G. Cattapan and V. Vanzani: ON A GENERALIZED DISTORTED-WAVE APPROXIMATION FOR NUCLEAR REARRANGEMENT REACTIONS.
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SUMMARY.

Starting from an exact three-body theory for the nuclear rearrangement reaction $a + (b + c) \rightarrow b + (a + c)$, we study a generalized distorted wave approximation in the momentum-space representation. Using suitable kinematic transformations we recast this representation in a form coinciding with the one obtained by means of the Feynman diagram summation method. The generalized potential responsible for the transition will be written in a compact form involving the resolvent operator for the a-b subsystem.

1. - INTRODUCTION. -

In recent years an increasing interest has been devoted to the three-body rearrangement scattering problem in the distorted-wave formalism. A generalized distorted-wave approximation (GDWA) has been formally derived from the Faddeev-Lovelace integral equations written in terms of symmetric transition operators\(^{(1, 2)}\). This GDWA involves the full interactions in both initial and final channels, not only in one channel as in the conventional DWBA. The occuring generalized transition potential contains the contributions of the basic rearrangement mechanisms which can be described by a polar and a triangular diagram.

It has been also recognized that the GDWA is equivalent to the Feynman-diagram summation method (FDSM) based on the above polar and triangular graphs\(^{(2, 3)}\). However an explicit GDWA momentum-space representation to be used as starting point for calculational purposes has never been investigated. The aim of the present paper is to study this explicit representation.
In performing this program one is faced by nontrivial difficulties arising from the complicated three-body kinematics. In order to deal with manageable formulae we shall use physical momenta and not the normalized ones. The compact procedure we follow allows to clarify the physical meaning of the GDWA and to perform a direct comparison with the FDSM.

In Sect. 2 we shall give a general outline of the GDWA in a three-body context. Section 3 deals with some kinematic transformations suitable for working in the momentum-space. In Sects. 4 and 5 we shall be concerned with the momentum-space representation of the GDWA and we shall give a detailed comparison between the GDWA and the FDSM.

2. GENERAL FORMALISM.

In a three-body context a nuclear rearrangement reaction \( A(a, b)B \) can be represented schematically as

\[
(a + (b + c)) \rightarrow b + (a + c),
\]

where \( A = b + c, B = a + c \) and \( a, b, c, \) are treated as inert entities interacting only via two-body interactions.

Let us introduce the transition operators \( U_{\beta \alpha}(z) \) from the initial channel \( \alpha \) to the final one \( \beta, \) in the symmetric form:

\[
(2.2a) \quad U_{\beta \alpha}(z) = \delta_{\beta \alpha}^{-1} G_\alpha^{-1}(z) + V_\beta + V_\beta^* G(z) V_\alpha,
\]

\[
(2.2b) \quad U_{\beta \alpha}(z) = \delta_{\beta \alpha}^{-1} G_\beta^{-1}(z) + V_\alpha + V_\alpha^* G(z) V_\beta,
\]

where \( \delta_{\beta \alpha} = 1 - \delta_{\beta \alpha}, V_\delta (\delta = \alpha, \beta) \) are the channel interactions, \( G_\delta \) the resolvents of the channel Hamiltonians and \( G(z) \) the resolvent of the total Hamiltonian. These operators are defined as in ref. (6).

The transition operators (2.2) satisfy the Faddeev-Lovelace-like integral equations

\[
(2.3a) \quad U_{\beta \alpha}(z) = \delta_{\beta \alpha}^{-1} G_\alpha^{-1}(z) + \sum_{\gamma \neq \beta} t_{\gamma}(z) G_\delta(z) U_{\gamma \alpha}(z),
\]

\[
(2.3b) \quad U_{\beta \alpha}(z) = \delta_{\beta \alpha}^{-1} G_\beta^{-1}(z) + \sum_{\gamma \neq \alpha} U_{\gamma \beta}(z) G_\gamma(z) t_{\gamma}(z),
\]

where \( G_\delta(z) \) is the resolvent of the free Hamiltonian, \( t_{\alpha}(z) \) is the two-body scattering operator for the \( \beta - \gamma \) subsystem acting in the three-body space.

