THE CP-VIOLATING ASYMMETRY in $K_L \rightarrow \pi^+\pi^- e^+e^-$ *

G. Ecker$^1$, and H. Pichl$^{1,2}$

$^1$ Institut für Theoretische Physik, Universität Wien
Boltzmanngasse 5, A-1090 Vienna, Austria
$^2$ INFN, Laboratori Nazionali di Frascati, P.O. Box 13, I-00044 Frascati, Italy

Abstract

We update the theoretical analysis of the CP-violating asymmetry in the decay $K_L \rightarrow \pi^+\pi^- e^+e^-$, relying on chiral perturbation theory and on the most recent phenomenological information. With the experimentally determined magnetic amplitude and branching ratio as input, the asymmetry can be calculated with good accuracy. The theoretical interpretation of the sign of the asymmetry is discussed.

* Work supported in part by TMR, EC-Contract No. ERBFMRX-CT980169 (EURODAΦNE)
1. A large CP-violating asymmetry in the decay $K_L \to \pi^+\pi^- e^+e^-$ was originally predicted by Sehgal and Wanninger [1]. The effect is almost entirely due to indirect CP violation in $K^0 - \bar{K}^0$ mixing. The predicted asymmetry in the angle between the $\pi^+\pi^−$ and the $e^+e^−$ planes has been confirmed experimentally [2, 3]. The purpose of this note is to update the theoretical analysis of the asymmetry [1, 4, 5, 6, 7, 8] using the most recent phenomenological input available and employing the methods of chiral perturbation theory. The latter will mainly be invoked to estimate higher-order corrections but also to interpret the observed sign of the asymmetry.

2. The amplitude for the decay $K_L(p) \to \pi^+(p_1)\pi^−(p_2)e^+(k_+)e^−(k_−)$ is expressed in terms of three invariant form factors $E_1, E_2$ (electric) and $M$ (magnetic):

$$A(K_L \to \pi^+\pi^- e^+e^-) = e q^2 \frac{\pi(k_-)\gamma^\mu v(k_+) V_\mu}{q^2}$$

$$V_\mu = iE_1 p_1\mu + iE_2 p_2\mu + M \varepsilon_{\mu\nu\rho\sigma} p_1^\nu p_2^\rho q^\sigma$$

(1)

We use the set of variables originally introduced by Cabibbo and Maksymovicz [9] for $K_{e4}$ decays. With this choice, the form factors $E_1, E_2, M$ depend on $s_{\pi}$ (invariant mass squared of the two pions), $q^2$ (invariant mass squared of the leptons) and $\theta_{\pi}$ (angle of the $\pi^+$ in the $\pi^+\pi^−$ center-of-mass system with respect to the dipion line of flight in the kaon rest frame). The two remaining Dalitz variables are $\theta_l$, the corresponding angle for the positron, and $\Phi$, the angle between the dipion and dilepton planes in the kaon rest frame.

After integration over four of these variables, the differential decay rate with respect to $\Phi$ assumes the general form$^1$ [10, 11]

$$\frac{d\Gamma}{d\Phi} = I_1 + I_2 \cos \Phi + I_3 \sin \Phi + I_4 \cos 2\Phi + I_5 \sin 2\Phi .$$

(2)

Under a CP transformation

$$\cos \Phi \longrightarrow \cos \Phi$$

$$\sin \Phi \longrightarrow - \sin \Phi$$

(3)

so that non-zero $I_3, I_5$ signify CP violation. It turns out that $I_3$ is very small in the standard model, being sensitive to direct CP violation only [4]. The quantity of interest here is $I_5$ that is almost exclusively due to indirect CP violation [4]. A convenient measure of this term is the asymmetry

$$A_{CP} = \langle \text{sgn}(\sin \Phi \cos \Phi) \rangle = \frac{4I_5}{\Gamma(K_L \to \pi^+\pi^- e^+e^-)} .$$

(4)

$^1$We use the conventions of Ref. [11], in particular $\varepsilon_{0123} = 1$ and the metric $(+−−−)$. 

