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ANALYTICAL CALCULATION OF THE IMPEDANCE OF A DISCONTINUITY

L. PALUMBO

Dipartimento di Energetica, Università di Roma,
Via A. Scarpa 14 – 00161 Rome, Italy
and
INFN, Laboratori Nazionali di Frascati dell'INFN,
Via E. Fermi 40 – Frascati, (Rome) Italy

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In this paper we describe a powerful analytical approach to the study of the electromagnetic interaction between a bunch of particles and the discontinuities of the vacuum chamber. The method applied to a single discontinuity leads to an exact analytical solution allowing a deeper understanding of the radiation process via the investigation of the impedance behavior at any frequency and the role of the relativistic factor γ.

1. INTRODUCTION

The electromagnetic interaction between a bunch of particles and an accelerating structure is successfully described, in the frequency domain, by the coupling impedance $Z(\omega)$, a quantity that allows for deriving the energy lost by the beam and the current thresholds set by the instability mechanism arising in the beam dynamics.

Energy losses and instability thresholds are strongly dependent on the length of the bunch in so far as the bunch can coherently excite electromagnetic fields only at frequencies within its own spectrum. Thus, although the impedance is strictly related to the Green function of the given geometry, being the response to a point-charge excitation, in practical cases only a limited frequency range in the low-frequency region is of real interest.

However, the recent growing interest in accelerators operating with extremely short bunches is demanding the investigation of the high-frequency impedance behavior.

In this paper we describe an analytical approach, so far applied to diffraction of plane within a discontinuous waveguide,[1,2,3] which has been shown to be very powerful in this context, providing a deeper understanding of the radiation process occurring in a discontinuous vacuum chamber.[4,5,6]

In the following sections we describe the main steps of the method that leads to a pair of general integral equations for a set of problems: single, double, and multiple discontinuities. Eventually the single-discontinuity solution, found by means of the Wiener-Hopf techniques, is presented and discussed in detail. It has the appealing feature of being an exact solution, and therefore it permits investigation of the impedance behavior at any frequency of the spectrum and the role of any parameter, in particular that of the relativistic factor γ.

2 GENERALITIES OF THE METHOD

2.1. The Integral Equations

Let us consider a charge q uniformly distributed within a δ-disk of radius a moving with constant velocity $v = \beta c$ on the axis of a coaxial complex formed by perfectly conducting cylindrical surfaces, as shown in Fig. 1. The radius of any inner discontinuous pipe is denoted by b, whereas d is the radius of the external pipe. The inner pipes exist over a domain D_z of the z-axis, where $\underset{\sim}{D}_z$ is the complementary domain where the pipe is missing.

The moving charge, during its flight, induces currents on the surrounding perfectly conducting walls. All the fields within the structure may be considered as being generated by the moving charge and by the currents induced on the walls at $r = b$. In the frequency domain, with $e^{-i\omega t}$ dependence, these source currents are referred to as $J_q(z)$ and $J_i(z)$, respectively.

The induced currents flowing on the inner surfaces are interrupted in the region $\underset{\sim}{D}_z$; therefore, we may impose the following condition:

$$J_i(z) = 0, \qquad z \in \underset{\sim}{D}_z \tag{1}$$

Resorting to the linearity of the problem, we consider all the fields as the sum of two terms produced by the corresponding charge current $J_q(z)$ and induced currents $J_i(z)$. These fields must satisfy the following boundary conditions:

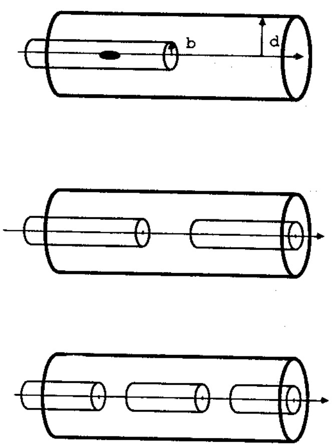

FIGURE 1 Discontinuities of an infinite circular pipe.

On the perfectly conducting surfaces there is no tangential component of the electric field. On the pipes at $r = b$ this condition implies that

$$[E_{zi}(r, z) + E_{zq}(r, z)]_{r=b} = 0, \qquad z \in D_z, \tag{2}$$

where E_{zi} and E_{zq} are produced by J_{zi} and J_{zq}, respectively.

Obviously the same condition has to be fulfilled also on the walls of the outer pipe:

$$[E_{zi}(r, z) + E_{zq}(r, z)]_{r=d} = 0, \qquad z \in]-\infty, +\infty[. \tag{3}$$

It is worth nothing that the above condition [Eq. (3)], unlike Eq. (2), is imposed on the entire z-axis.

