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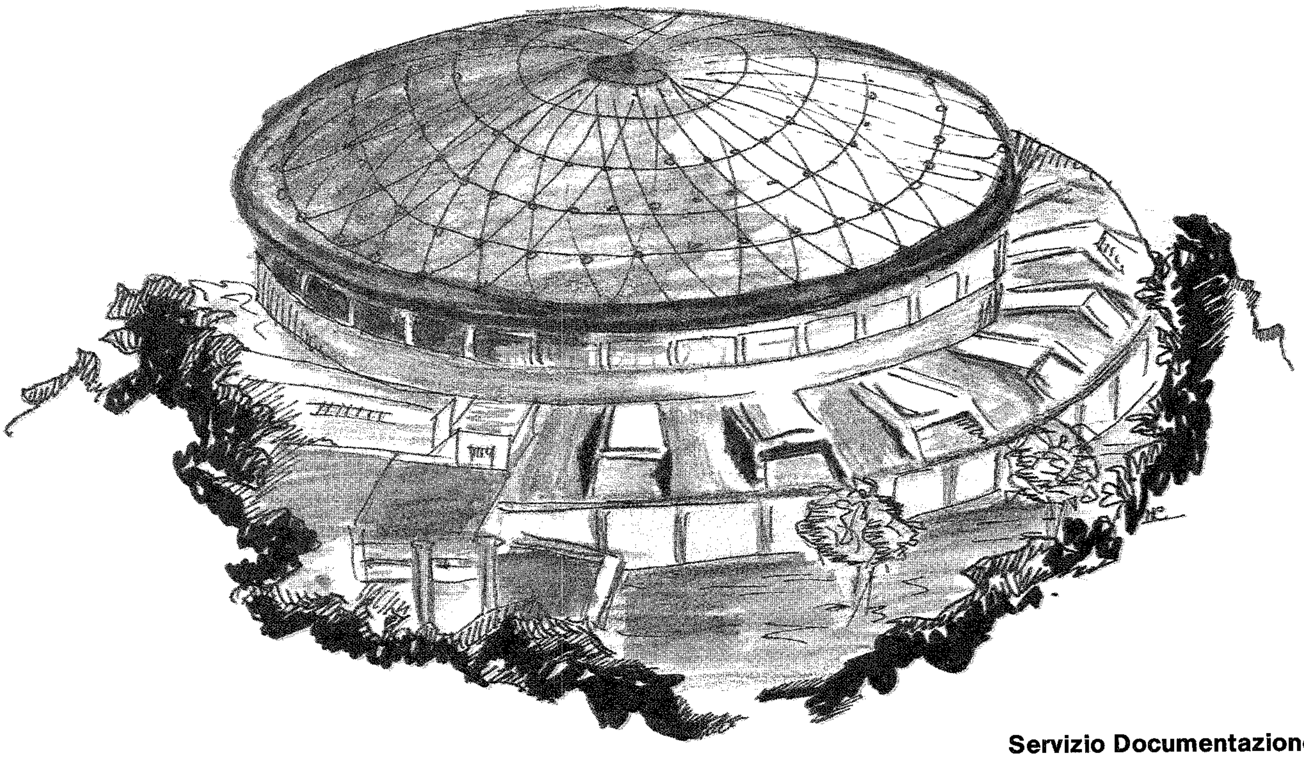

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WORMHOLES AND CHARGED PARTICLES

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ABSTRACT
We consider the modifications of the theory of scalar fields coupled to QED induced by large-scale topology-changing configurations. We compute the one-loop effective action taking into account the physically relevant corrections to lowest order. We show the existence of a critical scale which depends on the ratio of the coupling constants entering the formulation of the model. The gauge dependence of the wormhole corrections is discussed. In the case of a pure scalar theory, we also carry out the calculation of the higher-order corrections, up to the fourth power of the Riemann tensor.

1. INTRODUCTION

The problem of the smallness of the cosmological constant, Λ, can be considered to be one of the most outstanding difficulties in contemporary physics. The discrepancy between the observed result of $\Lambda<10^{-47}$ (GeV)^4 and the value estimated from quantum field theory of $\Lambda=10^{72}$ (GeV)^4 giving an impressive difference of 120 orders of magnitude deserves recording in the Guinness Book of Records!

A possible solution to this puzzling discrepancy was suggested by Coleman two years ago [1].This solution is based on the hypothesis that quantum gravity can be treated in Euclidean space. Separate universes can be connected by small-scale wormholes, with a size of the order of the inverse of the Planck mass.Such universes can provide the largest contribution to the functional integral of quantum gravity. In this case a vanishing value of Λ corresponds to the minimum of the effective action.

Wormholes are topology-changing configurations resulting in a multilply connected spacetime. The space that corresponds to the maximum probability of having a vanishing

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cosmological constant is S^4 (for $\Lambda>0$).The vacuum to vacuum transition probability is proportional to a double exponential [1].The sum over all topologies must be carried out. Just like Λ, all the parameters are driven to values maximizing the double-exponential probability density. As a consequence, the quantity ΛG^2 is driven to 0^+.

One can extract information about the parameters (i.e. masses and coupling constants) of the low-energy effective action by using a perturbative approach, in order to calculate quantum corrections to any given field-theoretic model in a S^4 background. The effect of Planck-size topology-changing configurations on the range of values of the physical constants at low-energy can be computed in this way. One-loop order results have been obtained for the simple theory of one scalar field and one Majorana fermion [2]. The physically meaningful corrections have been taken into account to lowest order, i.e. to order $O(R^3)$.

In the present work, we take up the calculation of the radiative corrections to scalar electrodynamics, in the presence of large-scale wormholes, taking into account $O(R^3)$ corrections. We derive the effective action in two different gauges for the vector field and compare our results with those contained in a letter appeared recently [3], where the wormhole corrections to an abelian gauge degree of freedom in one special gauge are discussed. In the case of a self-interacting scalar field, we carry out the calculation of the higher-order corrections, up to the fourth power of the Riemann tensor. These corrections modify the ranges for the physical constants of the low-energy theory.

This paper is organized as follows. In the next section we present the wormhole effect on the quantized theory of scalar electrodynamics in the physical gauge where no U(1) ghost contribution is present. The third section is taken up with quantizing the theory according to the procedure of ref. [3]. This section contains a careful treatment of the gauge dependence of the wormhole corrections. The results presented in this section correct an error published in ref. [3]. The analysis of a critical scale for the wormhole parameter space and the solution of the renormalization group equations are the object of the fourth section. The following section presents the higher-order radiative corrections for a pure scalar theory in the background of S^4. This section contains an accurate analysis of the Hadamard-Minakshisundaram-DeWitt (HaMiDew) coefficients up to the fourth order, mentioning, in passing, errors in the results of ref. [4]. We close with some concluding remarks. A discussion of the dependence of the wormhole corrections on the dimensional regularization scheme is demanded to an appendix.

