E. Recami: AN INTRODUCTION TO "EXTENDED", "PROJECTIVE", AND "CONFORMAL" RELATIVITIES.
E. Recami\(^{(4)}\): AN INTRODUCTION TO "EXTENDED", "PROJECTIVE", AND "CONFORMAL" RELATIVITIES\(^{(0)}\).

**PART A: INTRODUCTION TO "EXTENDED RELATIVITY": CAUSALITY; SUPERLUMINAL OBSERVERS**

1. - Foreword
2. - Revisiting the Postulates of Special Relativity (SR)
3. - Causality in SR
   3.1. - Reinterpretation Principle, Retarded Causality (and Antimatter)
   3.2. - Some consequences
4. - "Extended Relativity" (ER)
   4.1. - Historical remarks
   4.2. - Preliminaries (and warnings) about Tachyons
   4.3. - Duality Principle
   4.4. - Bradyons and Tachyons, Subluminal and Superluminal Lorentz transformations
   4.5. - The Generalized Lorentz Transformations (CLT)
   4.6. - Descriptions and Laws, Equivalence of bradyonic and tachyonic inertial frames
   4.7. - Causality and Tachyons
   4.8. - Classical Physics for Tachyons
   4.9. - Solving causal paradoxes for tachyons (and Digression)
5. - Matter and Antimatter
   5.1. - The Antimatter
5.2. - Sources and Detectors, Interactions and Objects
6. - The CPT theorem; Crossing Relations; and Advanced Solutions
   6.1. - The CPT theorem
   6.2. - Only Laws, and not Descriptions, have to be Covariant
   6.3. - Crossing Relations
   6.4. - Explaining Advanced Solutions
7. - Some Considerations on Tachyons
   7.1. - Black-holes and Tachyons
   7.2. - Various remarks
   7.3. - "Virtual" particles and Tachyons
   7.4. - Astrophysics and Superluminal Objects

**PART B: OTHER EXTENSIONS OF RELATIVITY.**

8. - The "Projective Relativity"
   8.1. - An Introduction to "Projective Relativity"
   8.2. - An alternative approach
9. - About "Conformal Relativity"
   9.1. - Introduction
   9.2. - A "hierarchy" of Universes and Unified Theory of Gravitational and Strong Interactions

**REFERENCES AND NOTES**

\(^{(4)}\) Also: Istituto di Fisica Teorica, Università di Catania, Catania, Italy; and C. S. F. N. e S. d. M., Catania.

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PART A: INTRODUCTION TO "EXTENDED RELATIVITY": CAUSALITY, SUPERLUMINAL OBSERVERS.

1. - FOREWORD.

Aims of the following pages are:

a) Reformulating the theory of Special Relativity (SR), essentially by adding to the usual two Postulates the Postulate of (Retarded) Causality, so to explicitly forbid information transmission into the past. Such a treatment of Causality within SR will lead us to predict existence of Anti-matter in a purely relativistic context;

b) Studying the extension of SR to faster-than-light inertial frames and to Super-luminal objects, thus introducing the theory of "Extended Relativity" (ER). The larger framework of ER will lead us to a better understanding of the ordinary relativistic physics (e.g., of the CPT-theorem, of the so-called "crossing relations" in elementary particle physics, of the connection between matter and antimatter, of the meaning of "advances solutions", etc.);

c) Briefly mentioning other possible extensions of relativistic theories (like "Projective Relativity" and "Conformal Relativity"), from a mainly intuitive point of view.

Before going on, let us immediately remember the following. In this starting paper on Special Relativity EINSTEIN(1) - after having introduced the Lorentz transformations - considered a sphere moving with speed \( v \) along the \( x \)-axis and noticed that (due to the relative motion) it appears in the frame at rest as an ellipsoid with semiaxes:

\[
\begin{align*}
a_x &= R \sqrt{1 - \beta^2} ; \\
a_y &= a_z = R. \\
\beta &= \frac{v}{c} 
\end{align*}
\]  

(1)

At this point, EINSTEIN added(2): "Für \( v = c \) schrumpfen alle bewegten Objekte - vom "ruhenden" System aus betrachtet - in flachenhafte Gebilde zusammen. Für Überlichtgeschwindigkeiten werden unsere Überlegungen sinnlos; wir werden übrigens in den folgenden Betrachtungen finden, dass die Lichtgeschwindigkeit in unserer Theorie physikalisch die Rolle der unendlich grossen Geschwindigkeiten spielt". Which means: "For \( v = c \) all moving objects - viewed from the "stationary" system - shrink into plane-like structures. For superlight speeds our considerations become senseless; we shall find, moreover, in the following discussion that the velocity of light plays in our theory the role of an infinitely large velocity"(3).

Einstein referred himself to the obvious fact that, for \( v > c \), quantity \( a_x \) becomes pure imaginary: if \( a_x = a_x(u) \), then(4)

\[
a_x(U) = i a_x. \\
U = \frac{c - v}{u} 
\]  

(2)

Einstein noticed also that, in the Einsteinian relativity, the speed of light \( v = c \) plays a rôle similar to the one played by the infinite speed \( v = \infty \) in the Galilean relativity(5).

When discussing point b), we shall see in which sense the above assertions are true, and consider if and how they can be (partly, at least) overcome.

The Part A of this contribution is largely based on work done by the author in collaboration with
2. - REVISITING THE POSTULATES OF SPECIAL RELATIVITY (SR).

The theory of SR is the typical framework for considering the problem of Causality in physics. It consider, as "background", a four-dimensional, pseudo-Euclidean space-time.

Let us remember that a suitable choice of Postulates for the theory of SR is the following (6, 7):

1) First Postulate: Principle of Relativity: "Physical laws of Mechanics and Electromagnetism are covariant ("invariant in form") when going from an inertial observer to another inertial observer". Notice that this postulate does not impose any constraint on the relative speed \( v \) of the two inertial observers. This first postulate is inspired to the consideration that all inertial frames should be (8) equivalent for a careful definition of "equivalence" see Sect. 4.6.

2) Second Postulate: "Space-time is homogeneous and space is isotropic". This second postulate is justified by the fact that from it the conservation laws of energy, momentum, angular-momentum follow, - which ones are well verified be experience, at least in our "local" spae-time region.

Since 1910 it was shown (9) that the postulate of light-speed invariance in vacuum is not strictly necessary, since it can be derived (9) from the above postulates 1) and 2). We shall come back soon to this point: here let us observe that the particular role of light-speed in SR is due to its invariance, and not to the fact that it is (or it is not) the maximal one.

Now, if we want - as we do - to avoid information transmission into the past, a third postulate is however necessary (6, 10):

3) Third Postulate: "Negative-energy objects or particles, travelling forward in time, do not exist (and physical signals are transported only by objects that appear to carry positive energy)". Such a form of the Third Postulate is clear also within Information Theory. This postulate will be shown to be equivalent to the principle of (Retarded)Causality: "For every observer, 'causes' chronologically precede their own 'effects' (for definitions of 'causes' and 'effects' see the following)".

Moreover, from Postulate 3) the existence of anti-matter will be inferred.

Let us go back to the postulates 1) and 2). From them it follows (7, 8) that one (and only one) quantity \( w^2 \) - having the physical dimensions of the square of a speed - must exist, which has the same value according to all the inertial frames:

\[ w^2 = \text{invariant}. \] (2)

If we assumed \( w = \infty \), as done in Galilean Relativity (5), than we'd get classical (Galilei-Newton's) physics. In such a case, the invariant speed would be the infinite one, and - if we symbolically indicate by \( \theta \) the operation of "speed composition" - we could write \( \infty \theta v = \infty \).

But the experience has shown to us that the invariant speed is finite (and real), namely that it is the speed \( c \) of light in vacuum. In this case, the invariant speed is actually \( c \):

\[ c \theta v = c, \] (4)

and we get immediately Einstein's Relativity and physics. Let us emphasize that, in this second case, the infinite speed is no more invariant: \( \infty \theta v = V^+ \neq \infty \). It simply means that the operation \( \theta \) is not the operation + of the arithmetic. Moreover, let us repeat that postulates 1) and 2) require the existence of an invariant speed, and not of a maximal speed; the light-speed will result to be in SR a limiting speed, but any limit is wellknown to possess a priori two sides.
If we add the assumption that, for any speed \( v \), it is \( v^2 < c^2 \), then the above postulates imply that the observations from an inertial observer \( O \) are transformed into the observations from another observer \( O' \), in uniform rectilinear relative-motion, by of the ordinary Lorentz transformations whose geometrical meaning is depicted — for future convenience — in Fig. 1.

3. - CAUSALITY IN SR.

3.1. - Reinterpretation Principle, Retarded Causality (and Antimatter).

In order to discuss our "Third Postulate", let us now consider Fig. 2, where for simplicity a two-dimensional space-time is depicted. When we are in the position \( x = 0 \) at time \( t = 0 \), we usually incline to consider as "existing" all the \( x \)-axis events. However, if another inertial observer, \( O' \), moving along the positive \( x \)-axis, overtakes us at the origin-event, then at the same time \( t = t' = 0 \) he will tend to consider as "existing" all the \( x' \)-axis events. Therefore, if we want to be able to start discussing and exchanging informations with him, we must first be prepared to consider that all chronotopical events "exist" (at least the ones outside the past-future zone of the light-cone). Then, nothing a priori prevents event \( A \) from influencing event \( B \) (see Fig. 2).

Exactly to forbid such a possibility, we introduced the "Third Postulate" (or "RIP" = Reinterpretation Principle). Our point is that, since we "explore" the Minkowski space-time going forward in time (along the direction determined by Thermodynamics and by the cosmological evolution), any observer will see the event \( B \) of Fig. 2 as the first one and the event \( A \) as the last one. Moreover, it can be shown that an object going backwards in time (Fig. 2) corresponds in the space dual of the chronotopical one, i.e. in the four-momentum space (see Fig. 3), to an object carrying negative energy. And, vice-versa, changing the energy-sign in one space corresponds to changing the sign of time in the other (dual) space. We can easily understand it, starting from the safe consideration of something that we already, surely know from common experience: of a positive-energy object going forward in time. If we now want to apply to it an operation turning its motion backwards in time, then Postulates 1) and 2) oblige us to use a non-orthochronous Lorentz transformation. But any Lorentz transformation changing the sign of the fourth-component of the chronotopical vector (i.e. of time) will change also the sign of the fourth-component of the four-momentum vec-
tor (i.e. of energy) and of any other 4-vectors associated to the same observed object.

This is true also in Quantum Field Theory (QFT), i.e. in relativistic quantum mechanics: for example, if \( f(p, E) = \frac{1}{(2\pi)^2} \int \mathcal{F}(x, t) \cdot \exp \left[ \frac{i p \cdot \vec{x} - i E t}{\hbar} \right] \cdot d^4x \), then\(^\text{(12)}\):

\[
f(p, -E) = \frac{1}{(2\pi)^2} \int \mathcal{F}(x, -t) \cdot \exp \left[ \frac{-i p \cdot \vec{x} + i E t}{\hbar} \right] \cdot d^4x.
\]

Let us now apply our Third Postulate. It is then easy to convince ourselves that those two paradoxical occurrences (motion backwards in time and negative energy) will be reinterpretated in a quite orthodox way by any observer, when they are - as they actually are - simultaneous. Namely, let us suppose (Fig. 4) that a particle P, with negative energy and e.g. charge\(^\text{(13)}\), traveling backwards in time, is emitted by A at time \( t_1 \) and absorbed by B at time \( t_2 < t_1 \). Therefore, at times \( t_1 \), object A "loses" negative energy and charge \(-e\), i.e. gains positive energy and charge \( +e \), i.e., loses positive energy and charge \(-e\). In fact, emission of a negative quantity is equivalent to absorption of a positive quantity, and vice-versa. The physical phenomenon here depicted will of course appear to be nothing but the exchange from B to A of a (standard) particle Q, with positive energy, charge \( +e \) and travelling forward in time.

\[
[t < t']
\]

\[
\begin{align*}
\text{ph} &= \begin{bmatrix} +q, E > 0; \vec{v}; p > 0 \\ (t_1, x_1) \end{bmatrix} \\
CPT(\text{ph}) &= \begin{bmatrix} -q, E > 0; \vec{v}; p > 0 \\ (-t_1, x_1') \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{"RIP"(ph)} &= \begin{bmatrix} +q, E > 0; \vec{v}; p > 0 \\ (t_2, x_2) \end{bmatrix} \\
\hat{C} \text{RIP}(\text{ph}) &= \begin{bmatrix} -q, E > 0; \vec{v}; p > 0 \\ (-t_2, x_2') \end{bmatrix}
\end{align*}
\]

\[\text{FIG. 4}\]

We have however seen that Q has the charge opposite to P; this means that our "reinterpretation procedure" operates\(^\text{(13)}\) - among other things - a charge conjugation, C. A closer inspection (see refs. (12), (14), (15)) of the "RIP" tells us that indeed Q will appear as the ANTI-PARTICLE\(^\text{(16)}\) of P:

\[Q = \overline{P}.\]

We are meaning that the concept of anti-matter is a purely relativistic one; and that, on the basis of the double sign (Fig. 3a)

\[E = \pm \sqrt{p^2 + m_0^2}, \quad \left[ c = 1 \right]\]

existence of antiparticles could have been predicted since 1905 – exactly with the properties they actually showed when later discovered –, provided that recourse had been made to the above "re-interpretation". We therefore mean that the points of the lower hyperboloid-sheet in Fig. 3a, since they correspond not only to negative-energy but also to motion backwards in time –, represent the kinematical states of the anti-particle \( \overline{P} \) (of the particle P represented by the upper hyperboloid-sheet).
Notice explicitly that our Third Postulate, together with the above reinterpretation-procedure, can assume the following form, that - after STUCKELBERG(17) and FEYNMAN(18) - we shall call "Reinterpretation Principle" (RIP): "Negative-energy objects traveling forward in time do not exist; any negative-energy object P traveling backwards in time can and must be reinterpreted as its anti-object \( \overline{P} \) going the opposite way in space (but endowed with positive energy and traveling forward in time)(6,15).

Notice that our three postulates imply that: Positive-energy objects traveling backwards in time do not exist; and that: Not only we can apply the "RIP", but rather we must apply it. In fact: since we belong to the macro-physical world and therefore we must "explore" space-time in the positive t-direction, then we cannot "see" a particle going backwards in time; we shall always see its "reinterpreted" antiparticle going forward in time. The Third Postulate requires, as we said, that "physical signals are transported only by objects travelling forward in time, or - equivalently - only by positive-energy objects".

It is now clear that our "RIP", by eliminating any information-transmission into the past, implements the validity of the Law of Retarded Causality ("causes happen before their own effects")(19). At this point, let us underline that two quite different statements are known under the name of "causality principle" in the standard physics-literature. The first statement bears that name very improperly, since it merely requires non-existence of faster-than-light signals: we shall give up such an arbitrary assumption. The second statement asserts that causes(19) must chronologically precede their own effects(19). This second statement is adopted(14) by us as the definition of causality(9) (or better of retarded causality(20)).

Let us also, explicitly observe that the reinterpretation procedure exchanges the roles of source and detector, and that - with reference to Fig. 2 - every observer will deem B to be the source and A the detector of the (reinterpreted) antiparticle \( \overline{P} \).

Here we want to anticipate that our Third Postulate allows solving even the Paradoxes connected with the fact that many physical problems admit, besides standard "retarded" solutions, also advanced solutions: such "advanced solutions" merely represent antiparticles travelling the opposite way(12,14,15). For instance, if Maxwell's equations admit solutions in terms of outgoing (polarized) photons of helicity \( \lambda = +1 \), then they will admit also solutions in terms of incoming (polarized) photons of helicity \( \lambda = -1 \).

Let us close this Section by mentioning that the "RIP" finds a more elegant, very natural formulation in a five-dimensional space, where the fifth axis corresponds to "proper-time", and is therefore related to rest-mass (see ref. (22)).

3.2. Some consequences.

Close inspection of Fig. 4b shows e.g. that the "RIP" does change - among other things - the 3-momentum sign, but doesn't affect the 3-velocity sign; i.e. it change the rest-mass sign. The "RIP" can be recognized(6,22) from Fig. 4b to be formally equivalent to changing the sign of all the additive charges(13) and of the rest-mass \( m_0 \) (besides changing emission into absorption and vice-versa); we shall call "strong conjugation" \( C \) the discrete operation(23)

\[
\overline{C} = C \ C_0
\]  

where \( C \) is the conjugation of all additive charges(13) and \( C_0 \) is the rest-mass sign-inversion. We can write (neglecting the operation X that effects emission \( \rightarrow \) absorption):

\["RIP" = \overline{C}.\]  

If we remember how antiparticles are "derived" from their particles, we conclude that antiparticles must be formally attributed negative rest-masses (but positive total, relativistic masses and energies, of course!). For clarity's sake, let us remember that in covariant from for any free particle:
\[ E = p_0 = m_0 u_0 c^2, \quad (10) \]

where \( u_0 \) is the four-velocity time-component. Now, let us consider a non-orthochronous Lorentz transformation \(-1 \), changing (for simplicity) only the sign of all time-components:

\[ E' = -E = m_0 (-u_0) c^2 = -m_0 u_0 c^2. \]

Afterwards, when applying the "RIP" so as to get the corresponding antiparticle, we finally have (for the antiparticle):

\[ E'' = -E' = (-m_0) (-u_0) c^2, \]

so that the antiparticle (still endowed – of course – with the 4-velocity component \(-u_0\)) remains with a negative rest-mass.

Therefore, we shall write

\[ \begin{cases} E = +m_0 c^2 & \text{for free particles} \quad (m_0 > 0) \\ E = -m_0 c^2 & \text{for free antiparticles} \quad (m_0 < 0) \end{cases} \quad (11) \]

so that always \( E = +|m_0| c^2 \). Notice that eqs. (11) do not violate covariance, since they both descend from the covariant equation (10).

It should be clear that nothing prevents us from introducing e.g. a new "proper mass" (as distinct from the ordinary "rest-mass") that is invariant when passing from particles to their antiparticles(24); what we wanted to notice is that – in the usual formalism – ordinary rest-mass has on the contrary the above-illustrates property. This consideration will help to clarify many points: for instance (shifting to quantum mechanics (Q.M.), if we correctly insist in associating positive energies to both electrons \( e^- \) and positrons \( e^+ \), then the Dirac equation yields(22) opposite intrinsic parities for \( e^- \) and \( e^+ \) – as required – only under the condition \( m_0(fermion) = -m_0 \) (antifermion).