Equations, (2) and (3) apply at any regular point of the conducting surfaces. However, a further condition must be fulfilled by the fields because of the "singular" surface points at the edges of the discontinuities: the edge condition. It requires that the electromagnetic energy stored in any finite volume near the edges stays finite.[7] In terms of induced currents this condition requires that $J_i(z)$ vanishes as the square root of the distance from the edge:

$$\lim_{z \to 0} J_i(z) \sim z^{1/2}. \tag{4}$$

We look for the expression of the induced currents $J_i(z)$ that is the solution of the Maxwell equations and fulfills the boundary conditions [Eqs. (1–4)]. For this purpose we make use of the Fourier integral representation of the currents and fields and write

$$J_i(z) = \int_{-\infty}^{\infty} F(\alpha) \ell^{i\alpha z}\, d\alpha, \tag{5}$$

where $F(\alpha)$ is the spectrum of the induced currents in the wavenumber domain α:

$$F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J_i(z) \ell^{-i\alpha z}\, dz. \tag{6}$$

Assume now that we are able to find, in the α-domain, the Green operator $G(r, \alpha)$ that will produce the induced electric field $E_{zi}(r, \alpha) = G(r, \alpha)F(\alpha)$ satisfying Eq. (3); then the boundary conditions [Eqs. (1) and (2)] may be rewritten as

$$\int_{-\infty}^{\infty} F(\alpha) \ell^{i\alpha z}\, d\alpha = 0; \qquad z \in \underline{D}z \tag{7}$$

$$\int_{-\infty}^{\infty} F(\alpha) G(b, \alpha) \ell^{i\alpha z}\, d\alpha = -E_{zq}(b, z); z \in D_z. \tag{8}$$

Equations (7) and (8) form a pair of integral equations for the unknown function $F(\alpha)$. They lead to a unique solution by resorting to the edge condition, which, in the α-domain, becomes

$$\lim_{|\alpha| \to \infty} F(\alpha) \sim \alpha^{-3/2}. \tag{9}$$

2.2. *Longitudinal Coupling Impedance*

The longitudinal Coupling Impedance of an infinite structure is defined by,[8]

$$Z(\omega) = -\frac{2\pi}{q}\int_{-\infty}^{\infty} E_z(0, z)\ell^{-i\alpha_0 z}\,dz, \tag{10}$$

where $\alpha_0 = k/\beta$, $k = \omega/c$, and $E_z(z)$ is the longitudinal electric field on the axis of the structure, synchronous with the charge q.

Using the integral representation of the electric field

$$E_z(r, z) = \int_{-\infty}^{\infty} F(\alpha)G(r, \alpha)\ell^{i\alpha z}\,d\alpha + E_{zq}(r, z), \tag{11}$$

we get

$$Z(\omega) = -\frac{2\pi}{q}\int_{-\infty}^{\infty} E_{zq}(0, z)\ell^{-i\alpha_0 z}\,dz - \frac{2\pi}{q}\int_{-\infty}^{\infty} F(\alpha)G(0, \alpha)\,d\alpha \int_{-\infty}^{\infty} \ell^{iz(\alpha-\alpha_0)}\,dz, \tag{12}$$

where we recognize the impulsive function $\delta(\alpha - \alpha_0)$ and write

$$Z(\omega) = -\frac{2\pi}{q}\left[\int_{-\infty}^{\infty} E_{zq}(0, z)\ell^{-i\alpha_0 z}\,dz + 2\pi\int_{-\infty}^{\infty} F(\alpha)G(0, \alpha)\delta(\alpha - \alpha_0)\,d\alpha\right]. \tag{13}$$

The above relation reveals a general characteristic of the radiation process. Each frequency component of the current $J_q(\omega)$ excites fields with a wide spectrum in the α domain; however, only those space harmonics with phase velocity synchronous with the charge may exchange energy with it.

2.3. *The Green Operator $G(r, \alpha)$*

In this section we will find the explicit expression of the Green operator $G(r, \alpha)$ required in the integral equations [Eqs. (7) and (8)].

The current density relative to the charge $q = 1$, uniformly distributed over a δ-disk of radius a and moving on the z-axis at constant velocity βc is

$$\bar{J}_q(z) = \begin{cases} 0, & r > a \\ \dfrac{\hat{z}}{2\pi^2 a^2}\,\ell^{i\alpha_0 z}, & r \leq a. \end{cases} \tag{14}$$

Owing to the cylinrdical symmetry of the geometry, the current density induced on the walls of the pipes located at $r = b$ may formally be expressed as

$$\bar{J}_i(z) = \hat{z}\,\frac{\delta(r-b)}{2\pi b} f(z), \qquad z \in D_z, \tag{15}$$

where $f(z)$ is an unknown function.

We will find the fields via the Hertz potential $\Pi_z(r, z)$ related to the current sources by the wave equation:

$$\nabla^2 \Pi_z(r, z) + \kappa^2 \Pi_z(r, z) = -\frac{iZ_0}{\kappa}[J_q(z) + J_i(z)], \tag{16}$$

where Z_0 is the vacuum impedance ($120\,\pi\Omega$).

Because of the symmetry, the sole longitudinal component of the potential $\Pi_z(r, z)$ is sufficient to derive all the field components by means of the following differential equations:

$$E_z(r, z) = \left(\frac{\partial^2}{\partial z^2} + \kappa^2\right)\Pi_z(r, z), \tag{17a}$$

$$E_r(r, z) = \frac{\partial^2 \Pi_z(r, z)}{\partial z\, \partial z}, \tag{17b}$$

$$H_\varphi(r, z) = i\kappa \frac{\partial \Pi_z(r, z)}{\partial r}. \tag{17c}$$

These are the only field components that may be excited by the on-axis current $J_q(z)$; they correspond to those propagating in a TM mode.