2. ADIABATIC EXPANSION OF THE EFFECTIVE ACTION

We begin with the classical action for scalar electrodynamics on Euclidean curved space If we specialize to a unitary gauge this may be written in the following form [5]

$$S_E[\phi(x),A_\mu(x)] = \int d^4x\sqrt{g}\{-\tfrac{1}{2}\phi\Box\phi + U(\phi)R + V(\phi)$$
$$+\tfrac{1}{2}A^\lambda[-\Box\, g_{\lambda\mu} + R_{\lambda\mu} + e^2\phi^2 g_{\lambda\mu} + \nabla_\lambda\nabla_\mu]A^\mu\} \quad . \tag{1}$$

We may expand the action about $\phi(x) = \phi_0(x)$, and the background value $A_\mu = 0$. Here $\phi_0(x)$ is the solution to the equation of motion for the charged scalar. We retain only terms of second order in $\phi(x) - \phi_0(x)$. Following ref. [2], $\phi_0(x)$ can be taken as a constant on the Euclidean four sphere which is relevant to the wormhole case. Define the operator

$$\hat{F}_\phi(\nabla)\,\delta(x-y) = \frac{\delta^2 S_E}{\delta\phi(x)\delta\phi(x)} \ . \tag{2}$$

Hence the effective action is given in the usual way by

$$\Gamma[\phi(x), A_\mu(x)] = S_E[\phi_0(x)] + \tfrac{1}{2}\hbar \,\mathrm{Tr}\ln \hat{F}_A(\nabla) + \hbar\,\mathrm{Tr}\ln \hat{F}_\phi(\nabla) \ . \tag{3}$$

In the present case we have the operators

$$\hat{F}_\phi(\nabla) = -\Box + U''(\phi_0)R + V''(\phi_0) \ , \tag{4}$$

$$\hat{F}_A(\nabla) = -\Box\, g_{\lambda\mu} + R_{\lambda\mu} + e^2\phi_0{}^2 g_{\lambda\mu} + \nabla_\lambda\nabla_\mu \ . \tag{5}$$

In general for quantum corrections of the form $\lambda\hbar\,\mathrm{Tr}\ln\hat{F}(\nabla)$, $\lambda = \frac{1}{2}$ for vector and real scalar fields, and $\lambda = -\frac{1}{2}$ for fermionic fields. For an operator of the form $-\Box + X$, in Euclidean space we have the DeWitt-Schwinger [6] as follows

$$\Delta\Gamma(x) = -\lambda\hbar\,\mathrm{Tr}\int_0^\infty \frac{ds}{s}\exp[-s\hat{F}(\nabla)] + \text{constant} \tag{6}$$

and

$$\exp[-s\hat{F}(\nabla)]\,\delta(x-x')$$

$$= \frac{1}{(4\pi s)^{d/2}}\left[\Delta(x,x')\right]^{1/2}\exp\left[-\frac{\sigma(x,x')}{2s}\right]\sum_{j=0}^{\infty} g_j(x,x')\, s^j \tag{7}$$

Here $\Delta(x,x')$ is the VanVleck-Morette determinant, which is unity in the coincidence limit, $x = x'$. The quantity $\sigma(x,x')$ is half of the geodesic distance from x to x', and this is zero in the limit $x = x'$. The coefficients $g_i(x,x')$ are the HaMiDew coefficients which are to be determined for the operator $-\Box + X$.

For an operator of the form $-\Box + X$, the first three coefficients have been determined by Gilkey [7]. They have been conveniently listed by Jack and Parker [8]. Only the traces of these coefficients in the coincidence limit will be required. We denote the trace of $g_i(x,x')$ as g. These traces will be calculated on S^4, where

$$R_{\mu\nu\rho\sigma} = \frac{1}{r_0{}^2}\left[g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right] \tag{8}$$

and $r_0{}^2$ is the radius of the four sphere. Following ref.[2] we choose to re-express the coefficients $g_i(x,x')$ in terms of alternative coefficients, $a_i(x,x')$, given by the expansion,

$$\sum_{j=0}^{\infty} g_j(x,x')\, s^j = \exp(-sX) \sum_{j=0}^{\infty} a_j(x,x')\, s^j \cdot \tag{9}$$

In this case X may have an R dependence, and there will be a logarithmic expansion of the curvature in the effective action which may easily be evaluated on S^4. Hence the contribution to the effective action for an operator of the minimal form $-\Box + X$ is

$$\Delta\Gamma = -\frac{\lambda\hbar}{(4\pi)^{d/2}} \mathrm{Tr} \sum_{j=0}^{\infty} \int d^dx \sqrt{g} \int_0^{\infty} ds\; s^{j-\frac{d}{2}-1} \exp(-sX)\, a_j(x) \ . \tag{10}$$

Here the coefficients $a_i(x,x')$ have to be determined for the operator in question using Gilkey's results.

For dimension $d = 2\omega$, and letting $\omega = 2-\varepsilon$ we find the general expression

$$\Delta\Gamma = -\frac{\lambda\hbar}{(4\pi)^{2-\varepsilon}} \mathrm{Tr} \left(\frac{X}{\mu^2}\right) \sum_{j=0}^{\infty} \int d^{2\omega}x \sqrt{g}\, a_j(x)\, X^{2-j}\, \Gamma(j-2+\varepsilon) \ . \tag{11}$$

The divergent part is therefore

$$\Delta\Gamma^{div} = -\mathrm{Tr} \frac{\lambda\hbar}{16\pi^2} \int d^4x \sqrt{g} \left[\frac{1}{\varepsilon} - \frac{1}{2}\gamma + \frac{1}{2}\ln(4\pi)\right]\left[a_0X^2 - 2a_1X + 2a_2\right] \ . \tag{12}$$

Here γ is the Euler's constant. In writing eq. (12), we have chosen a regularization prescription where the momenta of the expansion are analytically continued to dimension d, while the geometrical tensors $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R are taken in d=4. An appendix shows that another choice of the regularization scheme, where the gravitational tensors are considered in four dimensions, leads to some changes in the finite one-loop effective action.The divergent part will be seen to give an extra finite contribution to order R^2 which we may choose to include in the effective action to that order. This dimensional anomaly results from the use of dimensional regularization and the dimensional dependence of the curvature terms. For example, on S^4 the curvature scalar has a value $R = 12r_0^2$. In $d = 2\omega$, we have $R = 2\omega(2\omega-1)r_0^2$. Since terms up to order R^2 do not contribute to wormhole physics [9], the dimensional anomaly will not be required. Also the anomaly will not affect the renormalization group equations.

