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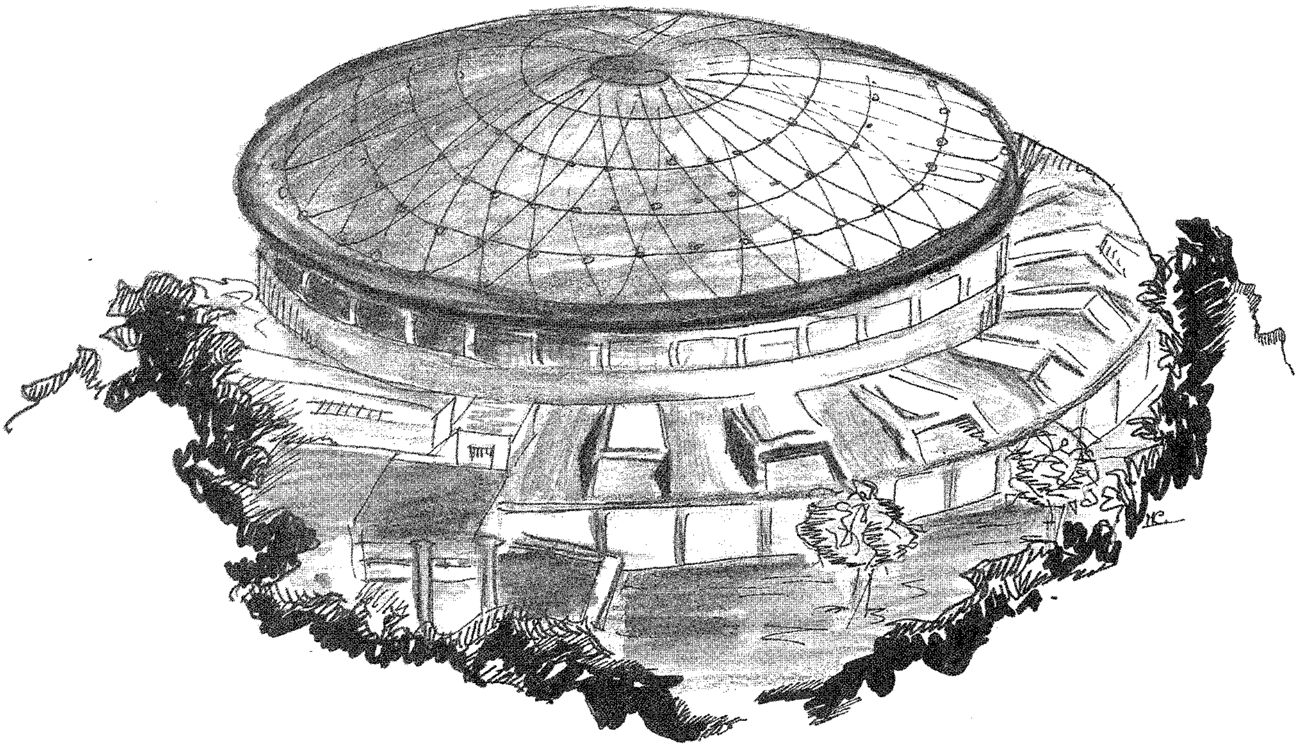

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EXACT EVALUATION OF THE FADDEEV-POPOV DETERMINANT IN A COMPLETE AXIAL GAUGE ON A TORUS

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ABSTRACT

We evaluate the Faddeev-Popov determinant in a complete axial gauge obtained by modifying the gauge $A_3 = 0$ on a torus. The Gribov horizon does not enclose a compact volume.

1. In his famous paper on the ambiguity of the gauge fixing Gribov[1] conjectured that in the Landau gauge the Faddeev-Popov determinant should vanish on closed surfaces in the space of gauge fields. We will call the surfaces where the Faddeev-Popov determinant vanishes Gribov surfaces irrespective of the fact that they are or are not closed. Gribov also conjectured that each gauge orbit should intersect each Gribov surface at least once, and that the integration on the gauge fields in the path integral could be confined to the interior of the smallest surface, called the Gribov horizon.