As already emphasized, we may apply the superposition principle and write the potential $\Pi_z(r, z)$ as the sum of two terms, each corresponding to a current source of Eq. (16):

$$\Pi_z(r, z) = \Pi_q(r, z) + \Pi_i(r, z). \tag{18}$$

Thus we solve the wave equation [Eq. (16)] for each current term. The potential $\Pi_q(r, z)$, derived in Appendix A, is

$$\Pi_q(r, z) = \frac{iZ_0 e^{i\alpha_0 z}}{2\pi\kappa I_0(\xi d)}[g_I(r)K_0(\xi r)I_0(\xi d) - g_I(a)K_0(\xi d)I_0(\xi r) + g_k(r)I_0(\xi d)I_0(\xi r)], \tag{19}$$

where $\xi = |k/\beta\gamma|$, $\gamma = (1-\beta^2)^{-1/2}$, $K_0(x)$ and $I_0(x)$ are the *I*- and *II*-type modified Bessel functions, and

$$g_I(r) = \frac{1}{\pi a^2}\int_0^r I_0(\xi r_0) r_0\, dr_0 = \frac{r I_1(\xi r)}{\pi a^2 \xi}, \tag{20}$$

$$g_k(r) = \frac{1}{\pi a^2}\int_r^a K_0(\xi r) r_0\, dr_0 = \frac{aK_1(\xi a) - rK_1(\xi r)}{\pi a^2 \xi}, \tag{21}$$

where $0 \leq r \leq a$.

The potential $\Pi_i(r, z)$, also derived in Appendix A, is expressed by the Fourier transform:

$$\Pi_i(r, z) = \int_{-\infty}^{\infty} \Pi_i(r, \alpha) e^{i\alpha z}\, d\alpha, \tag{22}$$

$$\Pi_i(r, \alpha) = -\frac{Z_0 F(\alpha)}{4k J_0(\Omega)} \begin{cases} J_0(\Omega r)[J_0(\Omega d)H_0(\Omega b) - J_0(\Omega b)H_0(\Omega d)] & 0 < r < b \quad (23a) \\ J_0(\Omega b)[J_0(\Omega d)H_0(\Omega r) - J_0(\Omega r)H_0(\Omega d)] & b < r < d, \quad (23b) \end{cases}$$

$\Omega = \sqrt{k^2 - \alpha^2}$ being the radial wavenumber.

Both potentials have been built up in such a way as to fulfill Eq. (3); in fact they vanish at $r = d$, as does the electric field E_z according to Eq. (17a):

$$E_{zq}(r, z) = -\xi^2 \Pi_q(r, z) \tag{24}$$

$$E_{zi}(r, z) = -\int_{-\infty}^{\infty} \Omega^2 \Pi_i(r, \alpha) \ell^{i\alpha z}\, d\alpha. \tag{25}$$

By comparing Eqs. (25) with Eq. (8) and making use of Eqs. (23) we may easily get the expression of the Green operator $G(r, \alpha)$:

$$G(r, \alpha) = \frac{Z_0 \Omega^2}{4\kappa J_0(\Omega d)} \begin{cases} J_0(\Omega r)[J_0(\Omega d)H_0(\Omega b) - J_0(\Omega b)H_0(\Omega d)], & 0 < r < b \quad (26a) \\ J_0(\Omega b)[J_0(\Omega d)H_0(\Omega r) - J_0(\Omega r)H_0(\Omega d)], & b < r < d. \quad (26b) \end{cases}$$

3. THE SINGLE DISCONTINUITY

The single-discontinuity problem has the peculiar feature of being asymmetric with respect to the velocity direction. Thus we will distinguish between the cases of charges moving toward the left or right.

The information about the particle velocity is in the term $e^{\pm i\alpha_0 \beta}$ of the potential $\Pi_q(b, z)$. For now we will assume the positive sign (charge moving toward the right).

The integral equation [Eqs. (7) and (8)], by use of Eqs. (23)–(26), may be written as

$$\int_{-\infty}^{\infty} F(\alpha) G(b, \alpha) \ell^{i\alpha z}\, d\alpha = -\xi^2 \Pi_q(b, z), \quad z < 0 \tag{27a}$$

$$\int_{-\infty}^{\infty} F(\alpha) \ell^{i\alpha z}\, d\alpha = 0, \quad z > 0 \tag{27b}$$

The above equations constitute a set of dual integral equations that can be solved for the unknown function $F(\alpha)$ using a procedure known as the Wiener–Hopf technique.[1,9]

3.1. The Wiener–Hopf Technique

The method looks for the function $F(\alpha)$ in the complex α-plane and exploits general properties of analytical functions of complex variables.

As a first step we transform the integration path of Eqs. (27) by adding infinite semicircles, the contributions of which are to be vanishingly small.

In the integral (27a), z is negative. Then by applying the Jordan's lemma, because of the exponential term $e^{i\alpha z}$, we can close the path of integration with a large semicircle below the real axis without changing the value of the integral. Since, however, the right side of Eq. (27a) is different from zero, we must require that $F(\alpha)G(b, \alpha)$ have an appropriate pole singularity, the residuum of which reproduces the quantity $-\xi^2\Pi_q(b, z)$.

In the integral [Eq. (27b)], z is positive; consequently, we can close the path of integration in the upper half-plane without affecting the value of the integral, provided that $F(\alpha)$ has no singularities in the lower half-plane.

