C. Bernardini: ON THE PARAMETRIZATION OF THE ANGULAR DISTRIBUTIONS OF HADRONS PRODUCED VIA THE ONE-PHOTON ANNIHILATION CHANNEL IN $e^+e^-$ REACTIONS.

1. - Most of the peculiarities of the angular distribution of hadrons from $e^+e^-$ annihilations in the center of mass are determined by the simple structure of the one photon channel. In an attempt to find model-independent parametrizations of the angular distribution of few observed charged tracks in events containing pseudoscalars with any multiplicity we got a simple answer for the single-track case, namely $a + b \cos^2 \theta_p$ where $a$ and $b$ are energy-dependent parameters and $\theta_p$ is the track angle with the beam line. Also, the two-track distribution has been studied and some simple property shown; in particular, the possibility to describe final state dynamics in terms of only one angle variable. We will sketch in the following the physics and the mathematical techniques for the one- and two-track cases. We did not study cases with more than two tracks.

2. - Cabibbo and Gatto\(^{(1)}\) have shown that, to lowest e.m. order the space part $\mathcal{J}(n)$ of the matrix element

$$<0|J_\mu|n> \equiv J_\mu(n)$$

distributed according to a $\sin^2 \theta$ law, $\theta$ being the angle of $\mathcal{J}(n)$ with the beam line (C.M. description is understood), because of it's coupling with the lepton current.

We will limit the analysis to the case in which $n$ contains only spin-zero particles.

Also it would not be difficult to account for a possible polarization.
of the beam electrons, but we will study the no-polarization case.

A reference system using the beam line as polar axis will be called "scanner system" (S. S.); the "current system" (C. S.) will have $J(n)$ as polar axis.

We do not know anything in general on how $J(n)$ is formed out of the particle 3-momenta in $n$. But we can say that, given a state $n$ (described by the nature and 3-momenta of each particle in $n$) the cross section for $e^+e^- \rightarrow n$ will be invariant under the following operations:

i) Rigid rotations around $J(n)$
ii) Inversion $J(n) \rightarrow -J(n)$
iii) Rotations with $J(n)$ at $\theta = \text{constant}$, around the beam line.

The more, keeping the particle configuration fixed in C. S., any change of the $J(n)$ direction in the S. S. will be accounted for by the $\sin^2 \theta$ law.

The single-track case corresponds to integration over all but one particle momenta. The two-track case concerns integrations over all but two particle momenta. Since usually the triggers require at least two tracks, the single-track analysis might be somewhat academic; nevertheless it illustrates well the power of the symmetries involved.

3. - We first consider the single-track case. Assume that a charged particle with momentum $\vec{p}$ is observed and no other particle is detected. Let us call $\theta_p$, $\varphi_p$ the polar angles of the track in the S. S.; also, $p = |\vec{p}|$. Assume for a moment that the direction of $\vec{J}$ for that event is known. Then, call $\theta$, $\varphi$ the polar angles of $J$ in the S. S. and $\Delta$, $\chi$ the polar angles of the track in the C. S. (see Fig. 1). The symmetries will be exploited by saying

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FIG. 1
that, should we know \( J \), the probability of the event would be of the form

\[
\frac{3}{8 \pi} \ W (\Delta, p) \sin \Delta \ d\Delta \ d\chi \ \sin^3 \theta \ d\theta \ d\psi
\]

where \( W \) is a function of \( \Delta \) normalized over the sphere. Because of ii, § 2:

\[
W (\Delta) = W (\pi - \Delta)
\]

Therefore \( W(\Delta, p) \) can be expanded in the form

\[
W (\Delta, p) = w_0 (p) + w_2 (p) P_2 (\cos \Delta) + \ldots.
\]

where \( P_n \) is a Legendre polynomial (even indexes only). Because of the normalization, \( w_0 (p) \) is known and equals \( 1/4\pi \).

Actually, we do measure \( \theta_p, \varphi_p \) so that integration over the remaining angles must be performed. By using

\[
\cos \Delta = \cos \theta \cos \theta_p + \sin \theta \sin \theta_p \cos (\varphi - \varphi_p)
\]

and the composition formula for Legendre polynomials, the following distribution is found after integration over \( \theta, \varphi \) for the track-angles in the S.S.:

\[
\frac{d\Omega_p}{4\pi} \left\{ 1 - \frac{4\pi}{5} w_2 (p) P_2 (\cos \theta) \right\}
\]

Therefore, the following theorem holds:

**Single-track theorem.** - Any state produced via the one-photon channel in \( e^+e^- \rightarrow (\text{spin zero hadrons}) \) will give a single track distribution in scatter-space of the form (1).