Still within the realm of Q.M., it is easy to observe (when we deal, as usual, with states of definite parity) that(22)

\[ \overline{\psi} P_5 \psi \]

quantity \( P_5 \) being the chirality operation (\( P_5^\dagger \psi P_5 = \gamma^5 \psi = \overline{\psi} P_5 \psi \)), so that(22):

\[ "RIP" = P_5. \quad (9bis) \]

See also Sect. 6.

4. "EXTENDED RELATIVITY" (ER).

All the considerations of Sect. 3 assume a more compact form when we allow room also for Super-luminal (=faster-than-light) frames of reference and for tachyons(12), so as to take account of all space-time "rotations" (for \( 0 < a < 2\pi \); see Figs. 1 and 5) as generalized Lorentz transformations.

In this connection, let us explicitly emphasize that it seems possible to extend(25) Special Relativity so as to consider Superluminal frames and objects without violating the principle of retarded causality(15,24). Namely, we shall start from the above-mentioned three Postulates (Sect. 2), without assuming a priori \( v < c \).

\[ \text{FIG. 5} \]
Let us remember that we had to assume the "RIP" (or equivalent principles) as the Third Postulate of SR in order to avoid there information transmission into the past. That Principle allowed also predicting the existence of antiparticles, and will permit us to understand better the connection between matter and antimatter. The very Third Postulate — or rather the same set of three postulates previously introduced — are enough for deriving a causal theory even in the presence of faster-than-light objects.

4.1. — Historical Remarks.

Faster-than-light objects have been given the name "Tachyons" (T) in ref. (26), from the Greek word ταχύς = fast. "Une particule qui a un nome possède déjà un début d'existence", will later be commented (27). We shall call 'Bradyons' (B) the ordinary, slower-than-light objects (28, 29), from the Greek word βράδυς = slow. At last, we shall call 'Luxons' (L) the objects — like photons — travelling exactly at the speed of light (30).

As regards tachyons, as far as we know, the first author mentioning faster-than-light particles was LUCRETIUS, as outlined by Corben (31). Here, let us explicitly quote another, different passage from De Rerum Natura (32).

"Quone vides citius debere et longius ire
Multiplexque loci spatium transcurrere eodem
Tempore quo Solis pervolgant lumina coelum?".

That means "Don't you see that they must go faster and farther / And travel a larger interval of space in the same amount of Time than the Sun's light as it spreads across the sky?". After Lucretius, we presently don't know about other progress until Thomson's (33), Heaviside's, Des Coudres' and particularly Sommerfeld's (34) works. In 1905, however, together with Relativity (1) the conviction spread over, that light-speed in vacuum was the upper limit of any speed, the early-century physicists being misled by the evidence that ordinary particles cannot overtake that speed. They behaved like Sudarshan's imaginary demographer studying the population patterns of the Indian subcontinent: "Suppose a demographer calmly asserts that there are no people North of the Himalayas since none could climb over mountain ranges! That would be an absurd conclusion. People of central Asia are born there and live there: They did not have to be born in India and cross the mountain ranges. So with faster-than-light particles". (Cf. Fig. 6).

Moreover, TOLMAN (35) believed to have shown, in his old "paradox", that the existence of superluminal particles allowed information transmission into the past (anti-telephone).

Therefore one had to wait until the fifties (36) for the pioneering works by the French ARZELIES (37) and practically the sixties before seeing the tachyon problem re-examined: by the Japanese TANAKA, the Sovietic TERLETSKY and the Indian SUDARSHAN with Coworkers (30). After ref. (30), a number of people started to study the subject, among whom e. g. in USA FEYNBERG (26) and in Europe Recami and Collagues (38).
Our present interest in Extended Relativity is due to the fact that it yields a better understanding of ordinary SR, even if tachyons would not exist in our cosmos as "asymptotic objects". However, no essential reason against their existence will be apparently met: and we might then get inspiration - following Murray Democritus and Gell-Mann – from the "principle"(39) asserting that "Anything not forbidden is compulsory". A strong objection to adopting such a view can be that tachyons do not seem to have been detected; but such an argument "amounts to nothing more than the convenient supposition that something that has not been observed does not exist. It predicates that we know everything", as Fred Hoyle put it on another occasion. Moreover, most experimental search looking for tachyons has been till now defective of a good theoretical background; actually, Galileo taught us that we cannot possess good ideas before having performed sensible experiments: but we have also learned that sensible experiments cannot be performed before possessing a good theory.

Furthermore, we shall see that tachyons (even when macro-objects) are typical carriers of mutual and "symmetric"-interaction between bodies, say A and B, and they cannot even be emitted by A (B) if B (A) is not yet ready to absorb them; this is similar to what happens for photons in WHEELER and FEYNMAN's theory.

In extended Relativity(12) both sub-luminal ("slower-than-light") and Superluminal frames are considered. The problem of finding out the "Superluminal Lorentz transformations" (SLT) connecting a frame of the former class to a frame S of the latter class has been first considered in the pioneering work by PARKER(40) (who studied the two-dimensional case) and, independently, in the works by OLKHOVSKY and RECAMI(41) and, then, by MIGNANI and RECAMI(42). Cf. also ref.(43).

The four-dimensional extension has been first attempted in refs.(41), with complex transformation.

4.2. - Preliminaries (and warnings) about Tachyons.

In thermodynamics, when physicists meet negative temperatures, it was proposed to redefine the temperature so that the "absolute zero" was shifted(44) from 0 K to -∞. Actually, also in a bi-dimensional space-time or in the case of purely collinear motions, it is possible to define rapidity, the quantity $R = c + \tanh^{-1} \beta$, so that $R = 0$ for $\beta = 0$ and $R \rightarrow \frac{1}{2} \infty$ for $\beta \rightarrow \frac{1}{2} c$ (and one gets an additive "rapidity composition law"). But this cannot be meaningfully done in more dimensions, so that space-like objects cannot be "squeezed away" from space-time in such a manner (cf.5).

Extended Relativity (ER) can be essentially based on postulates 1), 2), 3) of Sect. 2. To make our arguments simpler, let us however substitute postulate 2) with the more conventional one about light-speed invariance. Therefore, ER will be based on the following assumptions: (1) Principle of Relativity; (2) light-speed invariance in the vacuum; (3) Third Postulate: Principle of retarded Causality (or equivalent ones: see Sect. 3 and the following). Remember that the special role of light-speed SR followed from its invariant character in vacuum, and not - let us repeat - from the fact that was (or wasn’t) the maximal speed(45). We are here releasing the additional postulate that for all speeds $|v| < c$.

Extension of SR to Super-luminal objects and reference-frames would be straightforward if we had a symmetry between the numbers of space and time dimensions, like in the two-dimensional $(M^2)$ case(40) or in the case when one introduces three time-dimensions by means of a $M(3,3) = M^6$ space or a $C^3$ space. If we stick - as in the following - to the ordinary Minkowski space-time, the we shall see that one has(2) to deal with unusual imaginary quantities(25). We can understand this fact as follows. Let us premise that some Authors (as CORBEN(25,31), KÁLNAY, and SHAH(25)) are satisfied enough with the situation and the present interpretation-possibilities(25). Other authors(46), on the contrary, are looking for a larger interpretation on the basis of complex space-times or of real, multi-dimensional space-times. It is possible, when adopting a "conservative" view, to meet the requirements expressed in the papers in refs.(46) by considering the use of the SLT's as an analytic-extrapolation procedure, which implies - in the intermediate steps - dealing also with complex (or at least pure imaginary) space and time coordinates. This is not far from what done e.g. (in the high-energy elementary-particle physics) by the known T. REGGE's theory, where amplitudes were extrapolated to angular-momentum complex-values. The essential point is that
in ER one always succeeds – at last – in writing down equations in terms of real quantities only.

Let us choose throughout this article the signature (+---); natural units (c=1) will be adopted when convenient. However, we shall always avoid explicit use of a metric tensor(47) by making recourse to Einstein's notations and to $\eta_{\mu\nu} = \delta_{\mu\nu}$ ("Euclidean" metric), and by writing the generic chronotopical vector as $x \equiv (x_0, x_1, x_2, x_3)$, $s \equiv (ct, lx, ly, lz)$. Therefore, we shall not have to distinguish between covariant and contravariant components. The light-cone will be the "sphere" $x_{\mu} x^\mu = 0$.

Formally, we shall have:

$$\text{time } = i \cdot \text{space} \cdot \begin{bmatrix} c = 1 \end{bmatrix}$$

(12)

As noticed by MINKOWSKI himself, in natural units we may formally write(48):

$$1 \text{ second } = i (3 \cdot 10^8) \text{ meters.}$$

(12bis)

In space-time (see Figs. 1 and 5), our own world-line coincides with our time-axis; the world-line of a transcendent (=infinite-speed) tachyon moving along the x-axis will coincide, on the contrary, with our x-axis (according to us). That transcendent tachyon will then call time-axis (!) what we call x-axis, and analogously will consider our axes t, y, z as its three space-axes $x^t, y^t, z^t$. On the contrary, the structure of ER is such that to us it will seem to possess one space-axis and three time-axes. The same will be as regards clocks and rods!

Let us explicitly repeat that free bradyons always admit a particular class of subluminal reference-frames (the rest-frames) wherefrom they appear – in Minkowski space – as points in space extended in time along a line. On the contrary, free tachyons(49) always admit a particular class of subluminal(30) reference-frames (the "critical" frames) wherefrom they appear with divergent speed ($V^\infty$), i.e. as points in time extended in space along a line (cf. Fig. 1 or 5). Considerations of this kind are important for understanding the "localization" of tachyons with respect to us, and correspond to the fact that the "little groups" of the time-like and space-like representations of the Poincaré Group are $SO(3)$ and $SO(2, 1)$, respectively.(50bis). Moreover (cf. Sect. 7.4), a tachyon - observed by means of its light-signals - will in general appear as having two positions at the same time.

At this point, let us formally develop the ER, on the basis of the three starting postulates (1), (2), (3) chosen at the beginning of this Sub-section.

4.3. - Duality Principle.

From our present three postulates (Sect. 4.2), it follows that – given a certain inertial reference-frame – the class $\{I\}$ of the inertial reference-frames a priori consists of all the frames $f$ moving with constant relative-velocity $\vec{u}$, where $-\infty < \vec{u} < +\infty$.

When extrapolating usual Lorentz Transformations (LT) for angles $|\alpha| > 45^\circ$ (see Fig. 5, where for simplicity we consider the 2-dimensional case), i.e. considering also tachyonic reference-frames, we are led(12) to a new group, $G$, of "Generalized Lorentz Transformations" (GLT) which is constituted by all the "rotations" in Minkowski space-time for $0 \leq \alpha \leq 360^\circ$. The essential point for getting that result is the following.

Let us choose the particular inertial frames $s_0$. The light-speed c – because of its invariant-quantity character – allows an exhaustive partition of frames $f \in \{I\}$ in two subclasses $\{s\}$, $\{S\}$ of frames having speeds $u < c$ and $U > c$ relative to $s_0$, respectively. For simplicity, in the following we shall consider ourselves as "the observer $s_0". Frames s \in \{s\} are the sub-luminal ones and frames S \in \{S\} the Superluminal ones. The relative speed of two frames $s_1, s_2$ (or $S_1, S_2$) will always be smaller than c, and the relative speed between two frames s, S will be always larger than c. The important point is that the above, exhaustive partition is invariant when $s_0$ is made to vary inside $\{s\}$ (or inside $\{S\}$); on the contrary, when we pass from $s_0 \in \{s\}$ to a frames $s_0 \in \{S\}$, the subclasses $\{s\}$, $\{S\}$ are interchanged each other (cf. refs. 30, 31). At the present time, we neglect luminal frames ($u=$$U=c$) as "unphysical", even if mathematical use of "infinite-momentum frames" has spread recently in physics.
One can immediately deduce a "Duality Principle"\((42, 49)\), which may be briefly put in the form: "the terms B, T, s, S do not have an absolute meaning, but only a relative one". Let us notice that the opposite assumption, that the bradyonic/tachyonic character is absolute, would lead immediately to the impossibility of defining Superluminal frames\((42)\).

4.4. - Bradyons and Tachyons. Subluminal and Superluminal Lorentz Transformations.

We shall neglect space-time translations, i.e. consider only the so-called restricted Lorentz transformations. All frames are supposed to have the same event as their origin. Of course in Minkowski space bradyons are characterized by time-like world-line, luxons by light-like world line and tachyons by space-like world lines.

Now, the transformations L, effecting transition between two inertial frames \(f_1, f_2 \in \{l\}\), must be linear and must preserve the four-vector magnitudes, apart from the sign\((49,42)\). This point is proved e.g. in ref. (51), as a consequence of light-speed invariance. Therefore, transformations L between two inertial frames \(f_1, f_2\) must be such that

\[
x'^2 - x^2 = \pm (x'^0 - x^0)^2
\]

for every four-vector \(x = (x^0, \vec{x})\), where \(x\) means either 4-position, or 4-momentum, or 4-velocity, or 4-current, and so on. In the particular case of chronotopical vectors (and using our notations eq. (13) reads:

\[
c^2 t'^2 + (ix')^2 = \pm \left[ c^2 t^2 + (ix)^2 \right].
\]

It is easy to convince ourselves that the sign plus in eqs. (13), (14) refers to the usual case of subluminal relative speeds, whilst the sign minus has to be chosen for Superluminal relative speeds\((47)\).

The Postulates 1), 2) require considering frames \(s\) and \(S\) on an equivalent footing (cf. the following); therefore, even Superluminal observers \(S\) must be supposed to be able to fill their space (as seen by themselves) with meter-sticks and synchronized clocks, all at rest relative to \(S\), if it is to say, to build up - with respect to themselves - their "lattice-work" of meter-sticks and clocks\((47)\).

From the requirement that Superluminal frames are physical\((43)\), it follows of course that jects must exist which are at rest relative to \(S\) and tachyons relative to frames \(s\). From the fact that luxons \(\xi\) show the same velocity to any observer \(s\) or \(S\), it can be deduced that a bradyon \(\xi\) relative to an \(S\) will be a tachyon \(T(s)\) relative to any \(s\), and vice-versa:

\[
B(S) = T(s); \quad T(S) = B(s); \quad \xi (S) = \xi (s).
\]

This agrees with the Duality Principle, that we are going to complete by adding that "frames \(S\) are supposed to have at their disposal exactly the same physical objects as frames \(s\), and vice-versa".

In conclusion, when frames \(s, S\) observe the same event, "timelike" vectors transform into "spacelike" vectors, and vice-versa, in going from \(s\) to \(S\) or from \(S\) to \(s\). On the contrary, it is wellknown that usual LT's, from \(s_1\) to \(s_2\), or from \(S_1\) to \(S_2\), preserve the four-vector type. Or is therefore allowed to say that (subluminal) LT's are expected to be such that:

\[
c^2 t^2 + (ix)^2 = \pm \left[ c^2 t^2 + (ix)^2 \right], \quad \beta^2 < 1,
\]

while "Superluminal Lorentz Transformations" (SLT), from \(s\) to \(S\) or from \(S\) to \(s\), are expected to be such that \(\beta \approx (v/c)\)
Of course, also tachyons will possess real rest-masses\(^{(52)}\) since they are just usual particles with respect to their own rest-frames \(f\), where \(f\) are Superluminal frames to us. From eq. (16b), applied to 4-momentum vector, one can immediately derive for tachyons the relations

\[
E^2 - p^2 = -m_0^2 < 0, \quad \begin{bmatrix} m_0 \text{ real} \end{bmatrix}.
\]  

Therefore, one has:

\[
\begin{align*}
p^2 &= m_0^2 > 0 \quad \text{for bradyons (case I, or timelike),} \\
p &= 0 \quad \text{for luxons (case II, or lightlike),} \\
p^2 &= -m_0^2 < 0 \quad \text{for tachyons (case III, or spacelike).}
\end{align*}
\]  

In fourmomentum space (see Fig. 3), eqs. (18) represent respectively: (i) for \(B\)'s, a two-sheeted hyperboloid of rotation around the \(E\)-axis; (ii) for \(c\)'s, a double indefinite cone, having \(E\) as axis; (iii) for \(T\)'s, a single-sheeted rotation hyperboloid. In all cases \(m_0\) is real, and we have \(|v| < \frac{1}{c}\). For obvious reasons, in Fig. 3 only the 3-dimensional "space" \(p^2 > 0\) has been depicted; those hyperboloids are actually hyper-hyperboloids. Remember that any SLT maps the "interior" of the light-cone \(p^2 > 0\) into its "exterior", and vice-versa (as one can show e.g. within the mathematical "theory of catastrophes"\(^{(53)}\), even if such a mapping is one-to-one quasi-everywhere only\(^{(54)}\).

It may be noted that: a) the speed \(c\) preserves of course its character of limit kinematical-parameter of our four-dimensional cosmos\(^{(11, 55, 12)}\) (even if we know that such a limit has two "sides"), as well as its role for comparing the length and time units of different observers; b) tachyons will slow down when energy increases and accelerate when their energy decreases.

In particular, divergent energies are needed to slow down the tachyon speed towards the (lower) limit \(c\). On the contrary, when tachyon's speed tends to infinity, its energy tends to zero; this prevents violation of the common postulate that "energy can be transmitted only at finite speed", since a tachyon shows zero energy to the same observers to whom it presents divergent speed. Notice that a bradyon may have zero momentum (and minimal energy \(m_0 c^2\)), and a tachyon may have zero energy (and minimal momentum magnitude \(m_0 c\); however bradyons \(B\) (Fig. 3a) cannot exist at zero energy, as well as tachyons \(T\) (Fig. 3c) cannot exist at zero momentum – with respect to the observers to whom they appear as tachyons! -.

Incidentally, since transcendental tachyons do transport momentum, they allow getting the rigid-body behaviour even in SR. As a consequence, in elementary-particle physics, tachyons might a priori result as useful for interpreting diffractional scatterings, or the so-called pomeron-exchange reactions, and elastic scatterings\(^{(56)}\).