2
In terms of the invariant form factors defined in (1), the asymmetry is given by [10, 11]

\[ \mathcal{A}_{CP} = \frac{e^2}{128\pi^8 M_K \Gamma(K_L \rightarrow \pi^+\pi^- e^+e^-)} \int d_{LIPS} \varepsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu k_+^\rho k_-^\sigma (p_1 - p_2) \cdot (k_+ - k_-) \frac{\text{sgn}(\Phi\cos\Phi)\text{Im}[(E_1 - E_2)M^*]}{q^4} \]  

where \( d_{LIPS} \) is the invariant phase space integration measure. We will also consider the asymmetry for certain cuts (in \( q^2 \)). In this case, both the phase space integration in the numerator of (5) and \( \Gamma(K_L \rightarrow \pi^+\pi^- e^+e^-) \) in the denominator must be modified accordingly.

When CP is conserved we have \( E_2(p_1, p_2, q) = E_1(p_2, p_1, q) \). Therefore, only CP-violating contributions to \( E_1, E_2 \) matter for the numerator in (5).

The most recent published result for the asymmetry comes from the KTeV-Collaboration [2]:

\[ \mathcal{A}_{CP} = 13.6 \pm 2.5 \text{ (stat)} \pm 1.2 \text{ (syst)} \% . \]  

The preliminary result from NA48 [3] is fully compatible with this value.

Before delving into the theoretical analysis, we comment on the observed sign of the asymmetry (6). With our conventions [11] and with the assumption that \( \text{Im}[(E_1 - E_2)M^*] \) has a unique sign all over phase space, the theoretical expression (5) implies

\[ \text{sgn} \mathcal{A}_{CP} = \text{sgn} \text{Im}[(E_1 - E_2)M^*] . \]  

As we shall show in the sequel, \( \text{Im}[(E_1 - E_2)M^*] \) does in fact have a unique sign and must therefore be positive according to Eqs. (6) and (7).

3. At lowest order in the low-energy expansion, the electric amplitudes correspond to pure Bremsstrahlung:

\[ E_1 = \frac{-2ieA_0^{\text{tree}}(K_L \rightarrow \pi^+\pi^-)}{q^2 + 2p_1 \cdot q} = \frac{-2\sqrt{2}e\eta_{+-}A_0^{\text{tree}}}{q^2 + 2p_1 \cdot q} \]

\[ E_2 = -E_1(p_1 \rightarrow p_2) \].  

From the dominant \( \Delta I = 1/2 \) weak Hamiltonian, the \( I = 0 \) amplitude \( A_0^{\text{tree}} \) for \( K_S \rightarrow \pi\pi \) decays is given at tree level by

\[ A_0^{\text{tree}} = \sqrt{2}G_S F(M_K^2 - M_\pi^2) , \]  

with \( G_S = 9.1 \cdot 10^{-6} \text{ GeV}^{-2} \) and \( F = F_\pi = 92.4 \text{ MeV} \) at tree level. The current values for the CP-violating quantity \( \eta_{+-} \) are [12]

\[ |\eta_{+-}| = (2.276 \pm 0.017) \cdot 10^{-3} \]

\[ \Phi_{+-} = \text{arg} \eta_{+-} = (43.3 \pm 0.5)^\circ . \]  

\[ (10) \]
The magnetic tree-level amplitude starts at $O(p^4)$ where it is completely given in terms of low-energy constants of the nonleptonic weak Lagrangian of $O(p^4)$ [13, 14]. In the notation of Ref. [15], the leading-order magnetic amplitude is written as

$$ M = -\frac{eG_8}{2\pi^2 F}(a_2 + 2a_4) \quad (11) $$

with dimensionless coefficients $a_2, a_4$ of order one. As shown in Ref. [15], the chiral anomaly induces positive contributions to these coefficients. If the anomaly-induced contributions were the dominant ones, we would expect $a_2 + 2a_4$ to be positive.

It is easy to check for the tree-level amplitude that the CP-violating quantity $\text{Im}[(E_1 - E_2)M^*]$ has a definite sign that equals the sign of $a_2 + 2a_4$. From (6) and (7) we would therefore conclude that $a_2 + 2a_4$ is positive supporting the hypothesis that the contributions from the chiral anomaly dominate the magnetic amplitude (11).