The known cases of \( e^+e^- \rightarrow \pi^+\pi^- (K^+K^-) \) and \( e^+e^- \rightarrow \pi^+\pi^- \pi^0 \) correspond to \( W \sim \delta (1 - \cos^2 \Delta) \) and \( W \sim \delta (\cos \Delta) \) respectively. For them, \( w_2 = 5/4\pi \) and \( w_2 = -5/8\pi \). In general

\[-2 \leq \frac{4\pi}{5} w_2 (p) \leq 1\]

because of positivity requirement on (1).

When no information on the multiplicity is known, \( w_2 (p) \) must be understood as the sum over all possible hadronic states (we can safely assume that such states contain only \( \pi \) and \( K \)).

The cross section for the single track will be written in general
Integration over $p$ gives

$$4 \pi \int dp \ A(p) = \sigma_{\text{Total}}$$

the total cross section. Define

$$\bar{w}_2 = 4 \pi \int dp \ w_2(p) A(p)/\sigma_{\text{Total}}$$

then, when the momentum $p$ is not measured

$$\frac{d\sigma}{d\Omega_p} = \frac{\sigma_{\text{Total}}}{4 \pi} \left\{ 1 - \frac{4\pi}{5} \bar{w}_2 \ P_2(\cos\theta_p) \right\}$$

Note that, performing these integrations over the kinematical limits for $p$, the lower limit could be less than the threshold for an actual apparatus and, the worst, the threshold could be angle-dependent. Therefore, a word of warning must be said concerning the comparison with data.

4. - Let us consider now the two-track case. Labeling the parameters of the two particles by the index 1 and 2, we will call $\theta_i$, $\chi_i$ the polar angles in the S.S. and $\Delta_i$, $\chi_i$ those in the C.S., shown in Fig. 2. Also, $\theta_{12}$ is the angle between the two tracks. To lighten the notations, put

$$z_i = \cos \Delta_i , \ z = \cos \theta , \ \chi_i = \cos \chi_i$$

FIG. 2
and \( \chi = \chi_1 - \chi_2 \), \( z_{12} = x_1 x_2 - \sqrt{1 - x_1^2} \sqrt{1 - x_2^2} \cos (\varphi_1 - \varphi_2) \)

Assuming again the direction of \( \vec{J} \) to be known \((\theta, \varphi)\) then, because of the symmetries we would write for the probability of the \( \vec{p}_1 \), \( \vec{p}_2 \), \( \vec{J} \) configuration

\[
\frac{3}{8\pi} W (z_1, z_2, \chi; p_1 p_2) \frac{dz_1}{\text{d}z_1} \frac{dz_2}{\text{d}z_2} \frac{d\chi}{\text{d}\chi} (1 - z^2) \text{d}z \text{d}\varphi
\]

where \( W \) is a normalized function having the property ii), § 2

\[
W (-z_1, -z_2, \chi; p_1 p_2) = W (z_1, z_2, \chi; p_1 p_2)
\]

The analogous procedure to the single-track case would be to integrate over \( z, \varphi \) keeping \( \theta_1, \varphi_1 \) fixed for both particles:

\[
\frac{3}{8\pi} \int_0^{2\pi} \int_{-1}^{+1} (1 - z^2) \text{d}z \text{d}\varphi W (z_1, z_2, \chi; p_1 p_2)
\]

\( \text{d}\Omega_i \) being the solid angle for track \( i \) in the S.S.

That this would hardly produce simple results is seen in the circumstance that, while

\[
z_1 = z x_1 + \sqrt{1 - z^2} \sqrt{1 - x_1^2} \cos (\varphi - \varphi_1)
\]

is a simple relation, the formula

\[
\cos \chi = \frac{z_{12} - z_1 z_2}{\sqrt{1 - z_1^2} \sqrt{1 - z_2^2}}
\]

is exceedingly complicate. Therefore, the most natural expansion

\[
W (z_1, z_2, \chi; p_1 p_2) = \sum W_{rs}^m (p_1 p_2) P_r (z_1) P_s (z_2) \cos m \chi
\]

cannot be exploited in a simple way(2).

