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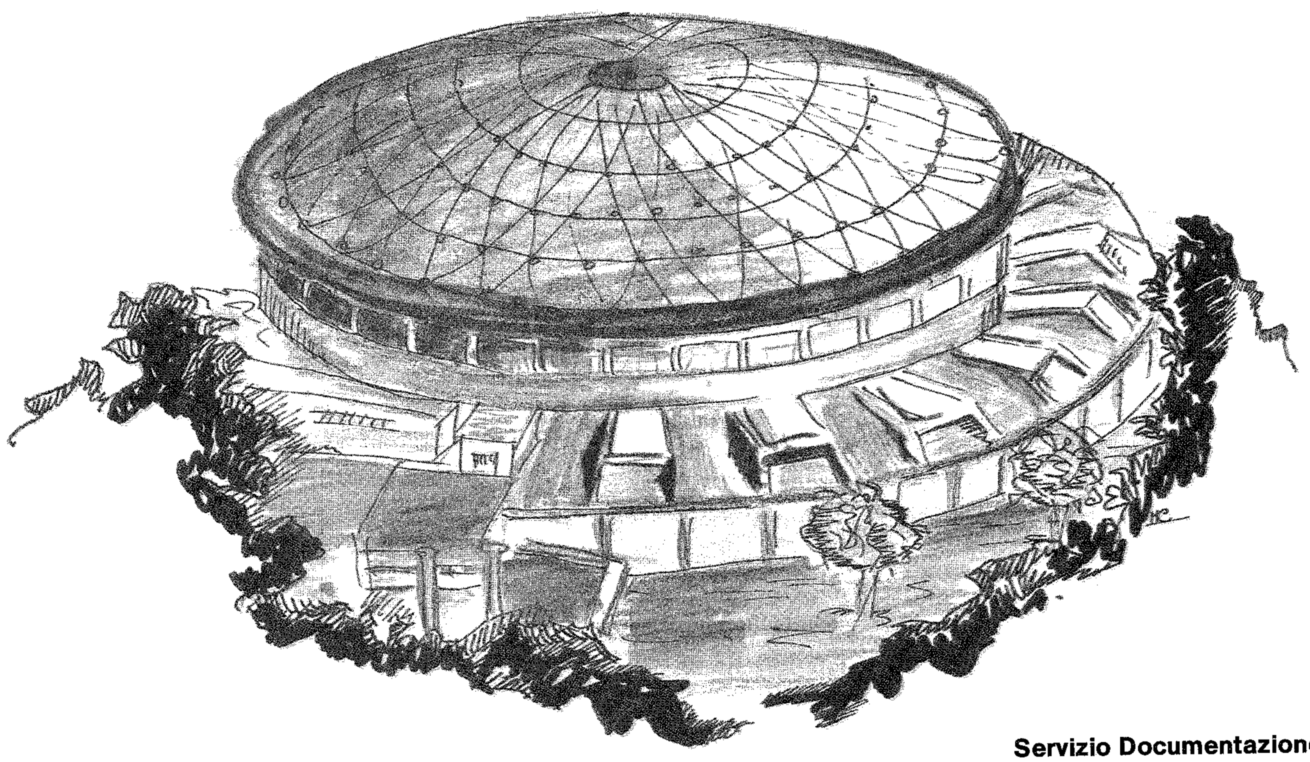

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DUALITY TRANSFORMATION OF THE ONE DIMENSIONAL SUPERSYMMETRIC STRING

by

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ABSTRACT

We solve, to all perturbative orders, the discretized representation of the one dimensional supersymmetric string, in terms of matrix variables on a circle of radius R. The compactified perturbative expansion does not possess duality symmetry, in contrast with the bosonic theory. Duality transformations map the exact solution for the supersymmetric string into the solution of the bosonic string in the $R = \infty$ case.

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1. - Introduction

String theory in $D = 1$, which is equivalent to gravity coupled to matter in $D = 1$, has been linked to random matrix models, through the discretization of the worldsheet [1]. The random matrix theory representation allowed to derive a solution to all perturbative orders, for both the case in which the target space is a real line [2,3] and the string theory compactified on a circle of finite radius R [4]. Also, the supersymmetric string on the real line [5,6] was solved to arbitrary genus, in the critical limit [7]. Here, we consider the representation of the supersymmetric string in terms of matrix variables on a circle.