The finite contribution is

$$\Delta\Gamma^{fin} = -\mathrm{Tr} \frac{\lambda\hbar}{16\pi^2} \int d^4x \sqrt{g} \left\{ a_0X^2\left[\frac{3}{4} - \frac{1}{2}\ln\frac{X}{\mu^2}\right] + a_1X\left[-1 + \ln\frac{X}{\mu^2}\right] - a_2 \ln\frac{X}{\mu^2} + a_3X^{-1} + \Sigma \right\} , \tag{13}$$

where

$$\Sigma = \sum_{j=4}^{\infty} a_j X^{2-j}\, \Gamma(j-2) \ . \tag{14}$$

In order to quickly read off the wormhole contribution, one could use the coefficients $g_j(x,x')$, and use the fact that the heat kernel for the operator $-\Box + X + m^2$ is equal to the heat kernel of the operator $-\Box + X$ multiplied by $\exp[-sm^2]$ where m is a constant that does not depend on the curvature. Then, the wormhole contribution, to lowest order, is simply the R^3 contribution to the effective action and, in this case, is simply

$$\Delta\Gamma^{(3)} = -\ \mathrm{Tr}\,\frac{\lambda\hbar}{16\pi^2}\, g_3\, m^{-2}. \tag{15}$$

The coefficients for the minimal operator found by Gilkey and listed by Jack and Parker are [7,8]

$$g_0\,(x) = \mathbb{1}\ , \tag{16}$$

$$g_1\,(x) = \frac{R}{6}\mathbb{1} - X\ , \tag{17}$$

$$g_2\,(x) = \frac{1}{2}\left[\frac{R}{6}\mathbb{1} - X\right]^2 + \left[-\frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{\Box R}{30}\right]\mathbb{1}$$

$$+\frac{1}{6}\Box X + \frac{1}{12}W_{\rho\sigma}W^{\rho\sigma}\ , \tag{18}$$

$$W_{\rho\sigma} = [\nabla_\rho, \nabla_\sigma]\ . \tag{19}$$

We take for the vector field $(W_{\mu\nu})^\rho{}_\sigma = R_{\mu\nu}{}^\rho{}_\sigma$ We use a Euclidean translation of the conventions by Jack and Parker. The coefficient g_3 has forty three terms and we choose not to list them.

Converting these coefficients to the $a_i(x,x')$ coefficients and specializing to spaces with constant Riemann tensor, such as S^4, yields

$$a_0 = \mathbb{1}\ , \tag{20}$$

$$a_1 = \frac{R}{6}\mathbb{1}\ , \tag{21}$$

$$a_2 = \frac{1}{360}\left[5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]\mathbb{1} + \frac{1}{12}W_{\rho\sigma}W^{\rho\sigma}\ , \tag{22}$$

Due to cancellations we find conveniently, on any space of constant curvature (e.g. S^4)

$$a_3 = P(R^3)\mathbb{1} + \frac{1}{30}W_{\alpha\beta}W^{\beta\rho}W_\rho{}^\alpha + \frac{1}{60}R_{\alpha\beta\gamma\delta}W^{\alpha\beta}W^{\gamma\delta} - \frac{1}{90}R_{\mu\nu}W^{\mu\alpha}W^\nu{}_\alpha + \frac{1}{72}RW_{\alpha\beta}W^{\alpha\beta}. \tag{23}$$

The polynomial $P(R^3)$ consists of eight terms

$$P(R^3) = \frac{1}{7!}\Big[\ \frac{35}{9}R^3 - \frac{14}{3}RR_{\alpha\beta}R^{\alpha\beta} + \frac{14}{3}RR_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + \frac{64}{3}R_{\alpha\beta\gamma\delta}R^{\alpha\gamma}R^{\beta\delta} - \frac{208}{9}R_{\mu\nu}R^{\mu\alpha}R^{\nu}{}_{\alpha}$$

$$- \frac{16}{3}R_{\mu\nu}R^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma} + \frac{44}{9}R_{\mu\nu\rho\sigma}R^{\mu\nu\alpha\beta}R^{\rho\sigma}{}_{\alpha\beta} + \frac{80}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^{\nu}{}_{\alpha}{}^{\sigma}{}_{\beta}\Big]\ . \qquad (24)$$

Evaluating $P(R^3)$ on S^4 gives

$$P(R^3) = \frac{1}{r_0^{\,6}}\frac{74}{63}\,\mathbb{1}\ . \qquad (25)$$

For the charged scalar contribution we consider the operator in equation (4) and apply equation (13). The result is simply twice that of ref. [2] for a real scalar field. We have the finite contribution, ignoring the dimensional anomaly,

$$\Delta\Gamma = -\ \mathrm{Tr}\ \frac{\hbar}{16\pi^2}\int d^4x\ \sqrt{g}\ \Big\{ V''^2\Big[\frac{3}{4} - \frac{1}{2}\ln\frac{V''}{\mu^2}\Big] + \frac{12V''}{r_0^{\,2}}\Big[U'' - \frac{1}{6}\Big]\Big[1 - \ln\frac{V''}{\mu^2}\Big]$$

$$+ \frac{1}{r_0^{\,4}}\Big[244\ U'' - 72\ U''^2 - \frac{29}{15}\Big]\ln\frac{V''}{\mu^2} + \frac{1}{V''r_0^{\,6}}\Big[144\ U''^2 - 288\ U''^3 - \frac{116}{5}U'' + \frac{74}{63}\Big]\Big\}\ , \qquad (26)$$

We introduce next the B-coefficients in the expansion of the effective action in powers of the Riemann tensor

$$\Delta\Gamma = -\int d^4x\ \sqrt{g}\left[B_0 + \frac{B_1}{r_0^{\,2}} + \frac{B_2}{r_0^{\,4}} + \frac{B_3}{r_0^{\,6}}\right]\ . \qquad (27)$$

Hence for $U'' = \xi$ the $O(r_0^{\,6})$ contribution is

$$B_3 = \frac{\hbar}{16\pi^2 V''}\Big[144\ \xi^2 - 288\ \xi^3 - \frac{116}{5}\xi + \frac{74}{63}\Big]\ , \qquad (28)$$

Had we defined $\hat{F}(\nabla) = -\Box + X + m^2$ we would also find B_3 to be the above using the coefficient g_3.

Next we consider the vector contribution. As it was shown by Coleman and Weinberg [10], in renormalized scalar electrodynamics the vector field develops a dynamical mass proportional to the (arbitrary) vacuum expectation value of the scalar field. At this point, we can resort to the method of ref. [11], in order to evaluate the trace of the operator for the vector field which appears in nonminimal form. In fact, in the unitary gauge, with no ghosts, we may write the vector contribution to the effective action (3) as follows [11]

$$\frac{1}{2}\hbar\ \mathrm{Tr}\ \ln\hat{F}_A(\nabla) = \frac{1}{2}\hbar\ \mathrm{Tr}\ \ln\hat{F}_0(\nabla) - \frac{1}{2}\hbar\ \mathrm{Tr}\ \ln\hat{F}_\chi(\nabla)\ . \qquad (29)$$

where

$$\hat{F}_\chi(\nabla) = -\Box + e^2\phi_0^{\,2}\ , \qquad (30)$$

$$\hat{F}_0(\nabla) = -\Box g_{\lambda\mu} + R_{\lambda\mu} + e^2\phi_0^2 g_{\lambda\mu} \; . \tag{31}$$

and $\hat{F}_A$ is given in (5). Hence we have the sum of the trace of an operator acting on a vector and one acting on a scalar, which we choose to call a compensating scalar, and both are in elliptical form. Therefore we can use Gilkey's coefficients to evaluate the traces. The result will reduce to the well known flat limit, with the required factor of $\frac{3}{2}e^2\phi_0^2$, in the form given, for example, in eq. (2.1) of ref. [12]. This agreement checks part of our calculation. Notice that, in order to have agreement with ref. [12] as for the finite part of the effective action, it is necessary to include the dimensional anomaly, i.e. to consider the gravitational tensors in dimension d. This is because, when the regularization of the divergent integrals entering the radiative corrections is carried out by analytically continuing the spacetime coordinates to d dimension, it is natural to analytically continue all the geometrical quantities. In the appendix we calculate the finite part using the dimensional regularization of ref. [12] and check the agreement of our "dimensional anomaly" with the results of Larsen and Nielsen.