We shall therefore consider a different parametrization in which the particle having momentum \( \vec{p}_1 \) is analyzed in the S.S. whereas particle 2 (\( \vec{p}_2 \)) is analyzed with respect to the polar axis \( \vec{p}_1 \), as shown in Fig. 3. We introduce now the angles:

\( \alpha \), between the \( (\vec{p}_1, \vec{J}) \) plane and the plane containing \( \vec{p}_1 \) and the beam line
\( \alpha_2 \), between the \(( \vec{p}_1, \vec{p}_2 )\) plane and the plane of \( \vec{p}_1 \) and the beam-line.

Thus, \( \alpha \) and \( \alpha_2 \) describe rotations around the \( \vec{p}_1 \) axis. A configuration is fully determined, for this two-track case, by giving \( z_1, z_{12}, \alpha - \alpha_2 \); that is, the dynamics will be contained in a function \( W (z_1, z_{12}, \alpha - \alpha_2; p_1 p_2) \) and the average over the directions of \( J \) will be performed by integrating over \( z_1 \) and \( \alpha \).

It follows that the probability to find \( p_1 \) in the solid angle \( d\Omega_1 \) and \( p_2 \) in the solid angle \( d\Omega_{12}(= dz_{12}d\alpha_2) \) from track 1 is

\[
\frac{3}{8\pi} d\Omega_1 d\Omega_{12} \int_0^{2\pi} d\alpha \int_{-1}^{+1} dz_1 (1 - z_1^2) W (z_1, z_{12}, \alpha - \alpha_2; p_1 p_2) =
\]

\[
\equiv d^4 Q (x_1, \psi_1; z_{12}, \alpha_2)
\]

Since

\[
z = z_1 x_1 + \sqrt{1 - x_1^2} \sqrt{1 - z_{12}^2} \cos \alpha
\]

we proceed again to expand \( W \):

\[
W (z_1, z_{12}, \alpha - \alpha_2; p_1 p_2) = \sum_r W_r m (z_{12} p_1 p_2) P_r (z_1) \cos m (\alpha - \alpha_2)
\]
Then, by integration

\[ d^4Q = d\Omega_{12} \left\{ W^0_{00}(\theta_1, \phi_2) + \frac{1}{10} W^0_{20}(\theta_1, \phi_2)(1 - 3x_1^2) - \right. \]

\[ - \frac{3}{5} \left[ \sum g_r W^1_r(z_1, p_1, p_2) \right] \sqrt{1 - x_1^2} \cos \lambda_2 - \frac{1}{2} \left[ W^2_0(z_1, p_1, p_2) - \right. \]

\[ - \frac{1}{5} W^2_2(z_1, p_1, p_2) \cos 2 \lambda_2 (1 - x_1^2) \} \]

where \( g_r = \int_{-1}^{+1} \sqrt{1 - z_1^2} P_r(z_1) dz_1 \).

The explicit form of \( g_r \) is:

\[ g_r = 2 \cos^2 \left( \frac{\pi r}{2} \right) \frac{(1 - r^2) (3 + r)}{1 - r^2} \]

with \( g_1 = 0 \) (by the definition formula). Therefore, \( g_r = 0 \) for \( r \) odd and

\[ g_0 = \frac{2}{3}, \quad g_2 = -\frac{2}{15}, \quad g_4 = -\frac{2}{105} \]

showing that only the first few terms in the sum \((W^1_0, W^1_2)\) contribute appreciably.

Also, by using the spherical harmonics

\[ Y_{mn}(\theta, \phi) = \cos m \phi P^m_n(\cos \theta) \]

the general formula can be rewritten:

\[ d^4Q = d\Omega_{12} \left\{ W^0_{00}(\theta_1, \phi_2) - \right. \]

\[ - \frac{1}{5} W^0_{20}(\theta_1, \phi_2) Y_{02}(\theta_1, \phi_2) - \frac{3}{5} \left( \sum g_r W^1_r \right) Y_{11}(\theta_1, \phi_2) - \]

\[ - \frac{1}{6} \left( W^2_0 - \frac{1}{5} W^2_2 \right) Y_{22}(\theta_1, \phi_2) \} \]

From this general formula we can deduce simpler formulas for special cases:
8.

a) Which is the probability that, given a track (track 1, say, pointing in the $\theta_1$, $\phi_1$ direction) we find another (track 2) making an angle $\theta_{12}$ with $\vec{p}_1$, irrespective of the orientation of the plane $(\vec{p}_1, \vec{p}_2)$? (remember $z_{12} = \cos \theta_{12}$).