The topological expansion of the discretized string partition function can be defined as

$$Z = \sum_G N^{2(1-G)} \sum_{S_N} g^A Z_N \,, \tag{1}$$

where G denotes the genus and N is the number of points in the discretization S_N, with area A. The coupling constant g can be expressed in terms of the cosmological constant Λ

$$g = e^{-\Lambda} \,. \tag{2}$$

In the string path integral above we have introduced

$$Z_N = \int [dx_k] exp\Big(- \sum_{<ij>} E_{ij}{}^{(N)} \Big) \,, \tag{3}$$

where the sum is over the nearest neighbour vertices of S_N. Recalling that the target space has $D = 1$, we can consider two cases. For a real line, where we integrate in the range $(-\infty, +\infty)$, we can assume a gaussian link factor

$$E_{ij} \sim \frac{1}{2}(x_i - x_j)^2 \,, \tag{4}$$

whereas for the case of a circle, where the integrations are carried out in the range $[0, 2\pi R)$, the link factor is taken to be periodic with period $2\pi R$

$$E_{ij} \sim R(x_i - x_j) \,. \tag{5}$$

The string partition function is related to that of a matrix model

$$Z = \int [d\Phi_{ab}] exp\Big[- N \int Tr\Big(\frac{1}{2}\dot{\Phi}^2 + V(\Phi)\Big) dt\Big] , \tag{6}$$

where Φ_{ab} is a $N \times N$ hermitean matrix and the potential can be expressed as

$$V(\Phi) \;=\; \frac{1}{2}\Phi^2 + \frac{1}{3}g\Phi^3 + \dots . \tag{7}$$

The action of the supersymmetric matrix model [5] reads

$$S \;=\; \int dt d\theta d\bar{\theta}\; Tr\Big[- \Phi D^2 \Phi + W(\Phi)\Big] . \tag{8}$$

Here we denote by $W(\Phi)$ the supersymmetric potential and by D the covariant derivative

$$D \;=\; \frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial t} . \tag{9}$$

The matrix $\Phi_{ab}(t,\theta,\bar{\theta})$ is defined on a $D=1$ superspace in terms of its components

$$\Phi_{ab}(t,\theta,\bar{\theta}) \;=\; \phi_{ab}(t) + \theta\psi_{ab}(t) + \bar{\theta}\bar{\psi}_{ab}(t) \; + \theta\bar{\theta}A_{ab}(t) . \tag{10}$$

Carrying out the integration over the fermionic coordinates θ, $\bar{\theta}$ and eliminating the auxiliary field A yields

$$S \;=\; \int dt Tr\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\bar{\psi}\dot{\psi} + \left(\frac{\partial W(\phi)}{\partial \phi_{ab}}\right)^2 + \bar{\psi}_{ab}\frac{\partial^2 W(\phi)}{\partial\phi_{ab}\partial\phi_{cd}}\psi_{cd}\right] . \tag{11}$$

We now go to the hamiltonian formalism

$$[p_{ab}, \phi_{cd}] = \frac{1}{N}\delta_{ad}\delta_{bc} , \tag{12}$$

$$\{\psi_{ab}, \bar{\psi}_{cd}\} = \delta_{ad}\delta_{bc} , \tag{13}$$

where $\frac{1}{N} \sim \hbar$. As $N \to \infty$, this reproduces the $D=1$ superstring. We must find the ground state of the hamiltonian

$$H = -\frac{1}{2N^2}\frac{\partial^2}{\partial {\phi_{ab}}^2} + \left(\frac{\partial W}{\partial \phi_{ab}}\right)^2 - \frac{1}{N}\frac{\partial^2 W}{\partial {\phi_{ab}}^2} . \tag{14}$$

