$j \left[ \left( \frac{\omega^2 - \omega_0^2}{\omega^2} \right) - \frac{1}{Q_w} \right] - \frac{1}{Q_w} = \frac{1}{\omega C} \left[ \frac{I_e}{V} - \frac{I_v}{V} \right]$$

where: $C = S/d$ is the capacity between the plates of the condenser part of the cavity, taking into account also the anode-grid capacity of the triode.

§ 4 - The two quantities in brackets at right side of (11) can be considered as the admittances of the system:

$$Y_e = \frac{I_e}{V} \quad \text{being the admittance of the beam load.}$$
\[ Y_V = \frac{I_V}{V} \] being the admittance of the triode.

Both the beam current and the anode current of the triode, supposed to be driven as Class C amplifier, are periodic short pulses of current appearing at some phase with respect to the radio frequency voltage \( V(t) \), as shown in fig.3.-

Supposing the \( V(t) \) to be a sinusoidal function of the time with angular frequency \( \omega \), both the periodic currents wave forms can be expanded in Fourier series, giving rise to a fundamental component of angular frequency \( \omega \), and to other higher harmonic terms.\(^{(8)}\) Taking into account the fundamental component only, we can represent the currents as vectors rotating with the angular frequency \( \omega \). In other words, the two admittances can be represented by complex numbers as follows:

\[
\begin{align*}
Y_o &= \frac{I_o}{V} = g_o + j b_o \\
Y_v &= \frac{I_v}{V} = g_v + j b_v
\end{align*}
\] (12)

Substituting (12) in (11), and separating the real and imaginary parts, we obtain the following two equations:

\[
\frac{\omega_c}{Q_w} = -(g_o + g_v) \quad (13)
\]

\[
\omega_c \left[ \left( \frac{\omega_o}{\omega} - \frac{\omega_v}{\omega_o} \right) - \frac{1}{Q_w} \right] = b_o + b_v \quad (14)
\]

§ 5 - As a first approach to the solution of the problem let we assume that the cavity is driven at a fixed angular frequency \( \omega' \), which is also the angular frequency of revolution

\(^{(7)}\) See the first reference of note (6), Ch.VI, pag.267-277.

\(^{(8)}\) J.P. Heyboer, P. Zijlstra: TRASMITTING VALVES (Philips' technical Library - 1951) - Ch.III, pag. 26-60.
of electrons of the beam.\(^{(9)}\) Moreover we shall assume that \(\omega'\)

is just the frequency of resonance of the cavity, taking into account the losses on the walls. In this case (14) reduces to:

\[
b_e + b_v = 0 \tag{14a}
\]

because the term:

\[
\omega' C \left( \frac{\omega_0}{\omega'} - \frac{\omega'}{\omega_0} \right) - \frac{1}{Q_w}
\]

in (14) vanishes.

In the following we shall assume as independent variable the peak value \(V\) of the radio frequency voltage at the gap of the cavity. The question we have to solve is:

"If \(V\) is the value of the voltage at the gap of the cavity without any beam load, what will be the value \(V'\) of the voltage when synchronous bunches of electrons are accelerated by the cavity?"\(^{-}\)

Let us consider the situation of the currents and voltages in the cavity, represented in fig. 4 as electrical vectors rotating at the angular frequency \(\omega'\).

![Diagram](image-url)

**Fig. 4** - Vectorial relationship between currents and voltages in the cavity.

According to (14a) the two components of the currents at \(90^\circ\) out of phase with the voltage have an equal amplitude and opposite sign:

\[
I_{V_1} = - I_{e_1}
\]

or:

\[
|I_v| \sin \Psi = - |I_e| \cos \Psi
\]

\[(15)\]

\((9)\) Or an upper harmonic of the revolution frequency. In our case that is the 4th harmonic.
According to (13), between the two components of the currents in phase with \( V' \) the following relation must be satisfied:

\[
\frac{\omega'_C}{q_w} |V'| = - \left( I_{V'} + I_{e_v} \right)
\]

or:

\[
\frac{\omega'_C}{q_w} |V'| = - \left\{ |I_{V'}| \cos \Psi + |I_e| \sin \Psi \right\}
\]  \hspace{1cm} (16)

Let us now introduce the following approximation: 'The wave form of the current of the triode is not modified by the reactive load (10) introduced by the beam'.

