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STOCHASTIC AND NON-STOCHASTIC SUPERSYMMETRY
Stochastic and Non-Stochastic Supersymmetry

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ABSTRACT

In this paper we review the supersymmetries discovered some time ago both in Langevin equations and in Hamilton's canonical equations. Of these two supersymmetries we mainly study the physical origins. For the Langevin supersymmetry (the stochastic one) we find its origin in the *Onsager principle of microreversibility*, while for the Hamilton's supersymmetry (the non-stochastic one) we find its origin in the geometry of the energy surfaces in phase-space. Of this second supersymmetry we also point out its connection with *ergodicity*. In the conclusions we speculate on the existence of a universal supersymmetry present in any quantum system and related to the "quantum" exterior derivative on the energy manifolds.

1. INTRODUCTION

In the last ten years, mainly thanks to the work of Parisi and Sourlas\[1]\[2]\[3], we have learned how to treat stochastic equations via path-integrals. This means that we can derive the correlation functions between solutions of stochastic equations, (Langevin eq. or similar), by means of a path-integral with an action in which the noise has disappeared and has been replaced by an "effective" dynamics. One of the advantages of this approach is that it puts stochastic processes on the same level as statistical or quantum processes that have always been treated via path-integrals.

A surprise of this formulation is that it presents an hidden supersymmetry (susy) that transforms the original bosonic variables into auxiliary Grassmannian variables introduced to obtain the "effective" action mentioned above. This supersymmetry was first used in studies of condensed matter systems\[4]\[5]. Later on it also appeared in the context of Nicolai\[6] maps\[7]\[8] and, surprisingly, even in the field of Stochastic Analysis\[9]. The field, anyhow, in which this supersymmetry has been used mostly is the one of stochastic quantization\[10]. Stochastic quantization is the topic of this volume and we do not like to expand on it here, but just remind the reader that the supersymmetry was first used in that context to masterfully simplify some perturbative calculations\[11] and also, later on, in some "tentative" non-perturbative treatment of stochastic quantization\[12]\[13].

In this flush of activities and applications, a physical understanding of the statistical-mechanical origin of this supersymmetry was very much neglected. A first attempt was presented in ref.[14]\[14]. That paper showed that the supersymmetry was a manifestation of the Onsager principle of microreversibility and this will be the topic of sect.2).

In 1984 it was still unclear which kind of stochastic processes would present this supersymmetry. It was widely believed that only the gaussian one would have this susy. Instead, surprisingly, two years later Zinn-Justin\[14] proved that a wide class of stochastic processes present BRS-like and supersymmetry-like
kind of invariances and not just the gaussian ones. We will report on this in sect. 3).

Inspired by this work\cite{11}, it has been easy to see\cite{14} that even totally deterministic systems, like classical Hamiltonian mechanics (CM), present this kind of supersymmetry once they are formulated via path-integrals. In particular in this case it has been possible to understand the geometrical nature of the BRS and of the susy involved. The BRS charges turned out to be nothing else than the exterior derivative in phase space, while the susy charge could be related to the exterior derivative on the energy surfaces. Moreover it was possible to show that the susy had a nice interplay with the concept of ergodicity\cite{17} \cite{14} This will be reviewed in sect.4). In the conclusions we will speculate about the presence of a universal supersymmetry in any quantum mechanical (QM) system on the line of the universal supersymmetry present in CM. This will not be the sort of supersymmetry we\cite{18} already found in any QM model related to Nelson stochastic quantization\cite{20}. It should be, instead, a supersymmetry related to the "quantum" exterior derivative on the energy surface on the lines of the "quantum" calculus recently developed for non-commuting systems\cite{21}.

2. STOCHASTIC SUPERSYMMETRY AND THE ONSAGER PRINCIPLE OF MICROREVERSIBILITY.

Many processes in nature are described, at least phenomenologically, by the Langevin equations

$$\frac{\partial \phi_i}{\partial t} = -F_i(\phi) + \eta_i \quad i = 1, \cdots , N$$  \hspace{1cm} (2.1)

$\phi_i$ indicate the physical macroscopic quantities of which we study the diffusion. The index "(i)" stands for the different variables that $\phi$ may represent: for example heat density, charge density, matter density, etc. $F_i$ are the drift forces
like temperature gradients, electric fields, gravitational fields, concentration gradients, etc. \( \eta \) is a Gaussian stochastic noise:

\[
\langle \eta_i(t) \eta_j(t') \rangle = 2 \delta_{ij} \delta(t - t')
\]  

(2.2)

We will restrict ourselves to quantities \( \phi_i \) and \( F \) even under time reversal, but it is not difficult to generalize our considerations to mixed systems that have both reversible and irreversible forces and both even and odd \( \phi_i \). It is also possible to generalize (2.2) introducing a diffusion matrix \( D_{ij} \) in (2.2) instead of the simple \( \delta_{ij} \), but we prefer to keep the formalism simple.

As the \( \phi_i \) are stochastic variables, the proper description of their evolution in time is through the concept of probability. We can define a set of "conditional probabilities":

\[
P_1 (\{\phi_i\}, t), \\
P_2 (\{\phi^0_i\}, \{\phi_i\}, t_1), \\
P_3 (\{\phi^0_i\}, \{\phi_i\}, t_1, t_2)
\]

(2.3)

where \( P_1 (\{\phi_i\}, t) \) is the probability to be in configuration \( \{\phi_i\} \) at time \( t \); \( P_2 (\{\phi^0_i\}, \{\phi_i\}, t_1) \) is the probability to be in configuration \( \{\phi_i\} \) at time \( t_1 \) if the system started at time \( t = 0 \) in configuration \( \{\phi^0_i\} \). In the same way \( P_3 (\{\phi^0_i\}, \{\phi_i\}, \{\phi^0_i\}, t_2) \) is the probability to be in \( \{\phi^0_i\} \) at time \( t_2 \), if it was in \( \{\phi_i\} \) at time \( t_1 \) and in \( \{\phi^0_i\} \) at time \( t = 0 \). As the process described by (2.1) is of the Markoff^{23} type, we only need \( P_2 \), because all the others \( P_n \) can be derived from it. Another important quantity that we will need is the "joint probability" \( W (\{\phi^0_i\}, \{\phi_i\}, t_1) \), that is the product of the probability to be in \( \{\phi^0_i\} \) at time \( t = 0 \) and the probability to be in \( \{\phi_i\} \) at time \( t = t_1 \):

\[
W (\{\phi^0_i\}, \{\phi_i\}, t_1) = P_1 (\{\phi^0_i\}) P_2 (\{\phi^0_i\}, \{\phi_i\}, t_1)
\]

(2.4)

For \( P_1 \) and \( P_2 \) an equation has been derived^{23} that describes their evolution.
in time: it is called the forward Fokker-Planck (FP) or (fp) equation:
\[
\frac{\partial P_2}{\partial t} = -\sum_i \frac{\partial}{\partial \phi_i} F_i P_2 + \sum_i \frac{\partial^2}{\partial \phi_i^2} P_2 \equiv \hat{L} P_2
\] (2.5)

with
\[
\hat{L} \equiv -\sum_i \frac{\partial}{\partial \phi_i} F_i + \sum_i \frac{\partial^2}{\partial \phi_i^2}
\]

From \( P_2 \), under time reversal, we can define a probability \( \tilde{P}_2 \)
\[
P_2(\{\phi^0_i\}0|\{\phi^1_i\}t_1) \rightarrow P_2(\{\tilde{\phi}^0_i\}0|\{\tilde{\phi}^1_i\}t_1) \equiv \tilde{P}_2(\{\tilde{\phi}^0_i\}0|\{\tilde{\phi}^1_i\}t_1) \] (2.6)

where we have indicated with \( \tilde{\phi}_i \) the time reversed \( \phi_i \). Remembering that we have assumed \( \phi_i \) and \( F_i \) even under time-reversal, we get
\[
\tilde{P}_2(\{\tilde{\phi}^0_i\}0|\{\tilde{\phi}^1_i\}t_1) = \hat{P}_2(\{\phi^0_i\}0|\{\phi^1_i\}t_1)
\]

The evolution equation for \( \tilde{P}_2 \) can be easily derived and it is called the backward Kolmogoroff-Fokker-Planck equation
\[
\frac{\partial \tilde{P}_2}{\partial t} = \sum_i F_i \frac{\partial \tilde{P}_2}{\partial \phi_i} + \sum_i \frac{\partial^2 \tilde{P}_2}{\partial \phi_i^2} \equiv \hat{L} \tilde{P}_2
\] (2.7)

Another approach to stochastic problems is to directly solve eq. (2.1) with some initial probability \( P_1(\{\phi^0_i\}) \) and then evaluate the correlation functions
\[
\langle \phi_i(0)_{\eta} \phi_j(t_1)_{\eta} \cdots \phi_l(t_m)_{\eta} \rangle_{\eta P_1}
\] (2.8)

\( \phi_i(t)_{\eta} \) is to indicate solutions of eq. (2.1) with some given function \( \eta(t) \) and \( \langle \cdot \rangle_{\eta P_1} \) is to indicate averages with respect to \( \eta \) and to the initial distribution \( P_1(\{\phi^0_i\}) \).
It has been shown in ref.[1,2] that these correlation functions can be derived from a generating functional in the usual field-theory fashion

\[ \langle \phi_i(0) \phi_j(t_1) \cdots \phi_i(t_m) \rangle_{\eta} = \left. \frac{\delta^n Z_{ss}^{fp}}{\partial J_i(0) \cdots \partial J_i(t_m)} \right|_{J=0} \]  

(\( f_p \) is for Fokker-Planck and \( ss \) for supersymmetric). As it is easy to understand, the form of \( Z_{ss}^{fp} \) is

\[ Z_{ss}^{fp} = N \int \prod_i D\phi_i \prod_i D\eta_i \ P_1(\{\phi_i(0)\}) \ \bar{\delta}(\phi_i(t) - \phi_i(t)_{\eta}) \exp \left\{ - \int_0^t \left( \frac{\eta_i^2}{4} + \sum_i J_i \phi_i \right) dt \right\} \]  

(2.10)

\( N \) is a normalizing constant and \( D\phi_i = \text{Lim}_{N \to \infty} \prod_{j=0}^N d\phi_i(t_j) \) is the usual definition of path-integral measure. The \( \bar{\delta}(\phi_i(t) - \phi_i(t)_{\eta}) \) is a "formal" functional Dirac-delta

\[ \bar{\delta}(\phi_i(t) - \phi_i(t)_{\eta}) = \text{Lim}_{N \to \infty} \prod_{j=0}^N \delta(\phi_i(t_j) - \phi_i(t_j)_{\eta}) \]  

that can be written as

\[ \bar{\delta}(\phi_i - \phi_i(\eta)) = \bar{\delta}(\phi_i + F_i - \eta_i) || \frac{\delta \eta_i}{\delta \phi_j} || \]  

(2.11)

The Jacobian \( || \frac{\delta \eta_i}{\delta \phi_j} || \) is

\[ || \frac{\delta \eta_i}{\delta \phi_j} || = \text{det} \left\{ \{ \delta i \delta j + \frac{\partial F_i(t)}{\partial \phi_j(t')} \} \delta(t - t') \right\} \]  

(2.12)

and it can expressed, using anticommuting variables \( \psi_i, \bar{\psi}_i \), as

\[ || \frac{\delta \eta_i}{\delta \phi_j} || = \int \prod_i D\psi_i D\bar{\psi}_i \ exp \left\{ \sum_i \bar{\psi}_i \left[ \partial t \delta i j + \frac{\partial F_i}{\partial \phi_j} \right] \psi_j dt' \right\} \]  

(2.13)

Inserting (2.13) and (2.11) into (2.10) and performing the integration over \( \eta, \)}
we get (after rescaling the time $t' \to \frac{t}{2}$):

$$Z^{P/J} = \int \prod D\phi_i D\psi_i D\bar{\psi}_i P_i \{\phi_i^0\} \exp\left[-\int_0^{2t} L^{P/J} dt' - \int_0^{2t} \frac{1}{2} \sum_i F_i \phi_i dt' - \frac{1}{2} \int_0^{2t} \sum_i J_i \phi\right]$$

where

$$L^{P/J} = \sum_i \left\{ \frac{1}{2} \dot{\phi}_i^2 + \frac{1}{8} F_i^2 - \bar{\psi}_i \left[ \partial_i \delta_{ij} + \frac{1}{2} \frac{\partial F_i}{\partial \phi_j} \right] \psi_j \right\}$$

(2.14)

In the single component case this Lagrangian reduces to

$$L^{P/J}_1 = \frac{1}{2} \dot{\phi}_i^2 + \frac{1}{8} F_i^2 - \bar{\psi}_i \left[ \partial_i \delta_{ij} + \frac{1}{2} \frac{\partial F_i}{\partial \phi_j} \right] \psi_j$$

(2.15)

In this case it is easy to check that this Lagrangian (plus the surface term $\int_0^{2t} \bar{\phi} \dot{i} \phi dt'$) exactly invariant under the transformation:

$$\delta \phi = e \phi$$

$$\delta \psi = 0$$

$$\delta \bar{\psi} = e (\dot{\phi} + \frac{1}{2} F)$$

(2.16)

while it is invariant, up to surface terms, under the following transformation

$$\delta \bar{\psi} = \bar{e} \bar{\psi}$$

$$\delta \bar{\psi} = (\dot{\phi} - \frac{1}{2} F) \bar{e}$$

$$\delta \bar{\psi} = 0$$

(2.17)

The two transformations combines to give a supersymmetry transformations, that means

$$i \delta \bar{\psi} + i \bar{\delta} \phi = \frac{\partial}{\partial t}$$

(2.18)

See the appendix for the "anticommuting" conventions we use.
Naively we would expect that, for the multicomponent case, (2.15) plus the surface term should be invariant under the following supersymmetry transformation:

\[
\begin{align*}
\delta \psi_i &= \epsilon \psi_i + \overline{\epsilon} \overline{\psi}_i \\
\delta \overline{\psi}_i &= \epsilon (\phi_i + \frac{1}{2} F_i) \\
\delta \phi_i &= (\phi_i - \frac{1}{2} F_i) \overline{\epsilon}
\end{align*}
\] (2.20)

Unfortunately this is not the case, even if we restrict ourselves to invariance up to surface terms, unless

\[
\frac{\partial F_i}{\partial \phi_j} = \frac{\partial F_j}{\partial \phi_i}
\] (2.21)

The above relations, new for us, are well known in the literature on stochastic processes and they are called potential conditions. They guarantee, in fact, the existence of a potential function \( V \) from which the \( F_i \) can be derived: \( F_i = \frac{\partial V}{\partial \phi_i} \). This potential \( V \) is what is called, in supersymmetric jargon, superpotential for the lagrangian (2.15).

