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THE SLOWLY VARYING ENVELOPE APPROXIMATION REVISED
The Slowly Varying Envelope Approximation Revised

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Abstract

We derive the limit of validity of the Slowly Varying Envelope Approximation (SVEA) as a function of the "bulk" velocity $v$ of the radiating system, which reads $t_p \gg \lambda(1 - v/c)$, being $t_p$ the radiation pulse length. This condition reduces to the usual SVEA in the limit $v/c \ll 1$, whereas it is sensibly relaxed in the relativistic limit. The example of a Free Electron Laser is discussed.
1 Introduction

In the study of the interaction of radiation with matter, the Maxwell wave equation is of fundamental importance. This is a second order partial differential equation in space and time coordinates which, under some conditions that we shall investigate, reduces to a first order partial differential equation.

Let us consider the one-dimensional case for one component of the transverse electric field \( \vec{E}(z,t) \) and for the transverse current density \( \vec{J}(z,t) \)

\[
\frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial \vec{J}}{\partial t} \tag{1}
\]

Usually, when a primary radiation wavelength \( \lambda \) arises in the study of a physical problem and the main propagation is in the +z direction, as the spontaneous emission in a FEL, it is useful to introduce the complex amplitudes \( E(z,t) \) and \( J(z,t) \) defined such as

\[
\vec{E}(z,t) = E(z,t)e^{ik(z-ct)} \tag{2}
\]
\[
\vec{J}(z,t) = J(z,t)e^{ik(z-ct)} \tag{3}
\]

where \( k = \frac{2\pi}{\lambda} \) and \( \lambda \) is the radiation wavelength.

\( E \) and \( J \) have an immediate physical meaning in the case in which they do not vary sensibly over a wavelength: they represent the envelope of the electric field and of the current respectively.

Replacing (2) and (3) into (1) we have

\[
\left( \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \right) + 2ik \left( \frac{\partial E}{\partial z} + \frac{1}{c} \frac{\partial E}{\partial t} \right) = \frac{4\pi}{c^2} \left( \frac{\partial J}{\partial t} - ickJ \right) \tag{4}
\]

Equation (4) can be simplified to, as a first approximation, a first order differential equation keeping only the largest terms on each side.

2 The usual SVEA

Let us briefly reconsider the hypotheses underlying the usual SVEA [1], which consist in assuming

\[
\left| 2ik \left( \frac{\partial E}{\partial z} + \frac{1}{c} \frac{\partial E}{\partial t} \right) \right| \gg \left| \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \right| \tag{5}
\]

and

\[
\left| ckJ \right| \gg \left| \frac{\partial J}{\partial t} \right|
\]

so that (4) becomes

\[
\frac{\partial E}{\partial z} + \frac{1}{c} \frac{\partial E}{\partial t} = -\frac{2\pi J}{c} \tag{6}
\]

Equation (6) is the so called SVEA counterpart of eq.(1). A sufficient condition for the validity of (5) is

\[
\left| \frac{\partial^2 E}{\partial z^2} \right| \ll 2k \left| \frac{\partial E}{\partial z} \right| , \left| \frac{\partial^2 E}{\partial t^2} \right| \ll 2ck \left| \frac{\partial E}{\partial t} \right| , \left| \frac{\partial J}{\partial t} \right| \ll ckJ \tag{7}
\]
neglecting the cases in which the terms on the l.h.s. cancel each other and the terms on the r.h.s. do not. Assuming the scaling argument

$$\frac{\partial E}{\partial z} \approx \frac{1}{c} \frac{\partial E}{\partial t} \approx \frac{1}{\ell_p} E, \quad \frac{\partial J}{\partial z} \approx \frac{1}{c} \frac{\partial J}{\partial t} \approx \frac{1}{\ell_J} J$$  \hfill (8)

where \( \ell_p \) and \( \ell_J \) define the scale of variation of the pulse and the current, we can write (7) as

$$\ell_p \gg \lambda; \quad \ell_J \gg \lambda$$  \hfill (9)

i.e. the radiation and the current pulse show a slow variation over a wavelength scale.

3 The generalized SVEA

Let us now write (4) in terms of \( z' \) and \( z_1 \) defined as

$$\begin{cases} 
    z' = z \\
    z_1 = z - vt
\end{cases}$$

where \( v \) is the bulk velocity of the radiating system.