By iterating eqs. (2.3) once and omitting the energy parameter \( z \) for the sake of simplicity, one gets
The inhomogeneous term of integral equations (2.4a) or (2.4b), characterized by a compact kernel, is expressed only in terms of \( G_0^{-1} \) and of the two-body scattering operators. As far as the process (2.1) is concerned, the two simplest terms which appear in the equation for \( U_{ba} \) give, in the channel state representation, the amplitudes for the polar and triangular diagrams, corresponding to basic rearrangement mechanisms (2). By using the relation

\[
G_c = G_0 + G_0^t G_0^c,
\]

the above terms can be rewritten in the compact form \( G_0^{-1} G_c G_0^{-1} \).

The integral equations (2.3) with compact squared kernel provide a rigorous mathematical basis for a quantitative formulation of the intuitive physical picture associated with the distorted-wave representation of the nuclear rearrangement amplitudes (1, 2).

Following the procedure of ref. (2), taking account of eq. (2.5) and neglecting the coupling terms between elastic and rearrangement channels in the equations expressing \( U_{aa} \) and \( U_{bb} \) in terms of \( U_{ba} \), one obtains

\[
(2.6a) \quad U_{ba} = G_0^{-1} G_c (G_0^{-1} + t \ G_0 \ U_{aa}) + t \ G_0^t G_0 U_{ba},
\]

\[
(2.6b) \quad U_{aa} = t + t \ G_0^t G_0 \ U_{aa},
\]

\[
(2.7a) \quad U_{ba} = (G_0^{-1} + U_{bb} G_0^t) G_0^{-1} + U_{ba}^t G_0 \ U_{ba},
\]

\[
(2.7b) \quad U_{bb} = t + U_{bb} G_0^t G_0^c.
\]

By analogy with the arguments of ref. (2), one can derive from eqs. (2.6a) and (2.7b) or (2.6b) and (2.7a) the following compact expression for \( U_{ba} \) on-the-energy-shell

\[
(2.8) \quad U_{ba} = (1 + U_{bb} G_0) v_\alpha v_b v_\gamma (1 + G_0 U_{aa}).
\]

In eq. (2.8) \( v_\alpha \) denotes the interaction between \( \beta \) and \( \gamma \) (\( \alpha \neq \beta \neq \gamma \)).
If the channel resolvent operators \( G_a \) \((a = a, b)\) are approximated by their dominant separable part \( |\psi_a^\ast| \psi_a \), which corresponds to the channel bound state \( |\psi_a^\ast\rangle \) and appears in their spectral representation, one obtains the generalized distorted-wave approximation

\[
T_{\text{GDWA}} = \langle \Phi_b \mid U_{\text{GDWA}} \mid \Phi_a \rangle = 
\]

\[
= \langle \Phi _b m\mid G a \mid \psi _b \rangle \langle \psi _b \mid G \psi _a \rangle (1 + g a m \mid \psi _a \rangle)
\]

In eq. (2.9) \( |\psi_a\rangle = |\tilde{p}_a m_a \rangle \) are the channel states, \( \tilde{p}_a \) (\( \tilde{p}_b \)) the initial (final) channel linear momentum, \( m_a \) the z-component of the spin \( s_a \) of the particle \( a \). All operators are evaluated for \( z = E_a + i\epsilon = E_b + i\epsilon \). The total channel energy \( E_a \) is given by \((\epsilon = 1)\)

\[
E_a = \frac{p_a^2}{2 v_a} - \epsilon a,
\]

where \( \epsilon a \) is the binding energy of the bound state in the channel \( a \) and \( v_a \) is the reduced mass for the system consisting of \( a \) and \( \beta + \gamma (\alpha \neq \beta \neq \gamma) \).