Let us now discuss the physics behind the conditions (2.21). In most of the phenomenological cases the drift forces \( F_i \) are linear in the variables \( \phi_i \):

\[
F_i = \sum_j M_{ij} \phi_j
\]

with \( M_{ij} \) a constant matrix. In this case the conditions (2.21) become:

\[
M_{ij} = M_{ji}
\] (2.22)

The (2.22) have a well-known physical meaning: they are the celebrated Kelvin reciprocity relations.

As we said at the beginning, the indices \( i,j \) stand for the different macroscopic physical quantities (heat, charge, matter, etc.) of which we study the
diffusion. Calling, for example $i = 1$ for heat-density and $j = 2$ for electric charge density, the relation (2.22) becomes:

$$M_{12} = M_{21}$$

This relation means that the coefficient $M_{12}$, that gives the response of the charge to a temperature gradient, is the same (in proper units) as $M_{21}$, that is the coefficient for the heat response to an electric field (Peltier and Seebeck effects). These identities have been verified experimentally long ago\([24]\).

Onsager\([25]\) in 1931, and before him Bohr\([24]\), asked himself which was the origin of the Kelvin relations (2.22) or, in general, of the potential conditions (2.21). He found the answer in the so-called principle of microscopic reversibility. This principle, as he showed, is the sole and unique hypothesis one needs to derive (2.21) and, in our framework, it becomes the hypothesis we need to have the supersymmetry of $L_{eq}^\phi$.

*It is for this reason that we can say that the principle of microreversibility is really at the origin of the supersymmetry of $L_{eq}^\phi$.*

Let us now look a little more carefully at this microreversibility. As we said before, the variables $\phi_i$ that we have used up to now are usually referred to as macroscopic variables. They are gross variables sufficient to describe the thermodynamics of the system and are a finite number. Of course, they can only give an average or gross behaviour for our system, and it is for this reason that their equation is a stochastic one. In principle, to have a better description of our system, we should not use the macroscopic variables but instead the microscopic ones $\{m_i\}$ associated to each single particle, and from their law of motion we should be able to derive all the features of the system as a whole. This task is, of course, beyond man and computer capacity but it is the basis of statistical mechanics both for equilibrium and non-equilibrium phenomena.

One of the properties of the microscopic dynamics, that might have a reflection also at the macroscopic level, is the fact that the fundamental equations
governing the motion of the individual particles are symmetric with respect to past and future: This is the Onsager principle of microreversibility. This microreversibility will give signals of itself even at the macroscopic levels despite the fact that the macroscopic dynamics is manifestly irreversible. The manner in which the microreversibility manifests itself at the macroscopic level is through some symmetry between phenomenological quantities: the Kelvin reciprocity relations.

Following ref. [25] and [26], let us introduce a complete set \{m_i\} of microscopic variables describing the whole system. As there are so many of these variables, we also need to introduce for them the concept of probability:

$$P_2(\{m_0\}|\{m\}t)$$

This is the conditional probability of finding the system in \{m\} at time t if it was in \{m_0\} at time t=0. Due to the reversibility of the microscopic equations of motion, this probability must be invariant under time-reversal:

$$P_2(\{m_0\}|\{m\}t) = P_2(\{\tilde{m}_0\}|\{\tilde{m}\} - t)$$ (2.23)

\{\tilde{m}\} are the transformed variables under \(t \rightarrow -t\). Having \(P_2\), we can of course derive an expression for the macroscopic joint probability \(W(\{\phi_0\}|\{\phi\}t)\). It is in fact easy to understand that

$$W(\{\phi_0\}|\{\phi\}t) \ d\{\phi\} \ d\{\phi_0\} =$$

$$= \int d\{m\} \int d\{m_0\} P_2(\{m_0\}|\{m\}t) P_1(\{m_0\})$$ (2.24)

In (2.24) \(P_1(\{m_0\})\) is the initial distribution for the variables \{m\}. The integration in (2.24) covers all values of the microscopic variables \{m\} which fulfill
the conditions

\[ \{ \phi_i^0 \} \leq \{ \phi_i([m]) \} \leq \{ \phi_i + d\phi_i \} \]
\[ \{ \phi_i^0 \} \leq \{ \phi_i^0([m_0]) \} \leq \{ \phi_i^0 + d\phi_i^0 \} \]

Changing the integration variables \( \{ m \} \rightarrow \{ \tilde{m} \} \) and \( \{ m_0 \} \rightarrow \{ \tilde{m}_0 \} \) in (2.24) and using the symmetry property (2.23), we easily obtain

\[ W(\{ \phi^0 \} | \{ \phi \}, t) = W(\{ \phi_0^0 \} | \{ \phi \} - t) \]

(2.26)

Remembering the definition (2.4) of joint probability, we can rewrite (2.26) as

\[ P_1(\{ \phi_i^0 \}) P_2(\{ \phi_i^0 \} | \{ \phi_i \}, t) = P_1(\{ \phi_i \}) P_2(\{ \phi_i^0 \} | \{ \phi_i \} - t) \]

that is

\[ P_1(\{ \phi_i^0 \}) P_2(\{ \phi_i^0 \} | \{ \phi_i \}, t) = P_1(\{ \phi_i \}) P_2(\{ \phi_i^0 \} | \{ \phi_i \} t) \]

(2.27)

These relations are the cornerstone for the proof of the potential conditions (2.21). From (2.27) we have for \( P_2 \):

\[ P_2(\{ \phi_i^0 \} | \{ \phi_i \}, t) = \frac{P_1(\{ \phi_i \}) P_2(\{ \phi_i^0 \} | \{ \phi_i \}, t)}{P_1(\{ \phi_i^0 \})} \]

(2.28)

Inserting the RHS of (2.28) in the forward Fokker-Planck equation (2.5), we get

\[ \frac{\partial P_1}{\partial t} \tilde{P}_2 + P_1 \frac{\partial \tilde{P}_2}{\partial t} = - \sum_i \frac{\partial}{\partial \phi_i} (P_1 P_2) \tilde{P}_2 - \sum_i P_1 P_2 \frac{\partial^2 \tilde{P}_2}{\partial \phi_i^2} \]
\[ + \sum_i \frac{\partial P_1}{\partial \phi_i} P_2 + P_1 \sum_i \frac{\partial^2 \tilde{P}_2}{\partial \phi_i^2} \]
\[ + 2 \sum_i \frac{\partial P_1}{\partial \phi_i} \frac{\partial \tilde{P}_2}{\partial \phi_i} \]

(2.29)

Here \( P_1 \) stands for \( P_1(\{ \phi_i \}) \) and \( \tilde{P}_2 \) for \( \tilde{P}_2(\{ \phi_i^0 \} | \{ \phi_i \}, t) \). Using now the fact
that \( P_2 \) itself satisfies the forward FP equation, we obtain

\[
P_2 \frac{\partial \bar{P}_2}{\partial t} = -\sum_i (P_i P_2) \frac{\partial \bar{P}_2}{\partial \phi_i} + P_1 \sum_i \frac{\partial^2 \bar{P}_2}{\partial \phi_i^2} + \sum_i 2 \frac{\partial P_1 \partial \bar{P}_2}{\partial \phi_i \partial \phi_i} \tag{2.30}
\]

As \( \bar{P}_2 \) obeys the backward FP equation, we can further simplify (2.30) and reduce it to

\[
2P_1 \left[ \sum_i F_i \frac{\partial \bar{P}_2}{\partial \phi_i} \right] + 2 \frac{\partial P_1 \partial \bar{P}_2}{\partial \phi_i \partial \phi_i} = 0 \tag{2.31}
\]

Let us now multiply this expression on the left for any arbitrary function \( G(\{\phi\}) \) and integrate over \( \phi \)

\[
\int G(\{\phi\}) \left[ P_1 \sum_i F_i \frac{\partial}{\partial \phi_i} + \frac{\partial P_1}{\partial \phi_i} \right] \bar{P}_2 \tag{2.32}
\]

This equation holds for any time \( t \) and even for \( t = 0 \). \( P_2 \) as well as \( \bar{P}_2 \) at \( t = 0 \) are equal to \( \delta(\{\phi_i\} - \{\phi^0_i\}) \) for the definition itself of conditional probability. Inserting this in (2.32) and integrating by parts, we obtain an equation involving \( G \) and \( G' \). To satisfy this equation, as \( G \) and \( G' \) are arbitrary, we have to have the coefficients of these functions separately equal to zero. The coefficient of \( G' \) is

\[
P_1 F_i + \frac{\partial P_1}{\partial \phi_i} = 0
\]

From this it is clear, as \( F_i = -\frac{\partial P_1}{\partial \phi_i} \) that

\[
\frac{\partial F_i}{\partial \phi_j} = \frac{\partial F_j}{\partial \phi_i} \tag{2.33}
\]

These are the potential conditions (2.21). To summarize this long proof, we want to remind the reader that we started from (2.23) that is the Onsager principle of
micro reversibility. This proves that the micro reversibility is a sufficient condition for the potential conditions. We do not know if it also a necessary one, but for sure in the last 60 years no proof of the potential condition has appeared that does not make use of the micro reversibility.

In the introduction of this paper, we stated our goal to be that of discovering which symmetry of the stochastic process is behind the supersymmetry of $Z_{\phi}^{P}$. After the analysis presented above, we can answer that that symmetry is the microscopic reversibility.

It is true, of course, that to generate the typical supersymmetric form (2.14), we needed other ingredients such as: 1) to start from an equation first order in $\frac{\partial}{\partial t}$ (Langevin eq.), and 2) with a Gaussian noise. They are two ingredients that the Nicolai theorem \cite{10} seems to indicate as necessary features of any supersymmetric action. They are, anyhow, structural properties and not symmetry features of the stochastic process. The previous analysis revealed that a third ingredient was needed, that is (2.21) and this condition brings in the signal of a symmetry. A stochastic Gaussian process without (2.21), and so without time reversibility, would not have generated the supersymmetry of (2.14).

We can at this point state with enough confidence that the supersymmetry is a manifestation, at the macroscopic level, of the time reversibility present at the level of microscopic variables. To further tie up micro reversibility with supersymmetry, we will analyze in the following the Ward identities that stem from the supersymmetry invariance of $Z_{\phi}^{P}$. These identities, once the Grassmannian variables have been integrated away, turn up to be symmetry relations between the forward and the backward Fokker-Planck dynamics, thus unveiling the real character of this symmetry as a time symmetry.