This is simply obtained by integrating over $\alpha_2$:

$$d^3Q(x_1, \phi_1, z_{12}) = 2\pi d\mathcal{N}_{12} dz_{12} \left\{ W_0^0(z_{12}; p_1 p_2) + \frac{1}{10} W_2^0(z_{12}; p_1 p_2)(1 - 3 x_1^2) \right\} = 2\pi d\mathcal{N}_{12} dz_{12} \left\{ W_0^0 - \frac{1}{5} W_2^0 p_2 (x_1^2) \right\}$$

a formula that reminds of the single track theorem.

It is also evident that

$$8\pi^2 W_0^0(z_{12}; p_1 p_2) dz_{12}$$

is the probability to find a second track at an angle $\theta_{12}$ with the first.

b) The general formula shows that the probability to find a track-pair whose ($\vec{p}_1$, beam) and ($\vec{p}_2$, ) planes form an angle $\alpha_2$ (integrate over $x_1, \phi_1, z_{12}$) is:

$$\frac{d\alpha_2}{2} \left[ 1 + A \cos \alpha_2 + B \cos 2 \alpha_2 \right]$$

where

$$A = -\frac{3\pi}{4} \sum g_r \int_{-1}^{1} dz_{12} W_r^1$$

$$B = -\frac{8}{3} \pi^2 \int_{-1}^{1} dz_{12} \left[ W_0^2 - \frac{1}{5} W_2^2 \right]$$

a) and b) are just examples of what can be done to analyze events; here we want to emphasize that the main result expressed by the general formula for $d^4Q$ is:

Two-track theorem. - The distribution in space of two-tracks from any $e^+e^- \rightarrow$ (spin zero hadrons) annihilation process depends on four unknown functions of the single variable $\theta_{12}$, the angle between the two tracks. The dependence on the other 3 angle variables is completely determined and given by the general formula (2).
Eventually as a check and example, it is easily verified that for \( e^+e^- \rightarrow 2\pi \) (2K), since

\[
W = \frac{1}{8\pi^2} \delta(1+z_{12}) \left[ \delta(1+z_1) + \delta(1-z_1) \right]
\]

one has \( W^m_r = 0 \) if \( m > 0 \); \( W^o_r = 0 \) if \( r \) odd. For even \( r \)

\[
W^o_r = \frac{2r+1}{8\pi^2} \delta(1+z_{12})
\]

whence \( d^4Q \sim d\Omega d\Omega_{12} \sin^2\theta_1 \delta(1+z_{12}) \).

Also, for \( e^+e^+ \rightarrow 3\pi \) (2K + \( \pi \)), write

\[
W = \frac{1}{4\pi} \delta(z_1) H(z_{12}; p_1 p_2)
\]

whence:

\[
W^m_r = 0 \quad \text{for} \quad m > 0
\]

\[
W^o_r = 0 \quad \text{for} \quad \text{odd} \quad r
\]

\[
W^o_r = \frac{2r+1}{8\pi^2} H(z_{12}; p_1 p_2) P_r(0)
\]

so that

\[
d^4Q \sim d\Omega_1 d\Omega_{12} H(z_{12}; p_1 p_2) (1+\cos^2\theta_1)
\]

5. - A full exploitation of the two-track theorem is made difficult by the presence of the momentum variables \( p_1 \) and \( p_2 \) and by the fact that one has to reconstruct unknown functions (the \( W^m_r \)) rather than to determine numerical values of parameters.

