M. Pavšič and E. Recami: CHARGE CONJUGATION AND INTERNAL SPACE-TIME SYMMETRIES
ABSTRACT.

We adopt the relativistic framework in which fundamental particles are regarded as extended objects. Then, we show that the (geometrical) operation which reflects the internal space-time of a particle is equivalent to the operation C which inverts the sign of all its additive charges.

In the present paper we critically comment on the discrete transformations of Minkowski space-time, namely on the effects of space-reflection $\mathcal{P}$ and time-reversal $\mathcal{T}$, by exploiting some results contained in previous papers $^{(1-3)}$. Our aim is to show the connection between the internal discrete transformations $^{(2)}$ and the charge-conjugation operator C. We assume fundamental particles to be extended objects, as many theoretical observations suggest to be the case, at least in relativistic theories $^{(4)}$.

First of all, let us recall that an inversion $\mathcal{I}_{ABC}(n)$ of the axes $x^A, x^B, x^C, \ldots$, in a n-dimensional space $M_n$ is equivalent to an appropriate 180°-rotation $^{(4)}$ $\mathcal{R}_{ABC,\ldots}(m)$ in the hyperplane $(x^A, x^B, x^C, \ldots, x^n)$ of the m-dimensional space $M_m$ with $m \geq n$. If the number

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(+) When M is Minkowskian, "rotation" will mean pseudo-rotation.
k of the inverted axes $x^A, x^B, x^C, \ldots$, is even, then it may be $m = n$; but if $k$ is odd, then $m > n + 1$. In particular, the total inversion (of all axes) in a $n$-dimensional space $E_n$ corresponds to a rotation either in $E_n$ (if $n$ is even) or in a $(n+1)$-dimensional space $E_{n+1}$ (if $n+1$ is even).

For instance, in the 2-dimensional plane the effect of the inversion $\mathcal{I}_x(2)$ (i.e., $x \rightarrow -x$, whilst $y \rightarrow y$) is equivalent to the effect of the $180^\circ$-rotation $\mathcal{R}_{xz}(3)$ in three dimensions around the $y$-axis:

$$\mathcal{I}_x(2) = \mathcal{R}_{xz}(3)|_{E_2},$$

where the subscript $E_2$ means that eq. (1) is true as far as we confine ourselves to the effect of its r.h.s. into the initial 2-dimensional space.

In Minkowski space there are the following discrete transformations (x) (we adopt the notation $x^E = (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$):

- **space-reflection**: $\mathcal{I}_1(4) \equiv \mathcal{P}$ (inversion of $x^1$);
- **time-reversal**: $\mathcal{I}_0(4) \equiv \mathcal{T}$ (inversion of $x^0$).

The product $\mathcal{P}\mathcal{T} = \mathcal{T}\mathcal{P}$ is equivalent to the $180^\circ$-(pseudo) rotation in $M_4$:

$$\mathcal{P}\mathcal{T} \equiv \mathcal{I}_0(4)\mathcal{I}_1(4) = \mathcal{I}_{10}(4) = \mathcal{P}_{01}(4).$$

Though the product $\mathcal{P}\mathcal{T}$ can be considered as a rotation in $M_4$, of course neither $\mathcal{P}$ nor $\mathcal{T}$ alone can be replaced by any rotation in $M_4$. However, if instead of the 4-dimensional space $M_4$ we consider the 5-dimensional space $M_5$, so that an event $e$ is described by the five coordinates

$$e: \ x^A = (x^0, x^1, x^2, x^3, x^4), \quad e \in M_5,$$

then the effect in $M_4$ of the reflection $\mathcal{P}$ is equal to the effect in $M_5$ of the $180^\circ$-rotation around the space $(x^0, x^2, x^3)$, i.e., of the $180^\circ$-rotation of $M_5$ "in the plane" $(x^1, x^4)$:

$$\mathcal{P}x^\mu \equiv \mathcal{I}_1(4)x^\mu = \mathcal{A}_{14}(5)x^A \bigg|_{M_4}; \quad \left[ \begin{array}{c} \mu = 0, 1, 2, 3 \\ A = 0, 1, 2, 3, 4 \end{array} \right]$$

and the effect of the time-reversal $\mathcal{T}$ is equal to the $180^\circ$-rotation of $M_5$ in the plane $(x^0, x^4)$:

(x) The inversion in $M_4$ of one of the space-axes $x^1, x^2$ or $x^3$, e.g., $\mathcal{I}_1(4)$, is called space-reflection. Applying, after $\mathcal{I}_1(4)$, also the $180^\circ$-rotation in the plane $(x^2, x^3)$ is equivalent to the inversion $\mathcal{I}_{123}(4)$ of all the three space-axes $x^1, x^2$ and $x^3$: $\mathcal{I}_{123}(4) = \mathcal{I}_1(4)\mathcal{R}_{23}(4)$. 
The subscript $M_4$ means that, after having performed the rotation, we take account only of the events in $M_4 \subset M_5$.