However, this cannot be the whole story. Assuming a constant magnetic amplitude, one can extract this amplitude from the branching ratio for $K_L \to \pi^+\pi^-\gamma$. Following Sehgal and Wanninger [1] (the same procedure is used in Refs. [4, 5]), one obtains in their notation

$$ M = -\frac{0.76|f_s|}{M_K^4} , \quad (12) $$

with $|f_s| = 3.9 \cdot 10^{-7}$ GeV denoting the absolute value of the $K_S \to \pi^+\pi^-$ amplitude. The resulting asymmetry at tree level is

$$ A_{CP}^{\text{tree}} = 7.7\% , \quad (13) $$

much smaller than the experimental result (6). Clearly, higher-order corrections are necessary to understand the asymmetry.

4. Chiral corrections to the tree-level amplitudes were already considered in Ref. [6]. Those authors included one-loop corrections to the absorptive parts of both electric and magnetic form factors. Whereas the corrections are small for the magnetic amplitude they are sizable for the electric form factors, mainly due to the large final state interactions of two pions with $I = 0$ in an $S$-wave, as first pointed out by Sehgal and Wanninger [1]. To some extent, Elwood et al. [6] compensated the neglect of dispersive contributions by using the tree-level value of the weak coupling constant $G_8$ also in the loop amplitude. On the other hand, this is partly double counting because $G_8$ should be reduced by about 30% [16, 17] when effects of $O(p^4)$ are included in the amplitude for $K_0 \to \pi^+\pi^-$. We propose here a different procedure for the CP-violating electric amplitude. We first decompose the electric form factors into a Bremsstrahlung part and a direct-emission piece:

$$ E_i = E_i^B + E_i^{\text{DE}} \quad (i = 1, 2) . \quad (14) $$

In the first part, we use the phenomenological $K_S \to \pi^+\pi^-$ amplitude,

$$ E_1^B = -\frac{2\sqrt{2}e\eta_{+-}}{q^2 + 2p_1 \cdot q} \left[ A_0 e^{\imath \delta_0} + \frac{1}{\sqrt{2}} A_2 e^{\imath \delta_2} \right] $$

$$ E_2^B = -E_1^B(p_1 \to p_2) , \quad (15) $$
with isospin amplitudes \(A_0 = 2.71 \cdot 10^{-7}\) GeV, \(A_2 = 0.12 \cdot 10^{-7}\) GeV taken from experiment. The pion-pion phase shifts \(\delta_0^0, \delta_2^0\) at \(s = M_K^2\) are taken from a new analysis of pion-pion scattering combining dispersion theory (Roy equations) with chiral perturbation theory \([18, 19]\):

\[
\delta_0^0 = 39.1^\circ, \quad \delta_0^2 = -8.5^\circ. \tag{16}
\]

For the CP-violating direct-emission part, we have calculated the amplitude of \(O(p^4)\) for \(K_1^0 \to \pi^+\pi^-\gamma^*\), including both a local amplitude derived from the nonleptonic weak Lagrangian of \(O(p^4)\) \([13, 14]\) and a loop amplitude restricted to the dominant pion loops. Of course, one has to project out the Bremsstrahlung part of \(O(p^4)\) that is already included in \(E_i^B\) in (15). Altogether, we have

\[
E_1^{\text{DE}} = E_1^{\pi-\text{loops}}(\mu = M_\rho)
\]

\[
+ \frac{2e\eta_+ - G_8}{3F} q^2 [N_{14}r(M_\rho) - N_{15}r(M_\rho) - 3(N_{16}r(M_\rho) + N_{17})]
\]

\[
+ \frac{4e\eta_+ - G_8}{F} p_2 \cdot q [N_{14}r(M_\rho) - N_{15}r(M_\rho) - N_{16}r(M_\rho) - N_{17}]
\]

\[
E_2^{\text{DE}} = -E_1^{\text{DE}}(p_1 \leftrightarrow p_2). \tag{17}
\]

The explicit form of the direct-emission pion-loop amplitude \(E_1^{\pi-\text{loops}}(\mu = M_\rho)\) can be found in Ref. \([20]\). Because we have only included the dominant pion loops there is a residual, but numerically unimportant scale dependence in (17). We have chosen the usual renormalization scale \(\mu = M_\rho\). The numerical values for the low-energy constants \(N_i^r(M_\rho)\) (\(N_{17}\) is scale independent) are taken from a recent analysis \([20, 21]\) of the branching ratio \(B(K_L \to \pi^+\pi^-e^+e^-)\). Anticipating the numerical discussion, the direct-emission form factors \(E_i^{\text{DE}}\) are negligible for \(A_{CP}\), especially when integrated over the whole phase space.