There is an ansatz for the wavefunction of the ground state $\chi(\{\lambda_i\})|0>$, where

$$\psi_{ab}|0>= 0 . \tag{15}$$

Here $\{\lambda_i\}$ are the eigenvalues of ϕ_{ab}

$$\phi_{ab} = \Omega^{-1}\Lambda\Omega , \tag{16}$$

with Ω a unitary and Λ a diagonal matrix. As a result of the change of variables, one has the Van der Monde determinant

$$[d\phi_{ab}] = [d\lambda_i][d\Omega]\Delta^2(\lambda) , \tag{17}$$

where

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) . \tag{18}$$

The ground state energy reads

$$E_{gs} = \frac{1}{N^2}\min_{\tilde{\chi}} \frac{\int[d\lambda](\tilde{\chi}H\tilde{\chi})}{\int[d\lambda](\tilde{\chi}\tilde{\chi})} , \tag{19}$$

in terms of the redefined ground state wavefunction

$$\tilde{\chi}(\lambda) \equiv \Delta(\lambda)\chi(\lambda) . \tag{20}$$

Hence, the problem can be reduced to that of N fermions in a $D = 1$ supersymmetric potential

$$V(\lambda) = [W'(\lambda)]^2 - \frac{1}{N}W''(\lambda) . \tag{21}$$

2. - The solution of the supersymmetric string

In a previous work we considered the discrete version of the supersymmetric string on the real line [7]. We considered the following case for the superpotential W

$$(W')^2 = -2\lambda^2 , \; W'' = i\sqrt{2} , \tag{22}$$

and solved the inverted supersymmetric harmonic oscillator. By taking the critical limit of the supersymmetric quantum mechanical theory, we derived an exact expression for the density of states. This is defined as

$$\rho(E) \equiv \frac{1}{\beta}\sum_{n}\delta(E_n - E) . \tag{23}$$

The coupling constant and ground state energy are introduced in terms of the density of states

$$g = \frac{N}{\beta} = \int_{0}^{E_F} \rho(E)\, dE , \tag{24}$$

$$E_{gs} = \sum_{n} E_n = \beta^2 \int_{0}^{E_F} E\, \rho(E)\, dE , \tag{25}$$

where N denotes the number of fermions that fill all levels up to the Fermi level.

We define $\mu = V_{max} - E_F$. The density of states and the coupling constant are singular functions of μ. Accordingly with this critical behaviour as $\mu \to 0$, the ground state energy depends non-analytically on the coupling constant. The renormalized cosmological constant is identified with the nonanalytic component of $E_{gs}(\Delta)$, where $\Delta = 1 - g$ [2]. The function $E_{gs}(\Delta)$ can be calculated from the equation

$$\frac{\partial E_{gs}}{\partial \Delta} = \beta^2(\mu - V_{max}) , \tag{26}$$

plugging the function $\mu(\Delta)$ obtained by solving

$$\frac{\partial \Delta}{\partial \mu} = \rho(E_F) . \tag{27}$$

In order to sum over higher-genus surfaces, we calculate the singular terms in the density of states near the maximum of the F^2-term in the potential [7]. In the scaling limit, this procedure leads to the analytic continuation to imaginary frequency of the hamiltonian for a supersymmetric harmonic oscillator. The asymptotic expansion of the density of states is found in ref. [7]

$$\rho(V_{max} - \mu) = -\frac{1}{2\pi}\log\mu - \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k}\left(\frac{2}{\beta\mu}\right)^{2k} . \tag{28}$$

The high degree of divergence of this asymptotic series is expressed by the fast growth of the Bernoulli numbers, reflecting a typical stringy behaviour [8]. The same higher-order expansion for the density of states is obtained using a modification of the WKB method, such that supersymmetry is maintained [7].

The exact result we obtained can be rewritten as an integral representation

$$\frac{1}{\beta}\frac{\partial\rho}{\partial\mu} = \frac{1}{2\pi\beta\mu}\, Im \int_0^\infty dt\; exp\,(-it)\; exp\left(\frac{t}{\beta\mu}\right)\frac{t}{\beta\mu}\left[sinh\left(\frac{t}{\beta\mu}\right)\right]^{-1}. \quad (29)$$

This should be regarded as an attempt to provide a nonperturbative definition of the supersymmetric string theory, in terms of the integral over the Borel transform of the asymptotic expansion (28), supplemented by a principle value prescription for integrating about the infinite number of poles on the real axis. We will have to keep in mind that the potential ambiguities associated with the infinite number of arbitrary parameters introduced by the principle value prescription, make the validity of the integral representation (29) beyond its asymptotic expansion (28) unclear.