Using equation (13), but this time taking care with the vector indices, we obtain for the finite minimal spin-one contribution

$$\Delta\Gamma_A = -\,\mathrm{Tr}\,\frac{\hbar}{32\pi^2}\int d^4x\,\sqrt{g}\,\Big\{e^4\phi_0^4\Big[3 - 2\ln\frac{e^2\phi_0^2}{\mu^2}\Big] + \frac{4e^2\phi_0^2}{r_0^2}\Big[1 - \ln\frac{e^2\phi_0^2}{\mu^2}\Big]$$
$$+ \frac{4}{15r_0^4}\ln\frac{e^2\phi_0^2}{\mu^2} + \frac{4}{e^2\phi_0^2 r_0^6}\Big[\frac{74}{63} - \frac{4}{5} + \frac{1}{5}\Big]\Big\} \; , \tag{32}$$

where we split the $O(r_0^{-6})$ contribution to the effective action into three terms. The first term represents the contribution of the expression $P(R^3)$ in eq. (25). The second term is produced by the part of $a_3(x)$ that vanishes when $W_{\rho\sigma} = 0$. The third term comes from the power series expansion of the logarithm $\ln\dfrac{V''+12r_0^{-2}U''}{\mu^2}$. We must add to the effective action (28) the contribution due to the compensating scalar field

$$\Delta\Gamma_\chi = -\,\mathrm{Tr}\,\frac{\hbar}{32\pi^2}\int d^4x\,\sqrt{g}\,\Big\{e^4\phi_0^4\Big[-\frac{3}{4} + \frac{1}{2}\ln\frac{e^2\phi_0^2}{\mu^2}\Big] + \frac{2e^2\phi_0^2}{r_0^2}\Big[1 - \ln\frac{e^2\phi_0^2}{\mu^2}\Big]$$
$$+ \frac{29}{15r_0^4}\ln\frac{e^2\phi_0^2}{\mu^2} - \frac{74}{63}\frac{1}{e^2\phi_0^2 r_0^6}\Big\} \; . \tag{33}$$

The dimensional anomaly has not been included. The full contribution of the matter (scalar plus vector) fields to the effective action is therefore

$$\Delta\Gamma = \int d^4x\,\sqrt{g}\Big[-B_0 - \frac{B_1}{r_0^2} - \frac{B_2}{r_0^4} - \frac{B_3}{r_0^6}\Big] \; . \tag{34}$$

Here

$$B_0 = \frac{\hbar}{32\pi^2}\Big\{2\Big[m^4 + \frac{\lambda\phi_0{}^4}{4} + m^2\lambda\phi_0{}^2\Big]\Big[\tfrac{3}{4} - \tfrac{1}{2}\ln\frac{m^2 + \frac{1}{2}\lambda\phi_0{}^2}{\mu^2}\Big]$$
$$+ e^4\phi_0{}^4\Big[\tfrac{9}{4} - \tfrac{3}{2}\ln\frac{e^2\phi_0{}^2}{\mu^2}\Big]\Big\} , \qquad (35)$$

$$B_1 = \frac{\hbar}{32\pi^2}\Big\{24\Big[\xi - \tfrac{1}{6}\Big]\Big[m^2 + \tfrac{1}{2}\lambda\phi_0{}^2\Big]\Big[1 - \ln\frac{m^2 + \frac{1}{2}\lambda\phi_0{}^2}{\mu^2}\Big]$$
$$+ 6e^4\phi_0{}^4\Big[1 - \tfrac{3}{2}\ln\frac{e^2\phi_0{}^2}{\mu^2}\Big]\Big\} , \qquad (36)$$

$$B_2 = \frac{\hbar}{32\pi^2}\Big\{\Big[- 144\xi^2 + 48\xi - \tfrac{58}{15}\Big]\ln\frac{m^2 + \frac{1}{2}\lambda\phi_0{}^2}{\mu^2} + \frac{11}{5}\ln\frac{e^2\phi_0{}^2}{\mu^2}\Big\} , \qquad (37)$$

$$B_3 = \frac{\hbar}{32\pi^2}\Big\{\Big[- 288\xi^3 + 144\xi^2 - \tfrac{116}{5}\xi + \tfrac{74}{63}\Big]\frac{2}{m^2 + \frac{1}{2}\lambda\phi_0{}^2} + \frac{118}{105}\frac{1}{e^2\phi_0{}^2}\Big\} . \qquad (38)$$

From this we obtain the renormalization group equations

$$\mu\frac{d\xi}{d\mu} = \frac{\hbar}{32\pi^2}\Big\{2\lambda\Big[\xi - \tfrac{1}{6}\Big] + e^2\Big\} , \qquad (39)$$

$$\mu\frac{d\lambda}{d\mu} = \frac{\hbar}{32\pi^2}\Big[6\lambda + 36e^4\Big] , \qquad (40)$$

$$\mu\frac{dm^2}{d\mu} = \frac{\hbar}{32\pi^2}\Big[2\lambda m^2\Big] . \qquad (41)$$

It is argued in ref. [9] that.the $O(R^2)$ corrections in the effective action for the large-scale topology of our four-dimensional euclidean de Sitter space do not contribute to any quantity having physical significance. After dropping the $O(R^2)$ terms from (34), and using the renormalized values of the constants Λ and G, the effective action for S^4 reads

$$\Delta\Gamma = \int d^4x\,\sqrt{g}\left[\Lambda - \frac{R}{16\pi G} - \frac{B_3}{r_0{}^6}\right] , \qquad (42)$$

with B_3 given by eq. (38) above.

In order to calculate the probability of a set of wormhole parameters having values between α and α+dα, one introduces the weighting factor $Z(\alpha)$

$$Z(\alpha) = \exp\Big\{\sum_{\text{topologies}} \exp\big[-\Gamma_\alpha(g)\big]\Big\} , \qquad (43)$$

where each possible topology is characterized by the background metric $g_{\mu\nu}$. We require the radius of the four sphere that minimizes $\Gamma_\alpha(g_{\mu\nu})$. Following ref. [2], and assuming constant ϕ_0 integration over the four sphere gives (omitting α, for convenience)

$$\Gamma(g) = \frac{8\pi^2}{3}\left[\Lambda r_0^4 - \frac{3r_0^2}{4\pi G} - \frac{B_3}{r_0^2}\right] , \tag{44}$$

The extremized form of the effective action for the background gravitational field is then

$$\Gamma(g) = -\frac{3}{8\Lambda G^2} - \frac{64\pi^3}{9} G\Lambda B_3 , \tag{45}$$

The validity of this expression, obtained by integrating out the matter fields (i.e. the scalar and gauge vector degrees of freedom), holds to the one-loop order and neglecting $O(R^4)$ corrections.