It might still be obscure to the reader the connection between the $Z_{\phi}^{P}$ of (2.14) and the Fokker-Planck equations. An analysis of this connection was presented, among other places, in the first and second reference of \cite{12} and in \cite{14} and we will follow the notation used there. Let us first observe that, if the poten-
tial conditions (2.21) hold, then we can introduce the following functions \( \Psi \) and \( \tilde{\Psi} \) in place of the forward and backward probability \( P \) and \( \tilde{P} \):

\[
\Psi \equiv P e^{\gamma}, \quad \tilde{\Psi} \equiv \tilde{P} e^{-\gamma}
\]

with \( F_i = \frac{\partial V}{\partial \phi_i} \). The equation of motion for \( \Psi \) and \( \tilde{\Psi} \) are easily derived from (2.5) and (2.7). They have the following Schrödinger-type form (after rescaling \( t \rightarrow \frac{t}{2} \))

\[
\frac{\partial \Psi}{\partial t} = -\hat{H}^{FP}_{\text{forward}} \Psi
\]

\[
\frac{\partial \tilde{\Psi}}{\partial t} = -\hat{H}^{FP}_{\text{backward}} \tilde{\Psi}
\]

where

\[
\hat{H}^{FP}_{\text{forward}} = \sum_i \left[ -\frac{1}{2} \frac{\partial^2}{\partial \phi_i^2} + \frac{1}{8} \left( \frac{\partial V}{\partial \phi_i} \right)^2 - \frac{1}{4} \frac{\partial^2 V}{\partial \phi_i^2} \right]
\]

\[
\hat{H}^{FP}_{\text{backward}} = \sum_i \left[ -\frac{1}{2} \frac{\partial^2}{\partial \phi_i^2} + \frac{1}{8} \left( \frac{\partial V}{\partial \phi_i} \right)^2 + \frac{1}{4} \frac{\partial^2 V}{\partial \phi_i^2} \right]
\]

We called these two operators the forward and backward Fokker-Planck Hamiltonians. They are positive semidefinite operators, in fact they can be written in the form

\[
\hat{H}^{FP}_{\text{forward}} = \frac{1}{2} \sum_i Q_i Q_i^\dagger, \quad \hat{H}^{FP}_{\text{backward}} = \frac{1}{2} \sum_i Q_i^\dagger Q_i
\]

with

\[
Q_i = \left[ \frac{\partial}{\partial \phi_i} - \frac{1}{2} \frac{\partial V}{\partial \phi_i} \right]
\]

Let us now look at the solutions of (2.35). They can be written in the usual
\[ \cos \theta = \frac{\beta}{m} \]

\[ m \beta \cos \theta = m \]

\[ \theta = \arccos \left( \frac{\beta}{m} \right) \]

\[ \frac{\beta}{m} = \frac{1}{2} \]

\[ \cos \theta = \frac{1}{2} \]
form
\[ \Psi = \sum_n c_n \psi_n e^{-E_n t}, \quad c_n = \text{constant} \]

with \( \hat{H}_{\text{forn}} \psi_n = E_n \psi_n \). The solutions of (2.36) can be written as
\[ \tilde{\Psi} = \sum_n c_n \tilde{\psi}_n e^{-E_n t}, \quad c_n = \text{const} \]

with \( \hat{H}_{\text{back}} \tilde{\psi}_n = \tilde{E}_n \tilde{\psi}_n \). In most cases we will be interested in the equilibrium probability \( P_{eq} \) and this is obtained from the solution (2.35) taking the limit \( t \to \infty \), that is
\[ \Psi_{eq} = \lim_{t \to \infty} \sum_n c_n \psi_n e^{-E_n t} = c_0 \Psi_0 \]

\( \Psi_0 \) is the ground state, at \( E_0 = 0 \), of \( \hat{H}_{\text{forn}} \) and it is \( \Psi_0 = e^{-Y} \). Remembering that \( \Psi = P e^Y \) we get
\[ \lim_{t \to \infty} P = P_{eq} = e^{-Y} \] (2.40)

This limit has a meaning only if \( \int P_{eq} d\phi = 1 \), that is equivalent to saying that \( \Psi_0 = e^{-Y} \) must be normalizable. If this is so, then the ground state i.e. \( (E_0 = 0) \), of \( \hat{H}_{\text{back}} \), that is \( \tilde{\Psi}_0 = e^{Y} \), cannot be a normalizable one and so it cannot be accepted as part of the spectrum of \( \hat{H}_{\text{back}} \). This means that the spectrum of \( \hat{H}_{\text{back}} \) is positive-definite i.e. \( \tilde{E}_n > 0 \).

Having re-written the FP equations in the form (2.35) and (2.36), let us go back now to the generating functional (2.14):
\[ Z^{FP}_{ss} = \int \prod_i D\psi_i D\bar{\psi}_i D\bar{\psi}_i \exp\left\{ -\int_0^t \sum_i \left[ \frac{\phi_i^2}{2} + \frac{F_i^2}{2} - \bar{\psi}_i (\delta_i \delta_{ij} + \frac{1}{2} \frac{\partial F_i}{\partial \phi_j}) \psi_j \right] dt \right\} \] (2.41)

In (2.41) we can perform the integration over \( \psi_i \) and \( \bar{\psi}_j \) and what we will
formally get is just
\[
\det \left[ \partial_t \delta_{ij} + \frac{1}{2} \frac{\partial F_i}{\partial \phi_j} \right] \delta(t - t')
\]
In the same way as in ref. [12] and [14], we can just evaluate this "det" as a product of eigenvalues
\[
\det \left[ \partial_t \delta_{ij} + \frac{1}{2} \frac{\partial F_i}{\partial \phi_j} \right] \delta(t - t') = \prod_{n=-\infty}^{\infty} \alpha^{(n)}
\]
where \( \alpha^{(n)} \) are defined by
\[
\sum_j \left[ \partial_t \delta_{ij} + \frac{1}{2} \frac{\partial F_i}{\partial \phi_j} \right] \psi_j^{(n)} = \alpha^{(n)} \psi_i^{(n)}
\]
The solutions of this equations are
\[
\psi_j^{(n)}(t) = [ \text{exp} \left\{ \int_0^t dt' (\alpha^{(n)} \delta - \frac{1}{2} \dot{\hat{O}}) \right\} ]_j \psi_i^{(n)}(0)
\]
where we indicated with \( \dot{\hat{O}} \) the matrix \( \delta_{ij} \) and with \( \dot{\hat{O}} \) the matrix \( \frac{\partial F_i}{\partial \phi_j} \). Imposing periodic boundary conditions: \( \psi_i(t) = \psi_i(0) \) we get for \( \alpha^{(n)} \)
\[
\alpha^{(n)} = i \left( \frac{2n\pi}{t} \right) + \frac{1}{N t} \int_0^t dt' \text{Tr} \hat{O}
\]  
(2.42)
where \( N \) is the number of components of \( F_i \). Using (2.42) we can immediately calculate the "det"
\[
\text{"det"} = \prod_{n=-\infty}^{\infty} \alpha^{(n)} = \prod_{n=-\infty}^{\infty} \left[ i \frac{2n\pi}{t} + \frac{1}{N t} \text{Tr} \int_0^t dt' \hat{O} \right] =
\]
\[
= C \sinh \frac{1}{4} \int_0^t dt' \sum_i \frac{\partial F_i}{\partial \phi_i}
\]  
(2.43)
where \( C \) is a normalizing constant independent of \( \phi \). Inserting (2.43) back
We get

\[ Z_{sp}^{f} = \frac{C}{2} \left\{ \int \prod_{i} D\phi_{i} \exp \left[ -\int_{0}^{2t} \sum_{i} \left( \frac{\phi_{i}^{2}}{2} + \frac{F_{i}^{2}}{8} - \frac{1}{4} \left( \frac{\partial F_{i}}{\partial \phi_{i}} \right) \right) dt' \right] + \right. \\
\left. - \int \prod_{i} D\phi_{i} \exp \left[ -\int_{0}^{2t} \sum_{i} \left( \frac{\phi_{i}^{2}}{2} + \frac{F_{i}^{2}}{8} + \frac{1}{4} \left( \frac{\partial F_{i}}{\partial \phi_{i}} \right) \right) dt' \right] \right\} \\
\equiv \frac{C}{2} \left\{ Z_{sp}^{f_{\text{forw}}} - Z_{sp}^{f_{\text{back}}} \right\} \]  

(2.44)

We see that \( Z_{sp}^{f} \) is just the sum of two generating functionals, the first one associated with the "Euclidean" lagrangian of \( \hat{H}_{sp}^{f_{\text{forw}}} \) of (2.35) and the second with \( \hat{H}_{sp}^{f_{\text{back}}} \) of (2.36). This is the reason we called \( Z_{sp}^{f} \) the generating functional of the overall Fokker-Planck dynamics.

The presence of both forward and backward dynamics in the supersymmetric form of the generating functional is very amusing and can also be understood at the operatorial Hamiltonian level\[ \text{[13]} \].

Before proceeding to discuss further applications of the supersymmetry, we would like to pause for a moment to take up the issue of the boundary conditions we used in our generating functional (2.14). There we inserted the initial distribution \( P_{1}(\phi_{0}) \) and this will explicitly break the supersymmetry unless we choose an explicitly supersymmetric invariant distribution. Of course the kind of breaking it causes is only a "surface-terms" kind of breaking because \( P_{1}(\phi_{0}) \) is just a surface-term. So it should not worry us too much. In fact "surface-terms" breaking were already produced even in the \( \delta^{\ast} \)-variation (2.18). To cancel these we should choose boundary conditions that satisfy the condition

\[ \{ \bar{\psi}(2t) [\phi(2t) + \frac{1}{2} F(\phi(2t))] \} - \{ \bar{\psi}(0) [\phi(0) + \frac{1}{2} F(\phi(0))] \} = 0 \]  

(2.45)

If, among the solutions of this equation we can find those that are \( \delta^{\ast} \) invariant, then our (2.14) would be perfectly supersymmetric invariant not just invariant up
One solution of the eq. (2.45) and which is also supersymmetric invariant is the periodic one

\[ P_1 = \delta(\phi(t) - \phi(0))\delta(\psi(t) - \psi(0))\delta(\bar{\phi}(t) - \bar{\phi}(0)) \]  \hspace{1cm} (2.46)

as it is easy to check. There might of course be more solutions. Anyhow we should not worry too much about these surface-terms breaking because it has been proven\(^{[m]}\) that in the equilibrium limit \(2t \to \infty\) the averages are independent of these surface terms, at it should be for any Markoff process. Moreover, even in the most delicate case, that is the diffusion of quantum fields\(^{[m]}\), these surface terms do not spoil the renormalizability of the theory. For a complete study\(^{[pa]} [29, 30]\) of the issue of the b.c. and of surface terms see ref.[27, 28, 29, 30].

The reader may worry of why we give two boundary conditions in the eq. (2.46) above and if this is not in contradiction with the Langevin eq. that, being first order in \(\phi\), need only one initial condition. The answer is simple. When we wrote the \(Z_2^{fp}[J = 0]\) (2.14), we actually could have written it as the integral of a joint probability \(W(\{\phi^0\}0;\{\phi\}2t)\) over the initial and final \(\phi\)

\[ Z_2^{fp}[J = 0] = \int d\phi^0 d\phi W(\{\phi^0\}0;\{\phi\}2t) \]  \hspace{1cm} (2.47)

(And that is actually equal to one!). We know that the joint probability \(W\) can be written as

\[ W(\{\phi^0\}0;\{\phi\}2t) = P_1(\{\phi^0\})P_2(\{\phi^0\}0;\{\phi\}2t) \]  \hspace{1cm} (2.47)

The object that has a very clear path-integral representation is the transition-probability \(P_2(\{\phi^0\}0;\{\phi\}2t)\) It could be written from (2.14) as

\[ P_2 = e^{-\frac{1}{\hbar}\left\{V(\phi(2t)) - V(\phi(0))\right\}} \int D\mu \int D\bar{\phi} \]  \hspace{1cm} (2.48)

The path-integral measure \(D\mu\) means the integral over all fields \(\phi, \psi, \bar{\psi}\). From this path-integral it is clear that we have to give two conditions, an initial and a
final one, and this is not in contradiction with the Langevin eq. It is exactly like
in quantum mechanics. This should clear up some misunderstanding related to
the Nicolai map. The path-integral part in (2.48) could be written, remember-
ing (2.44), and keeping only the $\hat{H}^{fp}_{\text{forw}}$, as

$$
\Psi_f^{(2t)} \int \mathcal{D}\mu e^{-\int L^{fp}_{\text{forw}}} = \langle \Psi_i | \Psi_f \rangle = \bar{\Psi}_i(0)\Psi_f(2t)
$$

(2.49)

Using now (2.34) and (2.49) in (2.48), we obtain "formally"

$$
P_z = \langle P(0)|P(2t) \rangle
$$

that is the correct thing to get. The careful reader will realize that this analysis
sheds light on the role of the surface term and on the similarity transformation
contained in (2.34).

In the first two papers of ref.[12], the presence of both $Z^{fp}_{\text{forw}}$ and $Z^{fp}_{\text{back}}$ in
$Z^{fp}_{\text{str}}$ (2.44) was further exploited, deriving symmetry relations between a limited
class of Green's functions calculated with $Z^{fp}_{\text{forw}}$ and the same Green's functions
calculated with $Z^{fp}_{\text{back}}$. These relations were successively generalized in the fourth
of ref.[7] in a very systematic way and proven to come from the Ward-identities
associated to the supersymmetry. To derive these relations, we have to introduce
fermionic currents $J_\psi, J_\bar{\psi}$ in (2.41) and also, in a perfect supersymmetric
fashion, an auxiliary field $\omega_i$ and its associated current $J_\omega$. The form of $Z^{fp}_{\text{str}}$ will then be

$$
Z^{fp}_{\text{str}} = \int \prod \mathcal{D}\phi_i \mathcal{D}\omega_i \mathcal{D}\psi_i \mathcal{D}\bar{\psi}_i \exp \left[ -\int (L^{fp}_{\text{str}} + \phi J_\psi + \omega J_\omega + \bar{\psi} J_{\bar{\psi}})dt \right]
$$

(2.50)

where we have indicated, in a compact way, $\phi J_\psi = \sum_i \phi_i J_\psi$, and the same for
the other fields. $L^{fp}_{\text{str}}$ in (2.50) is

$$
L^{fp}_{\text{str}} = \sum_{ij} \left[ F_{ij} \frac{\partial^2}{2} - \frac{\omega_i F_i}{2} + \frac{\omega_j F_j}{2} - \bar{\psi}_i (\partial_i \delta_{ij} + \frac{1}{2} \delta F_i) \psi_j \right]
$$