At this point let us stress that, if the considered space-time $M_4$ contains a particle $a$, we are going to assume that: (i) particle $a$ is - as we already mentioned - an extended object (4), so that the interior of its world-tube is a finite portion of space-time; (ii) our operations $\mathcal{P}$, $\mathcal{F}$ are to be regarded as acting both on the external space-time and on the internal one ("internal" and "external" with respect to the particle world-tube). Since the ordinary parity and time-reversal act on the contrary on only the external space-time, to avoid possible confusion we shall call $P = \mathcal{P}_E$ the ordinary space-reflection and $T = \mathcal{F}_E$ the ordinary time-reversal ($E = \text{external}$) (2).

Then, we shall show - among the others - that the charge-conjugation $C$ is equal to the product $\mathcal{P} \mathcal{I} = \mathcal{P}_I \mathcal{I}_I$, where $\mathcal{P}_I$ is the internal space-reflection and $\mathcal{I}_I$ is the internal time-reversal ($I = \text{internal}$) (2). So that $\mathcal{P} = \mathcal{C} = \mathcal{CPT}$.

Let us explicitly write:

$$\mathcal{P} = \mathcal{P}_E \mathcal{I}_I = \mathcal{P}_I \mathcal{P}_E;$$

$$\mathcal{F} = \mathcal{F}_E \mathcal{I}_I = \mathcal{I}_I \mathcal{F}_E,$$

where $\mathcal{P}_I (\mathcal{I}_I)$ is the internal, $\mathcal{P}_E (\mathcal{F}_E)$ the external, and $\mathcal{P} (\mathcal{F})$ the total space-reflection (time-reversal).

More precisely, the transformations $\mathcal{P}$, $\mathcal{P}_E$, $\mathcal{P}_I$, $\mathcal{F}$, $\mathcal{F}_E$ and $\mathcal{F}_I$ can be defined with the aid of the suitable rotations in $M_5$. The total space-reflection $\mathcal{P}$ is defined by eq. (3) and the total time-reversal by eq. (4). See Figs. 1, 2, where quantity $s^A$ is chosen to be a space-like vector lying inside the particle world-tube (x) and orthogonal to the world-tube axis (specified by its unit-vector $\tau^A$). The world-tube lies in the ordinary $M_4$.

The internal space reflection $\mathcal{P}_I$ can be defined as the $180^\circ$-rotation in $M_5$ of the particle world-tube around the space $\Sigma_p \approx (x^0, x^2, x^3)$ orthogonal to the plane $(x^1, x^4)$: See Figs. 1a. Notice that the space $\Sigma_p$ around which one has to perform the rotation in $M_5$ contains the time-axis $x^0$. When the particle $a$ is considered at rest, then the tube axis coincides of course with the time-axis; in such a particular case, therefore, $\Sigma_p$ contains $\tau^A$; See Figs. 1b.

\(x\) For simplicity, let us assume the particle $a$ to be spherical (even if with a non-spherically-symmetric structure).
FIG. 1 - The effect of the total space reflection $\mathcal{P}$, the internal space reflection $\mathcal{P}_I$ and the external space reflection $\mathcal{P}_E$ on the world-tube of a particle. Fig. a refers to a moving particle, and b to the simpler case of a particle at rest. The world-tube is characterized by the time-like 4-vector $x^\mu$ and the space-like 4-vector $s^\mu$ (see the text). The transformations $\mathcal{P}$, $\mathcal{P}_I$ and $\mathcal{P}_E$ change $x^\mu$ into $x'^\mu$, $x_1^\mu$ and $x_2^\mu$, respectively; and analogously for $s^\mu$.

FIG. 2 - The effect of the total time reversal $\mathcal{J}$, the internal time reversal $\mathcal{J}_I$ and the external time reversal $\mathcal{J}_E$ on the world-tube of a particle. Again, Fig. a refers to a moving particle, and b to the simpler case of a particle at rest. As to $x^\mu$ and $s^\mu$, the same notations are used as in Fig. 1.
The internal time reversal $\mathcal{F}_1$ can be defined as the $180^\circ$-rotation of the particle world-tube in $M_5$ around the space $\Sigma_T = (x_1, x_2, x_3)$ orthogonal to the plane $(x_0, x^4)$. See Figs. 2a. When the particle $a$ is in particular at rest, $\mathcal{S}^A$ can be chosen so to coincide with the $x_1$-axis; See Figs. 2b.