3. GAUGE DEPENDENCE OF THE EFFECTIVE POTENTIAL

Coming back to the gauge-fixing choice we adopted in the previous section, we start with the lagrangian for a complex scalar field $\Phi = \phi \exp(i\eta)$, $\Phi^* = \phi \exp(-i\eta)$, coupled to a U(1) gauge field

$$L = \frac{1}{2}\nabla_\mu\Phi^*\nabla^\mu\Phi + \frac{1}{2}e^2A_\mu A^\mu\Phi^*\Phi + U(\Phi^*\Phi)R + V(\Phi^*\Phi)$$

$$- \frac{1}{2}ie^2A^\mu\Phi^*\nabla_\mu\Phi + \frac{1}{2}ie^2A^\mu\Phi\nabla_\mu\Phi^* + \frac{1}{2}(\nabla_\mu A_\nu - \nabla_\nu A_\mu)\nabla^\mu A^\nu . \tag{46}$$

Then, we carried out a gauge transformation to a unitary gauge

$$\Phi' = \Phi \exp(-i\eta) , \tag{47}$$

where the phase of the field Φ is discarded as being irrelevant to our purpose. The vector field is subject to the constraint $\nabla_\mu A^\mu = 0$. This allows us to rewrite the lagrangian as follows

$$L' = L = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi) + U(\phi)R - \frac{1}{2}A^\mu\Box A_\mu + \frac{1}{2}e^2A^\mu A_\mu\phi^2 + \frac{1}{2}A_\mu\nabla^\nu\nabla^\mu A_\nu . \tag{48}$$

Here the gauge invariance of the classical action has been used. Recalling the expression

$$[\nabla^\mu,\nabla^\nu]A_\mu = R^{\mu\nu}A_\mu , \tag{49}$$

we can immediately obtain the action (1) from eq. (48). As we have seen in section 2, in this gauge there is no ghost contribution.

The calculation of the one-loop effective action can be carried out using gauge-fixing prescriptions which differ from the one above. The resulting effective potential depends on the prescription chosen. The gauge dependence of the effective action was observed by Jackiw [13]. However, this dependence can be removed, at least within the context of the so called R_ξ-gauges [14]. This can be achieved by compensating a gauge transformation connecting any two given R_ξ-gauges by a corresponding change in the value of $<\phi>$. The possibility of studying the gauge dependence of the resulting effective action makes important to calculate the radiative corrections to scalar QED in the S^4 background using a different gauge choice.

We consider the action

$$S_E[\phi(x),A_\mu(x)] = \int d^4x\sqrt{g}\{-\tfrac{1}{2}\phi\Box\phi + U(\phi)R + V(\phi)$$

$$+\tfrac{1}{2}A^\lambda[-\Box\, g_{\lambda\mu} + R_{\lambda\mu} + e^2\phi^2 g_{\lambda\mu} + \nabla_\lambda\nabla_\mu]A^\mu\} \quad . \tag{50}$$

where $\phi_1 = \mathrm{Re}\,\phi$, $\phi_2 = \mathrm{Im}\,\phi$. This action is obtained by adding to the classical action (1) the gauge-fixing term. The constants ρ_1, ρ_2 have been introduced, following ref. [15], in order to get rid of the vector-scalar couplings in the Feynman rules.

In this gauge there are ghosts. Their lagrangian reads

$$L_{ghost} = -\, c^*[\nabla^\mu\nabla_\mu - e^2\Phi^*\Phi]c \quad . \tag{51}$$

Hence the effective action obtained expanding about the background values of the fields, reads

$$\Gamma[\Phi(x),A_\mu(x)] = S_E[\Phi_0(x)] + \tfrac{1}{2}\hbar\,\mathrm{Tr}\ln\hat{F}_A(\nabla) + \tfrac{1}{2}\hbar\,\mathrm{Tr}\ln\hat{F}_{ij}(\nabla)$$

$$-\,\hbar\,\mathrm{Tr}\ln\hat{F}_{ghost}(\nabla) \quad , \tag{52}$$

where

$$\hat{F}_{ghost}(\nabla) = -\Box + e^2\Phi_0^*\Phi_0 \quad . \tag{53}$$

Since the operator $\hat{F}_A$ is in the standard minimal form $\hat{F}_0$ given by eq. (31)

$$\hat{F}_A = \hat{F}_0(\nabla) = -\Box\, g_{\lambda\mu} + R_{\lambda\mu} + e^2\Phi_0^*\Phi_0 g_{\lambda\mu} \quad , \tag{54}$$

the vector field contribution to the effective action in this covariant gauge can be obtained using Gilkey's coefficients and directly read from eq. (32)

$$\Delta\Gamma_A = - \mathrm{Tr}\,\frac{\hbar}{32\pi^2}\int d^4x\,\sqrt{g}\,\Big\{e^4(\Phi_0{}^*\Phi_0)^2\Big[3 - 2\ln\frac{e^2\Phi_0{}^*\Phi_0}{\mu^2}\Big]$$

$$+\frac{4e^2\Phi_0{}^*\Phi_0}{r_0{}^2}\Big[1 - \ln\frac{e^2\Phi_0{}^*\Phi_0}{\mu^2}\Big] + \frac{4}{15r_0{}^4}\ln\frac{e^2\Phi_0{}^*\Phi_0}{\mu^2} + \frac{4}{e^2\Phi_0{}^*\Phi_0 r_0{}^6}\Big[\frac{74}{63} - \frac{4}{5} + \frac{1}{5}\Big]\Big\}, \tag{55}$$

To this we must add the ghost contribution

$$\Delta\Gamma_{ghost} = - \mathrm{Tr}\,\frac{\hbar}{16\pi^2}\int d^4x\,\sqrt{g}\,\Big\{e^4(\Phi_0{}^*\Phi_0)^2\Big[-\frac{3}{4} + \frac{1}{2}\ln\frac{e^2\Phi_0{}^*\Phi_0}{\mu^2}\Big]$$

$$+\frac{2e^2\Phi_0{}^*\Phi_0}{r_0{}^2}\Big[1 - \ln\frac{e^2\Phi_0{}^*\Phi_0}{\mu^2}\Big] + \frac{29}{15r_0{}^4}\ln\frac{e^2\Phi_0{}^*\Phi_0}{\mu^2} - \frac{74}{63}\frac{1}{e^2\Phi_0{}^*\Phi_0 r_0{}^6}\Big\} . \tag{56}$$

Adding the contributions to the divergent effective action coming from the gauge vector and the ghosts yields

$$\Delta\Gamma^{div} = - \mathrm{Tr}\,\frac{\hbar}{16\pi^2}\int d^4x\,\sqrt{g}\,\Big[\frac{1}{\varepsilon} - \frac{1}{2}\gamma + \frac{1}{2}\ln(4\pi)\Big]\Big[e^4(\Phi_0{}^*\Phi_0)^2$$

$$+\frac{8e^2\Phi_0{}^*\Phi_0}{r_0{}^2} - \frac{62}{15r_0{}^4}\Big] . \tag{57}$$

It can be shown that this agrees with eq. (7.19) of ref. [16], after taking into account an overall sign difference in our definition of the effective action with respect to ref. [16]. In fact, calculating in the same covariant gauge that we employ in the present section, Parker and Toms obtained, for the pole part of the $O(R^2)$ contribution to the effective action

$$\mathrm{P.P.}(\Gamma^{vector}) = \frac{\hbar}{16\pi^2}\int d^4x\,\sqrt{g}\,\Big[\frac{1}{\varepsilon} - \frac{1}{2}\gamma + \frac{1}{2}\ln(4\pi)\Big]\Big[e^4(\Phi_0{}^*\Phi_0)^2$$

$$+\frac{8e^2\Phi_0{}^*\Phi_0}{r_0{}^2} - \frac{62}{15r_0{}^4}\Big] . \tag{58}$$

In writing the result of Parker and Toms, we considered only one vector field and took into account the difference in the conventions. After specializing to a S^4 background, this coincides with the $O(r_0{}^{-4})$ term in our eq. (57).