Consider now the contour shown in Fig. 2. A simple consideration will reveal that the term $-\xi^2\Pi_q(b, z)$ can be represented by a contour integral as follows;

$$-\xi^2\Pi_q(b, z) = \frac{\xi^2\Pi_q(b, z)}{2\pi i}\int_C \frac{G^-(\alpha)\ell^{i(\alpha-\alpha_0)z}}{G^-(\alpha_0)(\alpha-\alpha_0)}\,d\alpha, \tag{28}$$

where we have introduced some function $G^-(\alpha)$ that is regular in the lower half-plane so that the integral over C, evaluated by applying the Couchy theorems, yields $-2\pi i$.

Substituting Eq. (28) into Eq. (27a), with the understanding that the path of integration from $-\infty$ to $+\infty$ is transformed into the contour C, we conclude that

$$F(\alpha)G(b, \alpha) = \xi^2\frac{\Pi_q(b, z)}{2\pi i}\frac{G^-(\alpha)\ell^{-i\alpha_0 z}}{G^-(\alpha_0)(\alpha-\alpha_0)}, \tag{29}$$

where the function $G^-(\alpha)$ still remains to be found. To this end we may apply the factorization technique and express the function $G(b, \alpha)$ as the product of two terms that are holomorphic in the upper and lower half-planes, respectively, and choose $G^-(\alpha)$ to be one of them; we get, in Appendix B,

$$\omega b G(b, \alpha) = G^+(\alpha)G^-(\alpha), \tag{30}$$

with the asymptotic behavior

$$\lim_{|\alpha|\to\infty} G^{\pm}(\alpha) \sim \alpha^{1/2}. \tag{31}$$

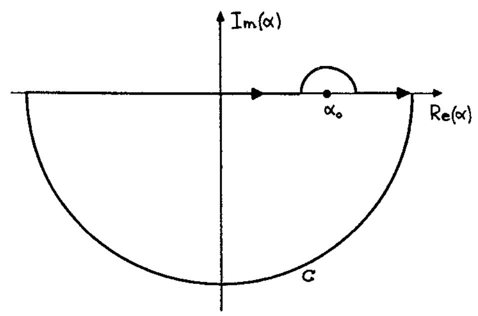

FIGURE 2 Integration path C in the complex α-plane.

Thus by inserting Eq. (30) into Eq. (29) and rearranging the terms we get

$$F(\alpha)G^{+}(\alpha)(\alpha-\alpha_0)=\frac{\omega b\xi^2\Pi_q(b,z)\ell^{i\alpha_0 z}}{2\pi i G^{-}(\alpha_0)}\equiv M(\alpha_0), \tag{33}$$

where the rhs is a constant, with respect to the variable α, that we will call $M(\alpha_0)$. From the above equation we derive the solution

$$F(\alpha)=\frac{M(\alpha_0)}{(\alpha-\alpha_0)G^{+}(\alpha)}, \tag{34}$$

which fulfils also the integral equations [Eq. (27)].

It is worth noting that the function $G^{-}(\alpha)$ of Eq. (28) has been chosen quite arbitrarily; in fact, a new solution can be found by replacing $G^{-}(\alpha)$ by $G^{-}(\alpha)\cdot X(\alpha)$, where $X(\alpha)$ is a function analytical on the whole complex α-plane. However, it is easy to show that by resorting to the edge condition, Eq. (34) turns out to be the unique solution to our problem. In fact, by introducing the regular function $X(\alpha)$, we obtain a new solution:

$$F'(\alpha)=\frac{F(\alpha)X(\alpha)}{X(\alpha_0)}. \tag{35}$$

For the edge condition to be satisfied, we require that this solution behave asymptotically as $\alpha^{-3/2}$. Since $G^{+}(\alpha)\sim\alpha^{1/2}$ at infinity, we conclude that $X(\alpha)$ is asymptotically constant. But being an analytical function, because of the Liouville theorem $X(\alpha)$ is constant everywhere; therefore, $X(\alpha)/X(\alpha_0)=1$, and $F'(\alpha)=F(\alpha)$.

3.2. Impedance of the Single Discontinuity

We now have all the elements we need to find the impedance according to Eq. (12) or, equivalently, Eq. (13). Since the function $F(\alpha)$ is singular at $\alpha=\alpha_0$ Eq. (12) is more suitable for our purpose. We first express the integrand function so as to isolate the singular point as follows:

$$\frac{G(0,\alpha)}{(\alpha-\alpha_0)G^{+}(\alpha)}=\frac{1}{(\alpha-\alpha_0)}\left[\frac{G(0,\alpha)}{G^{+}(\alpha)}-\frac{G(0,\alpha_0)}{G^{+}(\alpha_0)}\right]+\frac{G(0,\alpha_0)}{(\alpha-\alpha_0)G^{+}(\alpha_0)}. \tag{36}$$

Then we use the above expression in Eq. (12), splitting the impedance into two contributions,

$$Z(\omega)=Z_1(\omega)+Z_2(\omega), \tag{37}$$

that, after some manipulations, are given by

$$Z_1(\omega)=-2\pi\int_{-\infty}^{\infty}G(0,\alpha_0)\left[\frac{E_{zq}(0,z)}{G(0,\alpha_0)}-\frac{E_{zq}(b,z)}{G(b,\alpha_0)}\right]\ell^{-i\alpha_0 z}\,dz \tag{38}$$

$$Z_2(\omega)=-4\pi^2M(\alpha_0)\int_{-\infty}^{\infty}\left[\frac{G(0,\alpha)}{G^{+}(\alpha)}-\frac{G(0,\alpha_0)}{G^{+}(\alpha_0)}\right]\frac{\delta(\alpha-\alpha_0)}{(\alpha-\alpha_0)}\,d\alpha. \tag{39}$$