The quantities

\[
u_{aa} = \langle \psi_a \mid U_{aa} \mid \psi_a \rangle
\]

are the optical scattering operators acting on the plane-wave states \( |\tilde{p}_a m_a \rangle \). They are constructed as expectation values of the actual scattering operators \( U_{aa} \) in the subspace of the channel bound states. Then, the wave-operators \( 1 + g_{a} u_{aa} \) operating on \( |\tilde{p}_a m_a \rangle \) give the effective two-body distorted-wave states. In eq. (2.9) the operator \( \langle \psi_b \mid G \psi_a \rangle \), acting on channel distorted-wave states, has the meaning of a generalized transition potential. Its compact form explicitly exhibits the role of the two-body resolvent operator \( G_{c} \).

The generalized distorted-wave approximation (2.9) can be formulated in the language of the nonrelativistic Feynman diagrams. If the initial and final channel interactions are described by graphs involving the half-off-energy-shell optical scattering amplitudes

\[
\langle \tilde{p}_a m_a \mid U_{aa} \mid \tilde{p}_a m_a \rangle.
\]

one obtains the FDSM based on the polar and triangular diagram rearrangement mechanisms. We shall give a detailed and complete account of the GDWA-FDSM equivalence, by using complete sets of intermediate states and by resorting to the Feynman-diagram rules. The equivalence
of the two methods has been already noticed in ref. (8) for a particular triangular graph amplitude.

The transition amplitudes $T_{ba}$ introduced by us in the distorted-wave (Sect. 4) and Feynman diagram formalisms (Sect. 5) are related to the differential cross sections by the formula

$$\frac{d\sigma_{ba}}{d\Omega} = (2\pi)^4 \frac{\gamma_a \gamma_b}{p_a p_b} \frac{1}{(2s_a+1)(2s_A+1)} \frac{\Sigma}{m_A m_B} \left| T_{ba} \right|^2$$

3. - KINEMATIC CONSIDERATIONS.

As is well known, the three-body problem involves great complications arising from the number of kinematic variables. Then, it seems useful to give some kinematic transformations suitable for representing the GDWA in the momentum-space. Throughout this paper we shall use physical momenta, not the normalized ones as in ref. (4).

Let us denote by $p_a$ the momentum of particle $a$ in the total center of mass system and by $k_a$ the relative momentum between the particles $\beta$ and $\gamma$. By definition

$$k_a = \frac{M \gamma \beta - M \gamma \beta}{M \beta + M \gamma},$$

with $M_a$ the mass of particle $a$ and $(a, \beta, \gamma)$ a cyclic permutation of $(a, b, c)$. As is well known, in the total center of mass system only two of the six variables $k_a, k_b, k_c, p_a, p_b, p_c$ are linearly independent. Then, one can choose a pair $(k_a, p_a)$ as independent momenta for characterizing a three-particle state.

In order to pass from any pair $(k_a, p_a)$ to any other pair $(k_\delta, p_\delta)$ $(\delta = \beta, \gamma)$, one may use eq. (3.1) and the relation

$$p_a + p_b + p_c = 0.$$

One gets

$$\left\{ \begin{array}{l}
    k_\beta = - \frac{M_a}{M_a + M_\gamma} k_a - \frac{M \gamma}{(M_a + M_\gamma)(M_\beta + M_\gamma)} p_a \\
    p_\beta = k_a - \frac{M_\beta}{M_\beta + M_\gamma} p_a
\end{array} \right.$$

$$3.3a$$
4. - THE MOMENTUM-SPACE REPRESENTATION OF THE GDWA. -

By inserting in eq. (2.9) intermediate momentum integrations and magnetic sums over the complete sets of three-particle states

\[ |k^\leftarrow a p^\leftarrow a m^\leftarrow a m^\rightarrow b m^\rightarrow c > \quad \text{and} \quad |k^\leftarrow b p^\leftarrow b m^\leftarrow b m^\rightarrow a m^\rightarrow c > \]

one gets

\[ T_{G D W A} = \sum_{m_a m_c} \int \chi^{(+)p_b} (p_b^\leftarrow m_a, m_c) \matrix{0} (p_b^\leftarrow m_a, m_c) \times \]

(4.1)

\[ \times \chi^{(+)p_a} (p_a^\leftarrow m_a) d p_b d p_a \]

with

(4.2)

\[ T_{0} (p_b^\leftarrow a m_b^\rightarrow a m_a^\rightarrow c m_c^\rightarrow b) = \sum_{m_a m_c} \int F(k^\leftarrow b a c b m_a c m_c) d k_b d k_a \]

\[ F(k^\leftarrow b a c b m_a c m_c) = f_b (k^\leftarrow a, b, a) \times \]