The set of currents in (2.50) must transform in a special way for the overall
\[ Z_{ss}^{fp} \] to be supersymmetric invariant. It is easy to derive these transformations from (2.20)

\[
\begin{align*}
\delta^s J_\phi &= \epsilon J_\psi + \bar{J}_\psi \bar{\epsilon} \\
\delta^s J_\omega &= -\epsilon J_\psi + \bar{J}_\psi \bar{\epsilon} \\
\delta^s J_\phi &= \epsilon J_\psi - \frac{\bar{\epsilon}}{2} \bar{J}_\psi \\
\delta^s J_\psi &= -\epsilon J_\psi - \frac{\epsilon}{2} \bar{J}_\psi
\end{align*}
\]

(2.51)

Under this transformation we have

\[
\delta^s Z_{ss}^{fp}[J_\phi, J_\psi, J_\omega] = 0
\]

(2.52)

In (2.50) we can, as we did in (2.44), integrate away the fermionic variable, and we will be left with

\[
Z_{ss}^{fp} = Z_{forw}^{fp}[J_\phi, J_\omega] F[J_\phi, J_\psi] - Z_{back}^{fp}[J_\phi, J_\omega] F[J_\phi, J_\psi]
\]

(2.53)

where

\[
F[J_\phi, J_\psi] = \exp \left[ - \int \sum_{ij} J_{\phi_i}(t) \Delta^{-1}(t-t') J_{\phi_j}(t') dt dt' \right]
\]

and

\[
\Delta(t-t')_{ij} = \left[ \partial_t \delta_{ij} + \frac{1}{2} \partial_\phi \delta_{ij} \right] \delta(t-t')
\]

The invariance (2.52) can now be read as

\[
\delta^s Z_{ss}^{fp} = \delta^s (Z_{forw}^{fp} F) - \delta^s (Z_{back}^{fp} F) = 0
\]

or

\[
\delta^s (Z_{forw}^{fp} F) = \delta^s (Z_{back}^{fp} F)
\]

(2.54)

* Note that the transformations presented here are different than those in ref. [14]. There we unfortunately made some mistakes in signs and factors, but the overall result is not affected at all.
We can, at this point, apply any operator

\[ \hat{O} \left[ \frac{\delta}{\delta J_\phi}, \frac{\delta}{\delta J_\omega}, \frac{\delta}{\delta J_\psi}, \frac{\delta}{\delta J_\tilde{\psi}} \right] \]

on the LHS of (2.52) and we get

\[ \hat{O} \delta^{\alpha \beta} Z_{\alpha \beta}^F = 0 \]  

(2.55)

Using (2.51) in (2.55) and putting the currents to zero at the end, we obtain identities of the form

\[ \int G[\phi_i, \omega_j, \psi_k, \tilde{\psi}_l] \exp \left[ - \int L_{\text{back}}^F \right] = 0 \]

(2.56)

where \( G[\cdot] \) are combinations of Green's functions. In field theory language (2.56) are just the Ward-identities associated with the symmetry (2.20).

Let us now repeat the same steps we have done to derive (2.56), starting this time from (2.54). Applying the operator \( \hat{O} \) to both sides of (2.54), we have

\[ \hat{O} (\delta^{\alpha \beta} Z_{\text{forward}}^F) = \hat{O} (\delta^{\alpha \beta} Z_{\text{back}}^F) \]

and using, once again (2.51), we get

\[ \int G[\cdot] \exp \left[ - \int L_{\text{forward}}^F \right] = \int G[\cdot] \exp \left[ - \int L_{\text{back}}^F \right] \]

(2.57)

Due to the structure of (2.53), single fermion fields never enter in \( G[\cdot] \). In their place we will have \( \Delta^{-1} \) (entering as \( \psi \tilde{\psi} \)). So we can formally re-write (2.57) as

\[ \int G[\phi, \omega, \Delta^{-1}] \exp \left[ - \int L_{\text{forward}}^F \right] = \int G[\phi, \omega, \Delta^{-1}] \exp \left[ - L_{\text{back}}^F \right] \]

(2.58)

These relations are the most general and complete identities we can derive when we fully exploit the supersymmetry (2.20). We see that they express a time-reversal invariance that survives at the macroscopic level on some particular
functions $G[·]$. This invariance is a relic, at the macroscopic level, of the time-invariance present at the microscopic level (micro-reversibility). So, even if the macroscopic dynamics is irreversible, a small signal of the micro-reversibility shows up at the macroscopic level in the Kelvin reciprocity relations (as Onsager discovered) and in the relations (2.58). Further ideas on the interplay between macroscopic irreversibility and microscopic reversibility have been discussed in ref.[32][32].

The fact that a supersymmetry becomes a "quasi-discrete" symmetry has been explored also for the super-rotations introduced by Parisi and Sourlas in ref.[1]. The interplay we studied[23][24] was between this supersymmetry and the PCT invariance of relativistic field theory. We will not discuss it here the details but refer the interested reader to the relevant literature[23][24].

Let us now go back to (2.58). We want to analyze the Ward-identities above in the equilibrium limit $t \rightarrow \infty$ of the Fokker-Planck dynamics. Borrowing the formalism on the FP hamiltonians developed previously, (2.35), (2.36), we can write (2.58) as

$$Tr\{G[φ,ω,Δ^{-1}]e^{-2\hat{H}_\text{FP}}t\} = Tr\{G[φ,ω,Δ^{-1}]e^{-2\hat{H}_\text{FP}}t\} \quad (2.59)$$

In the limit of $t \rightarrow \infty$ we have

$$\lim_{t \rightarrow \infty} \sum_n \langle n|G[·]|n\rangle e^{-2E_n t} = \lim_{t \rightarrow \infty} \langle \hat{n}|G[·]|\hat{n}\rangle e^{-2\tilde{E}_n t} \quad (2.60)$$

where $|n\rangle$, $E_n$ and $|\hat{n}\rangle$, $\tilde{E}_n$ are eigenstates and eigenvalues of, respectively, $\hat{H}_\text{FP}^{\text{forw}}$ and $\hat{H}_\text{FP}^{\text{back}}$. Remembering that $E_n \geq 0$ while $\tilde{E}_n > 0$, we get from (2.60)

$$\langle 0|G[·]|0\rangle = 0 \quad (2.61)$$

where $|0\rangle$ is the ground state of $\hat{H}_\text{FP}^{\text{forw}}$ and, in the $\phi$ representation, is
\[ \langle \phi | 0 \rangle = e^{-\frac{1}{2}}. \]

In the path integral formalism, we can read (2.61) as

\[ \int G[\cdot] \exp \left[ \int_{0}^{\infty} L_{\text{F}}^{\text{pr}} dt \right] = 0 \]  

(2.62)

These relations (2.61) and (2.62) are extremely important: they indicate that, as a consequence of supersymmetry, there are a set of operators whose vacuum expectation value is zero. "Vacuum" means ground state of the forward Fokker-Planck Hamiltonian. Of course these functions \( G[\cdot] \) are not generic functions but very particular ones: they are only those combination which enter in the Ward-identities (2.56).

One particular interesting case of this relation (2.62) occurs when we choose the operator \( \hat{O} \) of (2.55) to be

\[ \hat{O} = \frac{\delta}{\delta J_{\phi_1}(t_1) \delta J_{\phi_2}(t_2)} \]

The relation (2.55) in this case turns out to be

\[ \frac{\partial}{\partial t_1} \langle 0 | \phi_k(t_2) \phi_j(t_1) | 0 \rangle = \frac{1}{2} \langle 0 | \phi_k(t_2) \omega_j(t_1) | 0 \rangle \]

(2.63)

This is the well-known fluctuation-dissipation theorem\(^{[24]}\). In fact the RHS of (2.63) is related to what is known as the "response function" of the system\(^{[24]}\), and the relation above relates the response function (the dissipation) to the two-point function (fluctuation) of the system. We will not expand on it more but refer the interested reader to ref.[14].

\[ * \text{ There is a factor } -\frac{1}{2} \text{ different than the same relation derived in ref.[14] but this does not change the conclusions.} \]
3. GENERAL BRS INVARIANCE FOR NON-GAUSSIAN SYSTEMS

In the previous section we have seen that to have Susy invariance in Gaussian processes, the potential conditions (2.21) were needed. To be precise the potential conditions (2.21) were just needed to have the $\delta$ invariance of (2.18), because the $\delta$-invariance of (2.17) would have been present anyhow. So the role of the form of the drift force is now "somehow" understood. What was less understood was the role of the Gaussian process. That means nobody had analyzed if any Grassmannian kind of symmetry would survive if we modify the nature of the stochastic process, changing it from Gaussian to other distributions. Zinn-Justin\textsuperscript{[10]} answered this question.

Let us suppose we have a general stochastic diffusion equation for a macroscopic field $\phi(x)$

$$F_\alpha(\phi(x)) = \eta_\alpha(x) \quad (3.1)$$

$\eta$ is the noise and $F$ is any function of $\phi$ and its derivatives. The distribution of the noise $\eta$ is not taken quadratic but of a general form $[d\rho(\eta)] = [d\eta]exp(-\sigma(\eta))$ Let us now, as in the previous section, write down the generating functional from which the stochastic correlations functions

$$\langle \phi(z_1)_\eta \cdots \phi(z_l)_\eta \rangle$$ could be derived. It is immediate to realize that it has the form

$$Z_F[J] = \int D\eta D\phi \delta[F_\alpha(\phi) - \eta_\alpha] \det M \exp(-\sigma(\eta)) \exp(\int dz J_\alpha \phi_\alpha(x)) \quad (3.2)$$

where, as before,

$$M_{\alpha\beta}(z, y) = \frac{\delta F_\alpha(\phi(x))}{\delta \phi_\beta(y)}$$

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Next we give a functional integral representation for the equation (3.1).

\[ \delta(F_\alpha(\phi) - \eta_\alpha) = \int D\omega_\alpha \exp\left\{ - \int dx \omega_\alpha(x) F_\alpha(\phi) + \int dx \omega_\alpha(x) \eta_\alpha(x) \right\} \]  
(3.3)

We also introduce the generating functional \( W(\omega) \) of connected \( \omega \) fields correlations, that is

\[ \exp(W(\omega)) = \int D\eta \exp\left( \int dx [\sigma(\eta) - \omega_\alpha \eta_\alpha(x)] \right) \]  
(3.4)

Finally we write the \( \det M \) in the usual way via Grassmannian fields \( \psi_\alpha, \bar{\psi}_\alpha \):

\[ \det M = \int D\bar{\psi}_\alpha(x) D\psi_\alpha(x) \exp\left( \int dz dy \bar{\psi}_\beta \bar{M}_{\alpha,\beta}(x,y) \psi_\beta(y) \right) \]  
(3.5)

Inserting (3.4) and (3.5) in the \( Z_F[J] \) of (3.2), we get

\[ Z_F[J] = \int D\phi D\bar{\psi} D\psi D\omega \exp\left\{ -\bar{S}(\phi, \psi, \bar{\psi}, \omega) + \int J_\alpha(x) \phi_\alpha(x) \right\} \]  
(3.6)

where \( \bar{S} \) is

\[ \bar{S} = -W(\omega) + \int \omega_\alpha(x) F_\alpha(\phi) dz - \int dz dy \bar{M}_{\alpha,\beta}(x,y) \psi_\beta(y) \]  
(3.7)

The action above is invariant under the following transformation that resembles the BRS invariance \(^{[7]} \) of gauge theory

\[ \delta^{brs}_\phi \phi_\alpha(x) = e \psi_\alpha(x) \]

\[ \delta^{brs}_\psi \psi_\alpha(x) = 0 \]

\[ \delta^{brs}_\bar{\psi} \bar{\psi}_\alpha(x) = e \omega_\alpha(x) \]

\[ \delta^{brs}_\omega \omega_\alpha(x) = 0 \]  
(3.8)

A natural question to ask is if there is also an analog of the anti-BRS invariance
of the form

\[ \delta \phi^{brs}_\alpha(x) = \bar{\delta} \psi^\alpha(x) \]
\[ \delta \phi^{brs}_\alpha(x) = - \bar{\delta} \omega^\alpha(x) \]
\[ \delta \phi^{brs}_\alpha(x) = 0 \]
\[ \delta \phi^{brs}_\omega(x) = 0 \]

(3.9)

The answer is negative, unless some "potential conditions" are satisfied

\[ \frac{\delta F_\alpha}{\delta \phi^\alpha} = \frac{\delta F_\beta}{\delta \phi^\beta} \]

in complete analogy with our analysis for the supersymmetry.

Once these conditions are satisfied, anyhow, the BRS and anti-BRS do not combine, like the (2.17) and (2.18), to give a supersymmetry like in (2.19). In fact the combination of the two does not give a time derivative like in (2.19), but it gives just zero:

\[ \delta \phi^{brs} \delta \phi^{brs} + \delta \phi^{brs} \delta \phi^{brs} = 0 \]

It would be nice to find out which combination of "noise" distribution \( W(\omega) \) and of drift force \( F_\alpha \) would produce an invariance of the action \( \tilde{S} \) also under the supersymmetry (2.17), (2.18). This question is too hard to investigate if we do not do any hypothesis on the structure of the time-derivatives, \( \dot{\phi} \), contained in \( F_\alpha \). In the next section we will find anyhow a striking example of a system that presents both the BRS and anti-BRS invariances of (3.8) and (3.9) and a supersymmetry of the type (2.19).