45. The Generalized Lorentz Transformations (GLT).

The (Generalized, Lorentz) transformations which are linear and satisfy either eq. (16a) or eq. (16b) form a new group\(^{(57)}\) \(G\), that reads - if we represent the GLT's by \(4 \times 4\) matrices:

\[
\begin{pmatrix}
G & = & \{ A_1 \} u \{ - A_1 \} u \{ - i A_2 \} u \{ + i A_3 \} \\
A_1 \cdot \Lambda = A \beta^2 < 1 & \Lambda = & A \beta^2 > 1
\end{pmatrix};
\]  

so that, if \(L \in G\), then also \(-L \in G\), \(V \in G\). In eq. (19), the set \(\{ + A_1 \} \) is the one of the usual proper, orthochronous, subluminal LT's; the set \(\{ - A_1 \} \) the one of the corresponding non-orthochronous LT's; and the sets \(\{ \pm i A_2 \} \) the ones of the SLT's, where the \(A_i\)'s are matrices formally identical to the \(A_i\)'s but containing\(^{(58)}\) values of \(\beta\) in the range \(\beta^2 > 1\). Notice that in four dimensions the symbol \(i\) represents (rather than the ordinary imaginary unit) a suitable quantity whose square equals \(-1\); for instance \(i\) can a priori be any suitable \(4 \times 4\) matrix \(\lambda\) such that \(\lambda^2 = -1\).
Notice also that: \[ \det L = +1, \forall L \in G; \]
in fact all GLT's are space-time rotations (cf. Figs. 1 and 3). Briefly speaking, SLT's are got usual LT's multiplying the latter ones by the imaginary unit i and simultaneously by changing \( \beta \) (cf. ref. (12)):

\[
\text{SLT}(1/\beta) = \pm i \left[ \text{LT}(\beta) \right], \quad \left[ \beta^2 > 1; \frac{1}{\beta^2} > 1 \right]
\]
where the operator \( \mathcal{P}^2 \) is the product of the operator \( \beta = e^{i/\beta^2} \) of the ordinary (pseudo-orthochronous) Lorentz group \( \mathbb{L}^4 \) by means of the two operations \( \mathcal{CPT} \) and \( \mathcal{F} \):

\[
G = \mathcal{E}(L'_4, \mathcal{CPT}, \mathcal{F}).
\]

At this point, let us choose (see note (4)) \( \beta^2 - 1 = \pm i/1 - \beta^2 \), differently from ref. (12). To a Superluminal boost with speed \( U \) along the positive x-direction, eqs. (20) yield:

\[
\begin{align*}
x' &= \pm \frac{c^2 - ux/c}{\beta^2 - 1}; & y' &= \pm iz; & \left[ u > c^2/U ; 1 \right] \\
t' &= \pm \frac{x - ut}{c^2 - (1/u/c)^2}; & z' &= \pm iz; & \left[ \beta > U/c > 1 \right]
\end{align*}
\]
where \( u > c^2/U \), and where careful attention must be paid to the relative signs, which depend on the above conventions. Notice that \( \beta = U/c > 1 \). Eqs. (22) first appeared in refs. (41), (42), (61).

In the two-dimensional case, eqs. (22) merely become:

\[
\begin{align*}
x' &= \pm \frac{x - Ut}{\beta^2 - 1}; & y' &= \pm iz; & \left[ \beta > 1 \right] \\
t' &= \pm \frac{1 - Ux/c^2}{\beta^2 - 1}; & z' &= \pm iz
\end{align*}
\]
eqs. (23), in such a form, appeared firstly in refs. (41), (42), and then in a number of subsequent papers (62). In refs. (42), (49) they were shown to be essentially equivalent to PARKER's eqs. (40), however in apparently complicated form and lacking in the double sign, which is on the contrary indispensable to getting the inverse transformations.

As an application of eqs. (22), let us consider a tachyon having rest-mass \( m_0 \) (with respect to its rest-frames) and moving with speed \( U \) relative to us; then we shall observe a relativistic mass

\[
m = \frac{m_0}{\sqrt{1 - \beta^2}} = \frac{-im_0}{\sqrt{1 - \beta^2}} = \frac{m_0}{\sqrt{1 - \beta^2}} = \left[ \beta > 1; m_0 \text{ real} \right]
\]

It is evident that tachyons are to be attributed real (and not pure-imaginary) rest-masses, provided that we take account of the suitable SLT's.


Before going on, let us show with some rigour that the usual definition of equivalence can actually be extended also to Superluminal inertial frames. Let us choose a set \( \mathcal{R} \) of certain well-defined reference-frames \( r \), the set \( \mathcal{P} \) of the phenomena \( p \) of Mechanics and Electromagnetism and the set \( \mathcal{S} \) of the descriptions \( \sigma \) of phenomena \( p \in \mathcal{P} \) from frames \( r \in \mathcal{R} \). All observers \( r \) are supposed to possess the same instruments, both physico-experimental and mathematico-theoretical (i.e., the same theory too). Strictly speaking, one has to deal with the "triads" \( \mathcal{d}_{rpr} \), elements of...
set \( \mathcal{A} \times \mathcal{B} \times \mathcal{C} \), the Cartesian product of the three sets considered. As depicted in Fig. 7 (which has only the task of supporting intuition), to the same \( p \) there correspond two descriptions \( d_1, d_2 \) in two different frames \( r_1, r_2 \), and so on. Given any two elements out of \( d, p, r \), the third one must be univocally fixed, in our assumptions. We must write (64):

\[
\begin{align*}
\{d_1, p_1, r_1 \} & \rightarrow \{d_2, p_2, r_2 \}, \quad \text{i.e., } (r_1 \rightarrow r_2, p \rightarrow p \Rightarrow d_1 \rightarrow d_2), \\
\{d_1, p_1, r_1 \} & \rightarrow \{d_2, p_2, r_2 \}, \quad \text{i.e., } (r_1 \rightarrow r_2, d \rightarrow d \Rightarrow p_1 \rightarrow p_2).
\end{align*}
\]

Let us define the subsets \( A_r \subset \mathcal{S} \):

\[
d \in A_r \iff \exists r \in \mathcal{S} \quad \mathcal{S} = \mathcal{A} \times \mathcal{B} \times \mathcal{C}.
\]

Then, following refs. (63), (64), we shall say that two frames \( r_1, r_2 \) are equivalent \( (\equiv) \) if \( A_r \) is mapped onto itself when passing from \( r_1 \) to \( r_2 \):

\[
r_1 \equiv r_2 \iff \exists A_r = A_{r_1} \rightarrow \mathcal{H} \in A_r \Rightarrow \mathcal{H} \in A_r \quad \text{and} \quad \mathcal{P} \in A_r \
\]

\[
\text{Such a condition operates an exhaustive partition} \quad (\ref{64}) \quad \text{of set } \mathcal{R} \text{ into subsets of equivalent frames. Conversely, given a frame } r \text{ and a set } \mathcal{P} \text{ of phenomena, it is possible to build up the set } \mathcal{R} \text{ of frames equivalent to } r. \text{ It is well-known that, given an inertial frame } r \text{ and the set } \mathcal{R} \text{ of usual mechanical and electromagnetic phenomena, a class of equivalent frames is the one } \mathcal{R}_S \text{ of the usual (subluminal) inertial frames } S, \text{ where } \mathcal{R}_S = \{S\}. \text{ It means that (loosely writing) we can write, given the set } \mathcal{A}_r:
\]

\[
\mathcal{A}_r = \mathcal{A}_r \equiv \mathcal{S}, \quad \forall r \in \mathcal{S}.
\]

Then, due to the Relativity and Duality principles, the set \( \mathcal{A} \equiv \mathcal{S} \) of descriptions will correspond \( - \) according to any Superluminal inertial frame \( S \in \mathcal{R}_S \), where \( \mathcal{R}_S = \{S\} \), \( - \) to another, new set \( \mathcal{P}_S \) of mechanical and electromagnetic phenomena; and, given any frame \( S \in \mathcal{R}_S \), all the other frames \( S \) of the set \( \mathcal{R}_S \) will be equivalent to it (with respect to the phenomena \( \mathcal{P}_S \)).

Now, according to SR, let us assume that the usual inertial frames \( S \in \mathcal{R}_S \) are equivalent also when considering all the mechanical and electromagnetic phenomena \( p \in \mathcal{P} \) associated to both subluminal and Superluminal bodies. In particular, \( \mathcal{P} \) will contain \( \mathcal{P}_S \cup \mathcal{P}_S \), plus other phenomena (i.e., the phenomena referred to both bradyonic and tachyonic sources and detectors; or, more generally, the phenomena to which both bradyons and tachyons participate, besides photons). In other words, let us assume that, given the set:

\[
\mathcal{A}_r = \mathcal{S}, \quad \forall r \in \mathcal{S}.
\]

At this point, we can say that also frames \( S \in \mathcal{R}_S \) are equivalent to the frame \( S \) (and to the other frames \( s \in \mathcal{R}_S \)) if the set \( \mathcal{P}_S \) is always mapped onto itself under all the transformations \( S \rightarrow S, S \rightarrow S, S \rightarrow S \), with respect to the whole set \( \mathcal{P} \) of (generalized) mechanical and electromagnetic phenomena; i.e., (loosely writing) if, given the set \( \mathcal{P} \):

\[
\mathcal{P}_r = \mathcal{P}, \quad \forall r \in \mathcal{P}_S.
\]
where 

\[ \mathcal{P} = \mathcal{P}_s \cup \mathcal{P}_s^*, \quad \text{with} \quad \mathcal{P}_s \cap \mathcal{P}_s^* = \emptyset. \]

In other words, the usual definition of equivalence works quite well also for defining the equivalence of Superluminal inertial frames to the usual subluminal ones. Therefore, our Postulate 1) of Sect. 2 has a clear meaning even when it refers – as it does, in our theory – also to Superluminal inertial frames.

Extended Relativity, being based on Postulate (1) – (3), does of course realize the validity condition (28). For instance, let us confine for simplicity to the descriptions \( \mathcal{S}_r, \mathcal{S}_r' \) (from same \( r \leq s_0 \)) of the phenomena \( p \) belonging to the set \( \mathcal{P} \) of the motions of single, free (bradyonic tachyonic) particles, respectively. Then, given the phenomena \( p \in \mathcal{P} \):

\[ \mathcal{S}_r = \mathcal{S}_r' \cup \mathcal{S}_r^*, \quad \mathcal{P} = \mathcal{P}_s \cup \mathcal{P}_s^*, \]

and, when passing from a \( s \) to a \( S, \mathcal{S}_r \) goes into \( \mathcal{S}_r' \) and vice-versa, but the whole \( \mathcal{S} \) is mapped onto itself, as required. (By the way, notice that the boundary \( \mathcal{S}_r^* \cup \mathcal{S}_r \), representing free photon goes into itself).

Let us now try to define physical laws. Given a phenomenon \( p \), if \( d_1 \) and \( d_2 \) are its descriptions in the frames \( r_1, r_2 \) respectively, and if the transformation \( L \) is such that

\[ Lr_1 = r_2, \]

we shall consequently use the convention of writing

\[ Ld_1 = d_2. \]

Let us suppose, now, that we have a criterion \( C \) for a given description \( d \) to belong to the set \( \mathcal{D} \) of the descriptions of phenomena \( p \) from the frame \( r \); we then write

\[ C(d) \text{ verified } \iff d \in \mathcal{D}. \]

We shall call \( C \) a "good criterion" if it holds for any \( d \):

\[ d \in \mathcal{D} \iff C(d) \text{ verified } \iff d \in \mathcal{D}. \]

It follows that the "good" criteria \( C \) are covariant (in form) under any \( L \):

\[ C(Ld) \text{ verified } \iff Ld \in \mathcal{D}. \]

We shall by definition call \( C \) (or rather the union of the various, possible good criteria \( C_1, C_2, \ldots \) the ensemble of the physical laws of phenomena \( p \) as seen by frames \( \mathcal{R} \). Conversely, a proposition will be considered a physical law if it is a part of \( C \). In other words, given \( \mathcal{Y} \) and \( \mathcal{S} \), we define "physical law" any proposition regarding a \( p \) which is covariant within \( \mathcal{Y} \).

Moreover, let us assume that we know, besides the class \( A \) of the ordinary physical laws (of Mechanics and Electromagnetism) for bradyons and antibradyons, also the class \( B \) of the physical laws for tachyons and antitachyons. When we pass from a subluminal frames \( s \) to a Superluminal frame \( S \), class \( A \) will have of course to transform into class \( B \), and vice-versa. In this sense, the total of physical laws \( (A \cup B) \) will be covariant under the whole group \( G \), i.e., \( G \)-covariant. And in this sense inertial frames (with relative speeds \( \| \mathbf{u} \| < c \)) are all equivalent.

Of course, – as better shown elsewher – the physical laws (of SR) may be written in a (universal) form valid for both \( B \)'s and \( T \)'s, a form obviously coinciding with the usual one in the bradyonic case. For example, the \( G \)-covariant expression eq. (24) in the form:

\[ m = \frac{m_0}{\sqrt{1 - \beta^2}} \quad [\beta^2 \geq 1, \ m_0 \ \text{real}] \]

has a "universal" validity.
From previous considerations a "Rule of extended relativity" immediately follows, generalizing Parker's principle[40] to the four-dimensional space-time. "The relativistic laws (of Mechanics and Electromagnetism, at least) for tachyons follow by applying a SLT - e.g. a "trancendent transformation"[12] - to the corresponding laws for bradyons".

At last, let us incidentally remark that the consideration of "Extended Relativity"[23] prompts us to choose a five-dimensional[65] space-time (at least) as a better background for Mechanics theories (see the following, and Sect. 3.2[63]).

4.7. - Causality and Tachyons.

In the case of tachyons, it is even clearer that our "Third Postulate" (asserting e.g. that negative-energy particles, travelling forward in time, do not exist) does easily eliminate any motion backwards in time. In fact (cf. Fig. 3c), to get transition from a (standard) tachyon A (with positive energy and moving forward in time) to a negative-energy tachyon A', it is enough a usual LT [a LT operates a movement on the same hyperboloid sheet]. The fact that such a LT will change sign not only to energy but also to time is easily seen by comparing Figs. 3c and 8.

Let us first look at Fig. 3c, and consider a frame s_0, and then a continuous succession of reference-frames moving with increasing positive speed v < c along the x-direction, which observe the same free tachyon T. When varying observer within that succession, the point K representing the kinematical state of the observed tachyon moves from its initial position A_{s_0} (representing e.g. tachyon T travelling along the positive x-direction with speed V > c - towards a final position A'. In order to go from the upper (E > 0) region to the lower (E < 0) one, the representative point K must cross the "plane" E=0. In such a position, since V/p/E, [c=1], the point K(E=0) refers to a transcendent tachyon, i.e. to a tachyon T endowed with infinite speed and minimal momentum m_0c. It is easy to calculate that, with respect to s_0, the critical frame s_0, where T appears to be transcendent is the one with relative-speed u = c^2/v < V < c. Incidentally, if we confine ourselves for simplicity - as before - to motions along x, then a one-to-one correspondence

\[ v \longleftrightarrow c^2/v \]  

(33)
can be set between subluminal frames (or objects) with speed v < c and Suproluminal frames (or objects) with speed c^2/v = V > c.

Any observer coming after s_0 in the above succession of frames should therefore see T endowed with a negative energy E (cf. Fig. 3c). Now, let us pass to Fig. 8. It is easy to realize that the frame s_0 will be represented by axes (x_{s_0}, t_{s_0}) rotated with respect to (x, t) by an angle \( \alpha \approx \alpha_{s_0} \) such that \( \alpha_{s_0} \) is superimposed to the "world-line" OT of the considered free tachyon T. The above frame-succession, in the chronotopical space, is got by increasing \( \alpha \) (from zero) with continuity: the frames attributing \( E > 0 \) to T correspond to \( \alpha < \alpha_{s_0} \), and the frames that should attribute \( E < 0 \) to T are rotated by \( \alpha > \alpha_{s_0} \). But inspection of Fig. 8 immediately confirms that the latter ones should also see tachyon T moving backwards in time (besides having negative energy!)

It is therefore straightforward to realize that (cf. Fig. 8), since point A' should represent a negative-energy tachyon T travelling backwards in time, then (owing to the "RIP") it actually represents nothing but an antitachyon T (travelling the opposite way, with positive energy, and forward in time).

We are left with no motion backwards in time.
4.3. Classical Physics for Tachyons.

Following the Rule of ER (Sect. 4.6), it is easy to derive the classical laws obeyed by tachyons by applying a SLT to the ordinary classical laws, obeyed by bradyons. As we anticipated, such laws for tachyons can be got in terms of purely real quantities.

For instance:

(i) The fundamental law of Dynamics for bradyons reads\(^{(66)}\):

\[ p^\mu = \frac{d}{ds} \left( m_0 c \frac{dx^\mu}{ds} \right), \quad \left[ \beta^2 < 1 \right] \]

and for tachyons will read\(^{(6)}\):

\[ p^\mu = - \frac{d}{ds} \left( m_0 c \frac{dx^\mu}{ds} \right), \quad \left[ \beta^2 > 1 \right] \]  

(34)

so that in G-covariant form we shall have

\[ p^\mu = \frac{d}{d\tau_0} \left( m_0 \frac{dx^\mu}{d\tau_0} \right), \quad \left[ \beta^2 > 1 \right] \]

where \( dx/ds \) is a four-vector only with respect to the group \( L^+ \) of ordinary LT's, whilst \( ds/d\tau_0 \) is a four-vector with respect to the whole group \( G \). Incidentally, let us mention that one can suitably choose the form of the action principle and of the Lagrangian, so that the tachyon 3-momentum \( c \) result to be opposite to the velocity: \( \bar{p}^\mu = - m_0 \bar{\nabla}^\mu / \beta^2 \); in such a case the space-part of eq. (3) would write \( \bar{F} = + d\bar{p}/d\bar{t} \) even for tachyons.

(ii) In a gravitational field a bradyon suffers the gravitational (attractive) 4-force:

\[ F^\mu = - m_0 \frac{d}{d\tau_0} \left( \frac{dx^\mu}{d\tau_0} \right), \quad \left[ \beta^2 < 1 \right] \]

but a tachyon will experience the repulsive 4-force\(^{(12)}\):

\[ F^\mu = + m_0 \frac{d}{d\tau_0} \left( \frac{dx^\mu}{d\tau_0} \right), \quad \left[ \beta^2 > 1 \right] \]  

(35)

where \( m_0 \) (so as in eq. (34)) is the tachyon (real) rest-mass. However, due to eq. (34), the equation of motion for both tachyons and bradyons in a gravitational field will still read\(^{(6)}\) (in G-covariant form):

\[ a^\mu + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0, \quad \left[ \beta^2 \leq 1 \right] \]

where \( u^\nu = dx/d\tau_0 \) and \( a^\nu = du/d\tau_0 \) are four-velocity and four-acceleration, respectively. In conclusion\(^{(12)}\): (a) from the energetical and dynamical point of view, tachyons appear to be gravitationally repulsed by ordinary matter, i.e., to be the 'anti-gravitational particles'; (b) from the kinematical viewpoint, however, tachyons appear as bending (or 'falling down') towards the gravitational-field source\(^{(67)}\).