The problem was immediately reconsidered by Bender et al.[2] and by Peccei[3], and later by Zwanziger[4], Siopsis[5] and Dell'Antonio and Zwanziger [6].

The latter authors were able to determine the Gribov horizon approximately in the Landau gauge on a torus. They confirmed that in this gauge the Gribov horizon encloses a compact domain. They found that, as a consequence of the restriction to such a domain in the functional integration, the two-point function of the gauge field deviates in essential way from perturbation theory both in the infrared and in the ultraviolet.

If the Gribov horizon has to play an essential role in the quantization of gauge theories, it is interesting to investigate the possibilities which may occur with different gauges. We consider in this paper a modification of the gauge $A_3 = 0$ on a torus. The reason why we do not consider the gauge $A_3 = 0$ itself is that, as observed by Bassetto et al.(7), in the presence of periodic boundary conditions it is inconsistent with Gauss's constraint. This is related to the fact that the gauge $A_3 = 0$ is incomplete, i.e. it leaves the freedom of gauge transformations independent of x_3. It is possible, however, by a natural modification to obtain a complete gauge(8) which is compatible with both periodic boundary conditions and Gauss's constraint. The price to be paid for is that it is nonlocal, but nonlocality is implied by a restriction in functional integration anyway.

Another feature of the gauge $A_3 = 0$ which is related to its incompleteness is the existence of the $\frac{1}{k_3}$ pole(9). Now giving a prescription for this pole ammounts to effectively alter the gauge. A prescription cannot therefore be given without appropriately modifying the Faddeev-Popov determinant. Completing the gauge gives at the same time the prescription (which in the present case turns out to be a principal value) and the necessary modification of the Faddeev-Popov determinant.

We use the canonical quantization in a formal way, hoping to provide a basis for a rigorous analysis. We formulate the path integral using the Hamiltonian which has already been constructed(8) in the complete gauge. We will evaluate exactly the Faddeev-Popov determinant and we will determine the Gribov surfaces. They divide the space of gauge fields into domains which are, unlike the Landau gauge, non compact. In the canonical formalism the restriction to one of such domains in the integration in the path integral comes out quite naturally. On Gribov surfaces the energy density is singular. If the singularity is strong enough the functional Schrödinger equation must be solved independently in each domain with vanishing boundary conditions and in its time evolution the gauge field is confined to the domain where it is localized initially. We will show that in the modified $A_3 = 0$ gauge the singularities are indeed strong enough.

The restriction of the gauge field to one of such domains solves a problem which was raised in ref. (8), where it shown (in a formal way) that the Hamiltonian of the gauge field coupled to quarks is unbounded from below. The proof was based on the assumption that the amplitude of zero momentum modes could be made arbitrarily large with respect to the components of the gauge field with nonvanishing momentum.

We will see that this assumption is in contrast with the above restriction on the gauge field.

2. Let us briefly review the problem of the consistency of the Gauss constraints with gauge fixing and periodic boundary conditions.

The Gauss constraints are

$$G^a = \mathcal{D}^a_{kb} E^b_k = 0, \tag{1}$$

where $\mathcal{D}^a_{kb}$ are the covariant derivatives in the adjoint representation

$$\mathcal{D}^a_{kb} = \partial_k \delta^a_b + g f^a_{bc} A^c_k, \tag{2}$$

f^a_{bc} being the structure constants and E^b_k the electric strength components. We adopt the convention of summation over repeated indices, although in some cases the sum will be explicitely indicated, for instance in order to establish the range of the index summed over.

Let us now see why the gauge fixing

$$A^a_3 = 0, \tag{3}$$

is inconsistent with periodic boundary conditions.

In the presence of such a gauge fixing the electric strength E^a_3 is not a dynamical variable but should be determined by the Gauss equations

$$\partial_3 E^a_3 + \mathcal{D}^a_{2b} E^b_2 + \mathcal{D}^a_{1b} E^b_1 = 0. \tag{4}$$

Due to periodic boundary conditions on E^a_3, integrating on x_3 over the periodic box edge of length L we get

$$\int_0^L dx_3 \left[\mathcal{D}^a_{2b} E^b_2 + \mathcal{D}^a_{1b} E^b_1\right] = 0. \tag{5}$$

The last equation appears as a new constraint on the dynamical variables A^a_1, A^a_2, E^a_1, E^a_2, which should instead be unconstrained, because the primary equations (1) have all been used to eliminate the variables E^a_3.