Under this assumption the amplitude of the vector \( I_{V'} \) is just the same as in the case of no reactive load:

\[
|I_{V'}| = I_v^0
\]

where \( I_v^0 \) is the current in the triode with a radio frequency voltage \( V' \). Putting this value in (16) instead of \( |I_{V'}| \) and dividing both sides by \( V' \), we get:

\[
\frac{\omega'_C}{q_w} = - g_v^0 \cos \Psi - g_e
\]  \hspace{1cm} (17)

where \( g_v^0 \) is the conductance of the triode without electron beam loading and \( g_e \) is the conductance of the electron beam; of course both of them are functions of the voltage \( V' \).

Still we have to determine the value of the electrical phase \( \Psi \) of the triode current, which is connected to the phases of the beam current \( \Psi_e \) by the (15).

Therefore we have:

\[
\cos \Psi = - \left[ 1 - \sin^2 \Psi \right]^{1/2} = - \left[ 1 - \left( \frac{|I_e| \cos \Psi_e}{|I_{V'}|} \right)^2 \right]^{1/2}
\]

Supposing that \( \sin \Psi << 1 \), we shall use the following approximate value:

\[
- \cos \Psi \approx 1 - \frac{1}{2} \left( \frac{|I_e| \cos \Psi_e}{I_v^0} \right)^2 \]  \hspace{1cm} (18)

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(10) This assumption is true only for pentodes, and quite good for the high \( \mu \) power television tubes. See the reference of note (8), Ch.IV.
§ 6 - For the numerical computation we shall use the following procedure:
- First we fix the working conditions of the triode employed:
  Vacuum tube: Siemens type RS 1001
  Anode supply voltage: $E_{bb} = 8.000 \text{ V}$
  Fixed grid bias: $E_g = -200 \text{ V}$
  Peak radio frequency vol. between grid and cathode: $v_g = 350 \text{ V}$
  Amplification coefficient: $\mu = 65$
  Plate resistance: $v_p = 1.600 \text{ ohm}$

From a graphical analysis (11) of the characteristic curves of this tube we have calculated the values of $g_v(V)$, referred in Table I°.

| TABLE I° - Values of the tube conductance v/s R.F. swing |
|------------------|-------|-------|-------|-------|-------|-------|
| $V$(volt)        | 2000  | 3000  | 4000  | 5000  | 6000  | 7000  |
| $g_v$ (mho)      | $10^3$ | 4.35  | 2.50  | 1.60  | 1.15  | 0.815 | 0.445 |

![Graph](image)

*Fig. 5 - $g_v$: conductance of the system v/s voltage $V$ at the gap.*

If we plot data as in the solid curve of fig.5, with the measured value of the R.F. voltage $V$ at the gap of the cavity without beam loading, we are able to find the value of:

(11) See the second reference of note (6), Ch.X, pag.372-378.
\[ \frac{\omega'_c}{Q_w} = \varepsilon^c_y, \]

which is a mechanical characteristic of the cavity employed, and therefore is not affected by any loading of the system.

We can now plot on the same diagram the values of the function of \( V \):

\[ -\varepsilon^c_y \cos \gamma - \varepsilon_y, \]

they are shown by the dotted curve in fig. 5. The abscissa of the point where this curve crosses the value \( \omega'_c/Q_w \), is just the voltage \( V' \) at which the cavity loaded by the electron beam works.

To draw the latter curve according to (17) and (18) we needed the values of:

\[ \gamma_s = \text{phase angle between the electrons current of the beam and the radio frequency voltage}. \]

\[ |I_e| = \text{the absolute value of the beam current, given by the number of electrons in the beam, and by the phase distribution of electrons in the bunches.} \]

Let us assume that the bunches of electrons are distributed around a phase \( \gamma_s \) which corresponds to the phase of the 'synchronous particle' accelerated by the synchrotron. The phase angle \( \gamma_s \) is defined by:

\[ \sin \gamma_s = \frac{u}{V} \]

where \( u \) is the minimum value of the R.F. voltage needed to accelerate electrons.\(^{(12)}\) In the following we shall assume

\[ u = \text{cost.} = 2,000 \text{ Volt}. \]

Moreover we have calculated the absolute value of the electrons current \( |I_e| \), for two different number of accelerated electrons; to say:

\[ N = 10^{11}; \quad N = 10^{12} \text{ electrons}. \]

For the distribution of the bunches of electrons around

\(^{(12)}\) E. Persico: TEORIA DELLA CATTURA RAPIDA IN UN SINCHROTRONE INIESSATO AD ALTA ENERGIA (relaz. n° T4 della Sezione Acceleratore dell'I.N.F.N. - 1953).
the synchronous phase $\psi_s$ we have assumed three different shapes:

a) All the electrons travel at phase $\psi_s$. The bunches have the shape of a delta function.

b) The electrons are uniformly distributed between the phase $\psi_s - 45^\circ$ and $\psi_s + 45^\circ$. The bunches have a rectangular shape around $\psi_s$.

c) The electrons are uniformly distributed between the fixed phases $110^\circ$ and $200^\circ$.