(iii) As a constant-speed bradyon in vacuum does not emit radiations, so a constant-speed tachyon in vacuum will emit no radiations: in particular, no Cherenkoff\(^{(6,68)}\).

(iv) As regards Doppler-effect for Superluminal sources, in the case of relative motion parallel to the \( x \)-axis, we shall have in both the sub- and Super-luminal cases\(^{(6)}\):

\[ v = \frac{\gamma}{\gamma - 1} \frac{1 - \beta^2}{\gamma + \beta \cos \alpha}, \quad \left[ \beta^2 > \frac{2}{c^2} \right] \]

where \( \gamma \equiv u \gamma \equiv \gamma c \) is the relative speed and \( \alpha \equiv u \gamma \), the vector \( \bar{u} \) going from the observer to the source. The same shift will be observed both for \( u = v \leq c \) and for \( U = c^2/v > c \) (remember also eq. (33)). For Superluminal approach, the radio-emission will be received in reversed chron
v) With regard to Maxwell equations, if one assumes the usual electromagnetic tensor $F_{\mu\nu}$ to be still a tensor under the new group $G$ of GLT's, then he gets (69) that Maxwell eqs. are $G$-covariant. However, if - more consistently - one firstly generalizes (6) the transformation-laws for electric, $E$, and magnetic, $H$, fields, then new generalized Maxwell equations are got. Namely, in presence of both subluminal, $j_\mu(s)$, and superluminal, $j_\mu(S)$, four-currents, we shall have (70) in Lorentz-covariant form:

$$\begin{align*}
V \cdot D &= \varrho(s), \\
V \cdot B &= \varrho(S), \\
\n\n\int \nabla E = j(s) - \partial B / \partial t, \\
\int \nabla H = j(s) + \partial D / \partial t.
\end{align*}$$

In other words, if we define the ordinary dual tensor $[\mu \nu \rho \sigma = 0, 1, 2, 3]$

$$F^{\#}_{\mu\nu} = \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}; \quad (F^{\#}_{\mu\nu})^{\#} = - F_{\mu\nu}$$

and introduce the complex quantities (70)

$$F^{\#}_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} - i \varepsilon_{\mu\nu\sigma\tau} B_{\sigma,\tau}, \quad F^{\#}_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu} - i \varepsilon_{\mu\nu\sigma\tau} A_{\sigma,\tau},$$

then eqs. (37) write:

$$\partial_\nu T^{\#}_{\mu\nu} = J_\mu; \quad T^{\#}_{\mu\nu} = i T_{\mu\nu}, \quad \left[\nabla^2 \geq c^2\right]$$

and a connection is established between the electromagnetic duality in eq. (38) and the "dual correspondence" (12) bradyons $\rightarrow$ tachyons. Moreover, if we introduce also the complex four-potential (70)

$$C_{\mu} = A_{\mu} + i B_{\mu}$$

where - following CABIBBO and FERRARI (71) -

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} - i \varepsilon_{\mu\nu\rho\sigma} B_{\rho,\sigma}; \quad F^{\#}_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu} - i \varepsilon_{\mu\nu\rho\sigma} A_{\rho,\sigma},$$

then the Generalized Maxwell Equations, eqs. (37'), will read (70)

$$\nabla \cdot C_{\mu} = J_\mu; \quad \partial_\mu C_{\mu} = 0 \quad \left[\nabla^2 \geq c^2\right]$$

(37''')

Of course, also eqs. (37'), (37'') can split into purely real equations. Notice that in our theory $A_{\mu}$ is only a Lorentz-vector and not a $G$-vector, since under GLT's it behaves so as $dx/ds$; e.g.: under a SLT = L:

$$A_{\mu} \rightarrow A'_{\mu} = -i L_{\mu\rho} A_{\rho}; \quad T_{\mu\nu} \rightarrow T'_{\mu\nu} = -i L_{\mu\rho} T_{\rho\sigma};$$

and analogously for $F_{\mu\nu}$ and $A_{\mu,\nu}$. Finally, the structure of our theory reveals (70) - however - that $F_{\mu} = - L_{\mu\nu}^{-1} A'_{\nu}$. For further details or comments, see refs. (70).
4.9 - Solving Causal Paradoxes for Tachyons (and Digression).

We have seen that in the case of tachyons it is even clearer that our Third Postulate does
minimize any causality violation. However, this success is obtained at the price of abandoning the
conviction that judgement about what is "cause" and what is "effect" is independent of the observer.
In fact, in the case above examined (Figs. 3, 4 and 5) the initial observer \( s_0 \) will judge the event B
as causing the event at A. Conversely, any observer \( s' \), which interpret the same phenomenon as
exchange of an antitachyon \( \bar{t} \) from B to A, will judge the event at B as the cause of the event
(see ref. (72)).

Nevertheless, all observers will always see the cause to precede chronologically its own
effect (12).

Once again, the Law of "Retarded Causality" is relativistically covariant, and holds for all
inertial observers, both subluminal and Superluminal. On the other hand, we have not to expect
at all covariance of the phenomenon-description and of the "description details" in the case
of the assignment of "cause" and "effect" names (73). Moreover, we shall see (74) that the
events "causally connected" according to an observer \( s_1 \) (e.g. via a tachyonic exchange) may
appear as totally uncorrelated to other observers \( s_2 \); so that not even the very existence of a
causal relation is relativistically covariant in ER. Relativity of judgement about cause and effect
and even more of existence of a "causal correlation" leads to a series of apparent "causal paradoxes" (75, 76) that - even if easily solvable (74, 77, 72) - gave rise to some perplexities.

We shall state and resolve here only a causal paradox which seems to be one of the most sophis-
cicated. It was proposed by Pirani (76) in 1970 and substantially solved by Parmentola and Y
(see ref. (72)) in 1971, on the basis of refs. (74), (72), (77).

Let us consider four observers A, B, C, D having given (76) velocities in the plane \((x, y)\) with
respect to a fifth observer \( s_0 \). Let us suppose that the four observers are given in advance the in-
struction to emit a tachyon as soon as they receive a tachy-on from another observer, so that the following chain of events takes place (see Fig. 10). Observer A initiates
the experiment by sending tachyon \( 1 \) to B; observer B immediately emits tachyon \( 2 \) towards C; observer C sends tachyon \( 3 \) to D, and observer D sends tachyon \( 4 \) back to A, with the result (see reference (76)) that A apparently receives tachyon \( 4 \) (event \( A_1 \)) before having initiated the experiment by emitting tachyon \( 1 \) (event \( A_0 \)). The sketch of this "gedanken experiment" is in Fig. 10, where oblique vectors represent observer velocities relative to \( s_0 \), and lines parallel to the Cartesian axes represent the tachyon paths.

It is important to notice that Fig. 10 does not represent the process actual description by any
observer (14). In fact, the arrow of each tachyonic line simply denotes its motion direction with re-
to the observer that emitted that particular tachyon. By the way, tachyons and observers velocity
can on the contrary be chosen (76) in such a way that all tachyons effectively appear to observer \( s_0 \)
move in directions opposite to the ones indicated in Fig. 10.

Since we cannot mix together observations by (four) different observers (78, 14, 79), it is neces-
sary to investigate how each observer describes the event chain.

Following ref. (78), let us pass, for this end, to Minkowski space and study the space-time de-
scription given e.g. by observer A. From a dynamical viewpoint, the observers may be replaced by
external force-fields that scatter the tachyons (or by atoms, able to absorb and emit tachyons).

In Fig. 11 it is clearly shown that the absorption of \( 4 \) happens before the emission of \( 1 \). It may
seem that one can send signals into the past of A. However, observer A will effectively see an or
thodox sequence of events as follows: event D consists in the creation of pair \( \bar{3} \) and \( 4 \) by the exter

\[ \text{FIG. 10} \]
fields; tachyon 4 is then absorbed at A1, while 3 is scattered at C (transforming into tachyon 2); event A2 is the emission, by A itself, of tachyon 1 that annihilates at B with tachyon 2. Therefore, according to A, one has essentially an initial pair-creation at D, and a final pair-annihilation at B; and tachyons 1, 4 do not appear causally correlated at all. In other words, according to A the emission of 1 does not initiate any chain of events that leads to the absorption of 4, and we are not in the presence of any effect preceding his own cause.

According to A, we have essentially an initial pair-creation at D, and a final pair-annihilation at B; and tachyons 1, 4 do not appear causally correlated at all. In other words, according to A the emission of 1 does not initiate any chain of events that leads to the absorption of 4, and we are not in the presence of any effect preceding his own cause.

Analogous, orthodox descriptions (i.e., the descriptions put forth by the remaining observers) may be obtained by Lorentz-transforming the description above by A.

Let us now formulate the same "paradox" in its "strong version" (12, 14, 78). Let us suppose that tachyon 4, when absorbed at A1 by A, blows up the whole laboratory of A, eliminating the physical possibility that tachyon 1 (believed to be sequence starter) is subsequently emitted (at A2), and thus originating a supposedly contradictory state of things – according to the paradox terms –. Following Root and Trefil (80), we can see on the contrary how e.g., observers s0 and A will really describe the phenomenon.

In particular, s0 will observe the laboratory of A blown up immediately after the emission (at A1) of tachyon 4 towards D. According to s0, therefore, tachyon 1 emitted by B will proceed beyond A (since it is not absorbed at A2) and will eventually be absorbed at some remote sink point U of the universe (80, 14). By means of a LT, starting from the description by s0, we can obtain the description given by A (14).

Observer A, after having absorbed at A1, tachyon 4 (emitted at D together with 3), will record his own laboratory explosion. At A2, however, A will realize that he has been bypassed by a tachyonic cosmic ray 1 (coming from the remote source U), which will annihilate at B with tachyon 3 scattered at C (i.e., with tachyon 2). (12, 14, 80).

Even if no problem about the "G-covariance" of the Retarded Causality Law is left open for tachyons, nevertheless we want to add the following "digression," without any reference to what precedes.

Let us notice that postulate 3) is a fundamental hypothesis of ours (in accord with statistical thermodynamics and with information theory), but a priori is not logically necessary (81). In fact: (i) Let us suppose that a statistical correlation exists between two series of events, in the sense that e.g., each second-series event happens about 1 second before a first series of events (see Fig. 12). Such a statistical correlation will be called a "causal connection"; (ii) Let us now suppose that first-series events are the "independent" ones, in the sense that we make them occur e.g., at instants chosen by consulting random-values' tables (maybe produced by a remote computer, having no reasonable relation with the events considered). Such events will be called the "causes"; (iii) The second-series events will then be called the "dependent" ones in the causal correla-
tion defined at point (i). They will be said to be the "effects"; (iv) One may therefore conclude, from the definitions above, that in this case effects do chronologically precede their own cause (Fig. 12). To conclude the present digression, let us shed some light on the possible nature of difficulties in conceiving effects chronologically preceding their causes, by reporting the following anecdote (81), which does not involve today prejudices. For ancient Egyptians (81, 12), who only the Nile and its tributaries, which all flow from South to North, the meaning of the word "South" coincided with the one of "up-stream", and the meaning of the word "North" coincided with the one of "down-stream". When Egyptians discovered the Euphrates, which unfortunately happens to flow from North to South, they passed through such a crisis that it is mentioned in the stele of Tuthmosis I, which tells us about "that inverted water that goes down-stream (i. e. towards North) in going up-stream" (81, 12).

Before closing, let us add that the present "digression" has nothing to do also with what follows.

5. - MATTER AND ANTIMATTER.

5.1. - The Antimatter.

We have seen that the "RIP" eliminates any information transmission into the past (even by tachyons); and simultaneously it allows us to predict — merely from Relativity — the existence of antimatter.

In fact, in our theory, antiparticles $\bar{P}$ are nothing but particles $P$ in the state with "negative" energy and motion backwards in time; "particles" $P$ in that state will indeed appear to us as antiparticles $\bar{P}$ (endowed with positive energy and motion forward in time) since we must explore space-time in a unique time-direction (that we called positive by definition). Our Third Postulate asserts that negative-energy particles moving forward in time (and then positive-energy particles moving backwards in time) do not exist.

Namely, in our theory we showed that, given a tachyon $T$, a usual LT can transform it into a particle $\bar{T}$ expected to have exactly all the properties that antiparticles actually showed to have in experiments.

In the case of bradyons $B$, however, (cf. Fig. 3a), by means of a usual LT one cannot leave the initial hyperboloid sheet. That is to say, a usual LT cannot bring an upper-hyperboloid point (representing a particle $B$) into a lower-hyperboloid point (representing the antiparticle $\bar{B}$).

It follows that — in the case when we confine ourselves to usual LT's — then the "matter" or "antimatter" character is invariant for bradyons, but is relative to the observer for tachyons. However, when eliminating the previous restriction, then — by means of GLT's, e. g. by means of two SLT's — we can indeed pass from particles to antiparticles even in the case of bradyons.

Thus, in Extended Relativity, the matter/antimatter character is relative to the observer also for usual particles and objects.

Namely — let us repeat it —, a particle $P$ in the kinematical state corresponding to a point $\bar{P}$ lower hyperboloid (Fig. 3a) has been shown to appear as the antiparticle $\bar{P}$ of $P$ in the usual SLT's. The fact is interesting that, once the notion of particle is introduces (as is usually done in SR), it rely from SR itself the concept of antiparticle follows (cf. eq. (7)).

At last, let us confine ourselves for simplicity to boost $\alpha$ along $x$, i.e. to collime Lorentz transformations along the x-direction: Then, the four subsets in eq. (19) of GLT's describe transitions from the initial frame $s_0$ (e. g. with right-handed space-axes, not only to all frames $f^R\alpha$ moving at $x$ with all possible speed $u$, where $-\infty < u < +\infty$, but also to all left-handed frames $f^L\alpha$, moving well with all possible speeds $u$ along $x$ (cf. Figs. 2-6 and 11 of ref. (6)).

In fact, when overtaking the transcendent frame (relative to $s_0$) $f(\infty)$, we pass $f^R\alpha$ to totally inverted frames $f^{\bar{T}}(\infty)$, with left-handed spatial frame, a reversed time-axis (83), etc. This could have been expected, since the total inversion, or strong reflection, $f^{\bar{T}}(\infty)$ (where $\bar{T}$ is the space-parity operation and $T$ the
time-reversal) is nothing but a particular "rotation" in 4-dimensional space-time; and we saw that GLT's coincide with all space-time "rotations" when we do not restrict ourselves to the usual, proper, orthochronous LT's, so that (83) PT \subseteq G (see the following).

Loosely speaking, we can say that (82) -- after applying the "RIP" -- if an ideal frame f could undergo a trip along the axis (circle) of the speeds, then it would come back (after having bypassed f(\omega)) with inverted spatial axes (space parity) and with particles transformed into antiparticles.

5.2. - Sources and Detectors. Interactions and Objects.

Let us confine ourselves to two-dimensional GLT's and consider:

a) a bradyonic object C in our rest-frame s_0; b) a continuous succession of Superluminal frames S, moving collinearly along the \( x \)-direction and geometrically represented (see Figs. 1 and 5) by rotations with angles \( \alpha \) ranging from \( 90^\circ - \alpha \) to \( 90^\circ + \alpha \). Let us call \( S_\alpha \) the transcendent frame, in which C becomes an infinite-speed tachyon. Due to "RIP", as we overtake the transcendent frame \( S_\alpha \) (see Fig. 13), the new frames will judge the observed object C as an antitachyon on C travelling in the opposite direction in space.

Vice-versa, we can put ourselves at rest (with respect to \( s_0 \)) and observe frames S considered as objects, or rather as one (tachyonic) object C that moves with speeds varying with continuity. We have already analysed that situations in Fig.4.

But close inspection of Fig.4 reveals that the Third Postulate cannot be applied if we (for each bradyon B or tachyon T) do not take account of the proper sources and detectors. More generally, since we have to deal with exchanges of the emission and absorption roles, the "reinterpretation procedure" loose-its meaning if we cannot refer our B's and T's to some (space-time) interaction regions.

For example, when a tachyon T overcomes the divergent speed, it passes from being e.g. a tachyon entering (a certain interaction region) to being an antitachyon T outgoing (from that interaction region). In conclusion, the "RIP" will be completed by saying (12): "Under a trans-critical GLT, when e.g. the roles of emitter and absorber happen to be interchanged, any negative-energy object in the initial "state" physically corresponds to its positive-energy antiobject in the final "state", and vice-versa!"

Of course, the Third Postulate -- in order to be used for reinterpreting the effects of GLT's -- requires considering processes with both initial and final "states".

Therefore, Extended (Special) Relativity imposes that physics must deal with interactions rather than with objects (in quantum-mechanical language, with "amplitudes" rather than with "states").

Now, let us go back again to Fig. 4. If the two macro-objects A and B are connected through the exchange of a particle P, then -- according to the particular class \{\epsilon_\alpha\} of observers -- object P is endowed with infinite speed: \( P=P_\infty \). But, for transcendent particles, the motion-direction along AB (see Fig.4) is not defined: In such a limiting case, we can for instance consider \( P_\infty \) either as a particle \( P(v=+\infty) \) going from A to B, or equivalently as an antiparticle \( P(v=-\infty) \) going from B to A; in the Quantum-Mechanics (Q. M.) formalism we could write for example:

\[
\begin{align*}
P_\infty \equiv P(v=\infty) & = a \left| P(v=+\infty) \right> + b \left| P(v=-\infty) \right> \\
a^2 + b^2 & = 1.
\end{align*}
\]
In other words, if A and B exchange a (bradyonic or tachyonic) object, for a certain class of events they are connected via a symmetrical, instantaneous interaction. (This perhaps helps justify the newly increasing use of action-at-a-distance descriptions, as equivalent to the star ones.)

But this clarifies also that no source can emit anything if a proper detector is not yet ready to absorb it, somewhere in the universe. At least in the case of tachyon exchange, this condition is strictly required by Relativity (if we confine ourselves only to subluminal LT's). It is not with meaning that, even when using standard SR, Wheeler and Feynman were able to build up the limiting case of photons, a theory (equivalent to usual Electromagnetism) where sources photons just only if their detectors are (already) ready to absorb them. We shall see that "black holes gravitational and "strong": see the following) seem to be suitable sources - and detectors of tachyons.

