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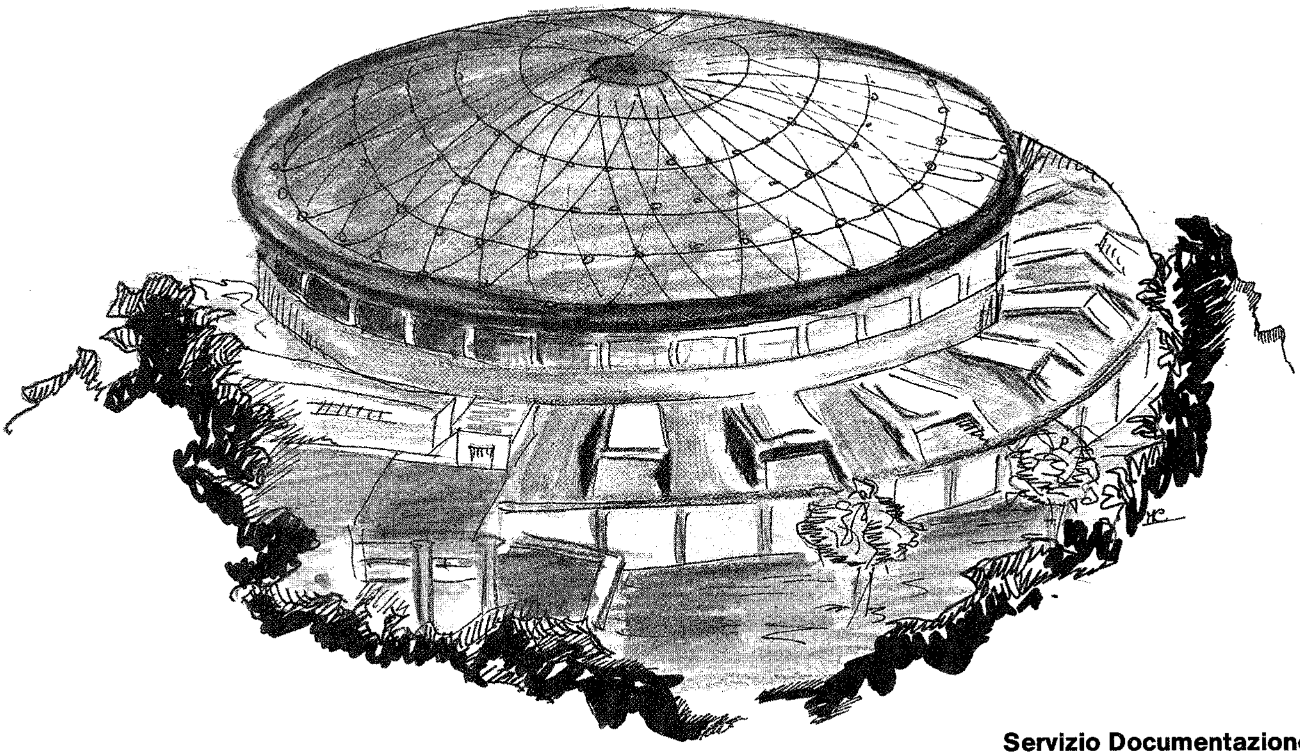
ON THE NONPERTURBATIVE SOLUTION OF d=1 SUPERSTRING



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On the Nonperturbative Solution of $d = 1$ Superstring

by

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Abstract

We construct nonperturbative solutions for the discretized Green-Schwarz superstring in one dimension. This is compared with a modification of the WKB series such that supersymmetry is preserved. The logarithmic divergences due to massless modes appear again, as in the bosonic string case. However, the density of states is different from the bosonic string theory, giving rise to a new series. We comment on the multicritical cases that could arise.