Anyhow, before jumping to the next session, we would like to expand a little bit more on the physical meaning of the BRS found in general systems. Zinn-Justin explains that the BRS is nothing else but the local version of a non-local transformation related to a shift in the noise. The noise, in fact, is integrated away in the generating functional and so we can do any transformation on it, provided we compensate for it at the level of the effective action \( \tilde{S} \) (3.7) that
"effectively" reproduce the dynamics of the noise. This is in perfect analogy with the BRS invariance of gauge theory\[^{[30]}\]: the BRS invariance is the local version of a non-local transformation called Slavnov-transformation.\[^{[33]}\] This Slavnov transformation is non-local but it does not require the introduction of anticommuting variables. In making it local we have to pay the price of having auxiliary anticommuting variables.

For the system of eq. (3.1),

\[
F(\phi) = \eta
\]  

(3.10)

the non-local transformation, (for simplicity we restrict to one field component), is the following:

\[
\eta \rightarrow \eta + \mu \\
\delta \phi = \int M^{-1} \mu
\]

(3.11)

with \( M = \frac{\delta F}{\delta \phi} \). It is easy to see that the eq.(3.10) is invariant under the transformation (3.11), in fact

\[
F(\phi) + \frac{\delta F}{\delta \phi} \delta \phi = \eta + \mu
\]

and using (3.11), we get

\[
F(\phi) + \frac{\delta F}{\delta \phi} M^{-1} \mu = \eta + \mu
\]

Now \( M \) is \( \frac{\delta F}{\delta \phi} \) and, inserting it in the above equation, we obtain

\[
F(\phi) + \mu = \eta + \mu
\]

that is

\[
F(\phi) = \eta
\]

So this proves that the (3.11) is an invariance of the eq. (3.10). But how does it relate to the BRS? Well, let us insert in \( \tilde{F} \) of (3.7) the currents associated

\[^{[30]}\text{For the reader not familiar with it we recommend ref.}[39]^{[30]} \]
to all the fields

$$\delta \mathcal{S}_J = -W(\omega) + \int \lambda F - \int \frac{\delta F}{\delta \phi} \psi + J_\phi \phi + J_\omega \omega + J_\psi \psi + \delta J_\phi$$  \hspace{1cm} (3.12)

Let us then derive the equation of motion for $\delta \psi$ from the action above:

$$\frac{\delta \mathcal{S}_J}{\delta \psi} = \frac{\delta F}{\delta \phi} = 0$$

From this we get

$$\psi = -\int J_\psi M^{-1}$$  \hspace{1cm} (3.13)

We put the integral sign considering that the $\phi$ can depend on $x$ and then $M$ is a matrix with $x, y$ as indices (besides the $\alpha, \beta$ indices) over which we have to integrate. Let us now remember the BRS transformation (3.8). The transformation of the $\phi$ was

$$\delta \phi = \epsilon \phi$$

inserting in $\psi$ the expression we got in (3.13), we obtain:

$$\delta \phi = -\epsilon \int J_\psi M^{-1} = \int M^{-1} \mu$$  \hspace{1cm} (3.14)

and this is the (3.11), where we have identified $\mu$ with $\epsilon J_\psi$.

From the steps above we see that we have identified the BRS transformations with the non-local shift (3.11). So the general rule is that it is in general possible to transform non-local variations in local variations but we have to pay the price of introducing anticommuting auxiliary variables.

On the line of the derivation above, it would be interesting to see which is the anticommuting symmetry associated to a rescaling of the noise, and not just a shift. This belongs to the same strategy as before: that we can do anything
We want to the noise provided this is compensated by a transformation on the effective action (3.7). So we want to rescale the noise, in eq. (3.10), in this way:

$$\eta \rightarrow (1 + \lambda)\eta$$  \hspace{1cm} (3.15)

The associated non-local transformation on (3.10) would be

$$\delta \phi = \lambda \int M^{-1} F$$  \hspace{1cm} (3.16)

we can see the non-local character if we explicitly indicate the $x$ or $t$ dependence of $\phi$: (3.16) would then becomes

$$\delta \phi(t) = \int M(t, t') F(t') dt'$$

Using the same tricks as before, and in particular eq. (3.13), it is not difficult to see which local Grassmannian transformation corresponds to (3.16). It is:

$$\delta \phi = \epsilon F(\phi) \psi$$ \hspace{1cm} (3.17)

where we have identified $\lambda = -\epsilon J_\phi$. From now on we will call the $\delta$ appearing in (3.17) as $\delta^d$. Beside the transformations (3.17) on $\phi$, it is not difficult to find the transformations on the other fields. They are

$$\begin{align*}
\delta^d \phi &= \epsilon F \psi \\
\delta^d \psi &= 0 \\
\delta^d \bar{\psi} &= \epsilon \omega F \\
\delta^d \omega &= 0
\end{align*}$$ \hspace{1cm} (3.18)

It is easy to check that the action (3.7) is invariant under it and that moreover:

$$\delta^d \delta^d + \delta^d \delta^d = 0$$

So $\delta^d$ has a true BRS-like character.
Another transformation that is also a symmetry of (3.7), and that is somehow similar to (3.18), is:

\[
\begin{align*}
\delta^d \phi &= \tau F \phi \\
\delta^d \psi &= -\tau \omega F \\
\delta^d \dot{\psi} &= 0 \\
\delta^d \omega &= 0
\end{align*}
\] (3.19)

Also for this one we have

\[\delta^d \delta^d + \delta^d \delta^d = 0\]

If we now try to see what is \(\delta^d \delta^d + \delta^d \delta^d\), we get that it is not any clearly recognizable operator.

The interest in this \(\delta^d\) and consequently in (3.15) and in (3.16) is related to quantum mechanics\(^{[46]}\) and work is in progress on it.

4. PATH-INTEGRAL APPROACH TO CLASSICAL MECHANICS

We have seen that in the formalism of Zinn-Justin we had a BRS invariance whatever might have been the weight given to the noise \(\eta\). That means that even if we give a weight equal to a Dirac delta

\[ [d\rho(\eta)] = \delta(\eta)d\eta \]

we should still have the BRS invariance. Giving a weight equal to the one above, means having a totally deterministic process, and we decided\(^{[46]}\) then to use, as a deterministic system, the Hamiltonian equation of motion of classical mechanics. In that manner, following the same steps as in the previous section, we shall obtain a path-integral for classical mechanics (CM).
The reader at this point might be puzzled at how can we have a path-integral for CM. Actually, if we have a path-integral, there must be a parallel operatorial formalism for CM. Actually this last one existed since 1931\textsuperscript{43} \textsuperscript{45}. Koopman and von Neumann, influenced by the invention of quantum mechanics, proposed this operatorial approach to classical mechanics (CM) that we will present in a short while.

The reader might anyhow insist and ask which is the motivation for developing such a tool as the classical path-integral. We feel that such a tool might help in studying some phenomena in the classical regime which, even to-day, are only very poorly understood: the example we have in mind is that of deterministic chaos in Hamiltonian systems\textsuperscript{44}.

In the next few pages we will proceed to present this path-integral, and, as before, we will show how the measure of this path-integral can be written in the standard form of the exponential of an action $\tilde{S}$. This action $\tilde{S}$ contains, as before, a set of anticommuting variables which can be understood as the Jacobi fields of classical mechanics\textsuperscript{45}. The action $\tilde{S}$ exhibits much more than just the BRS invariance: it exhibits a universal ISp(2) invariance-group. The conservation of one of its generators has been proved\textsuperscript{46} to be equivalent to the Liouville theorem of CM, which states that the phase-space volume is invariant under the Hamiltonian flow. In this review we will summarize some important features of this classical path-integral, in particular of its Hamiltonian form and interpret the remaining ISp(2) generators in terms of more familiar (geometric) objects\textsuperscript{46}. For example, the BRS-operator will be identified with the exterior derivative on the space of classical orbits, which, in turn, can be identified with phase-space. Similarly, from the action $\tilde{S}$ we can derive a "super-Hamiltonian" $\tilde{H}$ which turns out to be the Lie-derivative associated to the Hamiltonian flow. Here lies the real surprise and the power of the path-integral approach: it naturally generates geometrical objects, like the exterior derivative or the Lie-derivative, which do not have to be introduced in an abstract manner as it is usually done in the standard formulation of CM. In this way the standard Cartan calculus on phase-
space \cite{16} will be translated into a set of simple operatorial rules derivable from the operatorial content of the theory described by our path-integral. The crucial elements of the operatorial formalism mentioned above are the "classical commutation relations" which follow from the classical path-integral. This formalism naturally embeds the standard operator approach to CM pioneered by Liouville, Koopman and von Neumann \cite{16}. But the operator method discussed here goes beyond the standard one, in the sense that it not only deals with scalar probability density functions on phase-space, but also includes the dynamics of p-form fields on phase-space. The appearance of these higher fields is a consequence of the Grassmannian variables, and since these are related to the Jacobi fields, the p-forms contain information about the behaviour of nearby trajectories, for instance. This is the kind of information which is important for the study of chaotic phenomena.

We now present (but for more details see ref.\cite{16}) the classical path-integral for the Hamiltonian formulation of classical mechanics. We start from Hamilton's equations: 

$$\dot{\phi}^a(t) = \omega^{ab} \partial_b H(\phi(t))$$

where $$\phi^a \equiv (q^1, \ldots, q^n, p_1, \ldots, p_n)$$, $$a = 1, \ldots, 2n$$, is a coordinate on a 2n-dimensional phase-space $$\mathcal{M}_{2n}$$, $$H$$ is the Hamiltonian and $$\omega^{ab} = -\omega^{ba}$$ is the symplectic matrix. Another important concept we shall need is that of a probability density function $$\varrho(\phi^a, t)$$ on phase space. The time evolution of these distributions is given by

$$\frac{\partial}{\partial t} \varrho(\phi^a, t) = -\{\varrho, H\} \equiv -\hat{L} \varrho(\phi^a, t) \quad (4.1)$$

Here we have introduced the Liouville operator: $$\hat{L} = -\partial_a H \omega^{ab} \partial_b$$ which is the central element of the operatorial approach to classical dynamics \cite{16}. As we said before, due to the fact that there is an operatorial approach, there must exists a corresponding path-integral formulation. The simplest idea is to write a classical generating functional of the form:

$$Z_{cm} = \int \mathcal{D}\phi^a \tilde{\mathcal{D}}[\phi^a - \phi^a_{cl}] \quad (4.2)$$

where $$\phi^a_{cl}$$ are the classical solutions of Hamilton's equations. The delta-
functional forcing the system on its classical trajectories can be rewritten as:

$$\delta [\phi^a - \phi^a_c] = \delta [\phi^a - \omega^{ab} \partial_b H] \det \left( \partial_a \delta^a_b - \omega^{ac} \partial_c \partial_b H \right)$$  \hspace{1cm} (4.3)

We can now Fourier transform the delta function on the RHS of (4.3) and exponentiate the determinant using anticommuting variables. Thus we arrive at

$$Z_{cm} = \int D\phi^a D\lambda_a Dc^a D\bar{c}_a \exp i \int dt \{ \bar{L} + \text{source terms} \}$$  \hspace{1cm} (4.4)

with the Lagrangian

$$\bar{L} = \lambda_a \left[ \phi^a - \omega^{ab} \partial_b H(\phi) \right] + i \bar{c}_a \left[ \partial_t \delta^a_b - \omega^{ac} \partial_c \partial_b H(\phi) \right] c^b$$  \hspace{1cm} (4.5)

The source terms in the weight are not only for the bosonic variables but also for the anticommuting variables. This is crucial in order to have a non-trivial $Z_{cm}$ (for details see ref.[16]). Note that $\bar{L}$ contains only first order derivatives and therefore we can immediately read off the Hamilton function

$$\bar{H} = \lambda_a \omega^{ab} \partial_b H + i \bar{c}_a \omega^{ac} \partial_c \partial_b H c^b$$  \hspace{1cm} (4.6)

Having got to this point, we can then use the path-integral (4.4) to compute the equal-time (anti-) commutators of $\phi^a, \lambda_a, c^a$ and $\bar{c}_a$. Using standard techniques we find that

$$\langle [\phi^a, \lambda_b] \rangle = i \delta^a_b, \langle [\bar{c}_b, c^a] \rangle = \delta^a_b$$  \hspace{1cm} (4.7)

and that all other commutators vanish. In particular, $\phi^a$ and $\phi^b$ commute for all values of the index $a$ and $b$. In terms of the $q$'s and $p$'s (which were combined into $\phi^a$) this means $\langle [q^i, p_j] \rangle = 0$ for all $i$ and $j$. This shows very clearly that

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we are doing classical mechanics. The operator algebra (4.7) can be realized by differential operators

\[ \lambda_a = -i \frac{\partial}{\partial \phi^a}, \quad \bar{\lambda}_a = \frac{\partial}{\partial c^a} \]  

(4.8)

and multiplicative operators \( \phi^a \) and \( c^a \) acting on functions \( \bar{g}(\phi^a, c^a, t) \). Inserting the above operators into the operatorial form of \( \mathcal{H} \) we obtain:

\[ \mathcal{H} = -i \omega^{ab} \partial_b \partial_a + i \frac{\partial}{\partial c^a} \omega^{ac} \partial_c \partial_b H c^b \]  

(4.9)

Looking at (4.9), it is clear that the second part of \( \mathcal{H} \) is absent when \( \mathcal{H} \) acts on \( \phi \) which do not contain \( c^a \) variables. In that case only the first part of \( \mathcal{H} \) is left and it is equal to \( (-i) \) times the Liouvillian:

\[ \mathcal{H}|_{p=0} = -i \mathcal{L} \]

This is the same operator appearing in (4.1). So this analysis shows that the path-integral weight behind the Liouvillian is just a Dirac delta (4.2). This is somehow analogous to what Feynman did for Quantum mechanics: he asked himself which was the path-integral weight behind the Schrödinger operator and he found it to be \( \exp i S \).