(4.3)

\[ \times \langle k^\leftarrow b a c b m_a c m_c | G | k^\rightarrow c a a b c m_a m_b m_c \rangle f_a (k^\rightarrow a, b, c) \]

(4.4)

\[ \chi^{(+)p_b} (p_b^\leftarrow m_a) = \langle p_b^\leftarrow m_a | (1 + g_a u_a u_a) | p_b m_a \rangle \]

(4.5)

\[ f_a (k^\leftarrow a m_\beta m_\gamma) = \langle k^\leftarrow a m_\beta m_\gamma | v_a | \psi_a \rangle \quad (a \neq \beta \neq \gamma) \]

From the definition of \( g_a \) it follows

(4.6)

\[ g_a (E_a + i \epsilon) | p_a^\leftarrow m_a > = \left( \frac{p_a^2}{2 v_a} - \frac{p_a^2}{2 v_a} + i \epsilon \right)^{-1} | p_a^\leftarrow m_a > \]

Then, the channel distorted-wave states can be rewritten, in the momentum-space representation, in the explicit form.
Equation (4.5) gives the usual two-body form factors corresponding to the channel bound states $|\psi_a\rangle$. By means of standard angular momentum expansions (6, 9, 10), one may give an explicit momentum-space representation of $f_a$, involving spectroscopic factors and single-particle or single-cluster reduced widths.

In order to simplify the expression (4.2) for $T_0$, let us consider the relations (3.8) and change the integration variables $k_b$ and $i_a$ in $p_c$ and $p_c'$, respectively (see eq. (3.4a) with $(a, \beta, \gamma) = (b, c, a)$ and eq. (3.4b) with $(a, \beta, \gamma) = (a, b, c)$). Taking account of the property (3.9) for the matrix elements of $G_c$ and performing the integration over $p_c'$, one gets

\begin{equation}
T_0 (p^\prime b^\prime p^\prime m^\prime m) = \sum_{m^\prime a b c} \int f^* (q^\prime m^\prime m) \left| \frac{2}{p_c} \right| \frac{2}{2v_c} + i \varepsilon \cdot q^\prime m^\prime m \cdot (q^\prime m^\prime m) d p_c',
\end{equation}

where

\begin{equation}
S_c = E_a - \frac{p_c^2}{2v_c}
\end{equation}

and $q_a$, $q_c$, $q_b$, $q_c'$ stand for the momenta $k_a$, $k_c$, $k_b$, $k_c'$ respectively, evaluated for $p_c' = p_c$. Therefore, they are connected to the integration variables by the relations

\begin{align}
(4.10a) \quad q_a &= - \frac{M_c}{M_a + M_b} p_a - p_c, \quad q_c = - \frac{M_c}{M_a + M_b} p_c + p_a; \\
(4.10b) \quad q_b &= - \frac{M_c}{M_a + M_b} p_b + p_c, \quad q_c' = - \frac{M_c}{M_a + M_b} p_c - p_b'.
\end{align}

By expressing $\hat{G}_c$ in terms of $\hat{t}_c$ and $\hat{G}_0$ (see eq. (2.5) in the hat notation), one gets
\[ \langle \hat{q}'_{c} m'_{a} m'_{b} | \hat{G}(S + i\varepsilon) | \hat{q}_{c} m_{a} m_{b} \rangle = \] 
\[ \frac{\psi^{(+)}_{c}(\hat{q}'_{c} \hat{q}_{c} m'_{a} m_{a} m'_{b} m_{b}; S + i\varepsilon)}{S_{c} - \frac{q_{c}^{2}}{2\mu_{c}} + i\varepsilon}, \] 
(4.11)

where

\[ \psi^{(+)}_{c}(\hat{q}'_{c} \hat{q}_{c} m'_{a} m_{a} m'_{b} m_{b}; S + i\varepsilon) = \delta(q'_{c} - q_{c}) \delta_{m'_{a} m_{a}} \delta_{m'_{b} m_{b}} + \] 
(4.12)

is the momentum-space representation of the two-body scattering state for the a-b subsystem. Recalling the above kinematic transformations and the procedure followed in deriving eqs. (4, 8), one may write

\[ \langle \hat{q}'_{c} m'_{m} m'_{m}; S + i\varepsilon \rangle = \langle \hat{q}'_{c} \hat{q}_{c} m'_{m} m_{m}; S_{c} + i\varepsilon \rangle \] 
(4.13)