6. THE CPT-THOREM: CROSSING RELATIONS: AND ADVANCED SOLUTIONS.

6.1. The CPT theorem.

Let us add here the following. The GLT corresponding to \( \alpha = 180^\circ \) (see Figs. 1, 5) is the "strong reflection", or "total inversion" operation \( \mathcal{PT} = \mathcal{C} \), as already mentioned in Sect. 5.1. Actually, product of two SLT's (which one always yields a subluminal Lorentz transformation) can transformations both "orthochronous", i.e. of the type \( LT = + \Lambda \), and "non-orthochronous" of the type \( -LT = (\mathcal{PT}) \Lambda \). Let us in particular consider the "non-orthochronous" LT that is usually called since, in order to reach the value \( \alpha = 180^\circ \) (starting from \( \alpha = 0^\circ \)), we must by-pass the case \( \alpha = 0^\circ \) (see Figs.1 and 15), then we have to apply the reinterpretation principle (cf. Fig. 4): so that in the universe with charges the GLT that we called \( \mathcal{PT} \) does actually produce the exchange particle tiparticle and must be effectively considered a \( \mathcal{CPT} \) operation.

\[
\text{GLT}(\alpha = 180^\circ) = \mathcal{CPT},
\]

where here \( \mathcal{C} \) means inversion of all (additive) "charges". Since eq. (40) holds both for B's and T's, then under the "strong reflection" GLT, particles (B or T) in the initial or final state of an interaction process will be transformed into antiparticles (B or T) in the same state of the interaction process, and vice-versa. For instance, the two reactions

\[
\begin{align*}
\nonumber \text{a + b} & \rightarrow \text{c + d}, \\
\nonumber \bar{a} + \bar{b} & \rightarrow \bar{c} + \bar{d}
\end{align*}
\]

are the two different descriptions of the same phenomenon as seen by the two different inertial mes \( s_0 \) and \( -s_0 \), respectively.

It is worthwhile explicitly to notice that - if a GLT acts on a fourvector associated with an event - then it analogously acts also on the other fourvectors (as four-momentum, four-current, associated with that object. In particular, the "strong reflection" operation, which changes sign 3-position \( \vec{\mathbf{r}} \) and time \( \mathbf{t} \), will change sign also the 3-momentum \( \vec{\mathbf{p}} \) and to the energy \( E \). Thus we are led, within Extended Relativity, to introduce the new symbols \( \mathcal{P} \) (strong Parity) and \( \mathcal{T} \) (strong Time-Reversal) for meaning the sign-inversions of the first three components and of the fourth component of all fourvectors, respectively. Then, the meaning of eq. (40) is the following:

\[
\begin{align*}
\mathcal{P} & \Rightarrow \mathcal{CPT}, \\
\mathcal{T} & \Rightarrow \mathcal{RIP}.
\end{align*}
\]

\[
\begin{align*}
\mathcal{P} & = \mathcal{P} \mathcal{P} \ldots, \\
\mathcal{T} & = \mathcal{T} \mathcal{T} \ldots.
\end{align*}
\]
and where \( C, P, T \) are the usual discrete symmetries (except for the fact in our formalism \( C \) means conjugation - see Sect. 3.2). In such a new formalism, \( P \) is essentially equivalent to \( P \); but \( T \neq T \) (moreover, our \( T \) does not contain the operation \( X \) that effects the exchange absorption emission).

It follows that the \( CPT \) theorem, requiring covariance of all physical laws under \( CPT \), is a direct consequence of Special Relativity (when we do not confine ourselves to subluminal inertial frames).

Eqs. (40), (41) suggest also that, within Relativity, "the right way of doing \( PT \) is doing \( CPT \)" and this is one of the essential teachings of Lee and Yang (86). In fact, Extended Relativity says that we can safely - i.e. covariantly - reflect space-time (or, in the case when \( T \)-covariance is supposed to hold, reflect space) only if we simultaneously apply \( C \), so as to have furthermore particles changed into antiparticles.

Let us remember that the "RIP" can be given a more elegant formulation in a five-dimensional space (see Sect. 3.2), where the fifth axis is related to rest-mass.

We have seen that \( CPT \)-covariance is required by the mere SR. It is required, however, only for the physical laws of Mechanics and Electromagnetism (strictly speaking). Extension of the FR ("Principle of Relativity") also to nuclear and subnuclear phenomena (i.e. for strong and weak interactions) should possibly lead to a new wider theory (e.g. to "conformal relativity"), just as extension from Mechanics to include Electromagnetism led from Galilei-Newton's theory to Einstein's. Therefore, were e.g. a \( CPT \)-covariance violation found in sub-nuclear interactions, it could mean that Lorentz transformations are no longer precise enough to be used in building up strong and weak field theory.

6.2. - Only Laws, and not Descriptions, have to be Covariant.

Let us remember that the "Principle of Relativity" (i.e. our first Postulate) does not require at all that two different observers give the same description of the same phenomenon (for instance, due to Doppler effect, they can observe the same object as having different colours). It does require only that the two observers must find that phenomenon to be ruled by the same physical laws (generally speaking, conservation laws).

For instance, let us emphasize that the electric charge of an isolated system is not required to be a relativistic invariant, but only to be constant (as seen by any observer) during the transformation of the system.

It is instructive to analyse an explicit example (Fig. 14). Let us consider a positively charged particle that, with respect to a first observer \( O_1 \), decays into a neutral particle \( c \) and another positively charged particle \( b \) having in general different velocity (see Fig. 14a, where the time-arrow is represented too). It is possible to find another observer \( O_2 \), with respect to whom e.g. the outgoing particle \( b \) behaves as an incoming antiparticle \( b' \) having a negative charge. Observer \( O_2 \) will judge the process as a "resonance" formation or as an "annihilation" process (Fig. 14b). Frame \( O_1 \) observes a total electric charge +1, whilst \( O_2 \) a total electric charge zero. Both observers, however, will agree that the electric-charge conservation-law is verified in the observed process.

Moreover, before interaction, \( O_1 \) sees one particle, while \( O_2 \) sees two particles.

Therefore, the very number of particles (e.g. of tachyons, if we consider only subluminal LT's), at a certain instant, is not Lorentz invariant.
However, the total number of particles (e.g., of tachyons) participating in the reaction either the initial or the final states is Lorentz invariant, due to the very features of "RIP" and of GLT's. Again, we are encouraged to build the physical theories in terms of "reaction processes" rather than of "objects".

Before closing this Section, let us underline that the usual proofs of electric-charge invariance hold only for subluminal, orthochronous LT's applied to bradyons. We have already shown, on contrary, that the charge of a particle can change sign under GLT's (cf. Fig. 4).

6.3. - Crossing Relations.

Besides the CPT theorem, from the mere SR it is possible to derive also the so-called "crossing-relations" (even for usual elementary particles). Namely, it can be shown\(^{12,13}\) that within Extended Relativity - the same function must yield the "scattering amplitudes" of different processes like\(^{12}\):

\[
\begin{align*}
\text{a + b} & \rightarrow \text{c + d} \\
\text{a + c} & \rightarrow \text{b + d}
\end{align*}
\]

in correspondence, of course, to the respective (different) domains of the process variables. Such considerations may be performed at a classical (purely relativistic) level\(^{12}\), since cross sections and invariant (scattering) amplitudes can be defined also classically (cf. refs. 12, 1). For instance, let a, b, c, d be bradyonic objects. The two processes (42), (42') among B's are two different reactions \(p_1\), \(p_2\) as seen by us, but they can be seen as the same interaction \(d_1\) and \(d_2\) among T's (see the previous Sect. 6.2) by two suitable, different, Superluminal observers \(S_1\), \(S_2\). Due to the Extended Relativity, we can get the scattering-amplitude of \(p_1\), i.e. \(A(p_1)\), by applying the SLT\((S_1 \rightarrow s_0) = L_1\) to the amplitude \(A_1(d_1)\) found by \(S_1\) when observing the scattering \(p_1\):

\[
A(p_1) = L_1 \left[ A_1(d_1) \right]
\]

conversely, we may get the scattering-amplitude of \(p_2\), i.e. \(A(p_2)\), by applying the SLT\((S_2 \rightarrow s_0) = L_2\) to the amplitude \(A_2(d_2)\) found by \(S_2\) when observing the scattering \(p_2\):

\[
A(p_2) = L_2 \left[ A_2(d_2) \right]
\]

But, since by hypothesis

\[
A_1(d_1) = A_2(d_2) = A(d_S),
\]

it follows that:

\[
A(p_1) = A(p_2)
\]

for all reactions among B's of the kind (42) and (42').

Actually, in ordinary Q.M. (or rather quantum field theory), the requirement (43) is accomplished by assuming the amplitude \(A\) to be an analytic function, which can be (analytically) continued from the domain of the invariant variables relative to (42) to the domain relative to (42'), and so on. However, our requirement (43) - imposed by Relativity, or rather by ER, on processes (42), has a more general nature (and is purely relativistic in character).
6.4. - Explaining Advanced Solutions.

It is known since long that, in general, relativistic equations admit advanced (besides retarded) solutions. For instance, Maxwell equations predict both retarded and advanced electromagnetic radiation. Quite naively, "advanced" solutions have been considered as actually associated with motions backwards in time, forgetting the Stückelberg-Feynman "Reinterpretation Principle" and even the very structure of SR.

Within Extended Relativity, it is easy to explain why relativistic (both classical and quantal) equations are expected to admit both retarded and advanced solutions, i.e. solutions apparently relative to objects travelling forward and backwards in time, respectively.

In fact, when an equation admits a solution corresponding to (outgoing) particles or photons, then a procedure analogous to the one in the previous Sects. 6.3 will be able to connect that solution to another one corresponding to (incoming) antiparticles or (anti)photons.

Therefore, if that equation is relativistically covariant (or better G-covariant: i.e. is holds for all the inertial observers, both sub- and Super-luminal), then it must also admit solutions relative to incoming antiparticles or photons, whenever it admit solutions relative to outgoing particles or photons.

We have thus shown that all relativistic equations, covariant under the group G of GLT's must admit both retarded and "advanced" solutions (even if the latter ones are actually connected - because of the "RIP" - to oppositely-directed antiparticles, or photons, and NOT to objects travelling backwards in time).

The fact that, in general, relativistic equations actually satisfy this requirement - derived from Extended Relativity - is a point in favour (88) of Extended Relativity itself. But it does not mean that all relativistic equations are already written (in their present form) in G-covariant form.

It is obvious to conclude that no particle (in particular, no radiation) actually travelling backwards in time is either predicted by relativistic equations or expected to be experimentally detectable within the "world" of SR. We are, however, left with the problem: Why do we usually observe e.g. only the outgoing radiation, and not the incoming (anti)radiation? The whole clue of the answer is in taking into account boundary conditions (89). In usual macro-physics some boundary conditions are much more probable than other ones. For instance, (Mechanical) fluid-dynamics equations allow having on the sea-surface both outgoing circular, concentric waves and incoming, circular waves tending to a centre. However, the initial conditions yielding the first case are much more probable to be observed than the initial conditions yielding the second case.

7. - SOME CONSIDERATIONS ON TACHYONS.

7.1. - Black-holes and Tachyons.

Let us pass to consider for a moment General Relativity. Instead of allowing for completely general coordinates, let us - however - restrict the adoptable coordinates in the following way. Given a set of general coordinates \((a, b, c, d)\) and a space point \(P\), let us associate to them the (local) observer \(0\) which is at-rest at \(P\) with respect to those coordinates. Then, let us impose that we can pass from \((a, b, c, d)\) to other general coordinates \((a', b', c', d')\) only if the (local) observer \(0'\) - associated to \((a', b', c', d')\) at the same point \(P\) - locally moves with slower-than-light speed with respect to \(0\).

Let us remember that usually, in General Relativity, continuity and derivability of our space-time manifold are assumed, in such a way that the geodesics never change their type, and bradyons (tachyons) always remain bradyons (tachyons).

Our initial restriction, however, can lead us to release the manifold-smoothness requirement, at least on some "special" (not necessarily singular) surfaces, in such a way that the geodesics type can change there.
For instance, let us consider the Szekeres-Kruskal (SK) coordinates as a priori convenient for describing the (Schwarzschild) solution of the Einstein equations for a spherically-symmetric mass-distribution. Namely, let us consider the set of general coordinates defined outside the event-horizon (i.e., for \( r > 2M \)), and the set of coordinates defined for \( r < 2M \). We are assuming the definition of each set to be so extended as to cover the whole space-time manifold (both outside and inside the event-horizon). Then, it is immediate to realize that:

\[
\begin{align*}
\mathbf{v} \left( \frac{1}{r} \right) &= \mathbf{i} \left[ u(r) \right], \\
\mathbf{u} \left( \frac{1}{r} \right) &= \mathbf{i} \left[ v(r) \right],
\end{align*}
\]

where \( \mathbf{i} \) is here the operator changing \( r \rightarrow \frac{1}{r} \) and multiplying the whole function by the unit. The operator \( \mathbf{i} \) is formally identical to the operator entering eq. (20) and effecting the transition from sub-to Super-luminal frames. Incidentally, such an operator \( \mathbf{i} \) of eq. (20) is the transcendental boost \( \mathbf{K} = \lim_{\beta \to \infty} \text{SLT}(\beta) \), i.e., more precisely the "transcendent boost" that from eqs. (22) results to effect the transition:

\[
x \rightarrow x' = t ; \quad t \rightarrow t' = x.
\]

Moreover, let us notice that, if we define:

\[
\begin{align*}
u_\parallel &= \left[ \frac{r}{2M} - 1 \right] \cdot \exp \left[ \frac{r}{4M} \right] \cdot \cosh \left( \frac{t}{4M} \right) ; \\
v_\parallel &= \left[ \frac{r}{2M} - 1 \right] \cdot \exp \left[ \frac{r}{4M} \right] \cdot \sinh \left( \frac{t}{4M} \right),
\end{align*}
\]

so that for \( r > 2M \) it is

\[
u_\parallel \cdot v_\parallel = v_\parallel \cdot v_\parallel,
\]

then for \( r < 2M \) we have:

\[
u_\parallel \cdot v_\parallel = v_\parallel \cdot u_\parallel.
\]

Therefore, going (with \( t \) fixed) from \( r > 2M \) to \( r' = 1/r < 2M \) means exchanging the rôle of \( u, v \):

\[
u_\parallel \rightarrow u' = v_\parallel ; \quad v_\parallel \rightarrow v' = u_\parallel.
\]

A comparison of eqs. (46) with (45) again shows\(^{6,92}\) the formal analogy between going from \( S \) frames and going from the outside to the inside region of a black-hole horizon. In ref. (93) concluded that the internal SK-coordinates \((u_\parallel, v_\parallel)\) are associated to observers that move faster-than-light relatively to the observers associated (at the same point) to the external SK-coordinates \((u_\parallel, v_\parallel)\). In such a case, we would have a violation of our initial assumption and we should confine ourselves e.g. to choose everywhere either the 'external' SK-coordinates or the 'internal' SK-coordinates. The same could be said for other coordinate systems. Such considerations have been put on a more formal basis in the second ref. assuming a multiply-connected manifold and allowing for changes of topology in space-time.

With our choice of assumptions, and therefore with our restrictions about the adoptable Coordinates, it is easy to realize that a free-falling (outside the event-horizon) bradyon \( B \) would become a tachyon \( T \) inside the horizon\(^{93}\), and vice-versa. Black-holes will therefore be cis sources (and detectors) of tachyons.
Thus, ER seems to suggest that tachyonic objects can be classically exchanged between black-holes.

In any case, what we want here to stress in that the same mathematical problems, met in SR for extending LT's to Superluminal frames, are present also in General Relativity when one wants to pass from the exterior to the interior of a black-hole horizon. In particular for not spherically-symmetric mass-distributions, the same difficulties with imaginary units will be met as when dealing with SLT's. Again a good tool seems to be the "catastrophes" theory.

Actually, when analysing perturbed Schwarzschild problems, many authors (94) had to suggest the existence of coordinate-independent "anomalous" (or even singular) surfaces.

7.2. - Various Remarks.

A) Let us start with some "kinematical" considerations. First of all, it is easy to deduce from the four-momentum law that a body at rest cannot emit any tachyon in its rest-frame, unless it lowers—by a discrete "jump"—its own rest-mass: This is obvious, for instance, in the case of finite-speed tachyons. In fact, a trascendent tachyon is known to carry impulse but no energy. Namely, let us consider the case of two bodies A and B not changing their rest-masses during the electromagnetic exchange between A and B of a unique trascendent tachyon: Namely, either of a trascendent tachyon T emitted by B and absorbed by A, or of a trascendent antitachyon T emitted by A and absorbed by B. In conclusion, there will be an elastic interaction of A with B, caused by an infinite-speed impulse-transmission between A and B; and any such observer may adopt a description implying no vacuum instability.

Let us recall once more that the infinite speed is not Lorentz invariant, only the light-speed in vacuum being invariant in ER. Therefore, according to other observers s', s''... travelling along x, the abovementioned process will appear as due to the exchange of a finite-speed tachyon (or antitachyon).

C) With regard to the so-called "virtual particles", let us right now anticipate what follows (11, 97). Trascendent tachyons (carrying E=0, \( \gamma \neq m_0 \), \( c \neq 0 \)) can be the (classical) intermediaries of diffractive scattering, of the so-called "pomeron exchange" reactions, and of some elastic scatterings. For example, let us consider the case of two bodies A and B not changing their rest-masses during the tachyonic exchange, so that A and B elastically scatter (in the language of elementary particle physics). Then, in the center-of-mass system, the two bodies A, B appear as exchanging only momentum (and no energy); so that in a natural way we can regard them as connected via a trascendent tachyon exchange; but, if we apply Lorentz transformations, that means that elastic scattering can in general be regarded as due to the exchange of finite-speed suitable tachyons. More in general still, when redefining the requirement that rest-masses be unchanged, we find out that tachyon exchange can be useful to interpret (at a classical level, i.e. possibly without Q. M.) even the inelastic interactions between elementary particles (or between bodies). We shall come back to this point in Sect. 7.3.

D) Let us close this Section with another, often useful consideration. Namely, we want to illustrate (in a "coloured" way) a lesson coming to us from SR (tachyons, therefore, play no rôle in the following). Let us suppose (98) we are informed about a cosmic fighting between two different species of extraterrestrial living beings each on his own interplanetary rocket, where the rocket colours are
violet for the first species and green for the second one. Let us moreover suppose that we let the "green men" possess an inviolable natural instinct that makes them peaceful, so that they are inhibited to fire their guns; on the contrary, the "violet men" possess an aggressive, we instinct.