The term $Z_1(\omega)$ represents the usual impedance due to space charge and induced charges;[8] it diverges for an infinite tube so that it is commonly given (per unit length) as

$$\frac{dZ_1(\omega)}{dz}=\frac{iZ_0k}{(\beta\gamma)^2}\begin{cases} g_k(a)-g_I(a)\dfrac{K_0(\xi b)}{I_0(\xi b)}, & z<0 \\ g_k(a)-g_I(a)\dfrac{K_0(\xi d)}{I_0(\xi d)}, & z>0 \end{cases} \tag{40}$$

This term, as it is well known, is purely reactive and vanishes as $1/\gamma^2$ for ultrarelativistic particles.

The term $Z_2(\omega)$ is the real objective of our work since it is concerned with the radiation process due to the presence of the discontinuity in the pipe. From Eq. (39) we get:

$$Z_2(\omega)=4\pi^2M(\alpha_0)\left[\frac{\partial}{\partial\alpha}\frac{G(0,\alpha)}{G^+(\alpha)}\right]_{\alpha=\alpha_0} \tag{41}$$

The expression of the derivative is found in Appendix B; we get:

$$Z_2(\omega)=\frac{Z_0k}{(\beta\gamma)^2}g_I(a)\left[\frac{K_0(\xi b)}{I_0(\xi b)}-\frac{K_0(\xi d)}{I_0(\xi d)}\right]\left[\frac{(1+\beta)(\beta\gamma)^2}{\beta k}-\gamma b\frac{I_1(\xi b)}{i_0(\xi b)}-\sum\right], \tag{42}$$

where

$$\sum=\sum_1^{\alpha}n\left[\frac{1}{\alpha_n^b-\alpha_0}+\frac{1}{\alpha_n^c-\alpha_0}-\frac{1}{\alpha_n^d-\alpha_0}\right], \tag{43}$$

α_n^b, α_n^d, and α_n^c being the zeros of the Bessel functions $J_0(\Omega b)$, $J_0(\Omega d)$ and the cross product $[J_0(\Omega b)Y_0(\Omega d)-J_0(\Omega d)Y_0(\Omega b)]$, respectively.

The analysis of Eq. (42) shows that the real part of $Z_2(\omega)$, in the limit $\xi b\ll 1$ and $\xi d\ll 1$, is γ-independent, whereas the imaginary part vanishes as $k/(\beta\gamma)^2$:

$$R_e[Z_2(\omega)]\sim\frac{Z_0}{\pi}\ln\left(\frac{d}{b}\right) \tag{44}$$

$$I_m[Z_2(\omega)]\sim 0\left[\frac{k}{(\beta\gamma)^2}\right]. \tag{45}$$

In the high-frequency region, where $k\gg\beta b/\gamma$, Re (Z_2) drops exponentially. The real part of the impedance is plotted in Fig. 3 for several energies, whereas in Fig. 4 the same quantity is represented for values of the ratios d/b.

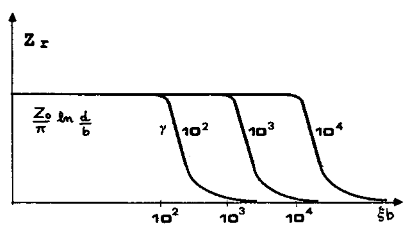

FIGURE 3 Real part of the impedance for several γ values.

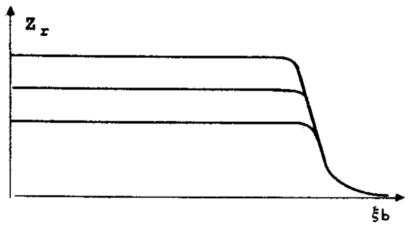

FIGURE 4 The real part of the impedance for various values of (d/b).

The impedance Z_2 for the case of a charge moving in the opposite direction is given by the same equation [Eq. (43)] by replacing β with $-\beta$:

$$Z_2^{\text{left}}(\omega) = Z_2^{\text{right}}(\omega, -\beta). \tag{46}$$

The real part of the impedance for this case is shown in Fig. 5.

3.3. *The Loss Factor*

The energy lost by a bunch of particles passing through our structure[8] can be calculated by

$$U = \frac{1}{\pi}\int_0^\infty Zr(\omega)\,|I(\omega)|^2\,d\omega, \tag{47}$$

where $Z_r(\omega)$ is the real part of the longitudinal impedance and $I(\omega)$ is the bunch spectrum. The energy lost is usually determined by the specific loss factor $k_l(\sigma)$ defined by

$$k_l(\sigma) = U/q^2. \tag{48}$$

In the case of Gaussian bunches, the loss factor and the real part of the impedance are related by

$$k_l(\sigma) = \frac{1}{\pi}\int_0^\infty Z_r(\omega)\ell^{-(\omega\sigma/\beta c)^2}\,d\omega, \tag{49}$$

where σ is the rms bunch length and βc the particle velocity.