Integrating eq. (27) using the perturbative series (28), we obtain

$$\Delta = \frac{\mu}{2\pi}\left[-\,log\;\mu + \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k-1)}\left(\frac{2}{\beta\mu}\right)^{2k}\right]. \quad (30)$$

This expression can be inverted to write μ as a function of Δ

$$\mu = -\frac{2\pi\Delta}{log\;\Delta}\left[1 + \frac{1}{log\;\Delta}\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k-1)}\left(\frac{log\;\Delta}{\pi\beta\Delta}\right)^{2k}\right], \quad (31)$$

where we neglect double logarithms, as well as terms that are suppressed by powers of $log\;\Delta$. Using $\mu(\Delta)$ from (31) and integrating eq. (26), yields the leading-logarithmic series for the ground state energy

$$E_{gs} = -\frac{\pi^2\beta^2\Delta^2}{log\;\Delta} - \frac{1}{6\pi}log\;\Delta + \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{|B_{2k+2}|}{k(k+1)(2k+1)}\left(\frac{log\;\Delta}{\pi\beta\Delta}\right)^{2k}. \quad (32)$$

For comparison, note that in the bosonic case one has

$$E_{gs} = -\frac{\pi^2\beta^2\Delta^2}{log\;\Delta} + \frac{1}{12\pi}log\;\Delta - \frac{1}{4\pi}\sum_{k=1}^{\infty}\left(2^{2k+1}-1\right)\frac{|B_{2k+2}|}{k(k+1)(2k+1)}\left(\frac{log\;\Delta}{2\pi\beta\Delta}\right)^{2k}. \quad (33)$$

In writing (33), we are correcting an error in the result given in eq. (3.13) of ref. [4]. The appearance of logarithmic divergences in our result deserves two comments. Firstly, these divergences can only be detected by the discretized version of the theory, though agreement is found with the scaling of the free energy like Δ^2 obtained from conformal field theory [9]. Secondly, the logarithmic dependance on the renormalized cosmological constant has been linked, in the [illegible] the presence of massless modes [10].

3. - Strings compactified on a circle

Following ref. [4] we define the matrix variable $\phi(t)$ on a circle of radius R, with the partition function

$$Z = Tr\ exp(-2\pi R\beta H) . \tag{34}$$

Note that $2\pi R$ can be interpreted as the inverse temperature. The coupling constant is defined in terms of the chemical potential μ_F as follows:

$$g = \int_0^\infty \rho(E)\{1 + exp[2\pi R\beta(E - \mu_F]\}^{-1}\, dE . \tag{35}$$

In order to calculate the free energy F as a function of Δ, one needs to consider the equations

$$\frac{\partial \Delta}{\partial \mu} = \int d\lambda\ \rho(\mu_c - \lambda)\frac{\partial}{\partial \mu}\{1 + exp[2\pi R\beta(\mu - \lambda]\}^{-1} , \tag{36}$$

$$\frac{\partial F}{\partial \Delta} = \beta^2(\mu - \mu_c) . \tag{37}$$

Here we have introduced $\mu \equiv \mu_c - \mu_F$, $\lambda \equiv \mu_c - E$. First, one uses (36) to determine the function $\mu(\Delta)$. Then, $F(\Delta)$ can be calculated by integrating eq. (37).

As in the bosonic case [4], we proceed by differentiating (36)

$$\frac{\partial^2 \Delta}{\partial \mu^2} = \int d\lambda \frac{\partial \rho}{\partial \lambda}\frac{1}{2}\pi R\beta\ \{cosh[\pi R\beta(\mu - \lambda)]\}^{-2} . \tag{38}$$

It is convenient to use the integral representation of ρ given in eq. (29) that provides the exact result for the supersymmetric string theory. The integral in λ can be carried out introducing the variable $x = \frac{t}{\beta\mu}$

$$\begin{aligned} \frac{1}{2}\pi R\beta \int d(\mu - \lambda)\ exp[i\beta(\mu - \lambda)x]\ \{cosh[\pi R\beta(\mu - \lambda)]\}^{-2} \\ = \frac{x}{2R}\left[sinh\left(\frac{x}{2R}\right)\right]^{-1} , \end{aligned} \tag{39}$$

and yields the result

$$\frac{1}{\beta}\frac{\partial^2\Delta}{\partial\mu^2} = \frac{1}{2\pi} Im \int_0^\infty dx\ x e^x \frac{x}{2R} e^{-ix\beta\mu} \left[sinh(x)\ sinh\left(\frac{x}{2R}\right)\right]^{-1} . \qquad (40)$$

Integrating in μ we find

$$\frac{\partial\Delta}{\partial\mu} = \frac{1}{2\pi} Re \int_\mu^\infty dt\ e^{-it} \frac{t}{2R\beta^2\mu^2} exp\left(\frac{t}{\beta\mu}\right)\left[sinh\left(\frac{t}{\beta\mu}\right) sinh\left(\frac{t}{2R\beta\mu}\right)\right]^{-1} , \qquad (41)$$

where we fix the integration constant, in order to agree with the WKB approximation.