A limitation on A^a_1, A^a_2, E^a_1, E^a_2 can only exist to compensate for a residual freedom in A^a_3 and E^a_3. Actually Eqs. (4) leave completely undetermined an electric field E^a_3 depending only on x_1, x_2. The gauge fixing should therefore not constrain a gauge field A^a_3 depending only on x_1, x_2.

In order to reformulate the gauge fixing taking into account the above observation we need some notations.

For any function f on the torus

$$f = \frac{1}{L^{3/2}} \sum_{\vec{n}} f_{n_1, n_2, n_3} \; e^{i \frac{2\pi}{L} \vec{n} \cdot \vec{x}} \tag{6}$$

we define the components

$$f_3 = \frac{1}{L^{3/2}} \sum_{\vec{n}} f_{n_1, n_2, n_3 \neq 0} \; e^{i \frac{2\pi}{L} \vec{n} \cdot \vec{x}}$$

$$f_2 = \frac{1}{L^{3/2}} \sum_{\vec{n}} f_{n_1, n_2 \neq 0, n_3 = 0} \; e^{i \frac{2\pi}{L} \vec{n} \cdot \vec{x}}$$

$$f_1 = \frac{1}{L^{3/2}} \sum_{\vec{n}} f_{n_1 \neq 0, n_2 = n_3 = 0} \; e^{i \frac{2\pi}{L} \vec{n} \cdot \vec{x}}$$

$$f_0 = \frac{1}{L^{3/2}} f_{000}. \tag{7}$$

We have

$$f = \sum_{h=0}^{3} f_h. \tag{8}$$

The notation just introduced can obviously be applied to vectors and tensors as well. So for instance $A_k^a = \sum_{h=0}^{3} A_{kh}^a$.

We will now show the consistency of the gauge fixing[8]

$$A_{kk}^a = 0, \quad k = 1, 2, 3. \tag{9}$$

We will not fix the gauge for the zero-momentum modes A_{k0}^a.

The dynamical variables are A_{kh}^a and E_{kh}^a, $k \neq h$, and we must show that the E_{kk}^a are determined by the Gauss constraints. To this end it is convenient to separate the Gauss equations into their components according to the notation just introduced

$$G_k^a = 0, \qquad k = 0, \ldots 3. \tag{10}$$

Rearranging the terms which appear in this equation we have for k > 0

$$\mathcal{D}^a_{kb}\,(\mathcal{A}_k)\,E^b_{kk} + R^a_k = 0, \qquad k = 1, 2, 3, \tag{11}$$

where

$$\mathcal{A}^a_k = \sum_{h=0}^{k-1} A^a_{kh}, \tag{12}$$

$$R^a_k = \sum_{h=1}^{k-1} \left(\partial_h\, E^a_{hk} + g\, f^a_{bc}\, A^c_{hk}\, E^b_{hh}\right) + g\, f^a_{bc} \sum_{h=1}^{3} \sum_{h_1,\, h_2 \neq h =0}^{3} \left(A^c_{hh_1}\, E^b_{hh_2}\right)_k, \tag{13}$$

and $(A^c_{hh_1}\ E^b_{hh_2})_k$ is the k-th component of the product $A^c_{hh_1}\, E^b_{hh_2}$ according to the notation of Eqs. (7). It should be noted that R_k does not depend on $\mathcal{A}_k$ and, as already emphasized in ref. (8), $\mathcal{A}^a_k$ depends only on x_h, $h < k$. It is this property which will enable us to evaluate exactly the Faddeev-Popov determinant.