However, our result (55) disagrees with eq. (16) of ref. [3], as far as the $O(r_0{}^{-6})$ term is concerned. The disagreement probably arises from the sum of the contribution coming from $P(R^3)$ and the term generated by that part a_3 of which goes to zero in the limit $W_{\rho\sigma}=0$. This gives us the result

$$- \mathrm{Tr} \frac{\hbar}{32\pi^2} \int d^4x \sqrt{g} \left\{ \frac{4}{e^2 \Phi_0{}^* \Phi_0 r_0{}^6} \left[\frac{74}{63} - \frac{4}{5}\right] \right\}$$

$$= - \mathrm{Tr} \frac{\hbar}{32\pi^2} \int d^4x \sqrt{g} \left\{ \frac{1}{e^2 \Phi_0{}^* \Phi_0 r_0{}^6} \left[\frac{472}{315}\right] \right\} , \qquad (59)$$

to be compared with the result by Gavilanes and Perez-Mercader

$$- \mathrm{Tr} \frac{\hbar}{32\pi^2} \int d^4x \sqrt{g} \left\{ \frac{1}{e^2 \Phi_0{}^* \Phi_0 r_0{}^6} \left[\frac{7394}{1575}\right] \right\} . \qquad (60)$$

The third term in the $O(r_0^{-6})$ contribution to our eq. (55) agrees with one of the two terms of the same order appearing in eq. (16) of ref. [3], which seemingly represents the contribution obtained expanding the logarithm $\ln \frac{V''+12r_0^{-2}U''}{\mu^2}$. Unfortunately, Gavilanes and Perez-Mercader do not further separate the different contributions, as we did in writing our result, hence a more detailed comparison is not possible.

Finally, we must add the scalar contribution with the results (55) and (56). Considering only the B_3 coefficient defined in (34), we obtain

$$B_3 = \frac{\hbar}{32\pi^2} \left\{ \left[- 288\xi^3 + 144\xi^2 - \frac{116}{5}\xi + \frac{74}{63} \right] \left[\frac{1}{m^2 + \frac{1}{2}\lambda\phi_0{}^2} + \frac{1}{m^2 + \frac{1}{6}\lambda\phi_0{}^2 + e^2\phi_0{}^2} \right] + \frac{16}{315} \frac{1}{e^2\phi_0{}^2} \right\} \qquad (61)$$

This should be compared with eq. (38) and shows explicitly how using different gauge-fixing prescriptions yields different results. This is in accordance with an observation made by Jackiw [13].

4. THE WORMHOLE CRITICAL SCALE AND PHASE TRANSITION

We now turn to the analysis of the consequences of our calculation on the low-energy wormhole physics. The value of the B_3 coefficient determines the one-loop correction to the radius of the four-sphere that minimizes the probability distribution of wormhole parameters

$$r_0{}^2{}_{max} = \frac{3}{8\pi\Lambda G} - \frac{32\pi^2}{9} G^2 \Lambda B_3 , \qquad (62)$$

The maximum of the function $Z(\alpha)$ is obtained for values of α that drive B_3 to its maximum ($B_3>0$) or minimum ($B_3<0$) value. For a pure scalar theory one finds a critical coupling ξ_C which is obtained from the real root of B_3, i.e. $\xi_C=0.10227$ [2].

The solution of the renormalization group equations for pure scalars is readily found

$$\frac{[\xi(\mu)-\frac{1}{6}]^3}{[\xi_0-\frac{1}{6}]^3}=\frac{1}{1-\frac{\hbar}{32\pi^2}12\lambda_0\ln\frac{\mu}{\mu_0}}$$

$$\frac{\lambda(\mu)}{\lambda_0}=\frac{1}{1-\frac{\hbar}{32\pi^2}12\lambda_0\ln\frac{\mu}{\mu_0}}$$

$$\frac{m^6(\mu)}{m_0^6}=\frac{1}{1-\frac{\hbar}{32\pi^2}12\lambda_0\ln\frac{\mu}{\mu_0}}, \qquad (63)$$

where we indicate by ξ_0, λ_0, m_0^2 the initial values at the Planck scale. Notice that ξ has an infrared-attractive fixed point at $\xi_{IR}=\frac{1}{6}$.

There is a scale-invariant quantity, namely the ratio $\frac{\xi(\mu)-\frac{1}{6}}{m^2(\mu)}$. This ratio has positive (negative) value for $\xi_0>\xi_{IR}$ ($\xi_0<\xi_{IR}$). If $\xi_0<\xi_C$, there appears a critical scale μ_C where the theory undergoes a phase transition from massless to massive scalar fields [2].

The situation of the spontaneously broken abelian gauge theory can be analyzed along similar lines. The critical parameters ξ_C, μ_C, defined by the vanishing of the coefficient B_3 in eq. (45), have values that depend from the mass of the gauge vector field $e^2\phi_0^2$. The calculation of the one-loop effective action carried out in section 3, choosing a covariant gauge-fixing prescription, adds a <u>negative</u> contribution, coming from the gauge field, to the ξ-dependent polynomial that comes from the charged scalar field. Hence, B_3 <u>cannot</u> be rendered always positive by adjusting the mass ratio of the scalar and gauge field. This is quite the opposite of what is found by Gavilanes and Perez-Mercader [3].

There is a need for the $O(R^4)$ contribution to the effective action, in order to calculate the effect of wormhole physics beyond the leading order.This procedure gives still the wormhole effect on the low-energy parameters, but for the case where r_0 is not so large. Then, our next task will be to find the $O(B_4)$ corrections to the radius of the four-spere that minimizes the effective action for the background gravitational fields.

5. HIGHER-ORDER WORMHOLE CORRECTIONS

In the previous sections we carried out the analysis of the quantum corrections to the field theoretic model in the S_4 background using a perturbative approach. The geometrical tensors have been expanded in powers of the geodesic distance and this procedure allowed us to calculate the one-loop effective action up to terms proportional to the third power of the Riemann tensor. Clearly, this procedure can be followed including the next order correction and this will improve the approximate validity of our result. In what follows, we will calculate the $O(R^4)$ contribution to the effective action of a self-interacting real scalar field. This is enough to give us a more accurate description of the impact of wormhole physics on the low-energy parameters of the field theory, in comparison with the results obtained above.