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1. - Introduction

In recent times it has become possible to study string theories in low dimensions, by discretizing the worldsheet and establishing a connection to matrix models [1]. Two-dimensional ($2d$) gravity has been obtained by carrying out random triangulations of arbitrary $2d$ surfaces [2]. The connection to random matrix theory led to summing the series to all orders, equivalently, to arbitrary genus surfaces in the critical limit [3]. This has also been achieved in the case of strings embedded in one dimension, which can be considered as one-dimensional matter coupled to gravity [4]. The reduction of this problem, through random matrix theory, to N fermions in one-dimensional quantum mechanics [5] can be exactly solved, in the critical limit, and analytic expressions can be given for the partition function and the correlation functions. The relationship to the continuum theory is exhibited by the presence of logarithmic divergences, which can be linked to the presence of massless modes [6]. The supersymmetric version has also been considered, through supersymmetric quantum mechanics, and some specific potentials have been analyzed [7]. In this letter, we take up supersymmetric quantum mechanics with arbitrary potentials and consider the critical limit, to all orders. When supersymmetry is unbroken, we give an exact expression for the density of states and the critical coupling, by adopting a modification of WKB method, suited to the supersymmetric case, called supersymmetric WKB (sWKB) [8]. We compare our results with the analytic continuation of the supersymmetric harmonic oscillator. We also consider multicritical points and compute the contribution of various terms in the sWKB series. Last, we present conclusions from our work *vis-à-vis* the results of other papers.

2. - The problem

Following Marinari and Parisi [7], we consider the mapping from a $2d$ surface to superspace. We discretize the $2d$ surface with polygons of the same size and associate to each polygon a point in superspace. The superstring theory is defined by integrating over all surfaces of arbitrary genus, imposing reparametrization invariance and supersymmetry, respectively, in parameter and target spaces. This can be considered to be a discrete version of the Green-Schwarz string. The arguments used for the bosonic string could be used here as well. In the large N limit (i.e. $N \to \infty$), we will reproduce the features of superstring theory in one dimension.

We start with a superfield matrix ϕ defined as

$$\Phi \equiv \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F . \tag{1}$$

Here ϕ, ψ, F are $N \times N$ hermitean matrices. The action, in superfield notation, reads

$$S = \int dt\, d\theta\, d\bar{\theta}\, Tr\left[- \Phi D^2 \Phi + W(\Phi) \right] , \tag{2}$$

where D is the covariant derivative and $W(\Phi)$ is the supersymmetric potential. Then, the partition function $Z(\beta)$ is given by

$$Z(\beta) = \int D\Phi\, e^{-\beta S} . \tag{3}$$

Integrating out the 'fermions' ψ, one obtains a hamiltonian which, in the sector with zero fermion number, can be written as

$$H = Tr\left(p^2 + F^2 - \frac{1}{\beta}\sigma_3 F' \right) , \tag{4}$$

where $p^2 = -\frac{1}{\beta^2}\frac{d^2}{dx^2}$. Here the trace is taken over $N \times N$ matrices. We have explicitly introduced β in the hamiltonian to keep track of $\frac{1}{N}$ terms. Also, we set $F = \frac{dW}{d\phi}$. The problem can be simplified by diagonalizing the matrix field and introducing an appropriate jacobian for the measure.

This procedure reduces the problem to that of N fermions at zero temperature. In the diagonal representation for σ_3 the hamiltonian reads

$$H = \begin{pmatrix} H_B & 0 \\ 0 & H_F \end{pmatrix} = \begin{pmatrix} p^2 + F^2 - \frac{1}{\beta}F' & 0 \\ 0 & p^2 + F^2 + \frac{1}{\beta}F' \end{pmatrix} . \tag{5}$$

N fermions fill all levels up to the Fermi level. In the large N limit, near criticality, we find that the levels become dense and the maximum of the potential just touches the Fermi level. In such a case, the problem can be analyzed by studying the singularity structure of the density of states, near the critical point. Following Gross and Miljković [5], we define the density of states

$$\rho(E) \equiv \frac{1}{\beta} \sum_n \delta(E_n - E) \tag{6a}$$

and the coupling constant

$$g = \frac{N}{\beta} = \int_0^{E_F} \rho(E)\, dE , \tag{6b}$$

where N is the number of fermions, which coincides with the order of the matrix.

The ground state energy is given by

$$E_{gs} = \sum_n E_n = \beta^2 \int_0^{E_F} E\, \rho(E)\, dE . \tag{7}$$

One defines $\mu = V_{max} - E_F$. As $\mu \to 0$, the density of states and the coupling constant become divergent, reflecting a critical behaviour. It is easiest to work with the derivatives of g and E_{gs}. We get [1]

$$\frac{\partial g}{\partial \mu} = - \rho \tag{8a}$$

and

$$\frac{\partial E_{gs}}{\partial \mu} = \beta^2 \mu \frac{\partial g}{\partial \mu} = - \beta^2 \mu \rho \ . \tag{8b}$$