When we look at the cosmic battle, it can well happen -- because of the Doppler effect -- that when a "violet man" fires his gun and strikes a green rocket, the violet appears to us as green (owing to violet-rocket recession and green-rocket approach). One has to be careful not to deduce that an (inviolable) natural law has been badly violated (the instinct-law) -- se extraterrestrial beings, in our case). At first sight, we actually "observed" an apparent violation of natural laws; but, if we know the physical theories (i.e., Relativity, besides the rocket velocity), we can calculate out the "proper colours" of the rockets, in their own rest-frames, and solve the wonderings. In other words, any observer is capable of understanding all physical happenings through his (only) observations provided that he properly uses his knowledge of the theory (activity). This means that we can scientifically look at nature only if we are equipped with theoretical -- besides experimental -- instruments.

7.3. "Virtual Particles" and Tachyons.

Let us go back to what said at point C) of the previous Sect. 7.2.

The four-momentum conservation law tells us that a body (or particle) \( \mathbf{A} \) cannot emit in its rest-frame any tachyon \( \mathbf{T} \) (whatever their rest-mass \( m \) is), unless the rest-mass \( M_A \) of \( \mathbf{A} \) is reduced to a lower value of \( M_A' \) such that\[ \Delta M = M_A - M_A' = m^2 - 2p^2 - m^2, \]
where \( p \) is the tachyon 3-momentum and \( E_T = \sqrt{p^2 + m^2} \); in fact, it must be:
\[ E_T = \sqrt{p^2 + m^2}. \]

In the particular case of infinite-speed tachyon emission, i.e. when \( E_T = 0 \) (in the rest-frame one has:
\[ \Delta M = -m^2. \]

Since emission of transcendent tachyons (antitachyons) is equivalent to absorption of transcendent antitachyons (tachyons), we shall get again eq. (48) also as a limit-case of tachyon absorption. Notice that a moving body \( \mathbf{B} \) can absorb tachyons only if it
\[ \Delta E = \sqrt{2p + m^2} - \sqrt{p^2 - m^2}, \]
where now \( \Delta = M_B' - M_B = 2p^2 - m^2 \). In the rest-frame of \( \mathbf{B} \), in particular, i.e. when we have that \( 2M_B |p| = m \sqrt{4M_B^2 + (m^2 + \Delta^2)/m^2} \); that is to say, a body \( \mathbf{B} \) at rest can absorb tachyons endowed with a velocity \( V \) having whatever direction, such that [e.s.1]:
\[ |V| = \sqrt{1 + 4M_B^2 m^2/(m^2 + \Delta^2)}/2. \]

Eq. (49) tells us e.g., that \( \mathbf{B} \) can absorb infinite-speed tachyons (no matter how directed) only if:
\[ \Delta = -m^2. \]
which agrees with eq. (48). Eq. (48') may immediately be derived also by observing that, if two bodies have infinite relative speed, then the product $p_\mu p_\nu$ of their 4-momenta is zero. Considerations of such a kind for elementary particles, although performed in the realm of quantum field theory, led CORBEN to explain many hadronic resonances as composed of bradyons and tachyons, thus putting forth a Lorentz-covariant "bootstrap" theory\cite{99}.

If we invade a field usually belonging to Q.M., i.e. the one of (strong) interactions among elementary particles, we find that the so-called "virtual" particles carry in general a negative four-momentum square:

$$t \equiv p^2 \geq E^2 - p^2 < 0$$

so as it happens for tachyons (cf. eq. (17)). This fact too suggests that "virtual particles" (i.e. the objects exchanged by sub-nuclear particles) can be classically regarded as tachyons\cite{97}. Actually, within some one-particle-exchange models (peripheral models "with absorption"), more than a decade ago it has been verified that the "virtual clouds" of hadrons should be associated to Superluminal speeds\cite{12}. Let us remember, then, that in the scattering processes with two initial and two final particles the quantity $t$ entering eq. (17') does change its sign and its meaning (from "momentum-transfer square" to "total-energy square") in the c.m.s., when we pass from the $g$-channel to the $t$-channel; this agrees with the fact that a SLT can transform a reaction (among bradyons) into the "crossed" reaction (Sect. 6.3) among tachyons. The previous considerations help us to understand better the derivation of "crossing relations" given in Sect. 6.3 (even if no GLT effectively transforms an interaction among bradyons into the "crossed" one still among bradyons).

If we want to adopt the common terminology, when everything is referred only to subluminal frames (and therefore eq. (24) is naively interpreted by associating imaginary rest-masses to tachyons), then "Resonances" - with their complex masses - may intuitively be considered as compounds of bradyons and tachyons\cite{97}.

The rôle of tachyons in hadron structure already appears to be confirmed (but we shall come back to this point) also by the relevance in particle physics of the "dual theories" (with their "string" models), of Higgs-type mechanisms, of "bilocal functions" for quarks, of "instantons", etc. With regard to the last ones it is instructive to study how a Superluminal observer sees a non-free bradyon, in particular a harmonically-oscillating bradyon (or, conversely, how a tachyon, which harmonically oscillates according to a Superluminal frame, will appear to us)\cite{97}. Moreover, we should not forget that the existence of space-like components always seemed to be a natural, and perhaps unavoidable\cite{100}, feature of interacting fields; for instance, it has been proved\cite{100} that, if a Fourier-transformed local field vanishes on a domain of space-like vectors in momentum-space, than the field is a generalized free field. Further: On the basis of eqs. (37), (38) one can show that the bradyon-tachyon duality is essentially equivalent to the electric-magnetic charges duality (cf. Sect. 4.8); but a serie of recent works are revealing connections between electromagnetism and dual theories (e.g., between "Dirac strings" - suitably modified and interpreted - and the "dual strings"), proposed in the scientific literature, of magnetic monopoles with quarks are also well-known; interesting results have been obtained by regarding quarks merely as quantized (and closed) fluxes of magnetic field\cite{102}. It is easy to realize, besides, that a tachyon subject to a central force can well move harmonically (by reversing its direction at the points where $\text{VI}=0$ or along closed paths. Such considerations are resumed also in the recent works regarding hadrons as possible "strong black-holes"\cite{103,104}. (Cf. also Part B, Sect. 9).

Let us spend some more words about the interesting, already quoted papers in refs.\cite{99}. Let us premise that, if a tachyon is bound by a repulsive, central force (as in the gravitational field inside our cosmos), and - by extension - in the strong field inside a hadron: see refs.\cite{103} and Sect. 9 in the following), then it reaches its minimal-potential energy-state when its speed diverges\cite{104} that is to say, the ground-state of the system corresponds to a transcendent, periodical motion of the tachyon. Moreover, if a bradyon having rest-mass $m_1$ absorbs a tachyon having rest-mass $m_2$ and $m_1 > m_2$, then the compound particle is always a bradyon\cite{97}. Now, within the realm of ER, CORBEN\cite{99} got quantum-numbers and rest-masses of a host of bradyon and meson resonances by considering them as composed of a hadronic bradyon and 1 to 3 hadronic tachyons. If hadrons, incidentally, can moreover be considered ad "strong black-holes" (cf. refs.\cite{103},\cite{104} and Sect. 9), then the
tachyonic constituent(s) would be emitted - when crossing the event-horizon - in the cor responding bradyonic form.

Furthermore, CORBEN(99) found the mass-differences among the members of various -multiplets by binding Superluminal leptons to the suitable (subluminal) hadrons. By gener al, to the quark level such an approach, the quark themselves could once more be regarded as or "loops" made of Superluminal leptons. (In such a philosophy, of course, quarks would be tures made of "partons" where partons would be nothing but tachyonic leptons).

Let us go back to eqs. (47)+(50). With regard to eq. (47), if \( \Delta (M^2) \) can assume only disc lues, then such equation yields a constraint for \( m \) (as a function of \( M_A \) and \( p^2 \) ), and vice-ve ke in the case, e.g., of the possible process \( \Delta_{33}(1232) \rightarrow p + t \) in the \( \Delta_{33}-\)resonance rest. By the way, if we compare that process with the electromagnetic decay of an excited atom, \( \rightarrow \Lambda + \gamma \), we meet again the hypothesis that the strong-field quanta can be (meson) tachyonic.

With regard to eq. (49), let us notice that - if \( \Delta=0 \) or if \( \Delta \) is "discrete" - the body B can (for every \( m \) ) only tachyons with a definite, discrete value of \( \vec{p} \), and vice-versa.

7.4. - Astrophysics and Superluminal Objects.

We already examined (Sect. 4.8) the Doppler effect for Superluminal cosmological objects eq. (36)),. We want here to add only what follows. Let us consider a macro-object C emitting rical electromagnetic waves. When we see it travelling with Superluminal, constant velocity cause of the "distortion" due to the large relative speed \( \sqrt{V^2 - c^2} \) we shall observe the electromagnetic waves to be internally tangent to an "enveloping" (double) cone \( \Gamma \) having as axis the n -line of body C (this cone has nothing to do with Cherenkov's: cf. point (iii) of Sect. 4.8). This analogous to what happens with an airplane moving at a constant, supersonic speed in the air.

A first observation is the following one. As we hear a sonic "boom" when we meet the st sound-contact with the supersonic airplane, so we shall analogously see an "optic boom" who first enter in radio-contact with body C, i.e., when we meet the \( \Gamma \)-cone surface. In fact, whis seen by us under the angle \( \alpha \) such that (see Fig. 15):

\[
v \cos \alpha = c
\]

all the radiations emitted by C in a certain interval around its position \( C_0 \) reach us simulta nely. If point \( C_0 \) is at cosmological distances, we can expect the "optic boom" conditions to hold (we hold) for a very long time.

Soon after the initial optic (or radio) contact with the emitting body C, we shall simultaneously receive the light emitted form suitable couples of points, one on the left and one on the right of respectively. We shall thus "see" the initial body, at \( C_0 \), to split in two luminous objects recce from each other with Superluminal (relative) speed \( U \). In the simple case when C moves with at most infinite speed along \( r \) (see Fig. 15b), the apparent relative speed of \( C_1 \) and \( C_2 \) varies in the initial stage as \( U \approx \sqrt{2d/c^2} \), quantity \( d \) being the distance \( OH = OC_0 \) and \( t=0 \) the time-instant when observer sees \( C_1 \) or \( C_2 = C_0 \).
Considerations of such a kind may be interesting in connection with the "experimental" fact that about 50% of certain strong radio-sources\(^{(106)}\) reveal a structure apparently interpretable in terms of Superluminal expansion. Typically, they just appear as constituted of two source collinearly receding from each other with (apparently) Superluminal relative speed; whilst "convergent" Superluminal motions have not been observed.

It is clear that phenomena of this type can catch the observer's attention only when the "angular separation" \(\theta \approx C_1 C_2\) between \(C_1\) and \(C_2\) is small, i.e., when \(C_1\) and \(C_2\) are still near the position \(C_0\). Fig. \(15\) makes it clear that — according to the interpretation here suggested — both bodies \(C_1\) and \(C_2\) should then show a Doppler-effect blue-shift, since they are the images of a unique, approaching body \(C\). However, if the Superluminal bodies \(C\) are situated only at cosmological distances (so as in the case of the abovementioned, presumed observations\(^{(106)}\)), then we have to take account of the cosmological red-shift, which can mask the initial; kinematical blue-shift.
PART B: OTHER EXTENSIONS OF RELATIVITY.

3. - THE "PROJECTIVE RELATIVITY".

3.1. - An Introduction to "Projective Relativity".

Special Relativity (in both its "ordinary" and "extended" forms) refers to a pseudo-Euclidean chronotopous which is supposed to be flat and infinite. One easily realizes that such a space background constitutes a very pushed extrapolation of the properties of our local space and time and does not adapt itself to be e.g. a good framework for the description of our cosmos. It is instance difficult to believe that physical laws are covariant also under time-translations of and millions years (in their ordinary form, at least).

An interesting step towards a space-time (s-t), that a priori is suitable to cosmological dies, is the following one\(^{107-109}\) (in this Sect. 8 we shall mainly refer to work by L. Fantappié, Arcidiacono, H.C. Corben, E. Pessa and others). Let us observe that the Galilei group \(G^1_{103}\) be obtained (through a "contraction") from the Poincaré one \(L^1_{103}\) as the "limit-case" when \(\lambda \to 0\). We can wonder whether the Poincaré group can be in its turn a "limit-case" of another, new remaining in a four-dimensional space (only considering 10-parameter groups), in 1954 Fantappié showed\(^{107-109}\) that a unique new group exists, depending with continuity on a parameter which reduces to Poincaré's for \(R \to \infty\) and which cannot be any more the "limit" of any of different group. Such a new group, \(F^1_{103}\), happens to be that of the motions into itself of deSitter space-time having constant curvature and with cosmological constant \(\Lambda = 3/R^2\). Not deSitter s-t is representable as a hypersurface with equation

\[
Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = 0
\]

embedded in a flat five-dimensional space (here and in what follows we shall admit that some binaries can be imaginary\(^{65}\)). From that point of view, then, the deSitter-Fantappié group becomes the group of rotations in a flat, five-dimensional space; and this clearly shows that \(F^1_{103}\) generalizes the Poincaré group (whose homogeneous - Lorentz - part, as well-known, isomorphism from the complex viewpoint to the group of rotations in a flat, four-dimensional space).

A useful, important physical interpretation of the deSitter-Fantappié group \(F^1_{103}\) has been forth in 1959 by Arcidiacono\(^{108,109}\), who distinguished the deSitter s-t from the "relativity of each observer, by taking into account the fact that every observer perceives the events as they happened in a flat s-t, - any geodesic appearing to it as a straight-line. In other words, "relative" space-time is a geodetic representation of deSitter s-t on a tangent hyperplane. The transformations of the deSitter-Fantappié group become projections, from the center of the conic \((52)\) and sections with the tangent hyperplane. Or, rather, the group \(F^1_{103}\) becomes\(^{108,1}\) the group of projectivities which transform into itself the quadric

\[
X_1^2 + X_2^2 + X_3^2 + X_4^2 + R^2 = 0
\]
that is to say
\[ R^2 = x^2 + y^2 + z^2 - c^2 t^2. \]  

(53')

When introducing homogeneous coordinates, by setting \( x_i = x_j / x_5 \), eq. (53) writes \( x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0 \). In conclusion, from the projective viewpoint, the ordinary "physical" space-time is the region external to the Cayley-Klein *absolute* with equation (53). But the projective space, defined as the region external to the quadric (53), is nothing but the Castelnuovo space-time\(^{110,109}\), and only in this space the mathematical expressions receive a physical interpretation. A "(Special) projective Relativity" follows, which\(^{108}\) reduces to the ordinary Relativity only when \( R \to \infty \). For its many, interesting features, we refer to quotation\(^{108}\).

Here let us add only the following\(^{108,109}\). In order to build up his General Relativity, EINSTEIN proceeded from a theory based on a group (*Minkowsky model*) to a theory constructed by gaining from the gravitational equations (and therefore from the \( ds^2 \)). Subsequently, to build up a "unitary theory", they tried to enlarge the Riemannian geometry. That is to say, before enlarging the General Relativity (GR) they did not try to "bring to perfection" the Special Relativity. According to ref.\(^{108}\), the ordinary unitary theories result to be unsatisfactory just because any theories build up by enlarging the Riemann geometry (or by passing to 5- or 6-dimensional manifolds) are still based on SR (set up in the Minkowski s-t) and on the Poincaré group. Such a group is not *simple*\(^{108}\) and therefore splits in the chronotopical (6 parameters) rotations and (4 parameters) translations. This leads to the partition of ordinary SR in two independent parts (Mechanics of continuous media , and Electromagnetism), where a sharp distinction exists between "matter" properties and "electricity" properties. In Projective Relativity\(^{108}\), on the contrary, rotations and translations merge together into the rotations of (a five-dimensional hypersphere) \( S^5 \) via the new fundamental length \( R \); as a consequence, a link is immediately found between "matter" and "electricity", while remaining - nevertheless - inside the realm of the "classical" theories founded upon groups ("Erlangen Program" for physical theories).

If, afterwards, one wants to erect a "general relativity" starting from Projective Relativity (which is based on the De Sitter-Fantappiè group), one expects that the new "general projective relativity" extends Einstein's gravitational theory on a cosmological scale, and therefore is particularly suited for astrophysical problems.

In order to blend the conceptions of those who want to rely only upon the \( ds^2 \) and of those who on the contrary appeal to group-considerations, we can take advantage of the unifying viewpoint by CARTAN\(^{111}\); who, by generalizing the idea of space, inserted the very Riemannian geometry in a group-context. In fact, following CARTAN\(^{111}\), a Riemannian variety \( V^q \) can be regarded as constituted of the infinite many - e.g. Euclidean - spaces tangent to it at each point of its, each one of those spaces having a geometry (in Klein's sense) grounded on the rototranslations group; such a geometry was called "holonomous" by CARTAN. Those infinite many, Euclidean space-elements are then linked together through a certain "connection" law (in this case called "Euclidean" by CARTAN), which allows deducing both curvature and torsion (local properties) of \( V^q \) by using infinitesimal closed cycles on the variety, and the "holonomy group" (global properties) of \( V^q \) by using finite closed cycles on \( V^q \). Vice-versa, once the holonomy group is known, the connection law can be univocally determined\(^{108,109}\). Of course, what precedes can be at once extended to the cases when the tangent spaces possess a non-Euclidean geometry, based on a group \( G^r \) with \( r \) parameters (still in the sense of the "Erlangen program"). Likewise, given any "holonomous" (= founded upon a group geometry, non-holonomous) geometries can be constructed corresponding to it. For instance: In Minkowski s-t the holonomy group is obviously the identity, and such a space-time is holonomous ; on the contrary, the Riemannian s-t of General Relativity is no more holonomous: hence ver it admits (it being devoid of torsion) as "holonomy group" the Lorentz one, i.e. the group of rotations in \( S^4 \) space.