One obtains easily

$$\left( \frac{\partial}{\partial z'} + (1 - \beta) \frac{\partial}{\partial z_1} \right) \left( \frac{\partial E}{\partial z'} + (1 - \beta) \frac{\partial E}{\partial z_1} + 2\beta \frac{\partial E}{\partial z'} + 2ikE \right) = -\frac{4\pi}{c} \left( \beta \frac{\partial J}{\partial z_1} + ikJ \right)$$  \hfill (10)

Let us now suppose that

$$\left| \frac{\partial E}{\partial z'} + (1 - \beta) \frac{\partial E}{\partial z_1} \right| \ll \left| 2\beta \frac{\partial E}{\partial z_1} + 2ikE \right|$$  \hfill (11)

With this condition, equation (10) can be written as

$$\left( \frac{\partial}{\partial z'} + (1 - \beta) \frac{\partial}{\partial z_1} \right) \left( 2\beta \frac{\partial E}{\partial z_1} + 2ikE \right) = -\frac{4\pi}{c} \left( \beta \frac{\partial J}{\partial z_1} + ikJ \right)$$  \hfill (12)

This is equivalent to the SVEA equation:

$$\frac{\partial E}{\partial z'} + (1 - \beta) \frac{\partial E}{\partial z_1} = -\frac{2\pi J}{c}$$  \hfill (13)

In fact, taking the derivative of (13) to respect to \( z_1 \), one obtains

$$2\beta \frac{\partial}{\partial z_1} \left[ \frac{\partial E}{\partial z'} + (1 - \beta) \frac{\partial E}{\partial z_1} \right] = -\frac{4\pi \beta}{c} \frac{\partial J}{\partial z_1}$$

This equation plus eq.(13) multiplied by \( 2ik \) gives back equation (12). Going back to the initial variables \( z \) and \( t \), equation (13) becomes the SVEA equation (6).

Hence a sufficient condition for the validity of (6) or (13) is the inequality (11).

For \( \beta = 1 \), our analysis becomes equivalent to that of Haselhoff [2].

For \( \beta = 0 \), since \( z_1 = z = z' \), inequality (11) reduces to

$$\left| \frac{\partial E}{\partial z} \right| \ll k|E|$$  \hfill (14)
which is the usual SVEA.

A sufficient condition for the validity of (11) is that

$$\left| \frac{\partial E}{\partial z'} \right| \ll 2k|E| ; \quad (1 - \beta) \left| \frac{\partial E}{\partial z_1} \right| \ll 2k|E|$$  \hspace{1cm} (15)

We are neglecting the case in which $2\beta \frac{\partial E}{\partial z_1}$ and $2ikE$ cancel each other. Defining a gain length $\ell_g$ and a pulse length $\ell_p$ such as

$$\frac{E}{\ell_g} \approx \frac{\partial E}{\partial z'} ; \quad \frac{E}{\ell_p} \approx \frac{\partial E}{\partial z_1}$$  \hspace{1cm} (16)

we have

$$\ell_g \gg \lambda ; \quad \ell_p \gg \lambda(1 - \beta)$$  \hspace{1cm} (17)

The physical meaning of these conditions is obvious. The first one implies that the field can not be sensibly amplified in a wavelength. The second condition can be derived imposing that the electron-photon interaction time is much larger than the optical period. Note that the last condition reduces to the usual SVEA condition for $\beta \ll 1$ and it is much less restrictive if $\beta \approx 1$, as in a FEL, since it can be written as $\ell_p \gg \lambda/\gamma^2$.

Furthermore, since in a FEL

$$\ell_p \approx \ell_c = (1 - \beta)\ell_g$$  \hspace{1cm} (18)

the two conditions (17) reduce to the single one

$$\lambda_w \gg \lambda \rho$$  \hspace{1cm} (19)

This can be seen easily using the normalisation of ref.[3]

$$\bar{z}_1 = \frac{z_1}{\ell_c} ; \quad \bar{z} = \frac{z'}{\ell_g}$$  \hspace{1cm} (20)

where

$$\ell_c = \frac{\lambda}{4\pi \rho} ; \quad \ell_g = \frac{\lambda_w}{4\pi \rho} ; \quad \lambda = \lambda_w(1 - \beta)$$  \hspace{1cm} (21)

In this way, inequality (11) becomes

$$\left| \frac{\partial E}{\partial \bar{z}} + \frac{\partial E}{\partial \bar{z}_1} \right| \ll \left| \frac{2\beta}{1 - \beta} \frac{\partial E}{\partial z_1} + 2i \frac{\lambda_w}{\lambda \rho} E \right|$$  \hspace{1cm} (22)

which gives immediately condition (19).

Finally let us note that in the steady-state regime, where $\frac{\partial E}{\partial z_1} = 0$, condition (19) becomes necessary and sufficient for the validity of the SVEA.

4 Conclusions

We have shown that the Slowly Varying Envelope Approximation is valid under conditions (17) which depend on the bulk velocity of the acting medium.

This condition gives a strongly relaxed limit of validity for the SVEA approximation in the case of the FEL, where the electrons move at relativistic velocity.
References


[2] E. H. Haselhoff (private communication);