When we consider the WKB approximation, which is good in the large N limit, we obtain to the zeroeth order

$$\rho = \frac{1}{\pi} \int \frac{dx}{\sqrt{(E-V)}} \ . \tag{9}$$

This expression becomes singular near the turning points which correspond to maxima of the potential, with the divergent part given by

$$\rho = - \frac{1}{2\pi} \log \mu \ . \tag{10}$$

While the same behaviour is obtained in the supersymmetric case, there are some differences, owing to the $\frac{df}{dx}$ term in the potential which is $O(\frac{1}{\beta}) \sim O(\frac{1}{N})$. Hence, to leading order in N, the maximum of the potential F^2 contributing to the singularity can be obtained from (9) by expanding $\sqrt{E - F^2 + \frac{1}{\beta}F'}$ and also taking into account the N dependance of the turning points carefully. To this order we obtain

$$E_{gs} \sim - \frac{N^2 (\Delta g)^2}{\log \, (\Delta g)} \ .$$

Once again, the logarithmic dependance on the renormalized cosmological constant appears.

In order to carry out the calculations to all orders and take the double scaling limit, we assume the zero of the energy scale to be at the maximum of the F^2 term, near the critical point. We impose that supersymmetry be preserved, when the energy is measured from this origin.[5] When we expand the potential near the maximum of F, using $F_{max} = 0$, we have

$$H = p^2 + a^2x^2\Big(1 + \sum_{n>0} c_n x^n\Big) - \frac{1}{\beta}\Big(a + \sum_{n>0} d_n x^n\Big), \tag{11}$$

where c_n and d_n are certain coefficients. Now, if we scale μ as $\frac{1}{\beta}$ and x^2 as $\frac{1}{\beta}$, the hamiltonian will scale as $\frac{1}{\beta} \sim \frac{1}{N}$, and the terms involving c_n and d_n in (11) will be suppressed by powers of $\frac{1}{N}$. In this scaling limit, our procedure leads to the potential of a supersymmetric harmonic oscillator, analytically continued to imaginary frequency, in analogy to the bosonic case.

The density of states can be obtained by analytic continuation from a supersymmetric harmonic oscillator, whose energy levels are given by $n\hbar$ rather than $(n + \frac{1}{2})\hbar$. This modification in our case gives

$$\rho(\mu) = \frac{1}{\pi} Re \sum_{n=0}^{\infty} \frac{1}{2n + i\beta\mu} . \tag{12}$$

Once again, one has to fix the divergent part, in order to agree with the $N \to \infty$ limit worked out in (10). This is achieved easily, by defining

$$\rho(\mu) = -\frac{1}{2\pi}\Big[Re\, \psi\Big(i\frac{1}{2}\beta\mu\Big) - log\, \Big(\frac{1}{2}\beta\Big)\Big], \tag{13}$$

where ψ is the digamma function

$$\psi(z) = \frac{d}{dz} log\, \Gamma(z) . \tag{14}$$

5 Normally, in a supersymmetric theory, one must fix the zero of the energy. Here we impose that supersymmetry is maintained when the zero of the energy coincides with the maximum of the potential.

We can express (13) in the form

$$\rho(\mu) = -\frac{1}{2\pi}log\ \mu + \frac{1}{\pi\beta\mu}Im\sum_{n=0}^{\infty}\frac{1}{1-i\frac{2n}{\beta\mu}}$$
$$= -\frac{1}{2\pi}log\ \mu - \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k}\left(\frac{2}{\beta\mu}\right)^{2k}, \qquad (15)$$

where B_{2k} are the Bernoulli numbers. Using the fact that

$$|B_2| = \frac{1}{6},\ |B_4| = \frac{1}{30},\ |B_6| = \frac{1}{42},$$

and so on, we can rewrite (15) as follows

$$\rho(\mu) = -\frac{1}{2\pi}\left(log\ \mu + \frac{1}{3\beta^2\mu^2} + \frac{2}{15\beta^4\mu^4} + \frac{16}{63\beta^6\mu^6} + ...\right). \qquad (16)$$

3. - The sWKB method

The same series can be obtained by using sWKB [8]. The potential in supersymmetric quantum mechanics has the form

$$V = f^2 - \hbar f'. \qquad (17)$$

The conventional WKB series is an expansion in powers of $\hbar$, through the assumption that $\hbar$ is small. We recall that in our case $\hbar$ corresponds to $\frac{1}{\beta}$ or $\frac{1}{N}$, where $N \to \infty$. In this sense, the second term in (17) is $O(\hbar)$ and can be ignored for the zeroeth order calculation. The charateristic difference in sWKB arises in two ways [8].

i) The integrand of sWKB is taken as f^2, instead of $f^2 - \hbar f'$ which is the full potential, including the contribution of fermionic loop integrals.

ii) The turning points are given by $E - F^2 = 0$, instead of $E - F^2 + \hbar F' = 0$.