For the magnetic amplitude, we must be less ambitious for the time being. Already at \(O(p^4)\), we cannot claim to be able to calculate the coefficients in Eq. (11). Moreover, \(\eta-\eta'\) mixing is known to produce a big contribution of \(O(p^6)\) \([22]\) that interferes destructively with the amplitude induced by the chiral anomaly. In addition, there are two apparently non-equivalent versions of implementing vector and axial-vector exchange in nonleptonic weak transitions \([23, 24]\).

On the other hand, there exists strong experimental evidence for higher-order effects parametrized in terms of a \(\rho\)-dominated form factor \([2, 25]\). Including final state interactions appropriate for \(P\)-wave \(\pi\pi\) scattering, we therefore adopt the magnetic amplitude measured by KTeV,

\[
M = \frac{e|f_s|}{M_K^4} \tilde{g}_{M1} e^{i\hat{a}_1(\pi^*)} \left[1 + \frac{a_1/a_2}{M_\rho^2 - M_K^2 + 2M_KE_\gamma^*}\right], \tag{18}
\]

with \(E_\gamma^*\) the total lepton energy in the kaon rest frame and with\(^2 [2]\)

\[
\tilde{g}_{M1} = 1.35 \pm 0.20 \quad a_1/a_2 = (-0.720 \pm 0.028) \text{ GeV}^2. \tag{19}
\]

\(^2\)The quantity \(a_2\) in (18) must not be confused with the same expression in (11).
The $I = J = 1$ phase shift is parametrized as

$$\delta_1^A(s_\pi) = \frac{1}{2} \arcsin \left[ \frac{4q_\pi^3}{s_\pi} (a_1 + b_1 q_\pi^2 + c_1 q_\pi^4) \right]$$  \hspace{1cm} (20)$$

with

$$a_1 = 0.038/M_{\pi^+}^2, \quad b_1 = 0.0057/M_{\pi^+}^4, \quad c_1 = 0.001/M_{\pi^+}^6,$$  \hspace{1cm} (21)$$

The slope parameters $a_1^b, b_1^b$ are taken from Refs. [18, 19] and the coefficient $c_1^b$ is introduced to reproduce the correct phase shift at $s = M_K^2$.

With the sign convention of (15), the magnetic coupling $\tilde{g}_{M1}$ must be positive in order to reproduce the measured sign of the asymmetry. Unfortunately, there is at present no unique way to infer from (18) and the experimental values (19) the sign of the lowest-order combination $a_2 + 2a_4$ in (11). A recent analysis of $K \to \pi\pi\gamma$ transitions by D’Ambrosio and Gao [26] in the framework of the vector-field representation for (axial-)vector resonances finds that both signs are possible depending on the coupling strength of spin-1 exchange. A deeper understanding of higher-order effects in nonleptonic weak interactions will be necessary before one could claim that the chiral anomaly is largely responsible for the measured sign of the asymmetry.

5. With electric form factors given in Eqs. (15) and (17) and with the magnetic amplitude (18) we obtain for the total integrated asymmetry

$$A_{CP} = 13.7 \%$$  \hspace{1cm} (22)$$

if all input quantities are taken at their respective mean values.