The curious reader might wonder at this point why in our path-integral the weight is concentrated on continuous and differentiable paths, while in QM we know that the paths that contribute are those continuous but nowhere differentiable. The reason is that in QM we have a quadratic kinetic piece (like \( \phi^2 \)) that acts as a gaussian noise\(^{[2]} \), rendering the paths that are continuous and differentiable of measure zero. This is what is called a Wiener process. In CM, instead, we do not have any quadratic term, \( \dot{\phi}^2 \), in the weight \( \mathcal{L} \) (4.5), so we do not have Wiener processes around and this explains while also continuous and differentiable paths contribute.
Going now back to (4.5), it is not surprising, from the analysis done in the previous section, that $\bar{S}$ has a set of BRS and anti-BRS-like symmetries. In fact, it is invariant under the BRS transformation

$$\delta \phi^a = e \varepsilon^a, \delta \bar{\varepsilon}_a = i \varepsilon \lambda_a, \delta \varepsilon^a = \delta \lambda_a = 0$$  \hspace{1cm} (4.10)$$

and an analogous anti-BRS transformations. These symmetries are generated by the charges

$$Q = i \, e \, \lambda_a, \bar{Q} = i \, \bar{\varepsilon}_a \omega^{ab} \lambda_b$$ \hspace{1cm} (4.11)

They anticommute among themselves and they are nilpotent. This is not the whole story: it is easy to verify that the following ghost bilinears are also conserved:

$$Q_g = e^a \varepsilon_a, K = \frac{1}{2} \omega_{ab} e^a e^b, \bar{K} = \frac{1}{2} \omega^{ab} \varepsilon_a \varepsilon_b$$ \hspace{1cm} (4.12)$$

The charge $Q_g$ can be identified as the ghost charge operator: it assigns ghost charge $+1$ to $e^a$ and $-1$ to $\varepsilon_a$ while $K$ and $\bar{K}$ act like a kind of ghost-charge conjugation. These five charges generate the inhomogeneous symplectic group $\text{ISp}(2)$. Acting on the functions $\bar{g}(\phi^a, e^a)$ the $\text{ISp}(2)$ generators have the following representation:

$$Q = e^a \delta_a, \bar{Q} = \frac{\partial}{\partial e^a} \omega^{ab} \delta_b, Q_g = e^a \frac{\partial}{\partial e^a}, K = \frac{1}{2} \omega_{ab} e^a e^b, \bar{K} = \frac{1}{2} \omega^{ab} \frac{\partial}{\partial e^a} \frac{\partial}{\partial e^b}$$ \hspace{1cm} (4.13)$$

Differently than in the stochastic models studied in the previous section, it is easy here to give a physical interpretation to the $\text{ISp}(2)$ charges and relate them to more familiar differential geometric objects. These charges, universal, they are present in any dynamical system. Hence they should represent something we know already, but under a different name. Let us first try to g
an interpretation of the ghosts. Their equation of motion is

\[ [\partial_t \delta_b^i - \omega^{ac} \partial_c \partial_b H] c^i = 0 \]

and they can be immediately identified with the first variation \( \delta \phi \) (Jacobi fields) around a classical trajectory\(^{[44]}\)\(^{[44]}\). For the charge \( K \), for example, this correspondence reads

\[ K \equiv \frac{1}{2} \omega_{ab} c^a(t) c^b(t) \leftrightarrow \frac{1}{2} \omega_{ab} \delta \phi^a(t) \delta \phi^b(t) \]  

(4.14)

The next point to be remembered is that the space of classical trajectories, which we will denote by \( \mathcal{P} \), is in one-to-one correspondence with the phase space \( \mathcal{M}_{2n} \)\(^{[44]}\)\(^{[44]}\)\(^{[44]}\). Symbolically \( \mathcal{P} \leftrightarrow \mathcal{M}_{2n} \). Due to this there must also be the correspondence \( T^* \mathcal{P} \leftrightarrow T^* \mathcal{M} \). The Jacobi fields \( \delta \phi^a(t) \) could be thought of as elements of \( T^* \mathcal{P} \), the cotangent bundle over \( \mathcal{P} \), while the elements of \( T^* \mathcal{M} \) are the usual forms \( d\phi^a \). Due to this identification we may now complete (4.14) as:

\[ K \equiv \frac{1}{2} \omega_{ab} c^a(t) c^b(t) \leftrightarrow \frac{1}{2} \omega_{ab} \delta \phi^a(t) \delta \phi^b(t) \leftrightarrow \frac{1}{2} \omega_{ab} d\phi^a \wedge d\phi^b \equiv \omega \]  

(4.15)

What we obtained on the RHS of (4.15) is the symplectic 2-form \( \omega \), which is known to be invariant under the Hamiltonian flow\(^{[44]}\). The same is true for all its exterior powers \( \omega \wedge \omega \wedge \cdots \wedge \omega \) and in particular for the phase-space volume form \( vol \equiv \omega^n \). The conservation of the latter is the content of the Liouville theorem. A similar interpretation can also be given to the ghost- charge and for \( K \)\(^{[44]}\). The BRS operator \( Q \) is most easily understood in the representation introduced in eq. (4.13): \( Q = c^a \partial_a \). If we again identify the ghosts with \( d\phi^a \) we have

\[ Q = c^a \partial_a \leftrightarrow d\phi^a \partial_a \equiv d \], i.e., the BRS operator can be put in correspondence with the exterior derivative on phase-space. Also the Hamiltonian \( \overline{H} \) has a very simple differential geometric interpretation. Introducing the Hamiltonian vector field\(^{[44]}\) \( h^a(\phi^c) \equiv \omega^{ab} \partial_b H(\phi^c) \) we have \( \overline{H} = -i\hbar \) where \( l_\hbar = h^a \partial_a + c^b(\partial_b h^a) \frac{\delta}{\delta \phi^c} \)
is the Lie-derivative operator along the vector field \( h^a \). The conservation of the BRS charge is easy to understand now if we remember that the exterior derivative \( d \) commutes with any Lie-derivative. Since \( Q \) can be identified with \( d \) and \( \Delta \) with \(-i\hbar \), respectively, \( Q \) is then conserved as a consequence of the well-known differential geometric property: \([d, I] = 0 \) \(^{[31]} \).

In the above discussion we have represented the \( ISp(2) \) generators by differential operators acting on functions \( \overline{g}(\phi^a, c^a) \), which we interpreted as (inhomogeneous) differential forms. Returning to the analogy with quantum mechanics, this kind of representation would be the analogue of the Schrödinger picture. In the rest of this section we describe the relation between the \( ISp(2) \) generators and standard differential geometry in what would be the analogue of the "Heisenberg picture". We shall not consider explicit realizations of "states" and their transformation laws, but rather equal time commutators of certain composite operators. We start from the following geometric objects: a vector field \( \delta \), a 1-form, \( \alpha \), a p-form field \( \mathcal{F} \) and an antisymmetric contravariant tensor field \( \bar{V}(\nu) \). To these objects we associate operators according to

\[
\delta = \phi^a \partial_a, \quad \alpha = \alpha_a c^a, \quad \mathcal{F}(\nu) = \frac{1}{p!} F_{a_1 \cdots a_p} c^{a_1} \cdots c^{a_p}, \quad \bar{V}(\nu) = \frac{1}{p!} V_{a_1 \cdots a_p} \overline{\epsilon}_{a_1} \cdots \overline{\epsilon}_{a_p} \quad \text{(4.16)}
\]

Here the caret "\( \cdot \)" means that we are dealing with operators containing ghosts. Because of the basic rules \((4.7)\), the objects \((4.16)\) have well defined (graded) commutators among themselves and with the \( ISp(2) \) generators \((4.11)\) and \((4.12)\). It turns out that all the tensor manipulations on symplectic manifolds, sometimes referred to as the Cartan calculus, can be reformulated in terms of such commutators. Let us give a few examples \[^{[44]}^{[45]} \quad ^{[46]}\]. The fact that \( Q \) acts like the exterior derivative \( d \) is now expressed by \([Q, \mathcal{F}(\nu)] = \left( d\mathcal{F}(\nu) \right)^\wedge \). There is an analogous relation between \( Q \) and the antisymmetric contravariant tensor fields:

\[
[Q, \bar{V}(\nu)] = \frac{1}{p!} \omega^{ab} \partial_b V_{a_1 \cdots a_p} \overline{\epsilon}_{a_1} \overline{\epsilon}_{a_2} \cdots \overline{\epsilon}_{a_p} \quad \text{(4.17)}
\]
In particular for \( p = 0 \), i.e., for functions, this reads \([\tilde{Q}, f] = \[(df)^\natural]^{\natural}\) where we introduced the map "\(\tilde{\cdot}\)" which associates a vector-field \(\alpha^\natural\) to any 1-form \(\alpha\) according to \((\alpha^\natural)^a = \omega^{ab}\alpha_b^{[b]}\). As a quick way to learn the Cartan-notation we recommend the appendix of ref.[52]^{[52]}. The same kind of dictionary can be built also for the interior contraction \(\iota_\alpha\) and other standard Cartan-calculus operations. In particular the Lie-derivative, \(l_\alpha = d\iota_\alpha + \iota_\alpha d\) has the following representation:

\[ (l_\alpha F(p))^\wedge = [\tilde{Q}, \tilde{F}] \]

As an example, the Lie-derivative of the Hamiltonian flow can be written as:

\[ (l_{dH} F(p))^\wedge = [\tilde{\mathcal{H}}, \tilde{F}] \]

The above equation confirms that our Hamiltonian \(\tilde{\mathcal{H}}\) is the lie-derivative of the Hamiltonian flow. Also the Poisson brackets of the original (2n-dimensional) phase-space can be expressed in terms of our commutators and the expression is given in ref.[16]

Having understood the geometry of all this construction, we tried^{[43]} to see if, via our path-integral, we could give a representation to some topological invariants of our manifold or of manifolds related to it. We succeeded in this way to give a path-integral representation to the Euler characteristic of the symplectic manifold and to the Maslov index. This approach is on the line of the "topological field theory" methods proposed by Witten,^{[44]} but differently from him, the theory we use is just CM with proper boundary conditions, and not a fancy field theory invented ad hoc.

Recently we have also studied^{[45]} the case of non-trivial phase-spaces in which the symplectic two-form is not globally constant. This slightly modifies some generators but the overall picture is maintained.
The generators (4.13) are not the only generators commuting with $\bar{H}$. Let us consider the quantities

\begin{equation}
N = e^a \partial_a H \\
\bar{N} = \bar{e}_a \omega^{ab} \partial_b H
\end{equation}

Using (4.7) it is easy to verify that they are conserved: $[\bar{H}, N] = [\bar{H}, \bar{N}] = 0$

Similarly, for all quantities $I_i$ conserved under $H$, we can define

\begin{equation}
N_i = e^a \partial_a I_i \\
\bar{N}_i = \bar{e}_a \omega^{ab} \partial_b I_i
\end{equation}

and we again find that these new charges are conserved: $[\bar{H}, N_i] = [\bar{H}, \bar{N}_i] = 0$

If we furthermore assume that the $I_i$'s are in involution, then the $N_i$ and $\bar{N}_j$ commute among themselves and with $N$ and $\bar{N}$.