The external space reflection $\mathcal{P}_E$ in $M_4$ affects a particle only by reflecting the world-line of its center-of-mass (the position of all other world-lines within the particle world-tube remaining unchanged relatively to the center-of-mass world-line). The external space reflection $\mathcal{P}_E$ is therefore nothing but the ordinary space-reflection $P$:

$$\mathcal{P}_E \equiv P.$$ (7)

The external time reversal $\mathcal{F}_E$ in $M_4$ is equivalent - with regard to a chosen particle $a$ - to the operation transforming its velocity $\vec{v}$ into $-\vec{v}$ (Figs. 2a), without affecting its internal structure. The external time reversal $\mathcal{F}_E$ is therefore nothing but the ordinary time reversal $T$:

$$\mathcal{F}_E \equiv T.$$ (8)

We shall also generalize to the case of extended particles the St"uckelberg-Feynman reinterpretation procedure(5).

Let us start by applying (from the active point of view) the total space-time reflection $\mathcal{P}_T$ to the world-tube $W$ of a particle $a$. We depict $W$ as consisting in a sheaf of world-lines $w$ which represent - say - its "constituents" (Fig. 3a); in Fig. 3 - besides the c.m. world-line - we show $w_1 \equiv A$; $w_2 \equiv B$. The operation $\mathcal{P}_T \equiv \mathcal{P}_E \mathcal{F}_E \mathcal{F}_Y \mathcal{F}_Y$ will transform $W$ into a new world-tube $\tilde{W}$ consisting of the transformed world-lines $\tilde{w}$ (Fig. 3b). The world-tube $\tilde{W}$ differs from $W$ in the fact that its world-lines $\tilde{w}$ point in the opposite time-direction and occupy - with respect to the center-of-mass world-line - the position symmetrical to the corresponding $w$.

By applying the Feynman procedure(5) each world-line $\tilde{w}$ transforms into the corresponding world-line $\overrightarrow{w}$ (Fig. 3c). Each world-line $\overrightarrow{w}$ points in the positive time-direction, but represents an anti-"constituent". We now identify the sheaf $\overrightarrow{W}$ of the world-lines $\overrightarrow{w}$ of the "anti-constituents" with the antiparticle $\overrightarrow{a}$; and therefore $\overrightarrow{W}$ with the world-tube of $\overrightarrow{a}$. This identification corresponds to assume that the overall time-direction of a particle $a$ (or $\overrightarrow{a}$) as a whole coincides with the time-direction of its "constituents". Such a procedure is an explicit generalization of Feynman procedure for extended particles.

A preliminary conclusion is that the antiparticle $\overrightarrow{a}$ of $a$ can be regarded (from the chronotopical, geometrical point of view) as derived from the reflection of its internal space-time.

Let us repeat what precedes in a more rigorous way, and recall that the St"uckelberg-
- Feynman reinterpretation procedure has been recently reformulated into one of the fundamental principles ("Third Postulate") of Special Relativity: See Refs. (3, 1, 2). Let us also recall that Special Relativity can be based on the whole proper group \( \mathbb{L}_4 \) of both ortho- and anti-chronous Lorentz transformations, \( \mathbb{L}_4 = \mathbb{L}_4^+ \cup \mathbb{L}_4^- \), since a clear physical meaning can be given also to antichronous (i.e., non-orthochronous) Lorentz transformations (3, 1).

The central elements of \( \mathbb{L}_4 \) are \( (+ \mathbb{I}, - \mathbb{I}) \), where \( \mathbb{I} \) is the identity matrix in four-dimensions. That is to say, in such a formalization of Special Relativity the operation \(-\mathbb{I}\) does represent an actual (even if antichronous) Lorentz transformation, corresponding to the 180° space-time "rotation":

\[
\mathbb{P}_\mathbb{T} = - \mathbb{I} .
\]  

(9)

Notice explicitly that in eq. (9) the operators \( \mathbb{P}, \mathbb{T} \) have a meaning different from the one of the ordinary space-parity \( P \) and time-reversal \( T \). Namely, for the very fact that eq. (9) represents a Lorentz transformation, quantities \( \mathbb{P}, \mathbb{T} \) and \( \mathbb{P}_\mathbb{T} \) will act not only on the chronotopical space, but also on the "dual" four-momentum space, etc. (This means that \( \mathbb{T} \), in particular, when acting on a four-momentum vector, will change also the sign of energy).

But let us go back to the mere chronotopical space.