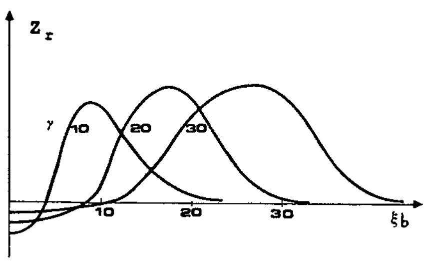

FIGURE 5 The real part of the impedance for a charge moving toward the left.

An approximate analytical evaluation of the loss factor can be obtained by inserting Eq. (44) into Eq. (49) and limiting the integration range to the frequency $\omega = \beta\gamma c/b$, beyond which $Z_r(\omega)$ vanishes exponentially. We obtain the following law for a bunch moving toward the wider region of radius d:

$$k_l(\sigma) \simeq \frac{\beta e Z_0}{2\pi^{3/2}\sigma} \ln\left(\frac{d}{b}\right) \Phi\left(\frac{\gamma\sigma}{b}\right), \tag{50}$$

where $\Phi(x)$ is the Fresnel integral.[10]

In all practical cases $\gamma\sigma/b \gg 1$; then we have $\Phi \sim 1$, and

$$k_l(\sigma) \simeq \frac{\beta e Z_0}{2\pi^{3/2}\sigma} \ln\left(\frac{d}{b}\right). \tag{51}$$

Expressing $k_l(\sigma)$ in the usual form $k_l(\sigma) = k_0 \cdot \sigma^{-1}$, where σ is measured in centimeters, for typical values of $d/b \sim 1.5$ we obtain $k_0 \sim 1$ V/pC.

For very short bunches we may fall into the other regime $\sigma\gamma/b \ll 1$; the Fresnel integral is then approximated by $\Phi(x) \sim 2x/\sqrt{\pi}$, and we get

$$k_l(\sigma) \simeq \frac{2\beta e Z_0}{\pi^2 b} \gamma \ln\left(\frac{d}{b}\right). \tag{52}$$

It is worth nothing that in this regime the loss factor becomes proportional to the particle energy.

These two different regimes can be explained by simple physical considerations. As long as the bunch is far from the edge and moving within the pipe of radius b, we expect that the field configuration is that within an infinite pipe. When the bunch crosses the edge discontinuity, this field configuration is perturbed, since new boundary conditions must be restored, causing the radiation process. Thus we may easily infer that the radiation process lasts the time

$$\Delta t \simeq \frac{2}{\beta c}\left(\sigma + \frac{b}{\gamma}\right), \tag{53}$$

or, equivalently, the power spectrum extends over a frequency range of bandwidth

$$\Delta f \simeq \frac{1}{\Delta t} = \frac{\beta c}{2\left(\sigma + \frac{b}{\gamma}\right)}. \tag{54}$$

From the above formula we may recognize two regimes: if the bunch is longer than the "critical length" b/γ, then the spectrum width is independent of the particle energy; if the bunch is shorter than the critical length, it behaves like a point charge, and the radiation spectrum extends over a frequency range proportional to the relativistic factor γ.

The loss factor k_l, evaluated for a charge moving leftward, is almost zero for long, high-energy bunches, whereas it is negative for low-energy long bunches.

From the analysis of these results we may derive some interesting properties of the radiation process.

It is well known that ultrarelativistic charges moving within a perfectly conducting infinite pipe produce electromagnetic fields identical to those in the free space, except for being confined within a δ-disk of finite radius. When the charge crosses the discontinuity, it behaves differently, depending on the velocity direction. The charge leaving the inner tube and entering the larger one has to rebuild its self-field consistently with respect to the new boundary conditions. Thus, part of its kinetic energy is transformed into electromagnetic energy.

Fields created by the charge travelling in the opposite direction already satisfy the new boundary conditions and are simply swept out, so that the charge doesn't lose its kinetic energy.

At low energies, the charge may also experience an electromagnetic attracting force due to the induced charges on the discontinuous pipe edge. This effect is the cause of the negative impedance values of Fig. 5 and explains the negative values of the loss factor.

CONCLUSIONS

In this paper we describe an analytical approach to the study of the electromagnetic interaction between a bunch of particles and the discontinuities of the vacuum chamber.

An analytical exact solution for the single-discontinuity problem has been carried out using the Wiener-Hopf techniques. The analysis of the results obtained shows a strong dependence of the longitudinal impedance on the particle energy and motion direction.

The loss factor has been calculated for a Guassian bunch as a function of the rms bunch length.

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APPENDIX A

Hertz Potentials

The current density related to the moving charge $q = 1$ has the following frequency spectrum:

$$\bar{J}_q(r_0, z_0) = \hat{z}\frac{\ell^{i\alpha_0 z_0}}{2\pi}h(r_0), \tag{A1}$$

where z_0 is the charge position during the motion and $h(r_0)$ is its radial distribution. In the case of uniform distribution within a disk of radius a we have

$$h(r_0) = \frac{1}{\pi a^2}, \qquad r \leq a. \tag{A2}$$

The current density induced on the pipe walls at $r = b$, because of the symmetry, can be described by

$$\bar{J}_i(b, z_0) = \hat{z}\frac{\delta(r-b)}{2\pi b}f(z_0), \tag{A3}$$

where $f(z_0)$ is an unknown function.