This value agrees with previous theoretical estimates and with the experimental results from KTeV and NA48. The main issue we want to address here is the theoretical uncertainty of this prediction. Electric and magnetic amplitudes are on quite a different footing in this respect. Except for the $P$-wave phase shift $\delta_1^A(s_\pi)$ that we will lump together with the phases occurring in $E_1^B$, we cannot ascribe a meaningful theoretical error to the magnetic amplitude. We will therefore scale the prediction (22) to the measured magnetic coupling $\tilde{g}_{M1}$ and include the experimental error of the ratio $a_1/a_2$ explicitly. Accounting for the measured branching ratio in the same way, we arrive at

$$A_{CP} = \frac{3.63 \cdot 10^{-7}}{B(K_L \to \pi^+\pi^-e^+e^-)} \cdot \frac{\tilde{g}_{M1}}{1.35} (13.7 \pm 1.3) \%$$  \hspace{1cm} (23)$$

where the given error is due to the error of $a_1/a_2$ in (19) only. Future experimental improvements and correlations between the measured values for $B(K_L \to \pi^+\pi^-e^+e^-)$, $\tilde{g}_{M1}$ and $a_1/a_2$ can easily be incorporated in this formula.

In contrast, the uncertainty of $A_{CP}$ related to the electric form factors is fully under control theoretically. For the total asymmetry under consideration here, the CP-violating direct-emission amplitude (17) is completely negligible and does not affect $A_{CP}$ to the
Given the magnetic amplitude and the branching ratio \( B \delta \) respectively in the combination \( q \) moves to larger of \( \Phi \) asymmetry for the cuts in question. As already mentioned, the low-energy constants \( q [21] \), about 13 % of all events (almost 16 % of those with constants. With the mean value \( Z \). [18, 19] we extract an uncertainty of 0 that the asymmetry is maximal for \( q \) corrections [28] have therefore no impact on \( A_{CP} \). Following the authors of Refs. [5, 6, 7], we consider the asymmetry also for different violating asymmetry can be calculated to a relative accuracy of about 2 %. Compared to the previous situation for the total asymmetry with \( q^2 \geq 4m^2_e \), there are now additional uncertainties that increase with the lower limit on \( q^2 \). In the numerator of the asymmetry, the direct-emission form factors \( E_i^{DE} \) in (17) become more important in comparison with the Bremsstrahlung amplitude. Likewise, the rate in the denominator becomes more sensitive to the inaccurately known weak low-energy constants \( N_i^r \) as one moves to larger \( q^2 \).

For large \( q^2 \), both the asymmetry and the rate decrease rapidly. For illustration, we therefore choose a realistic cut of \( q^2 > (40 \text{ MeV})^2 \). The CP-conserving amplitude of \( O(p^4) \) depends on the combination \( Z(\mu) = N_{14}(\mu) - N_{15}(\mu) - 3(N_{16}(\mu) - N_{17}) \) of weak low-energy constants. With the mean value \( Z(M_\mu) = 0.023 \) extracted from the total branching ratio [21], about 13 % of all events (almost 16 % of those with \( q^2 > (2 \text{ MeV})^2 \) as in the KTeV experiment [2]) satisfy \( q^2 > (40 \text{ MeV})^2 \).

However, these numbers are rather sensitive to the precise value of \( Z(M_\mu) \). In Table 2, the dependence of the rate on the electric amplitude is exhibited for \( q^2 > (40 \text{ MeV})^2 \). 

\[
\mathcal{A}_{CP} = \frac{3.63 \cdot 10^{-7}}{B(K_L \to \pi^+\pi^-e^+e^-)} \cdot \tilde{g}_{M1} \cdot \frac{\eta_{+}-\eta_{-}}{1.35} \ (13.7 \pm 1.3 \pm 0.3) \%.
\]
Table 1: The asymmetry $\mathcal{A}_{CP}$ for different cuts in $q^2$, the invariant mass squared of the lepton pair.

<table>
<thead>
<tr>
<th>$q^2 &gt; [\text{MeV}^2]$</th>
<th>$\mathcal{A}_{CP}$ in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>entire phase space</td>
<td>13.7</td>
</tr>
<tr>
<td>$2^2$</td>
<td>15.2</td>
</tr>
<tr>
<td>$10^2$</td>
<td>15.7</td>
</tr>
<tr>
<td>$20^2$</td>
<td>14.0</td>
</tr>
<tr>
<td>$30^2$</td>
<td>12.1</td>
</tr>
<tr>
<td>$40^2$</td>
<td>10.2</td>
</tr>
<tr>
<td>$60^2$</td>
<td>7.2</td>
</tr>
<tr>
<td>$80^2$</td>
<td>4.8</td>
</tr>
<tr>
<td>$100^2$</td>
<td>3.2</td>
</tr>
<tr>
<td>$120^2$</td>
<td>2.0</td>
</tr>
<tr>
<td>$180^2$</td>
<td>0.4</td>
</tr>
</tbody>
</table>