We start with the classical action for a pure scalar theory. This is obtained from eq. (1) by setting $A_\mu=0$. After renormalization, the finite effective action is given by eq. (13) to all perturbative orders in the Riemann tensor. What we need at this point is the value of the fourth Gilkey's coefficient g_4. Then, after integrating out the matter field ϕ, the expansion of the action for S_4, defined in terms of the B_4 coefficient, reads

$$\Delta\Gamma = -\int d^4x\,\sqrt{g}\left[B_0 + \frac{B_1}{r_0^{\,2}} + \frac{B_2}{r_0^{\,4}} + \frac{B_3}{r_0^{\,6}} + \frac{B_4}{r_0^{\,8}}\right] . \tag{64}$$

The value of the HaMiDew coefficient $g_4(x)$ is needed, in order to evaluate the $O(r_0^{-8})$ correction appearing in eq. (64). This coefficient has been calculated for a scalar field in a curved background spacetime in ref. [17]. All we have to do, in principle, is to specialize that result to the case of S_4. First, we want to check that the lower order result, i.e. the value of $g_3(x)$ provided in ref. [17], coincides with the coefficient calculated by Gilkey. This is quite a nontrivial check, since the method for computing the value of HaMiDew coefficients devised by the authors of ref. [17] is completely independent from Gilkey's method. In this way, we can be reassured about the correctness of the calculation of the higher order coefficient.

On a space of constant curvature, eq. (3.3) of ref. [17] gives

$$\begin{aligned} b_6 = \frac{1}{7!}\Big[&\left(-840\xi^3 + 420\xi^2 - 70\xi + \tfrac{35}{9}\right)R^3 + \left(28\xi - \tfrac{14}{3}\right)RR_{\alpha\beta}R^{\alpha\beta} \\ &+ \left(-28\xi + \tfrac{14}{3}\right)RR_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + \tfrac{8}{3}R_{\alpha\beta\gamma\delta}R^{\alpha\delta}R^{\beta\gamma} + \tfrac{8}{9}R_{\mu\nu}R^{\mu\alpha}R^{\nu}{}_{\alpha} \\ &+ \tfrac{56}{3}R_{\mu\nu}R^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma} + \tfrac{608}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\beta\sigma}R^{\nu}{}_{\alpha\beta}{}^{\rho} - \tfrac{256}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\beta\sigma}R^{\nu}{}_{\beta\alpha}{}^{\rho}\Big] . \end{aligned} \tag{65}$$

Specializing this expression to the case of the background space S^4, we get

$$b_6 = \frac{1}{r_0^{\,6}}\left(-288\xi^3 + 144\xi^2 - \frac{116}{5}\xi + \frac{74}{63}\right) . \tag{66}$$

This coincides with the result obtained evaluating, in the case of the four-dimensional sphere, the $O(R^3)$ coefficient calculated by Gilkey [7]. Next, we consider the higher-order coefficient computed by Amsterdamski et al..

We can write the $O(R^4)$ coefficient $b_8 = \frac{1}{9!}T_4$, where T_4 denotes all terms with four curvature tensors

$$T_4 = \left(15120\xi^4 - 10080\xi^3 + 2520\xi^2 - 280\xi + \frac{35}{3}\right)R^4$$

$$+ \left(1008\xi^2 - 336\xi + 28\right)R^2\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta}\right)$$

$$+ \left(2048\xi - \frac{1024}{3}\right)RR_{\mu\nu\rho\sigma}R^{\mu\alpha\beta\sigma}R^{\nu}{}_{\beta\alpha}{}^{\rho} + \left(-4864\xi + \frac{2432}{3}\right)RR_{\mu\nu\rho\sigma}R^{\mu\alpha\beta\sigma}R^{\nu}{}_{\alpha\beta}{}^{\rho}$$

$$+ \left(-1344\xi + 224\right)RR_{\mu\nu\rho\sigma}R_{\alpha}{}^{\nu\rho\sigma}R^{\mu\alpha} + \left(-192\xi + 32\right)RR_{\mu\nu\rho\sigma}R^{\mu\sigma}R^{\nu\rho}$$

$$+ \left(-64\xi + \frac{32}{3}\right)RR_{\mu\nu}R^{\mu\alpha}R^{\nu}{}_{\alpha} + \frac{12608}{5}R_{\mu\nu\rho\sigma}R^{\mu}{}_{\alpha\beta}{}^{\sigma}R^{\nu}{}_{\gamma\delta}{}^{\rho}R^{\alpha\gamma\delta\beta}$$

$$-\frac{4672}{5}R_{\mu\nu\rho\sigma}R^{\mu}{}_{\alpha\beta}{}^{\sigma}R^{\nu}{}_{\gamma\delta}{}^{\rho}R^{\alpha\delta\gamma\beta} + \frac{13632}{5}R_{\mu\nu\rho\sigma}R^{\mu}{}_{\alpha\beta}{}^{\sigma}R^{\nu}{}_{\gamma\delta}{}^{\alpha}R^{\rho\gamma\delta\beta}$$

$$-2944\,R_{\mu\nu\rho\sigma}R^{\mu}{}_{\alpha\beta}{}^{\sigma}R^{\nu}{}_{\gamma\delta}{}^{\beta}R^{\rho\gamma\delta\alpha} - \frac{6848}{5}R_{\mu\nu\rho\sigma}R^{\mu}{}_{\alpha\beta}{}^{\sigma}R^{\nu}{}_{\gamma\delta}{}^{\beta}R^{\rho\delta\gamma\alpha}$$

$$+\frac{96}{5}R_{\mu\nu\rho\sigma}R_{\alpha}{}^{\nu\rho\sigma}\left(-R^{\mu\beta\gamma\delta}R^{\alpha}{}_{\beta\gamma\delta} + 2\,R^{\mu\beta\gamma\alpha}R_{\beta\gamma}\right)$$

$$-1792\,R_{\mu\nu\rho\sigma}R^{\mu}{}_{\alpha\beta}{}^{\sigma}R^{\nu\gamma\alpha\rho}R_{\gamma}{}^{\beta} + 4352\,R_{\mu\nu\rho\sigma}R^{\mu}{}_{\alpha\beta}{}^{\sigma}R^{\nu\alpha\gamma\rho}R_{\gamma}{}^{\beta}$$

$$+\frac{8}{5}R_{\mu\nu\rho\sigma}R^{\mu}{}_{\alpha\beta}{}^{\sigma}\left(-173\,R^{\nu\alpha}R^{\rho\beta} + 447\,R^{\nu\beta}R^{\rho\alpha} + 28\,R^{\nu\rho}R^{\beta\alpha}\right)$$

$$+4\,R^{\rho\alpha}R_{\alpha}{}^{\sigma}\left[103\,R_{\rho\beta\mu\nu}R_{\sigma}{}^{\beta\mu\nu} + \frac{4}{5}\left(16\,R_{\mu\rho\sigma\nu}R^{\mu\nu} + R_{\sigma\nu}R_{\rho}{}^{\nu}\right)\right]$$

$$+\frac{28}{5}\left[R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\left(R_{\gamma\delta\rho\sigma}R^{\gamma\delta\rho\sigma} - 2\,R_{\rho\sigma}R^{\rho\sigma}\right) + R_{\mu\nu}R^{\mu\nu}R_{\rho\sigma}R^{\rho\sigma}\right] \quad . \tag{67}$$