Now, if we apply \( \mathbb{P}_\mathbb{T} = - \mathbb{I} \) from the active point of view to the world-tube \( W \) in Fig. 3a, we have to rotate it (by 180°, in four dimensions) into \( \widetilde{W} \) (Fig. 3b). Such a rotation will effect also a reflection of the internal 3-space of a particle \( a \), transforming it - among the others - into its mirror image. Analogously, from the passive point of view, if we apply \( \mathbb{P}_\mathbb{T} \) to the space-time in Fig. 3a, containing also \( W \), we shall pass to a \( \mathbb{P}_\mathbb{T} \)-ed frame whose space-time derives from the complete 180°-"rotation" of the initial space-time. Again, this will operate also the reflection of the internal space-time of particle \( a \) (relatively to the new observer).

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[FIG. 3 - Given a world-tube (Fig. a), we show the effect of the (antichronous) Lorentz transformation \( \mathbb{P}_\mathbb{T} = - \mathbb{I} \) before (Fig. b) and after (Fig. c) the application of the "Reinterpretation Principle" (5, 1, 2). See the text.]

(x) C.f. eq. (11) in the following.
Then, we extend the Reinterpretation Principle\(^{(3,1,2)}\) to the case of extended objects, i.e., we apply it (e.g., within the active point of view) to the world-tube \(\tilde{\mathcal{W}}\) of Fig. 3b. The world-tube \(\tilde{\mathcal{W}}\) represents an (internally reflected) particle not only going backwards in time, but also carrying negative energy. Therefore applying the Reinterpretation Principle\(^{(6)}\) will rigorously transform \(\tilde{\mathcal{W}}\) into \(\mathcal{W}\) (Fig. 3c), the anti-world-tube \(\bar{\mathcal{W}}\) representing the anti-particle \(\bar{\alpha}\).

In conclusion, as far as the chronotopical space is concerned, the (antichronous) Lorentz transformation \(\bar{\mathcal{T}} \equiv - \mathcal{T}\) can be considered as

\[
- \mathcal{T} = \bar{\mathcal{T}} = \mathcal{P}_* \mathcal{E}_* \mathcal{I}_* \mathcal{P} \mathcal{I} = \mathcal{P} \mathcal{T} \mathcal{I} \mathcal{P} \mathcal{I},
\]

so that in particular:

\[
\bar{\mathcal{T}} = \mathcal{P} \mathcal{I} \mathcal{P} \mathcal{I} = \mathcal{P} \mathcal{I} \mathcal{P} \mathcal{I} = \mathcal{C},
\]

(10)

(10')

At this point we have to recall that in Refs. (1, 3) we showed - by taking account also of the fourmomentum space and by applying the "Reinterpretation Principle" - that

\[
\bar{\mathcal{T}} = \mathcal{CPT},
\]

where \(\mathcal{C}\) represents the conjugation of all the additive charges\(^{(3,1)}\). Let us add, going back to eq. (9), that all known (relativistic) equations and (relativistic) interactions are actually CPT-covariant. From eqs. (10), (11) it is immediate to derive that

\[
\mathcal{P} \mathcal{I} \mathcal{P} \mathcal{I} = \mathcal{I} \mathcal{P} \mathcal{I} \mathcal{P} \mathcal{I} = \mathcal{C}.
\]

(12)

We have thus shown the (geometrical) operation of reflecting the internal space-time of the considered particle to be equivalent to the operation \(\mathcal{C}\) which inverts the sign of all its additive charges.

We have also seen that the internal transformations \(\mathcal{P}_1, \mathcal{I}_1\) do change the particle intrinsic state. If we convene to write \(\mathcal{P}_1 \mathcal{a}_{++} = \mathcal{a}_{++}\); \(\mathcal{I}_1 \mathcal{a}_{++} = \mathcal{a}_{--}\), then:

\[
\mathcal{P}_1 \mathcal{I}_1 \mathcal{a}_{++} = \mathcal{a}_{--},
\]

(12')

where the subscripts denote the internal parameters that transform under the action of \(\mathcal{I}_1\) and \(\mathcal{P}_1\), respectively; and where \(\mathcal{a}_{--}\) represents the intrinsic (i.e., internal) state of the anti-particle \(\bar{\alpha}\).

All what precedes can be applied also within the realm of quantum theories.

But let us here conclude by emphasizing that - in our opinion, and for the results in this paper and in Refs. (1-3) - we should advantageously substitute in theoretical physics the new operations \(\mathcal{P} \equiv \mathcal{P}_* \mathcal{E}_* \mathcal{I}_* \mathcal{P} \mathcal{I} = \mathcal{P} \mathcal{T} \mathcal{I} \mathcal{P} \mathcal{I}\) for the ordinary operations \(\mathcal{P}, \mathcal{T}\), which are merely external reflections (e.g., only the former do belong to the Full Lorentz Group).
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REFERENCES.


(6) - A new formalization of this principle ("RIP") has been very recently given by C. Schwartz, Phys. Rev. D23, 356 (1982).