The method of sWKB has numerous advantages over the conventional WKB series in dealing with quantum mechanical problems, particularly those which are in a supersymmetric form. In fact, sWKB explicitly maintains supersymmetry order by order, i.e. if ${E_n}^B$ and ${E_n}^F$ are the eigenvalues of the 'bosonic' and 'fermionic' hamiltonians $H_{B,F}$, then ${E_{n+1}}^B = {E_n}^F$ in every order. The ground state energy computed through a WKB approximation, normally, is a bad approximation, whereas in sWKB the ground state energy is automatically adjusted to be zero. Most interestingly, in all solvable potentials whose energy spectrum can be obtained by analytic methods, sWKB gives exact expressions already at first order, whereas one has to carry out the WKB series to infinite order. In short, sWKB is an effective resummation of the WKB series, convenient for dealing with supersymmetric problems. With this introduction to sWKB, let us see how the WKB and sWKB series look like, with respect to the treatment of supersymmetric quantum mechanical problems.

The conventional WKB series reads, up to $O(\hbar^6)$

$$\begin{aligned}
\sqrt{2m}\int_{x_1}^{x_2}\sqrt{E-V}\,dx \;-\; \frac{\hbar^2}{24\sqrt{2m}}\frac{d}{dE}\int_{x_1}^{x_2}\frac{V''}{\sqrt{E-V}}\,dx \\
+\; \frac{\hbar^4}{2880(2m)^{\frac{3}{2}}}\frac{d^3}{dE^3}\int_{x_1}^{x_2}\frac{7(V'')^2-5V'V'''}{\sqrt{E-V}}\,dx \\
-\; \frac{\hbar^6}{725760(2m)^{\frac{5}{2}}}\left[\frac{d^4}{dE^4}\int_{x_1}^{x_2}\frac{216(V''')^2}{\sqrt{E-V}}\,dx\right. \\
\left. +\; \frac{d^5}{dE^5}\int_{x_1}^{x_2}\frac{93(V'')^3-224V'V''V'''+35(V')^2V''''}{\sqrt{E-V}}\,dx\right] \\
= N\pi\hbar\ .
\end{aligned} \tag{18}$$

Here x_1 and x_2 are solutions of $E-V=0$. The sWKB series is given by [8]

$$
\begin{aligned}
&\sqrt{2m}\int_a^b \sqrt{E-f^2}\,dx \;-\; \frac{\hbar^2 E}{6\sqrt{2m}}\frac{d^2}{dE^2}\int_a^b \frac{(f')^2}{\sqrt{E-f^2}}\,dx \\
&+\; \frac{\hbar^4}{720(2m)^{\frac{3}{2}}}\left[\frac{d^2}{dE^2}\int_a^b \frac{30f'f'''}{\sqrt{E-f^2}}\,dx\right. \\
&+\; \left.\frac{d^3}{dE^3}\int_a^b \frac{-8(f')^4-31f(f')^2f''+7f^2(f'')^2-5f^2f'f'''}{\sqrt{E-f^2}}\,dx\right] \\
&+\; \frac{\hbar^6}{90720(2m)^{\frac{5}{2}}}\left[\frac{d^3}{dE^3}\int_a^b \frac{378(f''')^2}{\sqrt{E-f^2}}\,dx\right. \qquad (19)\\
&+\; \frac{d^4}{dE^4}\int_a^b \frac{[-2160ff'f''f'''+1674(f')^2(f'')^2-108f^2(f''')^2]}{\sqrt{E-f^2}}\,dx \\
&+\; \frac{d^5}{dE^5}\int_a^b \frac{1}{\sqrt{E-f^2}}[96(f')^6-1119f(f')^4f''+729f^2(f')^2(f'')^2 \\
&+\left.399f^2(f')^3f'''-93f^3(f'')^3+224f^3f'f''f'''-35f^3(f')^2f'''']\right] \\
&=\; N\pi\hbar\,.
\end{aligned}
$$