The Hertz potential Π_z can be found by means of the Green function as

$$\Pi_z(r, z) = \frac{iZ_0}{4\pi k}\int_\tau \frac{\ell^{ikR}}{R}[J_q(r, z) + J_i(r, z)]\,d\tau, \tag{A4}$$

where τ is the volume where the sources are located $(d\tau = r_0\,dr_0\,dz_0\,d\theta_0)$, Z_0 is the vacuum impedance $\sim 120\,\pi\Omega$, and $R = \sqrt{(z-z_0)^2 + r^2 + b^2 - 2rb\cos\theta_0}$.

Substituting Eqs. (A1) and (A3) into Eq. (A4), and making use of the following identity,[10] we get

$$\int_0^{2\pi}\frac{\ell^{ikR}}{R}d\theta_0 = i\pi\int_{-\infty}^{\infty}\begin{bmatrix} J_0(\Omega r)H_0(\Omega r_0) \\ J_0(\Omega r_0)H_0(\Omega r)\end{bmatrix}\ell^{i\alpha(z-z_0)}\,d\alpha, \qquad \begin{matrix} r < r_0 \\ r > r_0{}' \end{matrix} \qquad \begin{matrix}\text{(A5a)}\\ \text{(A5b)}\end{matrix}$$

where $\Omega = \sqrt{k^2 - \alpha^2}$, we get

$$\Pi_{zq}(r, z) = -\frac{Z_0}{8\pi}\int_0^\infty h(r_0)r_0\,dr_0\int_{-\infty}^{\infty}\begin{bmatrix} J_0(\Omega r)H_0(\Omega r_0) \\ J_0(\Omega r_0)H_0(\Omega r)\end{bmatrix}\ell^{i\alpha z}\,d\alpha\int_{-\infty}^{\infty}\ell^{iz_0(\alpha_0-\alpha)}\,dz_0 \tag{A6}$$

$$\Pi_{zi}(r, z) = \frac{1}{2\pi b}\int_{-\infty}^{\infty}\delta(r_0 - b)r_0\,dr_0\int_{-\infty}^{\infty}f(z_0)\ell^{-i\alpha z_0}\,dz_0\int_{-\infty}^{\infty}\begin{bmatrix} J_0(\Omega r)H_0(\Omega b) \\ J_0(\Omega b)H_0(\Omega r)\end{bmatrix}\ell^{i\alpha z}\,d\alpha. \tag{A7}$$

Since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(z_0)\ell^{-i\alpha z_0}\,dz_0 = F(\alpha) \tag{A8}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \ell^{iz_0(\alpha_0-\alpha)}\,dz_0 = \delta(\alpha-\alpha_0), \tag{A9}$$

we get

$$\Pi_{zq}(r,z) = \frac{iZ_0}{2\pi k}[g_I(r)K_0(\xi r) + g_K(r)I_0(\xi r)] \tag{A10}$$

$$\Pi_{zi}(r,z) = -\frac{Z_0}{4k}\int_{-\infty}^{\infty} F(\alpha)\begin{bmatrix} J_0(\Omega r)H_0(\Omega b) \\ J_0(\Omega b)H_0(\Omega r)\end{bmatrix}\ell^{i\alpha z}\,d\alpha; \quad \begin{matrix} r<b \\ r>b \end{matrix}. \tag{A11}$$

The functions $g_i(r)$ and $g_k(r)$ are

$$g_I(r) = \int_0^r h(r_0)I_0(\xi r_0)r_0\,dr_0$$

$$g_K(r) = \int_r^a h(r_0)K_0(\xi r_0)r_0\,dr_0.$$

In the case of uniform distribution [Eq. (A2)] we obtain Eqs. (20) and (21). The above expressions give the Hertz potential due to the current densities J_q and J_i as they were in the free space.

In our problem there is the outer pipe at $r = d$, which modifies the potential expressions [Eqs. (A10) and (A11)]. We may include the effect of the outer pipe by assuming that on its walls will flow some currents such to cancel the potential on it. Since we are interested in the fields within the region $r < d$, these currents, flowing on the surface at $r = d$, produce in both Eq. (A10) and (A11) a further term behaving like Eq. (A5a),

$$\Pi_{zq}(r,z) = \frac{iZ_0}{2\pi k}[g_I(r)K_0(\xi r) + g_k(r)I_0(\xi r) + C_1 I_0(\xi r)]\ell^{i\alpha_0 z}. \tag{A12}$$

$$\Pi_{zi}(r,z) = -\frac{Z_0}{4k}\int_{-\infty}^{\infty} F(\alpha)\begin{bmatrix} J_0(\Omega r)H_0(\Omega b) + C_2 J_0(\Omega r)H_0(\Omega d) \\ J_0(\Omega b)H_0(\Omega r) + C_2 J_0(\Omega b)H_0(\Omega r)\end{bmatrix}\ell^{i\alpha z}\,d\alpha. \tag{A13}$$

By setting the potentials at $r = d$ to be zero, we easily get

$$C_1 = -g_I(a)\frac{K_0(\xi d)}{I_0(\xi d)}; \qquad C_2 = -\frac{J_0(\Omega b)}{J_0(\Omega d)}. \tag{A14}$$

APPENDIX B

Factorization of the Function $G(b, \alpha)$

In this appendix we give the Wiener–Hopf factorization of the function $\omega b G(b, \alpha)$; namely, we show that

$$\omega b G(b,\alpha) = G^+(\alpha)G^-(\alpha), \tag{B1}$$

where $G^+(\alpha)$ is holomorphic in the upper half-plane of a and has no zeroes there, whereas $G^-(\alpha)$ has the same properties in the lower half-plane.