For k = 0 we have

$$G^a_0 = g\, f^a_{bc}\, A^c_{h0}\, E^b_{h0} + R^a_0 = 0, \tag{14}$$

with

$$R^a_0 = g\, f^a_{bc} \sum_{h=1}^{3} \sum_{h_1 \neq h =1}^{3} \left(A^c_{hh_1}\ E^b_{hh_1}\right)_0. \tag{15}$$

R^a_0 contains only unconstrained variables. Eqs. (14) constrain, therefore, the zero momentum modes.

Let us now consider Eqs. (11) for k = 1

$$\partial_1\, E^a_{11} + g\, f^a_{bc}\, \mathcal{A}^c_1\, E^b_{11} + R^a_1 = 0. \tag{16}$$

Integration on x_1 over the periodic box edge does not yield any new constraint because $\mathcal{A}_1$ does not depend on x_1 and $\int_0^L dx_1\, E_{11} = \int_0^L dx_1\, R_1 = 0$. Moreover R_1 contains only unconstrained variables. Eqs. (16) can therefore be solved for E^a_{11}

$$E^a_{11} = -\left[\mathcal{D}^{-1}_1\,(\mathcal{A}_1)\right]^a_b\, R^b_1, \tag{17}$$

whenever $\mathcal{D}_1(\mathcal{A}_1)$ does not have a null eigenvalue. We will come back to this essential point in a moment.

Let us now consider Eqs. (11) for k = 2. R_2 contains only unconstrained variables and E_{11} which has already been determined. Eqs. (11) for k = 2 can therefore be solved for E_{22} and analogously we get E_{33} for k = 3.

In general

$$E^a_{kk} = -\left[\mathcal{D}_k^{-1}(\mathcal{A}_k)\right]^a_b R^b_k. \tag{18}$$

The Hamiltonian density is

$$\mathcal{H} = \sum_a \sum_{k=1}^{3} \left\{ \frac{1}{2}\left[\left(\mathcal{D}_k^{-1}(\mathcal{A}_k)\, R_k\right)^a\right]^2 + \sum_{\substack{h\neq k \\ 0}}^{3} \frac{1}{2}\left(E^{a\,2}_{kh}\right)\right\} + \frac{1}{2}\sum_{i<j=1}^{3}\left(F^a_{ij}\right)^2, \tag{19}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^a_{bc} A^b_\mu A^c_\nu. \tag{20}$$

In order to study the singularities in Eqs. (18) it is convenient to expand R^a_k and E^a_{kk} with respect to the eigenfunctions of the Faddeev-Popov operators

$$\left(\delta^a_b\, \partial_k + g\, f^a_{bc}\, \mathcal{A}^c_k\right) u^b_{ns} = d_{ns}\, u^a_{ns}. \tag{21}$$

Writing the eigenvalue equation we have anticipated that there are two quantum numbers n, s. In fact, since $\mathcal{A}^a_k$ does not depend on x_k, $\delta^a_b\, \partial_k$ commutes with $g\, f^a_{bc}\, \mathcal{A}^c_k$, the u_{ns}'s are factorized and the d_{ns}'s are the sum of the corresponding eigenvalues

$$u^a_{ns} = L^{-1/2}\, e^{i\frac{2\pi}{L} n\, x_k}\, w^a_s$$

$$d_{ns} = i\left(\frac{2\pi}{L} n + g\, |\, \mathcal{A}_k\, |\, \varphi_s\right), \qquad n \neq 0 \tag{22}$$

where

$$g f^a_{bc} \mathcal{A}^c_k w^b_s = i g \mid \mathcal{A}_k \mid \varphi_s w^a_s , \quad \mid \mathcal{A}_k \mid = \left[\sum_a \left(\mathcal{A}^a_k \right)^2 \right]^{1/2} . \tag{23}$$

The restriction $n \neq 0$ comes from the fact that in Eq. (11) $\mathcal{D}_k (\mathcal{A}_k)$ acts on the space of functions which have no zero momentum modes with respect to x_k.