Note that all terms T_i (i=1,2,3) of ref. [17], with less than four curvature tensors, vanish in a constant curvature space, such as the four-dimensional Euclidean de Sitter space. In writing eq. (67) we are correcting a misprint in the second contribution to the r.h.s. of eq. (3.4d) of ref. [17]. Specializing to the case of S^4, we obtain

$$b_8 = \frac{1}{r_0{}^8}\left(864\xi^4 - 576\xi^3 + \frac{696}{5}\xi^2 - \frac{296}{21}\xi + \frac{149}{315}\right) \quad . \tag{68}$$

This yields the $O(r_0{}^8)$ contribution to the effective action

$$B_4 = -\frac{\hbar}{32\pi^2 V''(\phi)}\left[864\xi^4 - 576\xi^3 + \frac{696}{5}\xi^2 - \frac{296}{21}\xi + \frac{149}{315}\right] \quad . \tag{69}$$

The fourth coefficient of the heat kernel asymptotic expansion has been computed also in ref. [4], using a manifestly covariant technique. The use of this technique and his results look very promising for use in the case where a gauge vector particle is present in the theory, as in this case there are no results obtained following Amsteramski et al.. Hence, we turned our attention to the HaMiDew coefficients calculated in ref. [4] and compared those *vis-a-vis* the results discussed above. In the process, we have detected several errors in the formulae of ref. [4] and hence believe that a more careful application of the covariant method is needed, in order to calculate the $O(R^4)$ correction in the case of scalar electrodynamics.

APPENDIX

The Dimensional Anomaly

In order to calculate the dimensional anomaly, we consider the contribution up to order r_0^4 given by equation (12). We retain the dimensional dependence of terms of the form R, R^2, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ contained in X, $a_1(x)$ and $a_2(x)$. In so doing we obtain the extra finite contribution

$$\Delta\Gamma^{\text{anomaly}} = -\,\mathrm{Tr}\,\frac{\hbar}{32\pi^2}\int d^4x\,\sqrt{g}\,\Big\{-e^4\phi^4 - \frac{1}{r_0^2}\Big[3\,e^2\phi^2 + 28\,V''\Big(U'' - \frac{1}{6}\Big)\Big] - \frac{1}{r_0^4}\Big[336\,U''^2 - 112\,U'' + \frac{493}{90}\Big]\Big\}\,. \qquad \text{(A.1)}$$

As was mentioned, the divergent contribution

$$\Delta\Gamma\left(\frac{1}{\varepsilon}\right) = \frac{\lambda\hbar}{32\pi^2}\int d^{2\omega}x\,\frac{\sqrt{g}}{\varepsilon}\,\mathrm{Tr}\left[a_0\frac{x^2}{2} - a_1 x + a_2\right] \qquad \text{(A.2)}$$

splits up into a finite part and a divergent part when dimensional regularization is used on the S_4 manifold. We denote these

$$\Delta\Gamma\left(\frac{1}{\varepsilon}\right) = \Delta\Gamma\left(\frac{1}{\varepsilon}\right)^{\text{DIV}} + \Delta\Gamma\left(\frac{1}{\varepsilon}\right)^{\text{FIN}}\,, \qquad \text{(A.3)}$$

where $\Delta\Gamma^{\text{FIN}}$ is what we term the dimensional anomaly.

On S_4 we have the following results in 2ω dimensions, where $\varepsilon = 2-\omega$:

$$R = \frac{2\omega}{r_0^2}\,(2\omega-1) = \frac{1}{r_0^2}\,[12 - 14\,\varepsilon + 0\,(\varepsilon^2)]\,, \qquad \text{(A.4)}$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{1}{r_0^4}\,[36 - 66\,\varepsilon + 0\,(\varepsilon^2)]\,, \qquad \text{(A.5)}$$

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{1}{r_0^4}\,[24 - 28\,\varepsilon + 0\,(\varepsilon^2)]\,. \qquad \text{(A.6)}$$

When $\varepsilon = 0$ the above reduce to the usual results for $d = 4$. Substitution of the above into equation (A.2) gives the anomaly for the complex scalar contribution , $\Delta\Gamma_\phi$ and the vector contribution, $\Delta\Gamma_A$.

For the complex scalar field we find

$$\Delta\Gamma_\phi\left(\frac{1}{\varepsilon}\right)^{DIV} = -\frac{\hbar}{16\pi^2}\int d^{2\omega}x\,\frac{\sqrt{g}}{\varepsilon}\Big[\frac{V''^2}{2} + \frac{12V''}{r_0^2}\left(U'' - \frac{1}{6}\right) + \frac{1}{r_0^4}\left(72U''^2 - 24U'' + \frac{29}{15}\right)\Big] , \quad (A.7)$$

$$\Delta\Gamma_\phi\left(\frac{1}{\varepsilon}\right)^{FIN} = -\frac{\hbar}{16\pi^2}\int d^4x\,\sqrt{g}\Big[-\frac{14V''}{r_0^2}\left(U'' - \frac{1}{6}\right) - \frac{1}{r_0^4}\left(168\,U''^2 - 56U'' + \frac{401}{90}\right)\Big] . \quad (A.8)$$

For the vector field we have

$$\Delta\Gamma_A\left(\frac{1}{\varepsilon}\right) = \Delta\Gamma_A\left(\frac{1}{\varepsilon}\right) + \Delta\Gamma_\eta\left(\frac{1}{\varepsilon}\right) , \quad (A.9)$$

$$\Delta\Gamma_A\left(\frac{1}{\varepsilon}\right)^{DIV} = -\frac{\hbar}{32\pi^2}\int d^{2\omega}x\,\frac{\sqrt{g}}{\varepsilon}\left[\frac{3}{2}e^4\phi_0^4 + 6\,\frac{e^2\phi_0^2}{r_0^2} - \frac{11}{5}\frac{1}{r_0^4}\right] , \quad (A.10)$$

$$\Delta\Gamma_A\left(\frac{1}{\varepsilon}\right)^{FIN} = -\frac{\hbar}{32\pi^2}\int d^4x\,\sqrt{g}\left[-e^4\phi_0^4 - 3\,\frac{e^2\phi_0^2}{r_0^2} + \frac{103}{30}\frac{1}{r_0^4}\right] . \quad (A.11)$$

Hence we obtain the full extra finite contribution to the effective action. This does not affect the wormhole contribution, nor does it affect the renormalization group equations. The resulting new flat limit becomes

$$\Delta\Gamma^{flat}_{full}(R \to 0) = -\int d^4x\,\sqrt{g}\,\bar{B}_0$$

$$= -\frac{\hbar}{32\pi^2}\int d^4x\,\sqrt{g}\left(m^4 + \frac{\lambda\phi_0^4}{4} + \lambda m^2\phi_0^2\right)\left(\frac{3}{2} - \ln\frac{m^2 + \frac{1}{2}\lambda\phi_0^2}{\mu^2}\right)$$

$$+ e^4\phi^4\left(\frac{5}{4} - \frac{3}{2}\ln\frac{e^2\phi_0^2}{\mu^2}\right) . \quad (A.12)$$

This result is obtained in the unitary gauge described in section 2.

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