The result (41) is not symmetric under the duality transformations

$$2R \to \frac{1}{2R} \ , \ \beta \to 2R\beta \ . \qquad (42)$$

This is the opposite of what occurs in the bosonic case [4], whose solution can be obtained from (41) by simply dropping the factor $exp\left(\frac{t}{\beta\mu}\right)$ on the *r.h.s.* of (41)

$$\frac{\partial\Delta}{\partial\mu} = \frac{1}{2\pi} Re \int_\mu^\infty dt\ e^{-it} \frac{t}{2R\beta^2\mu^2} \left[sinh\left(\frac{t}{\beta\mu}\right)\ sinh\left(\frac{t}{2R\beta\mu}\right)\right]^{-1} . \qquad (43)$$

Remarkably, eq. (43) has a duality symmetry under the transformations (42). Although this is not a symmetry of the relation (41) holding for the supersymmetric theory, the transformations (42) still play a special role. Firstly, note that the exact supersymmetric solution (29) is obtained from (40), in the limit $R \to \infty$. Secondly, let us carry out a duality transformation of (40) and follow the fate of the exponential factor

$$exp\left(\frac{t}{\beta\mu}\right) \to exp\left(\frac{t}{2R\beta\mu}\right) . \qquad (44)$$

Then, taking the limit $R \to \infty$, this exponential factor goes to one and we recover the nonperturbative solution of the bosonic theory [2]

$$\frac{1}{\beta}\frac{\partial\rho}{\partial\mu} = \frac{1}{2\pi\beta\mu}\ Im \int_0^\infty dt\ exp\,(-it)\ \frac{t}{\beta\mu}\left[\ sinh\ \left(\frac{t}{\beta\mu}\right)\ \right]^{-1} . \qquad (45)$$

Hence, we conclude that duality is not a symmetry, rather it maps the nonperturbative solution of the $R = \infty$ supersymmetric string into the corresponding solution of the bosonic theory. One way to interpret this result is by speculating that the exact solution found in ref. [7] suggests itself as a candidate for a dual point of string theory. Summarizing, that is what we learned, having modified string theory to put it on a circle. Although the validity of (41) is based upon the assumption that the integral representation (29) that summarizes the asymptotic expansion of the density of states is correct, the asymptotic expansion of (41) holds unambiguously, just as the asymptotic expansion of ρ given in (28).

4. - Conclusions

The $R \rightarrow \alpha'/R$ duality of the bosonic solution [4] is broken for $T > T_{KT}$, where T_{KT} denotes the Kosterlitz-Thouless transition temperature [11]. We can study the Kosterlitz-Thouless phase transition on random surfaces. In this respect, it is useful to consider the theory of quantum mechanics with a discrete time step ϵ [12]. This problem can be reduced to finding the hamiltonian $H(\epsilon)$ such that

$$< x|exp\Big(-\epsilon N H(\epsilon)\Big)|y> = K(x,y) , \tag{46}$$

where $K(x,y)$ is the transfer matrix. The latter coincides with the propagator for the inverted harmonic oscillator. Hence, the free energy reads

$$E(\epsilon,\Delta) = \frac{1}{2}\omega(\epsilon)E(\Delta) , \tag{47}$$

where $E(\Delta)$ coincides with the N-fermion ground state energy of matrix quantum mechanics found in eq. (32). For $\epsilon > 1$, both $\omega(\epsilon)$ and $H(\epsilon)$ have complex values. This can be interpreted as an indication that the $c = 1$ phase of string theory is unstable. As $\epsilon \rightarrow 0$, we have $\omega(\epsilon) \rightarrow 2$. Hence, one recovers matrix quantum mechanics from the infinite chain of matrices with nearest neighbour couplings.

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