Next let us look at the algebra of the following linear combinations of our symmetry generators:

\begin{equation}
Q_H \equiv Q - \beta N, \quad Q_i \equiv Q - N_i \\
\bar{Q}_H \equiv \bar{Q} + \beta \bar{N}, \quad \bar{Q}_i \equiv \bar{Q} + N_i
\end{equation}

with $\beta$ an arbitrary parameter. In the differential operator representation of eqn. (4.8), they can be written as $^{[14]}$

\begin{equation}
Q_H = e^{\beta H} Q e^{-\beta H}, \quad Q_i = e^{I_i} Q e^{-I_i} \\
\bar{Q}_H = e^{-\beta H} \bar{Q} e^{\beta H}, \quad \bar{Q}_i = e^{-I_i} \bar{Q} e^{I_i}
\end{equation}

Using (4.7) one can easily verify that

\begin{equation}
[Q_H, Q_H] = [\bar{Q}_H, \bar{Q}_H] = [Q_i, Q_j] = [\bar{Q}_i, \bar{Q}_j] = 0
\end{equation}

\begin{equation}
[Q_H, Q_i] = [\bar{Q}_H, \bar{Q}_i] = 0
\end{equation}
\[ [Q_H, \bar{Q}_H] = 2i\beta \bar{\mathcal{H}} \equiv 2\beta l_{(dH)} \]
\[ [Q_i, \bar{Q}_j] = 2(l_{(dI_i)} + l_{(dI_j)}) = [Q_j, \bar{Q}_i] \]
\[ [Q_H, \bar{Q}_i] = (\beta l_{(dH)} + l_{(dI_i)}) = [Q_i, \bar{Q}_H] \]
\[ [l_{(dH)} \iota, l_{(dI_i)} \iota] = [l_{(dI_i)} \iota, l_{(dI_j)} \iota] = 0 \]

As before, here we have used the notation \((dH)_i \equiv \omega^{ab} \partial_a H \partial_b \) and \((dI_i)_i \equiv \omega^{ab} \partial_a I_i \partial_b \) for the Hamiltonian vector fields generated by the gradients of \(H\) and \(I_i\), respectively. Furthermore \( l_\iota \) denotes the Lie derivative along some vector field \( \iota \). The most interesting of these relations is eq. (4.26). It shows that the anticommutator of \( Q_H \) and \( \bar{Q}_H \) is the super-Hamiltonian and thus these operators, unlike \( Q \) and \( \bar{Q} \), are genuine supersymmetry generators. Note, on the other side, that all the \( Q_i, \bar{Q}_i \) and \( l_{(dI_i)} \), defined above, commute with the Hamiltonian \( \mathcal{H} \), as it is easy to check

\[ [Q_i, \mathcal{H}] = [\bar{Q}_i, \mathcal{H}] = [l_{(dI_i)} \iota, \mathcal{H}] = 0 \]

and so we find that each constant of motion \( I_i \) of the original Hamiltonian \( H(\phi) \) gives rise to a graded symmetry of the super-Hamiltonian \( \mathcal{H} \). The three generators of this graded symmetry \( Q_i, \bar{Q}_i, \) and \( l_{(dI_i)} \) form an \( S(2) \) superalgebra in the classification of De Crombrugghe and Rittenberg. If the \( I_i \)'s have vanishing Poisson brackets among themselves (they are in involution) and with \( H(\phi) \), then different \( Q_i \)'s commute (the same holds for different \( \bar{Q}_i \)'s and different \( l_{(dI_i)} \)'s), see eqs. (4.24), (4.27) and (4.29). Hence for integrable systems* the symmetry of \( \mathcal{H} \) is maximal and is \( S(2)^n \). For systems that are not integrable, but have some \( m < n \) constants of motion in involution, the symmetry is only \( S(2)^m \). If (as it is in most of the cases) they are not in involution, then each

* They have \( n \)-constants of motion \( I_i \) in involution.
of them makes an $S(2)$ symmetry but the overall symmetry is not the direct sum of $S(2)$'s. For ergodic systems, i.e., those that have only $H(\phi)$ as constant of motion, the only symmetry is the supersymmetry of eq. (4.26).

In the spectrum of the Liouvillian operator, discussed previously, are encoded many features of the dynamical system. The one we are most interested in here is that of ergodicity. It is characterized in the following way: a system is ergodic if, at fixed energy, the eigenstate $\varrho_o$ of $\hat{L}$ with eigenvalue zero is non-degenerate. This means that, for a given energy, the equation

$$\hat{L}\varrho_o = 0 \quad (4.31)$$

must have one and only solution for the system to be ergodic. Because of eq.(4.1), this solution $\varrho_o$ is time-independent and so it can only be function of the constants of motion $I_i(\phi^a), (i = 1, 2, \ldots, n)$, i.e., $\varrho_o = F(I_i)$ where $\{H, I_i\} = 0$. If the system is completely integrable, for instance, it has $n$ constants of motion (including the energy). If, on the other side, the system is ergodic, the only analytic constant of motion is the energy $E = H(\phi^a)$, so that $\varrho_o = \varrho_o(E)$. For a fixed energy this is a non-degenerate function: it is just a constant on the energy hyper-surface. If instead there exists a further constant of motion $I(\phi^a)$, we may build also the solution $\varrho_o = F(E, I(\phi))$, which is in general not constant for fixed $E$. Any $F$ of the form above, normalizable on the energy hyper-surface, is a zero-eigenstate of the Liouvillian and therefore the system is not ergodic because all these are degenerate zero-eigenstates of the Liouvillian. Of course it is not necessary to have analytic constants of motion to lose ergodicity, it is enough (like for KAM systems) to have invariant (invariant under the hamiltonian flow) tori of measure different from zero on the energy surface. The characteristic-functions of these tori will be zero-eigenstates of the Liouvillian degenerate with the constant function.

Up to now we have only shown that the action $\tilde{S}$ is invariant under the supersymmetry (4.26) and under the graded symmetries mentioned above, but
we have not addressed the question "Can these symmetries be spontaneously broken?". The question to ask is: "How does the ground state transform? Is it invariant under the same symmetries of the action?". We should remind the reader that supersymmetry can be spontaneously broken even in the case of finite degrees of freedom or finite volume. For the moment we will address this question just for the supersymmetry (4.26). For simplicity, let us assume that we are only interested in such observables "A" which do not depend on the ghosts: $A = A(\phi)$. Then in order to obtain a non-zero expectation value,

$$
(A)_t = \int d^{2n} \phi \, d^{2n} c \, A(\phi^a) \, \bar{\varrho}(\phi^a, c^a, t)
$$

(4.32)

the density $\bar{\varrho}$ has to contain $2n$ ghosts: $\bar{\varrho}(\phi^a, c^a, t) = \varrho(\phi^a, c^a) c^1 \cdots c^{2n}$. As a consequence, the classical average turns out to be of the standard form (without ghosts):

$$
(A)_t = \int d^{2n} \phi \, A(\phi) \, \varrho(\phi, t)
$$

(4.33)

Let us now ask the question "Which form the $\bar{\varrho}$ above has to have to be invariant under the supersymmetry". Since the above $\bar{\varrho}$ contains a maximum number of ghosts it is trivially annihilated by $Q_H$: $Q_H \bar{\varrho} = c^a (\partial_a - \beta \partial_a H) \bar{\varrho}(\phi, c) = 0$. Instead, invariance under $Q_H$ requires

$$
Q_H \bar{\varrho} = \frac{\partial}{\partial c^a} \omega^{ab} (\partial_a + \beta \partial_a H) \bar{\varrho}(\phi, c) = 0
$$

(4.34)

This equation can be satisfied only if

$$
\varrho(\phi^a) = ke^{-\beta H(\phi^a)}
$$

(4.35)

where $k$ is a constant. This result tells us that the supersymmetric invariant state has to be a Gibbs state i.e., a state that depends only on the Hamiltonian. The parameter $\beta$ naturally plays the role of $\frac{1}{K T}$ with $T$ the temperature and $K$ the Boltzmann constant. We feel that this result is a real striking surprise; it
tells us that the canonical, or better the Gibbs distribution, is selected among all other possible states by the requirement of being invariant under the universal supersymmetry present for any dynamical system. It has been a challenge for many years to find the physical reasons behind the Gibbs states, and many concepts like stability, locality, ergodicity, KMS conditions, have been explored. It might be that this supersymmetry embodies all these features.

Let us now proceed and go on to prove that, if the system is in the phase with the supersymmetry (4.26) unbroken, then it is in the ergodic phase, while systems which are in a phase characterized by ordered motion have, in that phase, the supersymmetry always broken. The proof goes as follows: let us assume that our system has another global constant of motion $I(\phi)$ besides the energy $H(\phi)$ so the system is not ergodic. Then, besides $\varepsilon_0(H)$, any normalizable function of $I(\phi)$, $\varepsilon'_0(I)$, would be an eigenfunction of $\bar{L}$ with zero eigenvalue and the corresponding $\tilde{\varepsilon}_0(\phi, \varepsilon) = \varepsilon'_0 c^1 \cdots c^{2n}$ would be a "ground state" of the super-Hamiltonian, because the super-hamiltonian is the lie-derivative of the Hamiltonian flow. $\bar{H}\tilde{\varepsilon}_0 = 0$. This equation just indicates that the state is invariant under the Hamiltonian flow because it is built out of constants of motion. Even if the superHamiltonian is un-bounded below, so that the state at zero eigenvalue is not the ground state, the considerations we will present are still correct, as it is proved in details in ref.[17]. Having both $\varepsilon_0(H)$ and $\varepsilon'_0(I)$ as ground states, not only ergodicity is spoiled but also supersymmetry. In fact, $\varepsilon'_0(I)$ is not supersymmetric invariant, i.e., $\bar{Q}_H \varepsilon'_0 \neq 0$ just because the solutions of the equation $\bar{Q}_H \tilde{\varepsilon} = 0$ are only states of the form $\tilde{\varepsilon} = k e^{-\beta H} c^1 \cdots c^{2n}$ and $\varepsilon'_0(I)$ is not of this form. So for systems with more than one constant of motion, the supersymmetry is broken. In fact the ground state could be any normalizable function of them

$$\tilde{\varepsilon}'_0 = F[\tilde{\varepsilon}_0, \varepsilon'_0] = \bar{F}[H, I, \varepsilon]$$

and this is not supersymmetric invariant! Therefore integrable systems or systems

---

* This is due to the fact that in $\bar{L}$ we have only first-order derivatives.
with few integrals of motion besides the energy, have this supersymmetry spontaneously broken. Furthermore the system has no “stability” reasons to choose the supersymmetric ground state instead of the most general state (4.36): both of them are ground states of the theory because both of them satisfy the operatorial condition $\tilde{H}_{\tilde{Q}} = 0$. In general it might be that we are not able to find analytic constants of motion, but still the system is non-ergodic presenting some invariant surfaces, of measure different from zero, on the energy manifold: KAM systems are of this kind. In this case the $\mathcal{H}'$ associated to this surfaces could be built out of the characteristic function of this surfaces and it is easy to see that they are not supersymmetric invariant. This proves the second part of the theorem, that is: “If the system is in an “ordered phase” then, in that phase, the supersymmetry is broken”.

The first part of the theorem is trivial. In fact, if we are in the phase with the supersymmetry unbroken, it means that the only state at zero eigenvalue of the super-Hamiltonian is the canonical Gibbs state, that implies that there are no other constant of motion beside the energy so the system is ergodic. In fact, if there were other constants of motion, then we could build, as we explained before, other states at zero of the super-Hamiltonian and these states would not be supersymmetric invariant contradicting, in this way, the hypothesis that the system was in a phase with the supersymmetry unbroken.

So far we have argued that unbroken supersymmetry implies ergodicity. It is important to note that the converse of this statement is not true. In fact, since the energy is not fixed in $Z_{cm}$, we could take $e^{-\beta H}$, as well as $F(H), G(H) \ldots$ as acceptable ground states for an ergodic system, where $F(H), G(H) \ldots$ are arbitrary normalizable functions. Clearly all these functions are constants on the energy hypersurface and thus proportional to each other, but not outside. Since in general $F, G \ldots$ are not supersymmetric invariant, we conclude that ergodicity is possible even if our supersymmetry (4.26) is spontaneously broken, i.e. unbroken

---

† Or invariant surfaces of measure different from zero on the energy surface.
supersymmetry is not necessary for ergodicity but only sufficient. At the same level we can say that ordered motion is only a sufficient condition for the breaking of supersymmetry, but not a necessary one.

The interplay between the susy broken/unbroken phases, ordered/ergodic phases and the relative sufficient/necessary inferences are indicated in the scheme below.

\[
\begin{align*}
\text{ordered motion} & \quad \downarrow \\
\text{ergodic motion} & \quad \uparrow \\
\text{broken susy} & \quad \rightarrow \quad \text{unbroken susy}
\end{align*}
\]

(4.37)

To make the scheme above works in both direction (sufficient/necessary) we need to study our generating functional with the constraint of constant energy, \(\delta(H - E)\), inserted. The symmetries of \(Z_{\text{em}}\) will be "slightly" different than before but it is their study that will allow us to find the symmetry whose spontaneous breaking is a sufficient and necessary conditions for ergodicity. In this manner, having a dependence of our generating functional on the energy, we could study at which energy the system is ergodic and at which energy it is not. The central tool to develop should be something like the Witten index that in this case would signal the transition from the ordered phase to the stochastic phase. Given a Hamiltonian, people could try to calculate that index on the computer via Monte-Carlo methods. Unfortunately it is hard to develop here the notion of Witten index, because of the lack of positive-definiteness of our \(\mathcal{H}\). Some work has been already presented but more work is in progress.

As we said before, the Gibbs nature of equilibrium states has been studied very extensively in the literature. In particular, a lot of work has gone into the study of the physical origin behind the Gibbs states and also a lot of work has been done in trying to replace the Gibbs-state condition by a more algebraic one. This algebraic characterization is known as KMS conditions. In this section we will derive it from the supersymmetry that we have seen as being responsible for the Gibbs nature of the states. To this end we consider the time-
evolution\textsuperscript{[3],[4]} of two observables $A_1(d^e)$ and $A_2(d^e)$ which are independent of the ghosts:

$$A_{1,2}(d^e, t) = e^{i\tilde{\eta}t} A_{1,2}(d^e, 0) e^{-i\tilde{\eta}t}$$  \hspace{1cm} (4.38)

The condition $\bar{Q} H \bar{\phi} = 0$ implies that the following expression is zero:

$$\int d^2n \ d^2n c \ A_1(d^c, 0) Q_H A_2(d^c, t) \ Q_H \bar{\phi}(\phi, c) = 0$$  \hspace{1cm} (4.39)

The above expression can be written explicitly as

$$\int d^2n \ d^2n c \ A_1(d^c, \phi) c \ A_2(\phi, t) \ \frac{\partial}{\partial \phi} \omega \{\partial_t - \beta \partial \phi H\} \bar{\phi}(\phi, c) = 0$$  \hspace{1cm} (4.40)

Doing integration by parts in $\frac{\partial}{\partial \phi}$ and in $\partial_t$ and remembering the form of the Poisson brackets, the expression (4.40) reduces to

$$\langle A_1(0) A_2(t) \rangle_{\bar{\phi}} = \frac{1}{\beta} \langle \{A_1(0), A_2(t)\} \rangle_{\bar{\phi}}$$  \hspace{1cm} (4.41)

This equation is known as the classical KMS condition\textsuperscript{[3]} From the eqs. (4.39), (4.40) and (4.41) it is clear that, in deriving the KMS condition from the supersymmetry, we did not use the explicit form of the ground state which usually is not known for complicated (infinite-dimensional) systems. Our derivation of (4.41) only embodies the algebraic character of the supersymmetry.