The wanted expansions are

$$R^a_k = \sum_{n \neq 0} \sum_s R_k (n,s) u^a_{ns} \tag{24}$$

$$E^a_{kk} = \sum_{n \neq 0} \sum_s E_{kk} (n,s) u^a_{ns}. \tag{25}$$

The amplitudes E_k (n,s) are obtained by Eqs. (18)

$$E^a_{kk} (n,s) = i \left(\frac{2\pi}{L} n + g \mid \mathcal{A}_k \mid \varphi_s \right)^{-1} R_k (n,s), \tag{26}$$

so that finally

$$\sum_{k}^{3} {}_1 \int dx^3 \sum_a \left(E^a_{kk} \right)^2 = \sum_{k}^{3} {}_1 \int \prod_{h \neq k}^{3} {}_1 \, dx_h \left(\frac{2\pi}{L} n + g \mid \mathcal{A}_k \mid \varphi_s \right)^{-2}$$

$$\cdot \mid R_k (n,s) \mid^2. \tag{27}$$

The Gribov surfaces are

$$\frac{2\pi}{L} n + g \mid \mathcal{A}_k \mid \varphi_s = 0. \tag{28}$$

On these surfaces the energy is infinite so that the fields belonging to them are dynamically avoided. If we euristically consider the functional Schrödinger equation as an ordinary Schrödinger equation we find that in the vicinity of a Gribov surface the state functional behaves like $\left[\frac{2\pi}{L} n + g \mid A_k \mid \varphi_s \right]^\beta$, $\beta > 1$. This behaviour is sufficient to make the solution to the left and to the right of a Gribov surface independent from each other.

In a domain bounded by two Gribov surfaces the gauge field components $\mathcal{A}_k$ are bounded. Since $\mathcal{A}_k = \sum_{h=0}^{k-1} A_{kh}$, the zero momentum components A_{k0} cannot be made arbitrarily large with respect to all the components with nonvanishing momentum. This solves the problem raised in ref. (8).

3. Having shown how to fix the gauge consistently with periodic boundary conditions we formulate the path integral

$$W = \int \prod_{xt} \prod_{a} \prod_{k=1}^{3} \prod_{h \neq k=0}^{3} d A^a_{kh}(x,t)\, d E^a_{kh}(x,t)\, \delta\left[g\, f^a_{bc}\, A^c_{io}\, E^b_{io} - R^a_o\right]$$

$$\exp i \int dx^3\, dt \sum_a \sum_{i=1}^{3} \sum_{j \neq i=0}^{3} E^a_{ij}\, \mathring{A}^a_{ij} - \mathcal{H} = \int \prod_{xt} \prod_{a} \prod_{k=1}^{3} d A^a_k(x,t)\, d E^a_k(x,t)$$

$$\delta\left(A^a_{kk}\right) \delta\left[g\, f^a_{bc}\, A^c_{io}\, E^b_{io} - R^a_o\right] \delta\left[E^a_{kk} + \left(\mathcal{D}_k^{-1}(\mathcal{A}_k)\, R_k\right)^a\right]$$

$$\exp i \int dx^3\, dt \left\{\sum_{k=1}^{3} E^a_k\, \mathring{A}^a_k - \frac{1}{2} E^a_k\, E^a_k - \frac{1}{2} \sum_{i<j=1}^{3} \left(F^a_{ij}\right)^2\right\}. \tag{29}$$

Now

$$\prod_{xt} \prod_{a} \prod_{k=1}^{3} \delta\left\{E^a_{kk} + \left[\mathcal{D}_k^{-1}(\mathcal{A}_k)\, R_k\right]^a\right\} \propto \int \prod_{xt} \prod_{k=1}^{3} \prod_{a}$$

$$\det \mathcal{D}_k\left(\mathcal{A}_k\right) d A^a_{ok}(x,t) \exp i \int dx^3\, dt \sum_a \sum_{h=1}^{3} \left(\mathcal{D}_h\, E_{hh} + R_h\right)^a A^a_{oh},$$

$$\prod_{t} \prod_{a} \delta\left[g\, f^a_{bc}\, A^c_{io}\, E^b_{io} + R^a_o\right] \propto \int \prod_{t} \prod_{a} d A^a_{oo} \exp i \int d^3_x\, dt \,\cdot$$