In fig. 6 the assumed distributions are indicated.

In figs. 7, 8, 9 the curves for the two values of N and for three cases a), b), c) are plotted. It is easy to see that our assumption about the shape of the bunches does not affect too much the results.

Fig. 6 - The assumed shapes of the beam current and corresponding phases.

For instance, with the unloaded cavity working at $V = 6$ kV, the value $V'$ of the voltage, reached when the load is of $10^{12}$ electrons, is in any case: $V' \geq 4.5$ kV. Moreover it is shown by the curves that in our case the system is quite insensitive to a load of $10^{11}$ electrons.

§ 7 - As a second approach to the solution of our problem let us consider the fact that during the first part of the accelerating cycle the cavity is driven at a frequency which
Fig. 7 - Case a)

Diagram for the calculation of the value $V'$ of the peak radio frequency voltage when the cavity is loaded by bunches of

- $N = 10^{11}$ electrons; curve ---
- $N = 10^{12}$ electrons; curve ---

The solid curve refers to the case of the unloaded cavity.
Fig. 8 - Case b)

Diagram for the calculation of the value \( V' \) of the peak radio frequency voltage when the cavity is loaded by bunches of

\[
N = 10^{11} \text{ electrons}; \quad \text{curve} - - - -
\]

\[
N = 10^{12} \text{ electrons}; \quad \text{curve} - - - -
\]

The solid curve refers to the case of the unloaded cavity.
Fig. 9 - Case c)

Diagram for the calculation of the value $V'$ of the peak radio frequency voltage when the cavity is loaded by bunches of

$N = 10^{11}$ electrons; curve ---

$N = 10^{12}$ electrons; curve ---

The solid curve refers to the case of the unloaded cavity.
is about 2\% lower than the resonating one.\(^{13}\) If we consider equation (14) of §4 we see that in this case the left side does not vanish.

Working at an angular frequency \(\omega\), which is \(\Delta \omega\) lower than the resonating frequency \(\omega'\), eq. (14) becomes:

\[
2 C \Delta \omega = b_e + b_v
\]

or by multiplying both sides by \(V'\):

\[
2 C \Delta \omega V' = I_{e_1} + I_{v_1}
\]

In other words the tube reactive current \(I_{v_1}\) must balance both the beam reactive current \(I_{e_1}\), and the detuned load reactive current \(2 C \Delta \omega \, V'\).

In our case the situation is represented in fig. 10 where the currents are indicated as vectors rotating at the frequency \(\omega\).

![Fig. 10 - Vectorial relationship between currents and voltages in the cavity.](image)

It is easy to show that also in this case it is possible to write an equation like the (17):

\[
\frac{\omega C}{Q_w} = - g_v^0 \cos \psi - g_e i
\]

(17a)

but the meaning of \(\cos \psi\) is now:

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\(^{13}\) See the reference of note (8), Ch. III, pag. 63-72.-
\[- \cos \varphi \approx 1 - \frac{1}{2} \left[ \frac{\frac{I_0}{V' (1 + 2 \frac{C}{V} \text{d} \omega)}}{\frac{I_0}{V'}} \right]^2 \]  \hspace{1cm} (18a)

By the (17a) and (18a) we calculated numerically the effect of detuning, in the following case:

- Detuning of the cavity $\Delta \omega = 2\%$ with $\omega' = 43 \times 10^6$ c/s,
- Capacity of the cavity and of the triode $C = 60$ pF.
- Shape of electron bunches as in the case a) of § 6.
- All other conditions as in § 6.

We obtained the curves of fig. 11. In that case it is shown that if the cavity works at 6 kV driven at the resonating frequency and without any beam load, with a beam loading of $10^{12}$ electrons the Radio Frequency voltage is lowered to about 3.7 kV, and with a loading of $10^{11}$ electrons it is lowered to about 5.2 kV, by the combined effect of loading and detuning.-
Fig. 11 - Effect of electrons loading and 2% detuning.

Diagram for the calculation of the value $V'$ of the peak radio frequency voltage when the cavity is loaded by bunches of

- $N = 10^{11}$ electrons; curve ---
- $N = 10^{12}$ electrons; curve -----

and is detuned by 2%. The solid curve refers to the case of tuned and unloaded cavity.