The curious reader may ask at this point if also the supersymmetry charge $Q_H$ has a geometrical meaning on the lines of the $Q$ that was interpreted as an exterior derivative in phase-space. We have for the moment only a "tentative" explanation. We have the feeling that $Q_H$ is somehow related to the exterior derivative on the energy surfaces (EDH) in phase-space.

In fact an EDH should be a derivative (let us call it $D_H$) that gives zero on any function $F$ of the energy:

$$D_H F(H) = 0$$

$F$ is a function that is constant on the energy surfaces but varies if we move it in direction "perpendicular" to the energy surface. That means $F(H)$ is only a
function of coordinates perpendicular to the energy surface and so $D_H$ should kill functions that do not depend on the coordinates of this surface. A simple solution of the equation above is

$$D_H = e^{i\theta} \left[ \frac{\partial}{\partial \phi^a} - \frac{\partial H}{\partial \phi^a} \frac{\partial}{\partial H} \right]$$  \hspace{1cm} (4.42)

It is immediate to verify that, as any good exterior derivative should be, we have

$$D_H^2 = 0$$

Let us now see which is the relation between $D_H$ and $Q_H$. First of all, let us do the Fourier transform of $\mathcal{F}(H)$

$$\mathcal{F}(H) = \int e^{i\beta H} \tilde{\mathcal{F}}(\beta) d\beta$$

and then apply on it the $D_H$

$$D_H \mathcal{F}(H) = D_H \int e^{i\beta H} \tilde{\mathcal{F}}(\beta) d\beta =
\int e^{i\theta} \left[ \frac{\partial}{\partial \phi^a} - \frac{\partial H}{\partial \phi^a} \frac{\partial}{\partial H} \right] e^{i\beta H} \tilde{\mathcal{F}}(\beta) d\beta =
\int e^{i\theta} \left[ \frac{\partial}{\partial \phi^a} - i\beta \frac{\partial H}{\partial \phi^a} \right] e^{i\beta H} \tilde{\mathcal{F}}(\beta) d\beta =
\int Q_H^{(i\theta)} e^{i\beta H} \tilde{\mathcal{F}}(\beta) d\beta$$  \hspace{1cm} (4.43)

From above we see that $D_H$ acts on function of $H$ as the $Q_H$ (with imaginary $\beta$) does inside the integral. Or, in a more precise language, we can say that the Fourier transform of $D_H$ and of $Q_H$ are the same. In fact, the Fourier transform of $D_H$ is

$$\int e^{-i\beta' H} D_H e^{i\beta H} dH \equiv \overline{D}(\beta', \beta)$$

and it is immediate to see that

$$\overline{D}(\beta', \beta) = \int e^{-i\beta' H} Q_H^{(i\theta)} e^{i\beta H} dH$$

So we can say that, at least the $Q_H$ with imaginary $\beta$, are related to the exterior derivative on the energy manifolds. That is reason why in the abstract
we said that the supersymmetry of CM is somehow related to the geometry of the energy manifolds.

Having understood the role of this supersymmetric Hamiltonian in CM, it is tempting to speculate that also the supersymmetric Fokker-Planck Hamiltonian entering the Langevin processes might somehow be a "Lie-derivative of the Langevin flow". So the overall supersymmetry would just be a manner to give a coordinate-free formulation of the Langevin-dynamics. To achieve that we need to introduce the so called stochastic-differential calculus and the associated stochastic exterior derivative but it would be too long to talk about it here.

5. NEW IDEAS AND CONCLUSIONS

In this conclusions we would like to speculate a little bit. The object of our speculation is the fact that we believe that also in quantum mechanics (QM) there should be a universal supersymmetry of pure geometrical origin. We found one in ref.[19] but we do not believe that that is the real supersymmetry of geometrical origin. In fact [19] in that paper, we just put into path-integral form the Nelson [20] stochastic quantization. This is a quantization without any additional fifth time and whose Langevin equation contains the logarithmic derivative of the ground state as a drift force. We do not believe that the supersymmetry discovered there is of any fundamental importance because the supersymmetric generating functional coincide with the standard Schrödinger generating functional only in the infinite-time limit. That supersymmetry, anyhow, is a very powerful tool and has been used in various context like: to get a better WKB limit [23], to improve variational techniques [24], to get iso-spectral potential [25] and also in the study of relativistic systems [26].

The supersymmetry we have in mind should stem from an attempt to give a quantum Lie-derivative of the Hamiltonian flow. The reason is that, as we have seen, any Lie-derivative of Hamiltonian flows seems to naturally present a supersymmetry. A first attempt in this direction has been proposed by Moyal in
1947[^1]. He proposed to do QM in phase-space using only c-functions and in this framework he calculated what we could call in modern language the quantum Hamiltonian vector field (let us indicate it as \((dH)^I\)) associated to the quantum mechanical time-evolution. This is the re-known Moyal brackets[^1] or sin-Poisson-bracket of the Hamiltonian

\[
(dH)^I(\cdot) \equiv \frac{2}{\hbar} \sin \frac{\hbar}{2} \{H, \cdot\}_{\text{pb}}
\]

There is no-space here for the details and we advise the interested reader to read ref.[67]. What should be done next, in analogy with CM, is to go from the quantum Hamiltonian vector field to the quantum Lie-derivative (let us indicate it as \(\hat{I}_A\)). The reason to do that is that the quantum-lie-derivative would give the evolution not only of normal distribution but also of higher-forms distribution in the same way as in CM. This formulation moreover would be the most coordinate-free formulation we could get of QM. We said the "most" because it is well-known that QM cannot be given a totally coordinate free formulation in phase-space.

To build the quantum lie-derivative we of course need the quantum exterior derivative (let us call it \(\hat{d}\)). This \(\hat{d}\) would of course be different from the classical exterior derivative \(d\) that we studied before: this is the reason why the quantum lie-derivative would not be totally coordinate free in phase-space.

A geometrical study of the Moyal formalism was started some time ago by the Lichnerowicz group[^4]. They understood that all the Moyal formalism boils down to a deformation[^4] of the normal dot-product "." of CM into a star"\(*\)-product". This \(*\)-product is still associative but not commutative and the Moyal bracket can be understood as a commutator under this \(*\)-product.

To go beyond that and build, as we indicated previously, the quantum \(\hat{d}\) and all the quantum Cartan calculus, we should "deform" also the wedge product \(\wedge\) into a \(\wedge^*\) product and the interior product \(\iota_v\) into a quantum one \(\iota_\varphi\).

[^1]: In the case of QM these are the Wigner functions.
This kind of work is on the line of the recent work\cite{21} on quantum group but it is made more difficult by the fact that we do not have a system with pre-assigned classical symmetries that we want to quantize. We want instead to quantize a general system, once it is formulated in a coordinate-free way. This seems to be a hard task plagued with a lot of ambiguities and a lot of freedoms and we do not know yet how to handle them, but work is in progress\cite{46}.

Among the many by-products we would expect to get, out of this study, we would like to highlight the following ones:

1). With the quantum Lie-derivative of the Hamiltonian flow in our hands, we could address the question if the form-number would still be conserved at the quantum-mechanical level. At the CM level it is and this is a cornerstone of the Cartan Calculus, but nobody guarantees that it should be so also at the QM level. An effective action has to be built and this one will tell if the form number is conserved. If it were not conserved, this would be a new quantum effect not predicted by the old QM. In fact Schroedinger and Feynman froze all the physics in the zero-form sector, that is the one of the normal wave functions $\Psi(q)$, and they never addressed the question of the dynamics of higher forms that is of distributions of the type $\tilde{\Psi}(q, dq)$. If the form number is not conserved, we can have tunneling between $\Psi(q)$ and $\tilde{\Psi}(q, dq)$ and this would be a new quantum effect.

2). The quantum exterior derivative $\hat{d}$ will have associated a cohomology problem on the line of the classical De-Rham cohomology\cite{49} associated to $d$. This "quantum" de-Rham cohomology will study the structure of some non-commutative space or of some complicated manifold in which QM lives.

3). Once we have managed to introduce the quantum exterior derivative $\hat{d}$, most probably it will be simple, on the same lines as in CM, to have the quantum $\hat{Q_H}$ which would be one of the two supersymmetry-charges that make up the quantum lie-derivative of the Hamiltonian flow. We expect that, as in CM,
these charges could be interpreted as exterior derivative on the energy manifolds that in this case will be some non-commutative space. We also expect that the non-breaking of this quantum supersymmetry be related to some of the definitions that are around of quantum ergodicity.

4). The classical exterior derivative $d$ was used in ref.[16] in a sort of Schroedinger picture. That means we had already eliminated the $\lambda$ field from our distribution in phase-space $\rho$ once we applied on it the $d$. If instead we had applied it on the full $\rho$ then it would have acted as a sort of "Classical" polarization. If we do the same with the quantum $\hat{d}$, then this object can be identified with the polarization\(^\dagger\) of geometric quantization\(^\dagger\).

5). In ref.[40] we studied the Schroedinger equation by introducing a non-local time rescaling (let us call it "mechanical similarity" MS) that is always anomalous in QM. We would like to make that transformation local, by introducing some auxiliary anticommuting variables, in the same fashion as we did with the Slavnov/BRS transformation. The anticommuting variables needed will be those contained in the quantum lie derivative of the Hamiltonian flow mentioned above. Once we have the BRS transformation associated to the MS, we can ask ourself the question of which are the "physical" states annihilated by the associated charge. We have the feeling they will be those states that, in a GNS construction\(^\dagger\), will give rise only to mixed density matrices. In QM mechanics we must not impose that condition on the states because the MS is anomalous, and so also pure states will enter. This would be the geometrical origin of interference in QM.

There are plenty of other issues that could be mentioned here and on which work is in progress\(^\dagger\), but for the moment we stop here.

\(^\dagger\) At least with some polarizations.
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APPENDIX

In this appendix we want to give the rules we use about the commutativity or anticommutativity of the various fields and of the various parameters and variations \( \phi, \psi, \bar{\psi}, \epsilon, \xi, \delta, \bar{\delta} \) present in this work.

If we call with capitola Latin Letters, like \( A, B \), the collections of fields and parameters and variations indicated above

\[
A \equiv \phi, \psi, \bar{\psi}, \omega, \epsilon, \xi, \delta, \bar{\delta}
\]

then the rules of commutativity are

\[
AB = BA(-1)^{F_A F_B}(-1)^{G_A G_B}
\]  \( (A1) \)

where \( F_A \) and \( G_A \) are something like the "Fermion" number and the "Grassmannian" number of the field \( A \). The list of the Fermion numbers and Grassmannian numbers of the various \( A \) around are listed below

<table>
<thead>
<tr>
<th>( A )</th>
<th>( G_A )</th>
<th>( F_A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \psi )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{\psi} )</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \omega )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \xi )</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \bar{\delta} )</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Due to this set of numbers, we have that all the various $A$ anticommute except the ones listed below:

\[
\begin{align*}
e\bar{e} &= \bar{e} e \\
\delta \psi &= \psi \delta \\
\bar{\delta} \psi &= \psi \bar{\delta} \\
\bar{\delta} \bar{\psi} &= \bar{\psi} \bar{\delta}
\end{align*}
\]  

(A2)

The reader might be a little puzzled by these set of commuting operations, but it is easy to see that, if we had assumed all $A$ anticommuting, we would have got into contradictions. For example, let us calculate $\delta(\bar{e} \bar{\psi})$ supposing everything anticommutes,

\[
\delta(\bar{e} \bar{\psi}) = -\bar{e} \delta \bar{\psi} = -\bar{e} \epsilon \omega
\]  

(A3)

Let us now change the order inside the first bracket:

\[
\delta(-\bar{\psi} \bar{e}) = -\delta \bar{\psi} \bar{e} = -\epsilon \omega \bar{e} = \bar{e} \epsilon \omega
\]  

(A4)

This last result in (A4) is in contradiction with the last result of (A3).

\* We use the $\delta$ of the BRS variation of formula (3.8).
REFERENCES

10. G.Parisi and Y.S.Wu, Scientia Sinica,24,483 (1891)
12. E.Gozzi: SLAC-PPF numbers IRN:1023489;IRN:970506; IRN:860948
   (July 1982)
24. N. Bohr, "Studier over metallernes elektroneori" Copenhagen, 1913
25. L. Onsager, Phys. Rev. 37, 405 (1931)
28. E. Gozzi, unpublished August 1987
31. B. Sakita, "Quantum theory of many variables systems and fields" World Scientific Publ. 1985
55. E. Gozzi, M. Reuter and W. D. Thacker, PARKS/PHY-91-19 preprint
61. E. Gozzi, M. Reuter and W. D. Thacker, ITP-UH 9/91 preprint
66. J. Gamboa and J. Zanelli, Ann. of Phys. (NY), 188, 239 (1988) and references therein
69. E. Gozzi, M. Reuter, W. D. Thacker, work in progress
70. N. Woodhouse, "Geometric Quantization", Oxford, Claredon Press. 1980

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