Then from (3, 7a) it follows

\[ S_{c} - \frac{q_{c}^{2}}{2\mu_{c}} = E_{a} - \frac{q_{a}^{2}}{2\mu_{a}} + \frac{p_{a}^{2}}{2\nu_{a}}, \quad S_{b} - \frac{q_{b}^{2}}{2\mu_{b}} = E_{b} - \frac{q_{b}^{2}}{2\mu_{b}} + \frac{p_{b}^{2}}{2\nu_{b}}. \] 
(4.14)

By inserting in (4.1) the expressions (4.8) and (4.11) with a correct number of primes for the intermediate momentum and magnetic variables, one obtains an explicit momentum-space representation of the GDWA in terms of two different types of two-body scattering waves: the initial and final channel distorted-waves \( \chi^{(+)}_{a} \) and the intermediate scattering wavefunction \( \psi^{(+)}_{c} \).

From eqs. (4.1), (4.11), (4.12) and (4.14) it is immediately seen that, owing to the presence of the half-off-energy-shell optical scattering amplitudes \( \langle \hat{p}_{a}^{m} \neq \hat{p}_{a}^{m} \rangle \), the GDWA amplitude has in \( p_{b}^{2} \) (or in \( p_{a}^{2} \)) two three-particle cuts running to the right of the normal three-particle threshold \( (p_{b}^{2})_{0} = 2\nu_{b} \epsilon_{b} \) (or \( (p_{a}^{2})_{0} = 2\nu_{a} \epsilon_{a} \)).

If only the plane-wave term \( \delta(\hat{p}'_{a} - \hat{p}_{a}^{m}) \delta_{m'_{a} m_{a}} \) is retained in the expression (4.7) for the distorted-waves \( \chi^{(+)}_{a} \), eq. (4.1) reduces to
the generalized plane-wave approximation (GPWA)

\[ T_{\text{GPWA}} = T_0 (p_{b}^{a} m_{b}^{a} m_{a}^{b} m_{a}^{a} ) , \]

where now the variables \( p_{a}^{a}, p_{b}^{b}, m_{a}^{a}, m_{a}^{b} \) are channel physical variables, not the intermediate state ones. In this particular case the three-particle cuts disappear and, in virtue of eqs. (2.10), (4.14), one gets

\[ T_{\text{GPWA}} = \]

\[ \sum_{m_{b}^{b} m_{c}^{c} m_{a}^{a} m_{a}^{a}} \int \frac{f_{b}^{a} m_{m}^{m} m_{a}^{a} m_{a}^{a} }{c_{b}^{b} a_{a}^{a} b_{a}^{a} c_{a}^{a}} \psi_{c}^{(0)} (q_{m}^{m} m_{m}^{m} m_{a}^{a} m_{a}^{a} ; S + i \varepsilon ) f_{a}^{a} (q_{m}^{m} m_{a}^{a} m_{a}^{a} ) d\varepsilon_{b}^{a} + \frac{\varepsilon_{b}^{a}}{2 \mu_{b}^{a}} \]