Let us summarize\(^{108,109}\): (i) For going beyond SR, Einstein changed from a theory founded upon the rotations group \( R_4 \) (Lorentz group) to theories which utilize Riemannian manifolds \( V^q , V^3, V^6, ... \) and in such a way he abandoned the path of groups; (ii) In the "theory of universes" by Fantappiè-Arcidiacono\(^{108}\), on the contrary, models of cosmoses (or of "universes") are build up on the basis of the rotations groups \( R_4, R_5, R_6, ... \), thus establishing a fundamental link among physical laws\(^{107,112}\), group, and cosmos (or universe) geometrical model. In fact, the chosen
group acquires in that way the geometrical task of representing the motions into itself of the corresponding "universe-model", and - from the physical viewpoint - of expressing in mathematics a "principle of relativity". Physics, for instance, can be build up by using the "topological group", i.e. $n(n-1)/2$ dimensional manifolds which possess both geometrical and group structure (112) (this is comparable (108) with that Lagrange did in his analytic mechanics, when he deduced a mechanical system in terms of its "Lagrangian parameters"); (iii) In order to conciliate the viewpoints of Einstein and of Fantappiè-Arcidiacono, we can make use of the link established by Cartan between group theory and differential geometry. From this "third" point of view (108) we can set up a series of "special relativities", based on the rotations groups $R_n$, and then associate with each of them a "general relativity" by making recourse to a "non-holonomous" geometry (that admits $R_n$ as its holonomy group, and therefore is a Riemannian geometry).

Here, let us hint for example at the construction of the "general relativity" when starting from Projective Relativity. We have then to introduce (108) a non-holonomous $X^4$ space (in general a variable-curvature Riemannian manifold) that admits the De Sitter-Fantappiè group as its holonomy group. Since that group is isomorphic to the one of $S^2$ rotations (where $S^2$ indicates -1 repeat - the n-dimensional hyperspherical space), we have to resort to the geometry of a Riemannian variety $V^5$ (which just admits as holonomy group the one of the rotations of $S^2$). Such a geometry of $V^5$ will have then to be interpreted in terms of projective differential geometry of a n-dimensional manifold $X^4$. It is known that the projective differential geometry of a $X^n$ allows fact a (n+1)-dimensional interpretation in terms of the Riemannian geometry of $N^{n+1}$.

Following again Cartan, a space $X^4$ with projective connection is a space having the characters of a projective space in the (infinitesimal) neighbourhood of each point $P$ of its, and endowed with a projective (homographic) connection law between the neighbourhoods of two infinitely-close points of its. To such a purpose, it is necessary to provide a suitable field of quadrics $Q$, placed in the spaces tangent to the single points $P$ of $X^4$ (cf. ref. (108)). Once fixed the point $P$, the corresponding quadratic (QP) constitutes the "absolute" of the local, non-Euclidean metric. The "parallel transport" in a Riemannian $V^4$ preserves the isotropous cones; analogously, the projective connection must yield a projective transport law that preserves the aforesaid field of quadrics. After having thus determined the projective connection (108), we can build up in the usual way the "curvature projective tensor" $P_{ab\gamma\delta} (a,\beta,\gamma,\delta = 1, \ldots, 5)$, whose vanishing is the necessary and sufficient condition for the given space to be "projectively flat" (i.e. with constant curvature), fact the constant-curvature varieties are locally representable onto the Euclidean space with preservation of the geodesics.

The vanishing of the curvature Riemann tensor leads, in GR, to get again the Minkowski on the contrary, the vanishing of the curvature projective tensor $P_{ab\gamma\delta}$ leads back - in projective general relativity (108) - to the De Sitter s-t with constant curvature. Finally, the tensor $P_{ab\gamma\delta}$ has the important property of including the torsion tensor (so that Cartan called it the "curvature-and-torsion tensor"). Actually, at variance with what happens in the ordinary spaces endowed with affine connection, now the curvature of a projective-curvature-space implies a torsion; this is due to the fact that the De Sitter-Fantappiè group (holonomy group of $X^4$) decouples - at the "relativistic" limit - in the rotations and translations of $S^3$, to which the "rotation curvature" is the "translation curvature" (= torsion) correspond, respectively (108).

8.2. - An alternative Approach.

Wanted we strictly to follow an "Erlangen program" in Physics, the following, alternative proof would be available (108, 113). The investigation of De Sitter Universe - projective relativity and of the corresponding, generalized Maxwell equations (108, 109) confirms the usefulness (best of a group-theoretical foundation of physics) of resorting to the rotations groups $R_n$ of n-dimensional spaces. We saw that, in such a way, a succession of "universe models" is obtained, represented by the hyperspheres $S^{n-1}$ embedded in n-dimensional spaces $E^n$, (n=4, 5, ...); and the problem arose of developing a "Relativity" just based upon the group $R_n$ of the motions into itself of the persphere $S^{n-1}$. Incidentally, in the groups $R_n$ (n > 2), with their projective coordinates $x_i$ (i = 1, ..., n), n-3 universal constants (56) appear, necessary for adding - complying with the physical dimension-homogeneity requirement - square lengths to the squares of the "new" coordinate (following the first three ones) (114).
If we set - so as in projective relativity - n-4 normalization conditions, one takes back the "n-dimensional Relativity" to a 4-dimensional formulation (in terms of the space-time coordinates). At the limit for R→∞, besides, every hypersphere Sn-1 is reduced to a flat space E11, and its n projective coordinates become n-1 Cartesian coordinates; consequently, the group Rn (with n(n-1)/2 parameters) decomposes into the product of rotations Rn-1 and translations Tn-1 (having (n-1)(n-2)/2 and (n-1) parameters, respectively), while the normalization conditions become n-5 independent equations with n-1 unknowns. For instance, for n=5 we get the projective transformations (projective geometry); for n=6 the Cremona-type transformations: In such a way, one succeeds in applying the algebraic geometry to physics.

At this point, the new "alternative" approach comes in. Let us notice, in fact, that in the aforesaid group-theoretical conception of physics a particular role is played by the generalized "Maxwell equations" of the various hyperspherical universes Sn-1, which are covariant under the group Rn. If we call Hik = -Hki (i,k = 1,2,...,n) the generalized "electromagnetic field", possessing n(n-1)/2 distinct components, the generalized Maxwell eqs. then write:

\[ \text{Curl } H_{ik} = J_{ikl} ; \quad \text{Div } H_{ik} = I_{ik} \]

where J_{ikl} and I_{ik} are the field "sources", and Curl, Div are understood to operate in n dimensions. A relation has been discovered (cf. refs. (108, 109)) between the enlarging of the basic group of physics and the possibility of unifying the various, physical interaction fields, such a synthesis being performed by the very algebraic structure of the various rotations groups.

Particularly interesting appears to be the extension from the group R5 (projective relativity) to the group R6 (conformal relativity), the latter comprehending also the uniform accelerations. In refs. (108) it has been shown, in this connection, that the corresponding, generalized Maxwell eqs. yield a unified theory of matter (gravitation plus "hydrodynamics" of continua) and of electromagnetism. In particular, for R→∞, one is taken back to a flat space E5 and the abovementioned, generalized Maxwell eqs. split, on one hand, in CORBEN'S equations (113) (of the unified gravitational-electromagnetic field) and, on the other hand, in the mechanical equations of the generalized "hydrodynamical" field. By using such a "Conformal Relativity" (n=6), therefore, there is no need of passing - as done on the contrary in General Relativity - to "non-holonomous" manifolds, but one succeeds in describing even gravitation without departing from a strict group-theoretical formulation of "physics".

9. - ABOUT "CONFORMAL RELATIVITY".

9.1. - Introduction.

Historically, when they took due account of the electromagnetic phenomena, besides of the mechanical ones, it was necessary to leave - as well known - Galilean relativity in favour of Einstein's. We could now wonder whether, once arrived at investigating also the nuclear and sub-nuclear forces, a further extension towards a new Relativity should be necessary. Actually, at the beginning of Sect. 8.1, we considered - roughly speaking - the following "chain" of groups:

\[ G_{10}^{10} \left( c \to \infty ; R \to \infty \right) \Leftarrow L_{10}^{10} \left( c ; R \to \infty \right) \Leftarrow F_{10}^{10} \left( c ; R \right) \]

where the final, De Sitter-Fantappiè group "contains" two universal constants (a fundamental length, R, and the light-speed in vacuum, c). But, in order to plan in a dimensionally correct way even on a mechanical (dynamical) theory, three universal constants are needed (113). To lengthen the chain (54) one has however to leave the 10-parameters groups (i.e., the fourdimensional Minkowski space) (114, 108). It is then easy to reach the conformal group C_{10}^{10}, with 15 parameters (which can be shown to be locally isomorphic to the rotations of a 6-dimensional space). That group will allow setting up the new "Conformal (Special) Relativity" (108, 115), a generalization (108) of the Project-
In the following, we shall discuss the

Before going on, let us suggest for a renewed meditation a passage from the last scientific writing by Einstein, i.e. from Einstein's preface to the book "Cinquant'anni di Relatività":

"...Andererseits muss man zugeben, dass der Versuch, die unbezweifelbare atomistische und Quanten-Struktur der Realität auf dem Boden einer konsequenten Feld-Theorie zu begründen, auf grosse Schwierigkeiten stößt, von denen ich hier nur einige erwähnen will. Ich will dies kurz erläutern an der Theorie des asymmetrischen Feldes (wie sie formuliert ist). Aus dem Bau der Feldgleichungen geht nämlich Folgendes unmittelbar hervor: Ist $g_{ik}(x)$ eine Lösung der Feldgleichungen, so ist auch $g_{ik}(x/a)$ eine Lösung, wobei $a$ eine positive Konstante ist ("ähnliche Lösungen"). Es mag das System der $g_{ik}$ $z.B.$ einen in einem flachen Raum eingebetteten Kristall von endlicher Ausdehnung darstellen. Es gäbe dann eine zweite "Welt" mit einem andern Kristall, der genau gleich konstituiert ist, dessen Linear-Dimensionen aber $a$ mal größer sind als die des ursprünglichen Kristalls.

Solange wir uns eine Welt denken, die nichts anderes enthält als eben diesen einen Kristall, so liegt hierin noch keine Schwierigkeit. Man sieht nur, dass die Ausdehnung eines solchen Kristalles ("Massstabes") durch die Feldgleichungen nicht bestimmt ist. Man denke sich aber nun, dass die von uns betrachtete "Welt" aus zwei solchen Kristallen besteht, die gemeinsam in einem flachen Raum eingebettet sind und die voneinander liegend weit entfernt seien. Für die Lösungen der Feldgleichungen gilt wegen deren Nicht-Linearität zwar nicht das "Superpositionsprinzip". Aber man ist doch wohl geneigt zu denken, dass es eine Lösung für das Gesamtfeld gebe, derart, dass das Feld innerhalb jedes der beiden Kristalle sich nur wenig unterscheidet von der Lösung für den Fall, dass dieser Kristall allein in der Welt vorhanden ist. Dann aber wäre dies eine Welt, in der es zwei körperliche Objekte gäbe, die zu einander "ähnlich" aber doch nicht kongruent wären..."

"...Damit also die Theorie annehmbar wäre, wäre es nötig, dass selbst weit voneinander entfernte "ähnliche" Objekte auf Grund der Feldgleichungen so stark aufeinander einwirken, dass eine irgendwie dauernde Koexistenz "ähnlicher" (nicht kongruenter) Objekte nicht möglich ist. Wir sind weit davon entfernt zu sehen, wie aus den Feldgleichungen eine derartige Folgerung gezogen werden könnte..."

This passage was written by Einstein at Princeton on the 4th of April, 1955.


We can also start from a different point of view (although within a more limited framework in one sense) for generalizing the Special Relativity according to the spirit of the beginning of Sect. 9.1. In the following, we shall essentially refer to work done by the author in collaboratio with P. Caldirola, M. Pavsic and P. Castorina (55, 103). Let us, in fact, observe that the symmetries of Maxwell equations have not been fully exploited by SR. Namely, Maxwell eqs. are known to be covariant - besides under Poincaré transformations - even under conformal transformations (116). As a first step, let us fix our attention in particular on the (space-time) dilatations ($\mu = 0, 1, 2, 3$):

$$x'_{\mu} = \theta x_{\mu},$$

(n)
and postulate that physical laws are covariant also under dilatations (55). We are however supposing that in nature only discrete values of \( q \) happen to have physical counterpart (53) (which can be e.g. obtained by imposing suitable boundary conditions in five-dimensional spaces (117)).

At this point, let us remember that the strengths of the gravitational and strong interactions are measured by the dimensionless squares of the corresponding coupling-constants, respectively:

\[
\frac{G m^2}{\hbar c} \simeq 1.3 \times 10^{-40}, \tag{56a}
\]

and

\[
\frac{N g^2}{\hbar c} \simeq 15, \tag{56b}
\]

where: (i) \( G \) and \( N \) are the gravitational and strong universal-constants in vacuum, respectively; (ii) quantities \( m \) and \( g \) represent the gravitational-charge (= mass) and the strong-charge (see the following) of one and the same hadron, e.g. of a nucleon. The value in eq. (56a) is calculated for the pion mass, \( m = m_\pi \); in eq. (56b) we typically used the value of the \( pp \pi \) coupling-constant square. Incidentally, with regard to the above expression "strong-charge of a hadron", let us regard the quarks as the actual sources of strong field, i.e., the real carries of strong-charge, and let us call "colour" the sign \( s \) of quark strong-charges (55); more precisely, the hadrons can be considred as endowed with a zero total-strong-charge (55), each quark possessing a strong charge \( g_i = \pm s_i |g_i| \) where \( \sum g_i = 0 \). The ordinary strong-interactions among hadrons should thus originate from Van-der-Waals-type forces (55). In correspondence to quantity \( m \) of eq. (56a), in eq. (56b) the quantity \( g = g_0 \) will enter, quantity \( g_0 \) being the average magnitude of the constituent-quark charges and \( n \) being the quark number.

Let us now put:

\[
q \equiv \frac{G m^2}{N g^2} \simeq 0.9 \times 10^{-41} \simeq 10^{-41}, \tag{57}
\]

and notice that, if we conventionally choose \( m \equiv g \), then the "strong universal-constant" \( N \) becomes:

\[
N = q^{-1} G \simeq 1.1 \times 10^{41} G \simeq 4 \pi \frac{\hbar c}{m^2} \simeq 7 \times 10^{30} \text{ m Kg}^{-1} \text{ s}^{-2}; \tag{58}
\]

conversely, if we choose units such that \( [N] = [G] \) and moreover \( N = G = 1 \), then we get (e.g., with \( n = 2 \) or \( n = 3 \)):

\[
g_0 = \frac{g}{n} = \frac{m}{n \sqrt{G}} \simeq \frac{5}{n} \times 10^{-33} \text{ cm} \simeq \frac{8}{n} \times 10^{-5} \text{ gr} \simeq \frac{\hbar c}{G} \simeq \text{Planck mass}, \tag{59}
\]

where eq. (59) tells us, by the way, that "Planck-mass" \( \sqrt{\frac{\hbar c}{G}} \text{ m} \text{ y}^{-1} \) is nothing but the quark "strong charge" (in suitable units). We do not expect, therefore, existence of further, new, small black-holes - as predicted by other Authors - with a mass of the order of Planck-mass, since we have already met hadrons (or, rather, quarks) with strong-charges of the order of Planck-mass (in suitable units).

The most important observation is, however, the following one. Let us regard both hadrons ("typically" the pions, or the nucleons), and our cosmos as finite objects. Then, relation (57) and the fact that, when calling \( R(U) \equiv R \) our cosmos radius (55) and \( r(h) \equiv r \) the hadron (pion) radius in strong interactions, one gets (103)

\[
\frac{r(h)}{R(U)} = \frac{10^{-15} \text{ m}}{10^{26} \text{ m}} \simeq 10^{-41} = q , \tag{60}
\]

suggest that our cosmos and hadrons can be considered as similar systems, i.e. systems (inter-
nally) governed by similar laws that differ only in a scale-factor \( q \) (which carries \( R \) into \( r \) a gravitational field into the strong one). Roughly speaking, we can imagine that - by shrinking a cosmos by the factor \( q = 10^{-41} \) - we can get the hadrons (see the following, and refs. (55)), to say that, by dilatating a hadron by the factor \( q^{-1} = 10^{41} \), we can get a cosmos. Indeed, after having called "universe" any almost-isolated system, essentially governed by one of fundamental forces, we have analogously introduced a "hierarchy of universes" (55), which can got through a series of suitable, discrete dilatations (or contractions).

Drawing our inspiration, as said above, from the hypothesis that physical laws are covariant under (discrete) dilations, we are led to assume, briefly, that (55):

A) inside our cosmos (gravitational case) the Einstein equations, with attractive cosmological constant \( \Lambda \), hold \([\mathcal{G} = 1]\):

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{8\pi}{c^4} T_{\mu\nu} = 2\Lambda \cdot (m_{G/c^4})^2 ;
\]

B) inside hadrons ("strong" case) the "scaled" Einstein equations hold \([N = G = 1]\):

\[
\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = - \frac{8\pi}{c^4} \tilde{T}_{\mu\nu} = 2\tilde{H} \cdot (m_{S/c^4})^2 ;
\]
Dimensional evaluations immediately tell us that (within our "conformal relativity")\(^{(55)}\):

\[ H = \theta^{-2} \Delta ; \quad m_G = \theta m_S , \quad (63) \]

where \( m_G \) and \( m_S \) are the average (small, but finite) mass of the "external" gravitons and the average mass of the "external" strong-quanta, respectively\(^{(55)}\). Moreover, the strong-charge tensor \( S_{\mu \nu} \) is essentially \( S_{\mu \nu} \neq \theta^{-1} T_{\mu \nu} \), where \( T_{\mu \nu} \) is a priori the ordinary matter-tensor (containing e.g. the Dirac spinorial functions, etc.). For example, if we require gravitational interactions to have a range\(^{(55)}\) of the order of \( R(U) \approx 10^{26} \text{ m} \), then we obtain at once\(^{(55)}\):

\[ m_G \approx 10^{-68} \text{ Kg} ; \quad \Delta \approx 10^{-56} \text{ cm}^{-2} ; \quad m_S \approx m_\pi . \quad (63') \]

as well as:

\[ H^{-1} \approx \theta^2 \Delta^{-1} \approx 10^{-25} \approx 0.1 \text{ barn} . \quad (64) \]

The present, elementary theory (by Caldirola, Pavšič, Recami and others\(^{(103)}\)) allows deriving (proving) in a systematic way all the empiric relations (which connect micro- with macro-cosmoses) heuristically discovered by Weyl, Eddington, Dirac, etc.; although our own "nume- rology"\(^{(103)}\) connects the gravitational interactions with the strong ones (that are - like the former - always attractive, non-linear, and eventually associable with non-abelian gauge theories; this Section, indeed, proposes an ante litteram geometrical interpretation of those theories), and not with the electromagnetic ones (as suggested, on the contrary, by Dirac). For instance, it is straightforward to prove from our "dilatation-covariant Relativity"\(^{(55, 103)}\) that the mass \( M \) of our cosmos and the mass \( m \) of the pion are linked as follows:

\[ M = \theta^2 m \approx 10^{54} \text{ Kg} ; \quad m = \theta^2 M \approx 10^{-28} \text{ Kg} . \quad (65) \]

Consistently with eqs.\(^{(61)}\), for the spatial part of our cosmos we can choose the simple model of the 3-dimensional hypersurface of a hypersphere. Analogously we can proceed for hadrons ("strong" universes), so as to be able to extend e.g. the Mach principle: in the sense that the inertia of every hadron-constituent (parton) will coincide with its strong-charge (and not with its gravitational-charge!). In such a way, we shall be able to consider the "equivalence principle" as locally valid even inside hadrons, so as to justify the present geometrization of the strong-field (inside hadrons) also from the point of view of the ordinary General Relativity.