Here a and b are solutions of $E-f^2=0$. For our case, $f\equiv i\sqrt{2}x$ and $f'=i\sqrt{2}$.[6]

We recall that, in order to obtain the density of states, one must differentiate (19) with respect to E. It is easy to verify that the divergent part for $\rho(\mu)$ is given by

$$\rho(\mu) \;=\; -\frac{1}{2\pi}\log\mu-\frac{1}{6\pi(\beta\mu)^2}-\frac{1}{15\pi(\beta\mu)^4}-\frac{8}{63\pi(\beta\mu)^6}-\cdots \qquad (20)$$

which reproduces the first few terms of Eq. (16).

6 One could be puzzled by the appearance of i in the potential. Indeed, this is the price to be payed in order to maintain supersymmetry, when the zero of the energy is set at the maximum of the F^2 term in the potential. In the final result of the calculation, though, there are no imaginary factors, owing to the fact that the f terms appear in pairs.

We now proceed to compute Δg. Integrating Eq. (8a) using Eqs. (13), (14), one obtains the exact expression

$$\Delta g \;=\; \frac{1}{\pi\beta}\left[-Im\;log\;\Gamma\left(i\frac{1}{2}\beta\mu\right)+\frac{\beta\mu}{2}\;log\;\left(\frac{\beta\mu}{2}\right)\right]-\frac{1}{2\pi}\mu\;log\;\mu. \qquad (21)$$

We could also have integrated the asymptotic expansion (15), from which we would have obtained the asymptotic expansion of (21), up to terms that vanish in either the planar limit or the double scaling limit. The integral representation of (21) can be used to investigate the non-perturbative aspects of the corresponding string theory, including the ground state energy. All correlation functions are related to $< Tr\phi^{2k} >$ and can be obtained by differentiating ρ with respect to β [4]. For the case of potentials which have vanishing k-th derivative at the maximum one finds the critical behavior is the same as in bosonic string theory [4]. Δg scales as $\mu^{(2+k)/2k}$ and E_{gs} goes as $N^2(\Delta g)^{2+\gamma_{st}}$, where $\gamma_{st}=\frac{k-2}{k+2}$. For the sWKB analysis of this case, one must take $f=ix^{k/2}$. In this case, for the nonperturbative analysis one should take

$$\mu \;\sim\; x^k \;\sim\; \frac{1}{\beta^2}\partial_x^2. \qquad (22)$$

From the sWKB series we find that the nth order term behaves as

$$\rho^{(n)} \;\propto\; \frac{1}{\beta^n}\int\frac{(f')^n}{(E-f^2)^{\frac{1}{2}+n}}dx. \qquad (23)$$

Hence

$$\rho \;=\; \mu^{\frac{2-k}{2k}}\sum_{n\;even}\frac{C_n}{\beta^n}\,\mu^{-n(\frac{k+2}{2k})} \qquad (24)$$

and

$$\Delta g \;=\; \mu^{\frac{k+2}{2k}}\sum_{n\;even}\frac{D_n}{\beta^n}\,\mu^{-n(\frac{k+2}{2k})} \qquad (25)$$

$$E_{gs} = \frac{1}{g_{st}^2}\left[1 + \sum_{n=1}^{\infty} \frac{a_{2n}}{\beta^{2n}(\Delta g)^{2n}}\right]; \tag{26}$$

where

$$g_{st}^2 = \frac{1}{\beta^2(\Delta g)^{\frac{3k+2}{k+2}}}.$$

From (22) we see that $\beta(\Delta g) \sim O(1)$, so that all the terms in the sum (26) are of the same order.

4. Discussion

We have derived a non-perturbative solution for the Green-Schwarz string. This has been done following the same procedure adopted for the bosonic string, with the additional imposition of supersymmetry. The calculation using sWKB yields the same result as the calculation using the expansion of the resolvent. The same logarithmic divergences appear in this case; their origin will be reported elsewhere. Though the series obtained is divergent and not summable, one can give an analytic continuation. Though the multicritical cases were also mentioned, it is difficult to give a closed expression for them. The D=1 theory considered here is effectively equivalent to a theory in D=2 when gravity fluctuations are included [9]. The methods adopted here can be considered in D=0 as well, and it would be interesting to study the changes induced in the nonlinear equations obeyed by the susceptibility. The scaling behavior for this model is the same as that in the bosonic theory, but the density of states is different.

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