Taking account of eq. (4.12) one may split the GPWA amplitude (4.16) in the following form

\[ T_{\text{GPWA}} = T_{\text{PWA}(0)} + T_{\text{PWA}(t)} \]

with

\[ T_{\text{PWA}(0)} = \sum_{m_{c}^{c} m_{b}^{b} m_{a}^{a} m_{a}^{a}} \int \frac{f_{b}^{a} m_{m}^{m} m_{a}^{a} m_{a}^{a} }{c_{b}^{b} a_{a}^{a} b_{a}^{a} c_{a}^{a}} \psi_{c}^{(0)} (q_{m}^{m} m_{a}^{a} m_{a}^{a} m_{a}^{a} ) d\varepsilon_{b}^{a} + \frac{\varepsilon_{b}^{a}}{2 \mu_{b}^{a}} \]

\[ T_{\text{PWA}(t)} = \]

\[ \sum_{m_{b}^{b} m_{c}^{c} m_{a}^{a} m_{a}^{a}} \int \frac{f_{b}^{a} m_{m}^{m} m_{a}^{a} m_{a}^{a} }{c_{b}^{b} a_{a}^{a} b_{a}^{a} c_{a}^{a}} \psi_{c}^{(0)} (q_{m}^{m} m_{a}^{a} m_{a}^{a} m_{a}^{a} ) f_{a}^{a} (q_{m}^{m} m_{a}^{a} m_{a}^{a} ) d\varepsilon_{b}^{a} + \frac{\varepsilon_{b}^{a}}{2 \mu_{b}^{a}} \]

The amplitude \( T_{\text{PWA}(0)} \) arises from the plane-wave term of eq. (4.12). In eq. (4.18) \( \xi_{a} \) and \( \xi_{b}^{f} \) stand for the momenta \( q_{a} \) and \( q_{b}^{f} \), respectively, evaluated for \( q_{c}^{f} = q_{c}^{f} \). They are given in terms of the channel physical momenta by the relations

\[ \xi_{a} = \frac{M_{b}}{M_{b} + M_{c}} p_{a}^{b} + p_{b}^{b} , \quad \xi_{b}^{f} = - \frac{M_{a}}{M_{a} + M_{c}} p_{b}^{a} - p_{a}^{a} . \]

From (4.14) it follows for \( q_{c}^{f} = q_{c}^{f} \).
12.

$$\varepsilon \frac{x_a^2}{2\mu_a} = \varepsilon_b \frac{x_b^2}{2\mu_b}$$

(4.21)

Obviously the amplitudes $T^{PWA(o)}$ and $T^{PWA(t)}$ can be directly obtained by starting from the formal expressions $\langle \Phi_b | G_0^{-1} | \Phi_a \rangle$ and $\langle \Phi_b | t_{c0} | \Phi_a \rangle$, respectively.

It is worthwhile noticing that in the amplitudes (4.8), (4.16), (4.18) and (4.19) there are no mass-dependent multiplicative factors, because we have used physical momenta and not the normalized ones (see, on the contrary, formula (3.19) or ref. (4)).

5. - THE GDWA IN THE FEYNMAN DIAGRAM LANGUAGE. -

The FDSM amplitude, based on the polar and triangular diagrams, can be written schematically as (3)

$$(5.1)$$

$$T_{b0}^{FDSM} = T_0 + T_{0a} + T_{b0} + T_{b0a},$$

where $T_0$ is the sum of the basic contributions (the polar and triangular amplitudes), $T_{0a}$ ($T_{b0}$) involves the initial (final) channel interactions, besides the basic contributions, and $T_{b0a}$ involves both the initial and the final channel interactions (see Fig. 2 of ref. (10) with $M_{f1}$ replaced by $T_{b0a}^{FDSM}$).

In order to derive transition amplitudes which are directly comparable with those of Sect. 4, we start from nonrelativistic Feynman-diagram rules written in a form slightly different from the usual one (11). We shall introduce the following factors:

a) a factor $-2M_{a}(p_a - 2M_{a}e_a - i\varepsilon)^{-1} \Sigma$ for each virtual particle characterized by the four-momentum $(p_a, e_a)$, ($a = a, b, c, A, B$);

b) a vertex amplitude $T_v$ for each vertex $v$;

c) an integration $\int dp_\xi de_\xi$ for each independent four-momentum $(p_\xi, e_\xi)$;

d) a general multiplicative factor $(-1)^{n+1}(2\pi)^{-1}$ where $n$ is the number of vertices and 1 the number of independent four-momenta.