Indeed, we shall consider the functions $\psi^\pm(\alpha)$, which have the same properties of the function $G^\pm(\alpha)$, to which they are related by the simple expression:

$$\psi^{\pm}(\alpha)=\frac{G^\pm(-\alpha)}{\sqrt{k\pm\alpha}}, \qquad \psi(\alpha)=\psi^+(\alpha)\psi^-(\alpha). \tag{B2}$$

The function $\psi(\alpha)$ and its components $\psi^\pm(\alpha)$ are asymptotically unity; thus the function $G^\pm(\alpha)$ asymptotically behaves as $\alpha^{1/2}$.

Assuming that k has a small positive imaginary part, the function $\psi(\alpha)$ is analytic and without zeroes within the strip $-\varepsilon<I_m(\alpha)<\varepsilon$, where $\varepsilon<I_mk$. For mathematical convenience we will instead factorize the logarithmic derivative of the function $\psi(\alpha)$, having the same properties within the strip of analyticity. The factorized functions may be represented by the Couchy integrals,[1]:

$$\frac{\psi'^+(\alpha)}{\psi^+(\alpha)}=-\frac{1}{2\pi i}\int_{\eta^+}\frac{d}{dt}[\ln\psi(t)]\frac{dt}{t-\alpha} \tag{B3}$$

$$\frac{\psi'^-(\alpha)}{\psi^-(\alpha)}=\frac{1}{2\pi i}\int_{\eta^-}\frac{d}{dt}[\ln\psi(t)]\frac{dt}{t-\alpha}, \tag{B4}$$

where

$$\begin{aligned}\frac{d}{dt}[\ln\psi(t)]=\frac{t}{b}&\left[-\frac{1}{\sigma}+\frac{bJ_1(\sigma b)}{J_0(\sigma b)}-\frac{dJ_1(\sigma d)}{J_0(\sigma d)}\right]\\&+\frac{tb}{\sigma}\left[\frac{Y_1(\sigma b)J_0(\sigma d)-Y_0(\sigma d)J_1(\sigma b)}{Y_0(\sigma b)J_0(\sigma d)-Y_0(\sigma d)J_0(\sigma b)}\right]\\&-\frac{td}{\sigma}\left[\frac{Y_1(\sigma d)J_0(\sigma b)-Y_0(\sigma b)J_1(\sigma d)}{Y_0(\sigma b)J_0(\sigma d)-Y_0(\sigma d)J_0(\sigma b)}\right],\end{aligned} \tag{B5}$$

and

$$\sigma=\sqrt{k^2-t^2}. \tag{B6}$$

Despite the formal complications, due to the presence of the multivalued functions σ and $Y_{0.1}(x)$, Eq. (B5) is single valued over the t-complex-plane. Furthermore, it vanishes for $|t|\to\infty$. Thus, we may close the integration path $\eta^\pm$ by adding a large circle of radius $k\to\infty$ without affecting the original values of the integrals [Eqs. (B3) and (B4)]. Equation (B4) is simply given by the sum of the residua at the simple pole singularities located at $\alpha=\pm k$ and at the zeroes α_n^b, α_n^d, and α_n^c of the Bessel functions $J_0(\sigma b)$, $J_0(\sigma d)$ and of the cross product $Y_0(\sigma b)J_0(\sigma d)-Y_0(\sigma d)J_0(\sigma b)$, respectively. The exact calculation at $\alpha=\alpha_0$, yields

$$\frac{\psi'^-(\alpha_0)}{\psi^-(\alpha_0)}=-\frac{1}{2(k-\alpha_0)}-\sum_1^\infty{}_n\left(\frac{1}{\alpha_n^b-\alpha_0}+\frac{1}{\alpha_n^c-\alpha_0}-\frac{1}{\alpha_n^d-\alpha_0}\right). \tag{B7}$$

We are able now to compute the derivative expression appearing in Eq. (41). Let's first express it in a more convenient form:

$$\frac{G(0,\alpha)}{G^+(\alpha)}=\frac{\sqrt{k-\alpha}\,\psi^-(\alpha)}{\omega bJ_0(\Omega b)}. \tag{B8}$$

Then deriving both the terms of the above equation with respect to the variable α we get

$$\left[\frac{\partial}{\partial\alpha}\frac{G(0,\,\alpha)}{G^{+}(\alpha)}\right]_{\alpha=\alpha_0}=\frac{1}{\omega b}\left[\frac{\psi'^{-}(\alpha)}{\psi^{-}(\alpha)}-\frac{1}{2(k-\alpha)}-\frac{\alpha b}{\Omega}\frac{J_1(\Omega b)}{J_0(\Omega b)}\right]_{\alpha=\alpha_0}, \qquad \text{(B9)}$$

and using Eq. (B8) we obtain

$$\left[\frac{\partial}{\partial\alpha}\frac{G(0,\,\alpha)}{G^{+}(\alpha)}\right]_{\alpha=\alpha_0}=\frac{1}{\omega b}\left[\frac{(1+\beta)(\beta\gamma)^2}{\beta k}-\gamma b\frac{I_1(\xi b)}{J_0(\xi b)}-\sum\right]. \qquad \text{(B10)}$$