About 25% of the rate is due to $Z(M_\rho)$ being different from zero. This also documents that $Z(M_\rho)$ can in principle be extracted with much higher precision from the partial rate than from the total branching ratio. In the sample accumulated by KTeV [29], several hundred events are expected to satisfy $q^2 > (40 \text{ MeV})^2$.

As already mentioned, the direct-emission amplitude of $O(p^4)$ also enters the numerator of the asymmetry. In contrast to the rate where the CP-conserving electric amplitude of $O(p^4)$ overtakes the CP-violating Bremsstrahlung amplitude (8) of $O(p^2)$ already at fairly small $q^2$ [21], only the CP-violating part of direct emission matters in the numerator. For the considered cut $q^2 > (40 \text{ MeV})^2$ this is still only a small correction to the leading-order Bremsstrahlung. Numerically, the correction is smaller than the uncertainty due to $A_0$ and it will be included in the theoretical error.

Consequently, for reasonable cuts in $q^2$ the asymmetry can still be predicted rather precisely in terms of the magnetic amplitude and the branching ratio. For $q^2 > (40 \text{ MeV})^2$, we obtain

$$\mathcal{A}_{CP} = \frac{4.75 \cdot 10^{-8}}{B(K_L \to \pi^+\pi^-e^+e^--; q^2 > (40 \text{ MeV})^2)} \cdot \frac{\tilde{g}_{M1}}{1.35} (10.2 \pm 1.1 \pm 0.3) \% \quad (25)$$

where the first error is again due to the error of $a_1/a_2$ in (19), the second one being the genuine theoretical error.

7. The electric direct-emission amplitude for the decay $K_L \to \pi^+\pi^-e^+e^-$ has been calculated in chiral perturbation theory. The CP-conserving part is important for the rate because it dominates the CP-violating Bremsstrahlung amplitude already for rather low invariant masses of the lepton pair. With appropriate cuts in $q^2$, it will be possible to
Table 2: Dependence of the rate $\Gamma_{\text{cut}}(K_L \to \pi^+\pi^-e^+e^-)$ on the electric amplitude [21] for $q^2 > (40 \text{ MeV})^2$. $E_i^{O(p^4)}$ denotes the complete electric amplitude to $O(p^4)$ that depends on the combination $Z(\mu) = N_{14}^r(\mu) - N_{15}^r(\mu) - 3(N_{16}^r(\mu) - N_{17}^r)$ of weak low-energy constants. The magnetic amplitude is given in (18).

<table>
<thead>
<tr>
<th>$E_i^{B}$</th>
<th>$\Gamma_{\text{cut}}/\Gamma_{\text{cut}}(O(p^4))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_i^{O(p^4)}$ with $Z(M_\rho) = 0.023$</td>
<td>1.</td>
</tr>
<tr>
<td>$E_i^{O(p^4)}$ with $Z(M_\rho) = 0$</td>
<td>0.75</td>
</tr>
</tbody>
</table>

extract the relevant combination of low-energy constants in the CP-conserving amplitude of $O(p^4)$ with good precision.

In contrast, the CP-violating asymmetry is quite insensitive to electric direct emission. This allows for an accurate calculation of the asymmetry once the magnetic amplitude and branching ratio have been determined experimentally. For realistic cuts in $q^2$, the asymmetry can be predicted with a relative accuracy of $2 \div 3\%$.

The contribution of the chiral anomaly to the magnetic amplitude of $O(p^4)$ leads to the observed sign of the asymmetry. However, higher-order terms in the magnetic amplitude, clearly required by experiment but not reliably calculable in chiral perturbation theory, make this connection less conclusive.

We thank T. Barker, B. Cox, G. D’Ambrosio, M. Jeitler, S. Ledovskoy and L.M. Sehgal for helpful correspondence.
References