By using these rules one gets the following expression for $T_{b0a}$

$$(5.2)$$

$$T_{b0a} = -\frac{4}{\pi} M_{a} M_{b} M_{A} M_{B} x$$

$$x \sum_{m_a^* m_b^*} \int A(p_{m_a^*} p_{m_b^*} e_a^* e_b^* m_a m_b) d\pi_{m_a^*} d\pi_{m_b^*} d\pi_{e_a} d\pi_{e_b},$$

where $A$ is the amplitude and $\pi$ the momentum. The integration is over the four-momenta $p_{m_a^*}$, $p_{m_b^*}$, $e_a^*$ and $e_b^*$. The sum is over the possible combinations of $m_a$ and $m_b$.
with
\[
A(p\null^me\null^np\null^nm\null^m) =
\]
\[
\frac{u\null^u(p\null^mp\null^nm\null^m)p\null^m(p\null^mp\null^nm\null^m)}{(p\null^m-2M_e\null^m-i\epsilon)(p\null^m-2M_e\null^m-i\epsilon)(p\null^m-2M_e\null^m-i\epsilon)(p\null^m-2M_e\null^m-i\epsilon)}
\]
\[
(p\null^m-2M_e\null^m)(p\null^m-2M_e\null^m)(p\null^m-2M_e\null^m)(p\null^m-2M_e\null^m)
\]
(5.3)

\[
e_a = \frac{2}{2\nu_a} - e^a, \quad e_B = \frac{2}{2\nu_a} - e^m.
\]
(5.4)

According to the notation of Sect. 4, we have denoted by \( u_{aa}, u_{bb}, T_0 \) the three four-ray vertex amplitudes appearing in the two-loop graph described by \( T_{b0a} \).

The integration over \( e_a^m \) and \( e_b^m \) in (5.2) can be performed in the complex \( e_a^m \) and \( e_b^m \)-plane. By writing explicitly the amplitude \( T_0 \) in terms of polar and triangular amplitudes, one can see that \( T_0 \) has no singularities in the lower half-plane of variables \( e_a^m \) and \( e_b^m \). Then, by means of the residue method and of some straightforward manipulations one obtains

\[
T_{b0a} = \sum_{m_a^m} \frac{x}{m_a^m} \int u_{bb}(x)_{b\to a} T_0 p^{m_a^m} p^{m_a^m} p^{m_a^m} u_{aa}(x)_{a\to b} p^{m_a^m} \frac{dp_a^{m_a^m} dp_b^{m_a^m}}{(2\nu_a^m - 2\nu_a^m + i\epsilon)(2\nu_a^m - 2\nu_a^m + i\epsilon)}
\]
(5.5)

where \( T_0(p\null^mp\null^nm\null^m) \) stands for \( T_0(p\null^mp\null^nm\null^m) \) evaluated for \( e_a^m = p_a^m/2M_a \) and \( e_b^m = p_b^m/2M_b \).

Let us now consider the amplitude \( T_0 \) appearing in (5.5). By definition it can be written in the form

\[
T_0(p\null^mp\null^nm\null^m) = T^{(o)}(p\null^mp\null^nm\null^m) + T^{(t)}(p\null^mp\null^nm\null^m)
\]
with

\[
T^{(o)}(p\null^mp\null^nm\null^m) =
\]
\[
-2M_c \sum_{m_c^m} \frac{f_{b\to a}(x)_{b\to a} f_{a\to c}(x)_{a\to c}}{(p_a^m + p_b^m)^2 - 2M_c(E - \frac{p_a^m}{2M_a} - \frac{p_b^m}{2M_b}) - i\epsilon}
\]
(5.6)

\[
= -2M_c \sum_{m_c^m} \frac{f_{b\to a}(x)_{b\to a} f_{a\to c}(x)_{a\to c}}{(p_a^m + p_b^m)^2 - 2M_c(E - \frac{p_a^m}{2M_a} - \frac{p_b^m}{2M_b}) - i\epsilon}
\]
(5.7)
\[ T^{(t)}(p_b \rightarrow p_a, m' m_a) = \]
\[ = \frac{4i}{\pi} M a b c \sum_{m b c a} B(p_b, p_e, p_{m' m m m' m}) \int dp_c \sum dp_d, \]