When momentum analysis is not done, the \( W \)'s must be integrated over the momentum spectrum for the two particles. This integration will generally require knowledge of the \( z_{12} \) dependence of the kinematical limits for \( p_2 \), given \( p_1 \). Nevertheless, integration over the momentum spectrum will give average functions \( \bar{W}_r^m(z_{12}) \) of the single variable \( z_{12} \).
such that \( d^4 \overline{Q} \), constructed by replacing \( \overline{W}_r^m(z_{12}; p_1 p_2) \) with \( \overline{W}_r^m(z_{12}) \) in (2), is related to the two track cross section by:

\[
\frac{d^4 \sigma}{d\Omega_1 d\Omega_{12}} = \sigma_{\text{Total}} \frac{d^4 \overline{Q}}{d\Omega_1 d\Omega_{12}}
\]

Now, it is a better procedure than to reconstruct unknown functions in \( d^4 Q \) to introduce empirical functions depending on few parameters in order to get fits.

We choose the following kind of empirical functions:

\[
\overline{W}^m_r(z_{12}) = \frac{1}{16 \pi^2} (a^m_r + b^m_r z_{12}^2)
\]

Note that when \( r = m = 0 \), because of (4) and the meaning of \( W^0_0 \) one has

\[
b^0_0 = 3(1 - a^0_0)
\]

Also, from (3) we get after integration over \( z_{12} \)

\[
a^0_2 + \frac{1}{3} b^0_2 = 4 \overline{W}_2
\]

that is a relation with the parameter appearing in the single-track formula (1).

Since the coefficients \( \overline{W}^1_r(z_{12}) \) appear in the general formula (2) as a linear combination:

\[
\sum_0^\infty g_r \overline{W}^1_r(z_{12})
\]

we only need the parameters

\[
a_1 = \sum g_r a^1_r \\
b_1 = \sum g_r b^1_r
\]
Also, the sum $W_o^2 - \frac{1}{5} W_2^2$ appears in the last term of 2, so that only
\[ a_2^o = a_2^o - \frac{1}{5} a_2^o \quad \text{and} \quad \frac{1}{5} b_2^o \]
have a role in the fit.

In conclusion, we propose to represent the two-track data by a formula containing 7 parameters:

In particular, formula (5) contains the parameters

\[ A = - \frac{3 \pi}{32} (a_1 + \frac{1}{3} b_1) \]
\[ B = - \frac{1}{3} (a_2 + \frac{1}{3} b_2) \]

and formula (3) shows that, when integrating over $\alpha_2'$, the distribution $d^3Q$ is given by a 3 parameter $(a_o^o, a_2^o, b_2^o)$ formula.

Positivity requirements impose some inequalities whose model independent form is of the following type:

\[ \bar{W}_o^o (z_{12}) \geq 0, \text{ any } z_{12} \]
\[ \bar{W}_o^o (z_{12}) - \frac{1}{5} \bar{W}_2^o (z_{12}) P_2 (x_1) \geq 0, \text{ any } z_{12}, x_1 \]
\[ B \leq 1, \quad A \leq 1 + B \quad \text{(from (5))} \]
and so on.

When we use the empirical formula, these inequalities become

\[ 0 \leq a_o^o \leq \frac{3}{2} \]
12.

\[
\begin{align*}
\alpha^0 &\geq -\frac{1}{10} \alpha^2 \quad \text{if } \alpha^0 \text{ is negative} \\
\alpha^0 &\geq \frac{1}{5} \alpha^2 \quad \text{if } \alpha^0 \text{ is positive} \\
\alpha^2 + \frac{1}{3} \beta^2 &> -3 \\
\alpha^2 + \frac{1}{3} \beta^2 - \frac{9\pi}{32} (\alpha^1 + \frac{1}{3} \beta_1) &< 3
\end{align*}
\]

and so on.

In conclusion, because of solid angle limitations in the experimental set-ups the reconstruction of the total cross section for hadronic annihilation events will require a knowledge of the geometrical efficiency and of the angular distribution of the produced particles. We have shown here how to proceed when one or two tracks are detected; the aim of the present work is therefore to help to judge of the sensitivity of an apparatus and, perhaps, to optimize its performances.

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REFERENCES AND FOOTNOTES.

(2) - Notice that, if \(S(p_1, p_2)\) is the momentum spectrum for \(p_2\) at a fixed \(p_1\)

\[
4\pi\int dp_2 S(p_1, p_2) W^0_{\rho_0}(p_1, p_2) = w_r(p_1)
\]

the single-track parameters. Since by ii) \(2r+s\) must be even, \(r\) even follows when \(s = 0\).