$$\cdot \left(g\, f^a_{bc}\, A^c_{io}\, E^b_{io} + R^a_o\right) A^a_{oo}, \tag{30}$$

so that

$$W \propto \int \prod_{xt} \prod_{a} \prod_{\mu=0}^{3} d A^a_\mu(x,t) \prod_{k=1}^{3} \delta\left(A^a_{kk}\right)$$

$$\det \mathcal{D}_k(\mathcal{A}_k) \exp i \int dx^3\, dt \frac{1}{4} \sum_a \sum_{\mu,\nu=0}^{3} \left(F^a_{\mu\nu}\right)^2. \tag{31}$$

The evaluation of the Faddeev-Popov determinants is straightforward

$$\det \mathcal{D}_k(\mathcal{A}_k) = \prod_{n>0}\prod_s d_{ns}\, d_{-ns} = \prod_{n>0}\prod_s \left(\frac{2\pi}{L}n\right)^2 \left[1 - \frac{1}{n^2}\left(\frac{Lg}{2\pi}\right)^2 |\mathcal{A}_k|^2 \varphi_s^2\right]$$

$$\propto \prod_{n>0}\prod_s \left[1 - \frac{1}{n^2}\left(\frac{Lg}{2\pi}\right)^2 |\mathcal{A}_k|^2 \varphi_s^2\right] = j_0\left[\frac{1}{2}L\, |\, g\, \mathcal{A}_k\, \varphi_s\, |\right], \tag{32}$$

where $j_0(x)$ is the Bessel function $\dfrac{\sin x}{x}$.

The Gribov surfaces, Eq. (28), do not enclose a compact volume. The complete axial gauge considered in this paper cannot therefore reproduce the results of Dell'Antonio and Zwanziger[6].

Finally we want to write down the propagator in momentum space. To this end it is convenient to define the four vectors

$$K_\mu^{(h)} : K_\mu^{(h)} = 0\,,\quad \mu > h\,,\quad K_h^{(h)} \neq 0,\quad h = 1, 2, 3. \tag{33}$$

The linearized equations of motion for the gauge field in momentum space are

$$\left[-(K^{(h)})^2\, g_{\mu\nu} + K_\mu^{(h)}\, K_\nu^{(h)}\right](1 - \delta_{\nu h})\, A_\nu(K^{(h)}) = j_\mu(K^{(h)}), \tag{34}$$

where the factor $(1 - \delta_{\nu h})$ accounts for the gauge conditions $A_{hh} = 0$.

The propagator is

$$G_{\mu\nu} = -\frac{1}{(K^{(h)})^2}\left[g_{\mu\nu} - \frac{1}{K_h^{(h)}} K_\mu^{(h)}\, g_{h\nu}\right],\quad \text{no summation over h.} \tag{35}$$

Notice that for $\mu = h$ $G_{\mu\nu} = 0$, in agreement with the gauge condition $A_{hh} = 0$.

Since $K_\mu = \dfrac{2\pi}{L} n_\mu$, in the limit $L \to \infty$ we get the principal value prescription for the poles. Such a prescription is accompanied by the prescription that the Faddeev-Popov operators $\mathcal{D}_k(\mathcal{A}_k)$ shuld contain $\mathcal{A}_k$ rather than A_k. This is a major difference which allowed us to evaluate exactly the Faddeev-Popov determinants.

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(8) F. Palumbo, Phys. Lett. 173B, 81 (1986); Nuovo Cim. 93A, 360 (1986). Unfortunately there are many misprints in this paper, especially in Eqs. (18) to (22). Eqs. (41) should be replaced by:

$$\bar{A}_i \rightarrow \frac{1}{\sqrt{\lambda}} \bar{A}_i \,, \quad \bar{E}_i \rightarrow \sqrt{\lambda}\, \bar{E}_i \,, \quad i = 1,2$$

$$\bar{A}_3 \rightarrow \sqrt{\lambda}\, \bar{A}_3 \,, \quad \bar{E}_3 \rightarrow \frac{1}{\sqrt{\lambda}} \bar{E}_3 \,.$$

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