\[ B(p_b, p_e, p_{m' m m m' m}) = \]
\[ = \frac{f(q p m m m m)}{(p_{p p e} - 2 M e - i \epsilon)(p_{p b c} - 2 M e - i \epsilon)(p_{p c} - 2 M e - i \epsilon)} \]

\[ E = E_a - \frac{p^2_c}{2(M_a + M_b)} , \]
\[ e_a = E_b - \frac{p^2_b}{2M_b} - e_c , \quad e_b = E_a - \frac{p^2_a}{2M_a} - e_c . \]

In eqs. (5.9) and (5.7) \( t \) stands for the four-ray vertex amplitude and \( f_a, f_b \) are the two three-ray vertex functions appearing in the basic diagrams. The relative momenta at the vertices \( p_a, p_b, \) and \( q_a, q_b, q_c \) are given in terms of the intermediate momenta \( p_a, p_b, p_c \) by relations formally identical to (4.20) and (4.10), respectively.

By passing from the form (3.7b) to the form (3.7a) for the total kinetic energy expressed in terms of \( \vec{p} \)- and \( \vec{p} \)-type variables, one gets

\[ T^{(o)}(p_b \rightarrow p_a, m' m_a) = \sum_{m c} \frac{f(q x m m)}{E_a - (\frac{x^2}{2\mu_a} + \frac{p^2_a}{2\nu_a}) + i\epsilon} . \]

Since for \( q_c = q_b \) the relations (4.14) coincide, the propagator in (5.11) can also be written in the form \( \left[ E_b - (\frac{p^2_b}{2\mu_b} - \frac{p^2_b}{2\nu_b} + i\epsilon) \right]^{-1} . \)

After suitable kinematic transformations the denominator \( D \) in eq. (5.9) takes the form
D(p'_{b}p'_{c}p_{a}E) = \frac{2M_{c}(M_{a} + M_{b})}{\mu_{c}} x \tag{5.12}

x (q_{c}^{2} - 2\mu_{E}E - i\varepsilon)(E - S_{c} - i\varepsilon)(q_{c}'^{2} - 2\mu_{E}E - i\varepsilon)

with S_{c} defined by (4.9). By changing in eq. (5.8) the energy integration variable from e_{c} to E, we may perform the integration over E in the upper half-plane by means of the method used in ref. (10). Notice that the bound state poles of the amplitude t_{c} lie on the lower half-plane. Then, taking account of the kinematic relations (4.14), we obtain

\[ T_{c}(p'_{b}p'_{a}m'_{m}m_{a}) = \sum_{m_{b}m_{c}m'_{c}} x \tag{5.13} \]

By summing up the amplitudes (5.11) and (5.13) and taking into account that \( q_{a} \) and \( q_{b}' \) reduce to \( q_{a}' \) and \( q_{b}' \), respectively, for \( q_{c} = q_{c}' \), one gets eq. (4.8) with (4.11), (4.12) and (4.14).

If the variables \( p_{a} \), \( p_{b} \), \( m_{a} \), \( m_{b} \) are interpreted as channels physical variables, the propagators appearing in (5.11) and (5.13) simplify. In this case eqs. (5.11) and (5.13) reduce to (4.18) and (4.19) respectively; they give the polar and triangular amplitudes without channel interactions. By summing them one gets the first term in (5.1).

The terms \( T_{0a} \) and \( T_{0b} \) can be evaluated in a similar way. Performing the sum (5.1) and taking account of (4.7), one obtains just the \( T_{GDWA} \) amplitude considered in Sect. 4. Therefore

\[ T_{FDSM} = T_{GDWA} \tag{5.14} \]

In conclusion, the results of Sects. 4 and 5 explicitly exhibit the equivalence between the FDSM based on the polar and triangular diagram mechanisms and the GDWA momentum-space representation.

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REFERENCES AND FOOTNOTES.

(4) - C. Lovelace, Phys. Rev. 135B, 1225 (1964).

(XX) - As usual, the channel interactions are assumed to be independent of the spins of A and B(9).