Let us now find out the exact solution of eqs.\(^{(62)}\) for a spherically-symmetric distribution of strong-charge. The geodesic equation for a small test-"parton" with strong-charge \( g' \) in the vacuum gives in the radial case\(^{(55)}\) \((l, i) = 1, 2, 3; N = 1\):

\[ \frac{d^2 r}{dt^2} = -\frac{c^2}{2} \left( 1 - \frac{2g}{c^2}, \frac{Hr^2}{3} \right) - \left( \frac{2g}{c^2}, \frac{2H}{3} \right) . \quad (66) \]

Since quarks are not small hadron-costituents, eq.\(^{(66)}\) will hold only approximately for quarks. Nevertheless, it yields the quark (or rather parton) so-called "asymptotic freedom" for small distance \( r \), as well as the quark (parton) "confinement" - the so-called "infrared divergency" - for large values of \( r \).

Let us first examine the case of small values of \( r \), when the attractive term \( -1/r^2 \) dominates (so as in the gravitational case). Notice that the repulsive term \( \alpha + 1/r^3 \) effectively works only at extremely small values of \( r \), so that the radial acceleration vanishes only for \( r \approx 10^{-33} \text{ cm} \) (and, in the gravitational case, we'd get the same result only for \( r \approx 10^4 \text{ cm} \)). However, if we attribute a kinetic energy (and an angular momentum \( J \)) to the considered parton (quark), i.e., if we add the "kinetic-energy term" to the radial potential corresponding to eq.\(^{(66)}\), then - with the choice\(^{(58)}\) for the measure-units - we can write for small \( r \) (\( r \ll r(h) \)):

\[ V = -\frac{(J/g')^2}{r^2} - \left( \frac{N_G g^2}{r^2} - \frac{c^2 H}{3} + \cdots \right) \approx -\frac{N_G g^2}{r^2} + \frac{(J/g')^2}{r^2} . \quad (66') \]
In the quark case \( g' \cong (g - g')/n' \), with \( n' = 1, 2 \), one gets \( V \approx 0 \) for \( r \approx 10^{-3} / (N g^3) \); if we bear in mind quantum mechanics the suggestion that \( J \approx n g \), then we obtain \( V \approx 0 \) for \( r \approx n^{-2} \times 10^{-1} \). Incidentally, that suggestion corresponds to attributing a speed \( v = c \) to the considered, moving quark. Conversely, if we assume the "stability-radius", i.e., in the baryons case, \( N \cong 10^4 G \) - to be of the order of \( r_0 = 0.00 \text{ Nm} / c^2 \cong 10^{-15} \text{cm} \) (i.e., of the order of one hundred "strong Schwarzschild radius") of our hadron, considered as a strong black-hole \( (55) \), then we get the Regge-like relation \( J \cong (N/1000) \text{m}^3 \), where \( m \) is the hadron mass in Kgs; this relation also be written \( J/m \approx m^2 \) with \( m \) now measured in \( \text{GeV} / c^2 \).

For large distances, when \( r \gg r(h) \), one gets\( (55) \) the radial, confining force \( (N = 1; \ [N] \)

\[
F \propto - \frac{g}{3} \left( c^2 Hr - 2H \right) \cong - g c^2 Hr / 3 \propto r.
\]

In other words, from our "classical" theory we get in a very natural way also a confining \( (11) \) potential \( V \propto r^2 \) for large \( r \). At last, we would get, for very large values of \( r \) (obtainable e.g., when the considered hadron starts to get deformed during a high-energy collision), an even stiffer confining-force, proportional - for \( r \gg r(h) \) - to \( r^3 \):

\[
F \propto - g c^2 \left( \frac{1}{3} r^3 + \frac{1}{2} r^2 + \cdots \right). \quad (r > r(h))
\]

We may regard the spatial parts of our cosmos and of hadrons (time aside) as embedded in a four-dimensional, flat space. The problem of strong interactions between two hadrons requires considering the intersection of hadrons with our cosmos: such intersections being 2-dimensional spherical surfaces, that we just call "hadrons" tout court. Since (in our cosmos) two "hadrons" interact strongly - e.g., via Van-der-Waals-like forces \( (55, 11) \) - we need therefore to describe (strong) interactions between the aforesaid "intersections". To this end, when considering the interaction of a hadronic test-particle - possessing both strong and gravitational charges - a "hi-so" theory is required for our cosmos in the surroundings of hadrons and in presence of sub-nucleonic interactions: In other words, we need to modify the gravitational Einstein equations by introcutting the micro-neighbourhood of the above-mentioned intersections (hadrons), a strong metric-determination affecting (only) the objects with strong-charge (i.e., with scale-factor \( k = 0 \cong 10^{-41} \)), not affecting the ones with gravitational-charge only (i.e., with scale-factor \( k = 1 \)). Around a hadron we can assume the gravitational metric-tensor to be \( g_{\mu \nu} \cong \eta_{\mu \nu} \) (in suitable coordinates) set

\[
\tilde{g}_{\mu \nu} = f_{\mu \nu} + \tilde{h}_{\mu \nu} = \eta_{\mu \nu} + \tilde{h}_{\mu \nu}
\]

where the components of the strong metric-tensor \( \tilde{h}_{\mu \nu} \) have to vanish for \( r \gg 1 \) Fermi. In ref.\( (103) \) we proposed the following field-equations (for test-objects having both gravitational and strong-charge, in the surroundings of a hadron, in our cosmos):

\[
R_{\mu \nu} + \tilde{H}_{\mu \nu} = - \frac{8 \pi \hbar}{c^4} \left( S_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} S^0 \right)
\]

with \( N = G^{-1} ; S_{\mu \nu} = NT_{\mu \nu} \); and where the "cosmological (strong) term" with the hadronic charge \( H \) takes care of the geometric properties of the strong field around the "source hadron". We refer for details to refs.\( (55) \). Here, let us briefly put forth what follows: (i) at the static limit, we get\( (55) \) the Yukawian behaviour \( \eta_{\mu \nu} \cong - (2g / (c^2 r^2)) \exp(-r \text{mg} c / \hbar) \); (ii) if, in our space, we consider spherically-symmetric "strong-charge" distributions with the aforesaid "intersection", according to Sect. \( 7.3 \) - we can regard hadrons as "strong black-holes" \( (103) \), and the "strong Schwarzschild radius" \( r_0^H \) can be calculated\( (103) \). The results appear to yield the "effective radii" of strong interactions; for instance, \( r_0^H = 0.8 \text{ Fermi} \) for nuclei. In such a context, the "Steen event-horizon" \( (103) \) plays for hadrons - at a classical level - the same rôle of the MIT "bag". Let us remember that "black-holes" can carry (besides mass, charge, and spin) other quantum numbers; particularly the "strong" black-holes.
At this point, let us add that the "classical confinement" here obtained for hadron constituents can be violated by quantum effects so as Hawking's results(120) to be of the order of $T \approx 2 \times 10^{-11}$ eV, and corresponds a priori to an "evaporation time" of the order of $\Delta t \approx 10^{-\infty}$ s, unless we do impose some stability-conditions of the kind of Bohr's(55). In any quantum theory, however, quarks may be again "totally confined" if you want - by associating with their classical (strong, Schwarzschild) "horizon" a suitable barrier of *super-selection rules* and of "conservation super-laws".

Let us conclude with three last observations. First: if our cosmos is similar to a hadron, it might e.g. be regarded - following the calculations of the present Sect. 9.2 - as a Super-pion, and therefore as constituted by one matter half-cosmos (or "Meta-galaxy") and by one antimatter half-cosmos (so as each pion consists of a quark and an antiquark). Second: if neutrons can be considered as "strong black-holes", we can imagine the "Second law of black-hole Thermodynamics"(12) to hold even for them when they melt together during the final period of cosmos-contraction; thence we must have a process that builds up a new cosmos with radius $R > 10^{25}$ m, and this consideration may be a hint to investigating the big-bang "explosions". Third: if hadrons are similar to our cosmos, they too could perform successive cycles of expansion and contraction, with a period - however - of about $\Delta t \approx 10^{10}/10^{41}$ s $\approx 10^{-23}$ s. We should thus get that elementary particles can be regarded as point-like only at certain successive, discrete positions along their trajectory (associable with a fundamental chronon), and we'd meet again considerations analogous to the ones developed by P. CALDIROLA(117) elsewhere.

At last, let us close this contribution by calling attention to the verse by Goethe quoted in our last reference(122).

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REFERENCES (AND NOTES).


(4) - 0 priori, $\sqrt{1 - \beta^2} = 1$, since $(1 - \beta^2) = -1$. See the following. It will always be understood that $\sqrt{1 - \beta^2}$ represents the upper half-plane solution, for $\beta^2 > 1$.

(5) - G. Galilei, "Dialogo... sopra i due massimi sistemi del mondo: Tolemaico e Copernicano (G. B. Landini, Florence, 1632). See e.g. G. Galilei, "Dialogue Concerning the Two Chief World Systems," translated by S. Drake (Univ. of California Press, Berkeley, Calif. 1953). We would like to take advantage of the present occasion to notice - however - a mistake in the above translation. At page 186 you can read: "... Shut yourself up with some friends in the main cabin below decks on some large ship,... hang up a bottle that empties drop by drop into a wide vessel beneath it... Have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that... The droplets will fall as before into the vessel beneath without dropping towards the stern, although while the dr are in the air the ship runs many spans...". This picture of the Galilean Relativity is obscured by the adjective "wide" (see the word in italics, above). Actually, it should be substituted by narrow, since Galileo originally wrote "angusta" (narrow), and not "augusta" (= wide). The correct translation, which reads "... into another narrow-necked bottle placed underneath..." is (thus restoring Galileo's example to its full, exact physical meaning) can be found in G. Galilei, "Dialogue on the Great World Systems," in the Salusbury translation, by G. De Santillana (Univ. of Chicago Press, Chicago, Ill., 1953), p. 199.


(8) - Even if we assumed as "privileged" the inertial frames in which the cosmic radiations of cosmological origin (so as the 2.70K "background radiation") happen to be isotropically distributed, nevertheless the theory of SR - apparently - would not be essentially affected. Cf. e.g. C. Gordon, Found. of Phys. 5, 173 (1973); S. C. Barrowes, Found. of Phys. 7, 617 (1974).

(9) - W. V. Ignatowski, Phys. Zeits. 21, 972 (1910); P. Frank and H. Rothe, Ann. der Phys. 34, 825 (1911); E. Hahn, Arch. Math. Phys. 21, 1 (1913); Among the most recent works, see e.g. F. Severi, in "Cinquant'anni di Relativitá", edited by M. Pantaleo (Editorice Universitaria, Firenze, 1953); A. Agodi, (unpublished); M. Di Jorio, Novo Cimento B22, 70 (1974).

(10) - L. Fantappié was the first to notice that (ordinary) Special Relativity, when deprived from the "Third Postulate", would allow sending signals into the past. See also ref. (14), and K. Godel, in "A. Einstein, Philosopher-Scientist", edited by P. A. Schilpp (Open Court, La Salle Ill., 1973), p. 558.


(12) - See E. Recami and R. Mignani, ref. (6) and references therein; and the book "Tachyons, Monopoles, and Related Topics", edited by E. Recami (North Holland, Amsterdam, 1978).

(13) - Here, and in the following, "charge" means any additive charge, like electric, magnetic, barionic, leptonic, ... charges. Therefore, our definitions of "charge conjugation" etc, will be different from the usual ones, but in agreement with ref. (12). Cf. refs. (12, 14, 15).


(16) - Actually, the mere "RIP" yields the antiparticle except for the helicity sign. But the full result is immediately got when considering the action of the complete Lorentz transformation (together with "RIP").

(19) - For definitions of "causes", "effects" and "causal connection" see Sect. 4, 9.
(21) - In fact, antiphotons essentially coincide with photons; see refs. (12).
(22) - E. Recami and G. Zilio, ref. (15), and references therein. See also M. Pavšič, Obz. Mat. Fiz. 19, 20 (1972, Ljubljana).
(23) - In quantum mechanics, our operator C will be unitary when acting on the space-state: see refs. (22).
(31) - In this paper, the reinterpretation principle (17, 18) has been applied to tachyons for the first time. See also G. Gregory, Phys. Rev. 125, 2136 (1962).
(32) - H. C. Corben, Nuovo Cimento A29, 415 (1975); In this paper a passage from the second "book" of Lucretius' De Rerum Natura has been quoted, mentioning faster-than-light speeds.
(34) - J. J. Thomson, Phil. Mag. 28, 13 (1889).
(36) - R. C. Tolman, "The theory of relativity of Motion" (Berkeley, Cal., 1917), p. 54. This apparent "paradox" has been recently proposed again by many physicists, unaware of the already existing literature.
(37) - Apart from the purely mathematical considerations by E. Majorana, Nuovo Cimento 2, 335 (1932) and by E. Wigner, Ann. of Math. 40, 149 (1939).
(40) - Cf. "Anything not forbidden is compulsory" in T. H. White, "The once and future king" (Berkeley, Cal., 1939). The same concept was previously expressed in a clear way by Democritus of Abdara, according to whom everything that was thinkable without meeting contradictions existed somewhere in the unlimited universe (see e.g., A. Virieux-Reymond, in "Scienziati e Tecnologi" (Mondadori, Milano, 1975)). We thank E. Henig and G. Schifferer for calling our attention to these references.
See e. g. A. Danielian, Phys. Letters A51, 61 (1975),


- With respect to us,


- See e. g. W. Rindler, Special Relativity (Oxford University Press, 1996), p. 16.

- See also L. Parker, Phys. Rev. A40, 1230 (1989); see also E. Recami and R. Mignani, ref. (42).

- See also L. Parker, ref. (40), and D. Leiter, Lett. Nuovo Cimento 1, 395 (1971).

- See refs. (46). For instance, if the chosen mapping is the symmetry with respect to the light cone, the whole 3-space E=0 goes into the E-axis; but we can restore a one-to-one correspondence e. g. by associating a direction with every object at rest (namely, the limit-direction of its motion when arriving at rest), or by adding a unique "improper" point - at infinity - to the 3-velocity space. In any case, see also refs. (53).


(66) See e.g. P. Caldiroli, Istituzioni di Fisica Teorica (Milano, 1966).


(73) See ref. (6); and e.g. M. C. Camenzind, General Relat. Gravit. 1, 41 (1970).


(77) E. Recami, ref. (14); P. L. Csonka, ref. (74); see also refs. (72).

(78) J. A. Parmentola and D. H. Yee, ref. (72).

(79) E. Recami, Scientia 109, 721 (1974); see also E. Recami, ref. (74).

(80) E. Recami, ref. (72).

(81) P. L. Csonka, ref. (74); R. Newton, Phys. Rev. 162, 1274 (1967); E. Recami, ref. (14); E. Recami and R. Migiani, ref. (12).

(82) Since we have to deal with divergent speeds, we may usefully borrow a bit from projective geometry, where e.g. straight lines are considered as large (infinite)-radius circles.

(83) Before applying the "RIP" we shall see, in Sect. 6, that - after application of "RIP" - our PT operation is essentially the usual CPT. In our formalism, the CPT operation is a linear (classical) operator in the pseudo-Euclidean space (since it is an element of G), and a unitary (quantum-mechanical) operator when acting on the space of Q. M. states.


(87) If we want that O2 is subluminal with respect to O1, then particle b (at least) must be a tachyon: cf. Fig. 8.


(89) E. Torsello di Francia, private communication.

(90) G. Szekeres, Publ. Math. (Debrécz) 7, 285 (1960); M. D. Kruskal, Phys. Rev. 119, 1743 (1960). The same considerations could be developed e.g. with regard to Finkelstein's coordinates; etc.
(91) For simplicity, here we write down only the radial coordinate and the time coordinate.

(92) E. Recami and R. Mignani, ref. (6), page 221.


(95) Let us remember that transcendent tachyons do not have a definite direction along their motion-line (in the sense that emission of an infinite-speed tachyon is completely equivalent to absorption of an infinite-speed antitachyon, and vice-versa).

(96) Of course, the same description would be put forth by s', s", ..., even when s was chosen to describe its own observations in terms of a "vacuum decay" of the above-mentioned kind.


(99) H. C. Corben, Lett. Nuovo Cimento 20, 645 (1977); 22, 116 (1978); three preprints (Scarborough College, West Hill, Ontario, Aug., Sept. and Nov. 1977); and in 'Tachyons, monopoles, and Related Topics', edited by E. Recami (North Holland, Amsterdam, 1978), p. 31. Let us observe that, if two bodies have divergent relative speed, their four-momenta are orthogonal. See also the last ref. (69) and ref. (101).


(101) See e.g. A. O. Barut, ref. (100).


(109) E. Pessa, Collectanea Mathematica 24, 151 (Barcelona, 1973).


(114) See E. Recami and C. Spitaleri, ref. (65); see also, e.g., T. G. Pavlopoulos, Nuovo Cimento B60, 93 (1969).
(115) - See refs. (65); e.g. R.L. Ingraham, Nuovo Cimento 12, 825 (1954); 1, 82 (1955); B146, 217 (1978).


(120) - See e.g. S.W. Hawking, Comm. Math. Phys. 25, 152 (1972); 43, 199 (1975); Nature 243, 30 (1974).


(122) - 'In jeden Quark begrüßt er seine Nase!', J.W.v. Goethe, "Faust", 292. For this quotation thanks are due to G. Schifferer.